U. P. Rajarshi Tandon Open University, Prayagraj

Master of Science/

Master of Arts

PGMM-104N/MAMM-104N
Numerical Analysis

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PGMM –104N: NUMERICAL ANALYSIS

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Master of Science
PGMM -104N
Numerical Analysis

**U. P. Rajarshi Tandon
Open University**

Block: 1 Calculus of Finite Differences

Unit-1: Finite Differences

Unit-2: Application of Finite Differences

Block: 2 Interpolation

Unit-3: Newton's Interpolation formula with Equal Intervals

Unit-4: Gauss' and Stirling Interpolation formula with Equal Intervals

Unit-5: Lagrange's Interpolation Formula for Unequal Intervals

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Unit-13: Numerical Solution of Ordinary Differential Equations-I

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Syllabus

PGMM-104N/MAMM-104N:

Numerical Analysis

Block-1: Calculus of Finite Differences

Unit-1: Finite Differences
Introduction, finite differences, forward differences, backward differences, central differences, shift operator, relations between the relations.

Unit-2: Application of Finite Differences
Fundamental theorem of the difference calculus, factorial function, properties of factorial function.

Block-2: Interpolation

Unit-3: Newton's Interpolation formula with Equal Intervals
Introduction, to find one missing terms, to find two missing terms, Newton's forward and backward interpolation with equal intervals.

Unit-4: Gauss' and Stirling Interpolation formula with Equal Intervals
Introduction, Gauss's forward and backward interpolation with equal intervals, Stirling's difference formula.

Unit-5: Lagrange's Interpolation Formula for Unequal Intervals
Introduction, Lagrange's interpolation with unequal intervals.

Block-3: Solution of Linear Simultaneous Equations

Unit-6: Solution of Linear Simultaneous equations-I
Introduction, Linear equations, Gauss elimination method, Gauss-Seidel method.

Unit-7: Solution of Linear Simultaneous equations-II
Introduction, LU Decomposition method or triangular method, Crout's method and Choleski's method.

Block-4: Solving Algebraic and Transcendental Equations

Unit-8: Numerical Method for solving Algebraic and Transcendental Equations-I

Introduction, Polynomial, algebraic and transcendental equations, Bisection method and Newton Raphson Method.

Unit-9: Numerical Method for solving Algebraic and Transcendental Equations-II

Introduction, Regula-Falsi method and Secant method.

Block-5: Numerical Differentiation and Integration

Unit-10: Numerical Differentiation-I

Introduction, derivatives using forward difference formula, derivatives using backward difference formula.

Unit-11: Numerical Differentiation-II

Introduction, derivatives using Stirling difference formula, derivatives using Newton's Divided difference formula.

Unit-12: Numerical Integration

Introduction, general quadrature formula for equally spaced arguments, Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule.

Unit-13: Numerical Solution of Ordinary Differential Equations-I

Introduction, Euler's method, Euler's modified method, Taylor Series method.

Unit-14: Numerical Solution of Ordinary Differential Equations-II

Introduction, Picard's method. Runge-Kutta method for fourth order, Milne's predictor-corrector method.

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Block

1 Calculus of Finite Differences

Block-1

Calculus of Finite Differences

Numerical analysis holds significant importance in various domains such as Engineering, Science, and Technology. It involves obtaining results in numerical form through computational methods applied to given data. The foundation of numerical analysis lies in the calculus of finite differences, a branch that addresses alterations in the dependent variable resulting from changes in the independent variable. Finite differences can be computed in both forward and backward directions, depending on whether values ahead or behind are used in the calculations. Finite differences have various applications in numerical analysis, providing a versatile tool for approximating derivatives, solving differential equations, and interpolating functions.

Finite differences are frequently used to approximate derivatives of a function. By expressing derivatives as finite difference quotients, such as the forward difference or central difference, numerical approximations can be obtained. This is particularly useful when dealing with functions for which analytical derivatives are challenging to compute. Finite differences play a crucial role in interpolation methods. Newton's divided difference interpolation formula relies on finite differences to construct polynomial interpolants. This technique is used to estimate values between known data points. Finite difference methods are employed to numerically solve differential equations. Discretizing the differential equation using finite differences transforms the problem into a system of algebraic equations, which can be solved using numerical techniques like the Euler method, the Runge-Kutta method, or finite difference schemes for partial differential equations. Finite differences are used in numerical methods for root finding, such as Newton's method. The finite difference quotient helps approximate the derivative in the iterative process of finding roots of equations.

In the first unit, we shall discussed the Finite differences, operators and relations between the operators. A finite difference table is a systematic way of organizing finite differences at different orders. It helps identify patterns and relationships in the data. In second unit we shall discuss the fundamental theorem difference calculus, factorial function, properties of factorial function.

UNIT-1: Finite Differences

Structure

1.1 Introduction

1.2 Objectives

1.3 Finite Differences

1.4 Forward Differences

1.5 Backward Differences

1.6 Central Differences

1.7 Shift Operators E

1.8 Relations between the Operators

1.9 Summary

1.10 Terminal Questions

1.1 Introduction

Numerical analysis plays a crucial role in Engineering, Science, and Technology by providing numerical results through computational methods applied to given datasets. At its core, numerical analysis relies on the principles of the calculus of finite differences, which explores how changes in the independent variable lead to corresponding changes in the dependent variable. The Calculus of Finite Differences constitutes a mathematical discipline concerned with discrete quantities and the distinctions between successive values. Finite differences are used in optimization algorithms, where gradients or partial derivatives are approximated numerically to find extrema of functions.

Finite differences provide a powerful and intuitive approach to solving various numerical problems, making them a fundamental tool in the field of numerical analysis. In contrast to classical calculus, which revolves around continuous functions and limits, the calculus of finite differences directs its attention to the discrete characteristics of data or functions defined at specific points. In the present unit we shall discuss about the finite differences, forward differences, backward differences, central differences, shift operators and relations between operators.

A finite difference table is a systematic way of organizing finite differences at different orders. It helps identify patterns and relationships in the data. Finite differences can be computed in both forward and backward directions, depending on whether values ahead or behind are used in the calculations. Finite differences find applications in diverse fields, including numerical analysis, computer science, physics, engineering, and discrete mathematics. Finite differences are employed in interpolation to estimate values between known points and extrapolation to predict values beyond the given data points. Finite differences are used in error analysis of numerical methods. Understanding the behavior of finite differences helps assess the accuracy and convergence of numerical algorithms.

1.2 Objectives

After reading this unit the learner should be able to understand about the:

- Finite Differences
- Forward differences
- backward differences
- Central Differences
- Shift Operators E
- Relations between the operators

1.3 Finite Differences

Finite Differences refers to a mathematical concept that involves the computation of discrete changes or differences between consecutive values of a function or sequence. Unlike traditional calculus, which deals with continuous and infinitesimal changes, finite differences focus on the specific, discrete variations in values at distinct points. Finite Differences is a numerical analysis technique used to approximate derivatives, integrals, and other mathematical operations.

This method is particularly useful when dealing with functions or equations for which analytical solutions are difficult to obtain. The basic idea behind finite differences is to approximate the derivative of a function by considering the differences in function values at discrete points.

The concept of finite differences is particularly useful when dealing with discrete data sets or functions defined at specific points, providing a practical and numerical approach to understanding changes in values.

Let $y = f(x)$ be a function of x, the value of independent variable x ($x_0, x_1, \ldots\ldots\ldots, x_n$) are called arguments and corresponding values of dependent variable y ($y_0, y_1, \ldots\ldots\ldots\ldots, y_n$) are called entries. To find the values of y and $\frac{dy}{dx}$, for some intermediate value of x, is based on principle of finite difference.

1.4 Forward Differences

Consider the function

$$y = f(x) \qquad \text{….. (1)}$$

where the function is not known only the data set are given.

The first forward differences is defined by

$$\Delta y_0 = y_1 - y_0, \text{ where } h = 1 \text{ is interval difference.}$$

The differences $y_1 - y_0,\ y_2 - y_1, \ldots\ldots\ldots\ldots\ldots\ldots, y_n - y_{n-1}$ are known as the first forward differences of equation (1) and denoted by $\Delta y_0, \Delta y_1, \ldots\ldots\ldots\ldots, \Delta y_{n-1}$, respectively, where Δ is known as the forward difference operator.

Generally, the first forward differences is defined by

$$\Delta y_x = y_{x+h} - y_x, \text{ where } h \text{ is interval difference.}$$

or

$$\Delta f(x) = f(x+h) - f(x), \text{ where } h \text{ is interval difference.}$$

The differences of the first forward differences are known as the second forward differences and denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots\ldots$ etc.

Therefore, we have

$$\begin{aligned}\Delta^2 y_0 &= \Delta[\Delta y_0] \\ &= \Delta[y_1 - y_0], \text{ where } h = 1 \text{ is interval difference.} \\ &= \Delta y_1 - \Delta y_0 \\ &= (y_2 - y_1) - (y_1 - y_0)\end{aligned}$$

$$= y_2 - 2y_1 + y_0 \qquad \ldots(2)$$

Similarly, we have

$$\Delta^2 y_1 = \Delta[y_2 - y_1]$$

$$= \Delta y_2 - \Delta y_1$$

$$= (y_3 - y_2) - (y_2 - y_1)$$

$$= y_3 - 2y_2 + y_1 \qquad \ldots(3)$$

Generally, we have

$\Delta^2 y_x = \Delta y_{x+1} - \Delta y_x$, where h = 1 is interval difference.

Forward Difference Table:

Argument x	**Entry** $y = f(x)$	**First Differences** Δy	**Second Differences** $\Delta^2 y$	**Third Differences** $\Delta^3 y$	**Fourth Differences** $\Delta^4 y$
x_0	y_0	$y_1 - y_0 = \Delta y_0$			
$x_0 + h$	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta y_1 - \Delta y_0 = \Delta^2 y_0$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	
$x_0 + 2h$	y_2	$y_3 - y_2 = \Delta y_2$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$	$\Delta^3 y_1 - \Delta^3 y_0 = \Delta^4 y_0$
$x_0 + 3h$	y_3	$y_4 - y_3 = \Delta y_3$	$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$		
$x_0 + 4h$	y_4				

Again, the differences of second forward differences are known as third forward differences and denoted by $\Delta^3 y_0$, $\Delta^3 y_1$ etc.

Thus we have $$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) \quad \text{[from equation (2) and (3)]}$$

$$= y_3 - 3y_2 + 3y_1 - y_0 \text{ and so on.}$$

In general, the *n*th forward difference is given by

$$\Delta^n y_x = \Delta^{n-1} y_{x+h} - \Delta^{n-1} y_x,$$ where h is interval difference.

1.5 Backward Differences

Consider the function

$$y = f(x) \qquad \text{….. (1)}$$

where the function is not known only the data set are given.

The differences $y_1 - y_0$, $y_2 - y_1$, ………………, $y_n - y_{n-1}$ are known as the first backward differences of equation (1) and denoted by $\nabla y_1, \nabla y_2, \ldots\ldots\ldots\ldots\ldots, \nabla y_n$, respectively, where ∇ is known as the backward differences operator.

Generally, the first backward difference is defined by

$$\nabla y_x = y_x - y_{x-1},$$ where $h = 1$ is interval difference.

or $$\nabla f(x) = f(x) - f(x-h),$$ where h is interval difference.

The differences of the first backward differences are known as second backward differences and denoted by $\nabla^2 y_2, \nabla^2 y_3, \ldots..$ etc.

Therefore we have

$$\nabla^2 y_2 = \nabla(\nabla y_2)$$

$= \nabla(y_2 - y_1)$, where $h = 1$ is interval difference.

$$= \nabla y_2 - \nabla y_1$$

$$= (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

Generally, we have

$\nabla^2 y_x = \nabla y_x - \nabla y_{x-1}$, where $h = 1$ is interval difference.

Backward Differences Table:

Argument x	**Entry** $y = f(x)$	**First Differences** ∇ **y**	**Second Differences** $\nabla^2 y$	**Third Differences** $\nabla^3 y$	**Fourth Differences** $\nabla^4 y$
x_0	y_0				
$x_0 + h$	y_1	$y_1 - y_0 = \nabla y_1$			
			$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$		
$x_0 + 2h$	y_2	$y_2 - y_1 = \nabla y_2$		$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	
			$\nabla y_3 - \nabla y_2 = \nabla^2 y_3$		$\nabla^3 y_4 - \nabla^3 y_3 = \nabla^4 y_4$
$x_0 + 3h$	y_3	$y_3 - y_2 = \nabla y_3$		$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
			$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
$x_0 + 4h$	y_4	$y_4 - y_3 = \nabla y_4$			

Again the differences of second backward differences are known as third backward differences and denoted by $\nabla^3 y_3, \nabla^3 y_4$, …… etc.

Thus we have

$$\nabla^3 y_x = \nabla^2 y_x - \nabla^2 y_{x-1}$$

In general, the n^{th} backward differences is given by

$$\nabla^n y_x = \nabla^{n-1} y_x - \nabla^{n-1} y_{x-1}$$

1.6 Central Differences

The differences $y_1 - y_0 = \delta\, y_{1/2}$, $y_2 - y_1 = \delta\, y_{3/2}$, ...,$y_n - y_{n-1} = \delta\, y_{n-1/2}$ are known as central differences and δ is known as central differences operator.

Similarly, we have

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2$$

$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2}$ and so on.

Central Difference Table:

Argument x	**Entry** $y = f(x)$	**First Differences** δy	**Second Differences** $\delta^2 y$	**Third Differences** $\delta^3 y$	**Fourth Differences** $\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^2 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

In generally, the central differences is given by

$$\delta y_x = y_{x+h/2} - y_{x-h/2}, \text{ where } h \text{ is interval difference.}$$

1.7 Shift Operator E

The shift (increment) operator E is defined as

$$Ey_x = y_{x+h}, \text{ where } h \text{ is the interval difference.}$$

$$E^2 y_x = y_{x+2h}$$

$$\vdots \qquad \vdots$$

$$E^n y_x = y_{x+nh}$$

Also the inverse operator E^{-1} is defined as

$$E^{-1} y_x = y_{x+(-h)}$$

$$= y_{x-h}, \text{ where } h \text{ is the interval difference.}$$

Check your Progress

1. What do you mean by Finite Differences?

2. Explain the forward and backward differences.

3. Define the central differences

4. What is shift operator?

1.8 Relations between the Operators

There are several relations between the operators. Some of the important relations are following:

(i) To show that $\Delta = E - 1$ or $E = \Delta + 1$.

We know that $$\Delta y_x = y_{x+h} - y_x$$
$$= Ey_x - y_x$$

$$\Delta y_x = (E-1)y_x$$

Thus we have

$$\Delta = E - 1$$

or $$E = \Delta + 1.$$

(ii) To show that $\nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$.

We know that $$\nabla y_x = y_x - y_{x-h}$$
$$= y_x - E^{-1} y_x$$
$$= (1 - E^{-1}) y_x$$

Thus we have

$$\nabla = 1 - E^{-1}$$

or $$E^{-1} = 1 - \nabla.$$

(iii) To show that $E\nabla = \nabla E = \Delta$.

We have $$E\nabla y_x = E(y_x - y_{x-h})$$
$$= Ey_x - Ey_{x-h}$$
$$= y_{x+h} - y_x$$

$$E\nabla y_x = \Delta y_x$$

$$E\nabla = \Delta \qquad \text{..... (1)}$$

Again we have $\nabla E y_x = \nabla y_{x+h}$

$$= y_{x+h} - y_x$$

$$\nabla E y_x = \Delta y_x$$

$$\nabla E = \Delta \qquad \text{….. (2)}$$

Using equations (1) and (2), we have

$$E\nabla = \nabla E = \Delta$$

(iv) To show that $\delta = E^{1/2} - E^{-1/2}$.

We know that $\delta y_x = y_{x+h/2} - y_{x-h/2}$

$$= E^{1/2} y_x - E^{-1/2} y_x$$

$$\delta y_x = (E^{1/2} - E^{-1/2}) y_x$$

Thus we have

$$\delta = E^{1/2} - E^{-1/2}.$$

(v) To show that $\Delta = \delta E^{1/2}$.

We know that $\Delta y_x = y_{x+h} - y_x$

$$= E y_x - y_x$$

$$= (E-1) y_x$$

$$= (E^{1/2} - E^{-1/2}) E^{1/2} y_x$$

$$\Delta y_x = \delta E^{1/2} y_x$$

Thus we have

$$\Delta = \delta E^{1/2}.$$

(vi) To show that $E = e^{hD}.$

We know that $Ef(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots.. \text{ (Using Taylor's Theorem)}$$

$$= f(x) + hDf(x) + \frac{h^2}{2!} D^2 f(x) + \ldots.$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 \ldots..\right) f(x)$$

$$Ef(x) = e^{hD} f(x)$$

Thus we have

$$E = e^{hD}.$$

(vii) To show that $(1+\Delta)(1-\nabla) = 1.$

We know that $(1+\Delta)(1-\nabla) f(x) = (1+\Delta)(f(x) - \nabla f(x))$

$$= (1+\Delta)[f(x) - (f(x) - f(x-h))]$$

$$= (1+\Delta) f(x-h)$$

$$= Ef(x-h)$$

$$(1+\Delta)(1-\nabla) f(x) = f(x)$$

Thus we have

$$(1+\Delta)(1-\nabla)=1.$$

Examples

Example.1. Show that $\Delta^3 = E^3 - 3E^2 + 3E - 1.$

Solution: Using the definition of Δ, we have

$$\Delta f(x) = f(x+h) - f(x)$$

and $$Ef(x) = f(x+h)$$

Therefore, we have

$$E^n f(x) = f(x+nh)$$

$\therefore$ $$\Delta^2 f(x) = \Delta[f(x+h) - f(x)]$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

and $$\Delta^3 f(x) = \Delta[f(x+2h) - 2f(x+h) + f(x)]$$

$$= f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

$$= E^3 f(x) - 3E^2 f(x) + 3Ef(x) - f(x)$$

$$\Delta^3 f(x) = (E^3 - 3E^2 + 3E - 1) f(x)$$

or $$\Delta^3 = E^3 - 3E^2 + 3E - 1.$$

Example.2. Prove that $\Delta[\log x] = \log\left(1+\frac{h}{x}\right)$.

Sol: Using the definition of Δ, we have

$\Delta f(x) = f(x+h) - f(x)$, where h is the interval difference.

Now we have $\Delta[\log x] = \log(x+h) - \log x$

$$= \log\left(\frac{x+h}{x}\right)$$

$$= \log\left(1+\frac{h}{x}\right).$$

Example.3. Prove that $\Delta \tan^{-1} 5x = \tan^{-1}\left(\frac{5h}{1+25x^2+25xh}\right)$.

Sol: Using the definition of Δ, we have

$\Delta f(x) = f(x+h) - f(x)$, where h is the interval difference.

Now we have

$$\Delta \tan^{-1} 5x = \tan^{-1} 5(x+h) - \tan^{-1} 5x$$

$$= \tan^{-1} \frac{5(x+h)-5x}{1+5(x+h)5x}$$

$$= \tan^{-1} \frac{5h}{1+25x(x+h)}$$

$$= \tan^{-1}\left(\frac{5h}{1+25x^2+25xh}\right).$$

Example.4. Prove that $\nabla\Delta \equiv \delta^2 \equiv (\Delta - \nabla)$.

Sol: We have $\nabla\Delta\ f(x) = \nabla[\Delta f(x)]$

$$= \nabla[f(x+h) - f(x)]$$

$$= \nabla\ f(x+h) - \nabla f(x)$$

$$= f(x+h) - f(x+h-h) - [f(x) - f(x-h)]$$

$$= f(x+h) - 2f(x) + f(x-h) \qquad \ldots(1)$$

Now we have

$$\delta^2 f(x) = \delta[\delta f(x)]$$

$$= \delta\left[f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)\right]$$

$$= \delta f\left(x+\frac{h}{2}\right) - \delta f\left(x-\frac{h}{2}\right)$$

$$= f\left(x+\frac{h}{2}+\frac{h}{2}\right) - f\left(x+\frac{h}{2}-\frac{h}{2}\right) - f\left(x-\frac{h}{2}+\frac{h}{2}\right) + f\left(x-\frac{h}{2}-\frac{h}{2}\right)$$

$$= f(x+h) - 2f(x) + f(x-h) \qquad \ldots(2)$$

Now we have

$$(\Delta - \nabla)\ f(x) = \Delta f(x) - \nabla f(x)$$

$$= [f(x+h) - f(x)] - [f(x) - f(x-h)]$$

$$= f(x+h) - 2f(x) + f(x-h)$$

From equations (1), (2), (3), we have

$$\nabla\Delta \equiv \delta^2 \equiv (\Delta - \nabla).$$

Example.5. Prove that $\Delta \ = \frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}}.$

Sol: We know that

$$\delta = E^{1/2} - E^{-1/2}.$$

Here we have

$$\frac{1}{2}\delta^2 + \delta\sqrt{1+\frac{\delta^2}{4}} = \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{1+\frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{4+E+E^{-1}-2}{4}}$$

$$= \frac{1}{2}\left(E + E^{-1} - 2\right) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{\left(E^{1/2} + E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E + E^{-1} - 2\right) + \left(E^{1/2} - E^{-1/2}\right)\frac{\left(E^{1/2} + E^{-1/2}\right)}{2}$$

$$= \frac{1}{2}\left[\left(E + E^{-1} - 2\right) + \left(\mathrm{E}\text{-}\mathrm{E}^{-1}\right)\right]$$

$$= \frac{1}{2}\left[\left(2E - 2\right)\right]$$

$$= \frac{1}{2}\left[2\left(E - 1\right)\right]$$

$$= E - 1$$

$\equiv \Delta.$

Example.6. Prove that $\Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$

Solution: Using the definition of Δ, we have

$\Delta \log f(x) = \log f(x+h) - \log f(x)$, where h is the interval difference.

$$= \log \frac{f(x+h)}{f(x)}$$

$$= \log\left[\frac{Ef(x)}{f(x)}\right]$$

$$= \log\left[\frac{(1+\Delta) f(x)}{f(x)}\right] \qquad [\because \quad E = 1 + \Delta]$$

$$= \log\left[\frac{f(x) + \Delta f(x)}{f(x)}\right]$$

$$= \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

Example.7. Evaluate the following

(i) $\Delta^2(2e^x)$, where h =1 (ii) $\Delta[Sin(ax+b)]$

Solution: Using the definition of Δ, we have

$\Delta f(x) = f(x+h) - f(x)$, where h is the interval difference.

(i) We have $\Delta^2\left(2e^x\right) = 2\Delta^2\left(e^x\right)$

$$= 2\Delta.\left(\Delta\left(e^x\right)\right)$$

$= 2\Delta\left(e^{x+1} - e^x\right)$, where h = 1 is the interval difference.

$$= 2\left(e^{x+2} - e^{x+1} - e^{x+1} + e^x\right)$$

$$= 2\left(e^2 - 2e + 1\right)e^x$$

$$= 2(e-1)^2 e^x.$$

(ii) We have

$$\Delta[\sin(cx+d)] = \sin(c(x+h)+d) - \sin(cx+d)$$

$$= 2\cos\left(\frac{c(x+h)+d+cx+d}{2}\right)\times\sin\left(\frac{c(x+h)+d-cx-d}{2}\right)$$

$$= 2\cos\left(cx+d+\frac{ch}{2}\right)\sin\frac{ch}{2}.$$

Example.8. Evaluate $\Delta(3x + e^{2x} + \sin x)$.

Solution: Using the definition of Δ, we have

$\Delta f(x) = f(x+h) - f(x)$, where h is the interval difference.

$$\Delta(3x + e^{2x} + \sin x) = [3(x+h) + e^{2(x+h)} + \sin(x+h)] - [3x + e^{2x} + \sin x]$$

$$= 3h + e^{2x}[(e^{2h} - 1) + 2\cos\left(\frac{x+h+x}{2}\right)\sin\frac{x+h-x}{2}$$

$$= 3h + e^{2x}(e^{2h} - 1) + 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

Example.9. Evaluate $\Delta[e^{2x}\log 3x]$.

Solution: We know that

$$\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$$

Take $\qquad f(x) = e^{2x},\ g(x) = \log 3x.$

Then we have

$$\Delta(e^{2x}\log 3x) = e^{2(x+h)}\Delta \log 3x + \log 3x\,\Delta\, e^{2x}$$

$$= e^{2(x+h)}[\log 3(x+h) - \log 3x] + \log 3x.(e^{2(x+h)} - e^{2x})$$

$$= e^{2(x+h)}\left(\log\frac{3(x+h)}{3x}\right) + \log 3x.e^{2x}(e^{2h}-1)$$

$$= e^{2x}\left[e^{2h}\log\left(1+\frac{h}{x}\right) + (e^{2h}-1)\log 3x\right].$$

Example.10. Evaluate $\Delta^n(e^{ax+h}), h = 1.$

Solution. We know that

$\Delta f(x) = f(x+1) - f(x)$, where h = 1 is the interval difference.

$\therefore \qquad \Delta e^{ax+b} = e^{a(x+1)+b} - e^{ax+b}$

$$= e^{ax+b}(e^a - 1)$$

Again $\qquad \Delta^2 e^{ax+b} = \Delta(\Delta e^{ax+b})$

$$= \Delta[e^{ax+b}(e^a - 1)]$$

$$= (e^a - 1)\Delta e^{ax+b}$$

$$= (e^a - 1)e^{ax+b}(e^a - 1)$$

$$= (e^a - 1)^2 e^{ax+b}$$

Proceeding in the same way, we get

$$\Delta^n(e^{ax+b}) = (e^a - 1)^n e^{ax+b}$$

Example.11. Show that $\Delta^r y_x = \nabla^r y_{x+r}$

Solution: We have $\nabla^r y_{x+r} = (1 - E^{-1})^r y_{x+r}$ $\quad [\because \quad \nabla = 1 - E^{-1}]$

$$= \left(\frac{E-1}{E}\right)^r y_{x+r}$$

$$= (E-1)^r E^{-r} y_{x+r}$$

$$= (E-1)^r y_x \qquad [\because \quad \Delta = E - 1]$$

$$= \Delta^r y_x.$$

Example.12. Evaluate $\Delta \cos(2+3x)$**.**

Solution: Using the definition of Δ, we have

$\Delta f(x) = f(x+h) - f(x)$, where h is the interval difference.

We have

$$\Delta \cosh(2+3x) = \cosh(2+3(x+h)) - \cosh(2+3x)$$

$$= 2\sinh\frac{2+3(x+h)+2+3x}{2}\ \sinh\frac{2+3(x+h)-2-3x}{2}$$

$$= 2\sinh\left(2+3x+\frac{3h}{2}\right)\sinh\frac{3h}{2}.$$

Example.13. Find the value of $\left(\frac{\Delta^2}{E}\right)x^3,$ where $h=1.$

Solution: We have $\left(\frac{\Delta^2}{E}\right)x^3 = \left[\frac{(E-1)^2}{E}\right]x^3 \qquad [\because \quad \Delta = E-1]$

$$= \left[\frac{E^2+1-2E}{E}\right]x^3$$

$$= [E+E^{-1}-2I]x^3$$

$$= (x+1)^3+(x-1)^3-2x^3$$

$$= x^3+3x^2+3x+1+x^3-3x^2+3x-1-2x^3$$

$$= 6x.$$

1.9 Summary

The first forward differences is defined by

$\Delta y_0 = y_1 - y_0$, where h =1 is interval difference.

The first backward difference is defined by

$$\nabla y_x = y_x - y_{x-h}$$, where h is interval difference.

The differences $y_1 - y_0 = \delta\, y_{1/2}$, $y_2 - y_1 = \delta\, y_{3/2}$, ..., $y_n - y_{n-1} = \delta\, y_{n-1/2}$ are known as central differences and δ is known as central differences operator.

The shift (increment) operator E is defined as

$$Ey_x = y_{x+h}$$, where h is the interval difference.

The inverse operator E^{-1} is defined as $E^{-1} y_x = y_{x+(-h)}$

Some of the important relations are following:

(i) $\Delta = E - 1$ or $E = \Delta + 1$

(ii) $\nabla = 1 - E^{-1}$ or $E^{-1} = 1 - \nabla$.

(iii) $E\nabla = \nabla E = \Delta$.

(iv) $\delta = E^{1/2} - E^{-1/2}$.

(v) $\Delta = \delta\, E^{1/2}$.

(vi) $E = e^{hD}$.

(vii) $(1+\Delta)(1-\nabla) = 1$.

1.10 Terminal Questions

Q.1. What do you means by Finite Differences?

Q.2. Explain the shift operators.

Q.3. Write a short note on Central difference operators.

Q.4. Prove that $\Delta^2 \equiv E^2 - 2E + 1$.

Q.5. Evaluate the following:

(i) $\Delta[\sinh(a+bx)]$

(ii) $\Delta[\tan ax]$

(iii) $\Delta[\cot 2^{x}]$

(iv) $\Delta(x+\cos x)$

(v) $\Delta(x^{2}+e^{x}+2)$

(vi) $\Delta[e^{ax}\log bx]$

(vii) $\Delta\left[\frac{x^{2}}{\cos 2x}\right]$

Q.6. Evaluate the following:

(i) $\Delta^{2}\cos 2x$

(ii) $\Delta^{2}(ab^{cx})$

(iii) $\left(\frac{\Delta^{2}}{E}\right)x^{3}$

(iv) $\Delta^{2}\left[\frac{5x+12}{x^{2}+5x+6}\right]$

Q.7. Evaluate the following:

(i) $(2\Delta^{2}+\Delta-1)(x^{2}+2x+1)$

(ii) $(\Delta+1)(2\Delta-1)(x^{2}+2x+1)$

(iii) $(E+2)(E+1)(2^{x+h}+x)$

(iv) $(E^2+3E+2)2^{x+h}+x$

Q.8. Prove that if $f(x)$ and $g(x)$ are the function of x then

(i) $\Delta[f(x)+g(x)]=\Delta f(x)+\Delta g(x)$

(ii) $\Delta[a\, f(x)]=a\,\Delta f(x)$

(iii) $\Delta[f(x)g(x)]=f(x)\Delta g(x)+g(x+h)\Delta f(x)=f(x+h)\Delta g(x)+g(x)\Delta f(x)$

(iv) $\Delta\left[\dfrac{f(x)}{g(x)}\right]=\dfrac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)g(x+h)}$

Answers

5. (i) $2\sinh\left(\dfrac{b}{2}\right)\cosh\left(a+\dfrac{b}{2}+bx\right)$

(ii) $\dfrac{\sin a}{\cos ax\cos a(x+1)}$

(iii) $-\operatorname{cosec} 2^{x+1}$

(iv) $h-2\sin\left(x+\dfrac{h}{2}\right)\sin\left(\dfrac{h}{2}\right)$

(v) $2hx+h^2e^x(e^h-1)$

(vi) $e^{ax}\left[e^{ah}\log\left(1+\dfrac{h}{x}\right)+(e^{ah}-1)\log bx\right]$

(vii) $\dfrac{h(2x+h)\cos x + 2x^2 \sin h \sin(2x+h)}{\cos(2x+2h)\cos 2x}$

6. (i) $-4\sin^2 h\cos(2x+2h)$

(ii) $(b^c-1)ab^{cx}$

(iii) $6x$

(iv) $\dfrac{4}{(x+2)(x+3)(x+4)} + \dfrac{6}{(x+3)(x+4)(x+5)}$

7. (i) $5h^2+2hx+2h = x^2-2x-1$

(ii) $5h^2+2hx+2h-x^2-2x-1$

(iii) h

(iv) h

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT- 2: APPLICATION OF FINITE DIFFERENCES

Structure

2.1 Introduction

In the present unit we shall discuss the fundamental theorem difference calculus, factorial function, properties of factorial function with examples. The Fundamental Theorem of Difference Calculus is a principle in mathematics that provides a fundamental relationship between summation and differencing operations. It is a counterpart to the Fundamental Theorem of Calculus, which connects integration and differentiation. This fundamental theorem plays a crucial role in discrete mathematics and is applicable in various areas, including numerical analysis and computer science.

In numerical analysis, the factorial function often plays a role in various computations, especially in problems involving combinatory, series expansion, and algorithms. In numerical algorithms, factorials may appear in computations involving series expansions, especially in contexts where precise numerical evaluation is required. When dealing with large factorials, it's important to consider numerical precision and computational efficiency. For very large factorials, numerical libraries or specialized algorithms may be employed to avoid overflow issues and enhance computational performance.

2.2 Objectives

After studying this unit, the learner will be able to understand:

- the fundamental theorem on difference calculus
- the factorial function
- the properties of factorial functions

2.3 Fundamental Theorem of the Difference Calculus

The fundamental theorem of the difference calculus in numerical analysis is a principle that establishes a connection between the process of differencing a sequence and the original sequence itself. It plays a crucial role in understanding and manipulating discrete data.

This theorem is foundational in numerical analysis, especially in techniques involving finite differences, interpolation, and the construction of numerical algorithms. It provides a theoretical basis for understanding the relationship between cumulative sums and differencing in the discrete domain. Let *f*(*x*) be a polynomial of n^{th} degree in *x*, then the n^{th} difference of *f*(*x*) is constant and $\Delta^{n+1} f(x) = 0.$

Proof: Consider the n^{th} degree polynomial

$$f(x) = A_0 + A_1 x + A_2 x^2 + ... + A_n x^n$$

Where $A_0, A_1, A_2, ..., A_n$ all are constants and *n* is a positive integer.

$$\Delta f(x) = f(x+h) - f(x)$$

$$= [A_0 + A_1(x+h) + A_2(x+h)^2 + ...A_n(x+h)^n] - [A_0 + A_1 x + A_2 x^2 + ... + A_n x^n]$$

$$= A_1 h + A_2[(x+h)^2 - x^2] + A_3[(x+h)^3 - x^3] + ... + A_n[(x+h)^n - x^n]$$

$$= A_1 h + A_2[x^2 + {}^2C_1 xh + h^2 - x^2] + A_3[x^3 + {}^3C_1 x^2 h + {}^3C_2 x^2 + h^3 - x^3] + ...$$

$$+ A_n[x^n + {}^nC_1 x^{n-1} h + {}^nC_2 x^{n-2} h^2 + ... + {}^nC_n h^n - x^n)$$

or $$\Delta f(x) = B_1 + B_2 x + B_3 x^2 + ... + B_{n-1} x^{n-2} + nA_n h x^{n-1} \quad .....(1)$$

Where $B_1, B_2, ..., B_{n-1}$ all are constants.

Using equation (1), we see that the first difference of a polynomial of degree *n* is given a polynomial of degree (*n*-1).

Again we have

$$\Delta^2 f(x) = \Delta f(x+h) - \Delta f(x)$$

$$= B_1 + B_2(x+h) + B_3(x+h)^2 + ... + B_{n-1}(x+h)^{n-2} + nA_n h(x+h)^{n-1}$$

$$-[B_1 + B_2 x + B_3 x^2 + ... + B_{n-1} x^{n-2} + nA_n h x^{n-1}]$$

$$= B_2 h + B_3[(x+h)^2 - x^2] + B_4[(x+h)^3 - x^3] + ...$$

$$+ B_{n-1}[x+h)^{n-2} - x^{n-2}] + nA_n h[(x+h)^{n-1} - x^{n-1}]$$

$$= B_2 h + B_3[x^2 + {}^2C_1 xh + h^2 - x^2] + B_4[x^3 + {}^3C_1 x^2 h$$

$$+ {}^3C_2 xh^2 + h^3 - x^3] + ... + B_{n-1}[x^{n-2} + {}^{n-2}C_1 x^{n-3} h$$

$$+ {}^{n-2}C_2 x^{n-3} h^2 + ... + {}^{n-2}C_{n-2} h^{n-2} - x^{n-2}]$$

$$+ nA_n h[x^{n-1} + {}^{n-1}C_1 x^{n-2} h + {}^{n-1}C_2 x^{n-3} h^2 + ... + {}^{n-1}C_{n-1} h^n - x^{n-1}]$$

or $$\Delta^2 f(x) = C_2 + C_3 x + C_4 x^2 + ... + C_{n-1} x^{n-3} + n(n-1) h^2 A_n x^{n-2} \quad .....(2)$$

where $C_2, C_3, ..., C_{n-2}, C_{n-1}$ are constants.

Using equation (2), we see that the second difference of a polynomial of degree *n* is again a polynomial of degree (*n*-2).

Proceeding in the same way, we will get a zero degree polynomial for the *n*th difference *i.e.*,

$$\Delta^n f(x) = n(n-1)(n-2)........1 \ h^n a_n x^{n-n}$$

$$= n!h^n a_n.$$

Therefore the *n*th difference is constant.

Now we have

$$\Delta^{n+1} f(x) = \Delta[\Delta^n f(x)]$$

$$= \Delta[n!h^n a_n]$$

$$= 0 \qquad [\because \Delta C = 0]$$

Hence the n^{th} difference of *f*(*x*) is constant and $\Delta^{n+1} f(x) = 0.$

2.4 Factorial Function

A product of the form $x(x-h)(x-2h)..........(x-(n-1)h)$ is known as factorial function and denoted by $x^{(n)}$.

We have

$$x^{(n)} = x(x-h)(x-2h)..........(x-(n-1)h)$$

If the interval of differencing is unity. Then we have

$$x^{(n)} = x(x-1)(x-2)(x-3)...........(x-(n-1)).$$

Check your Progress

1. What do you mean by Fundamental theorem of the difference calculus?

2. Define the factorial function.

2.5 Properties of Factorial Function

(i) To show that $\Delta^n x^{(n)} = n!h^n$ and $\Delta^{n+1} x^{(n)} = 0.$

Proof. By the definition of Δ, we have

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)(x+h-h)(x+h-2h)....(x+h-(n-1)h)$$

$$- x(x-h)(x-2h)....(x-(n-1)h)$$

$$= (x+h)x(x-h)(x-2h)....(x-(n-2h)h)$$

$$-x(x-h)(x-2h)....(x-(n-2)h)(x-(n-1)h)$$

$$= x(x-h)(x-2h)....(x-(n-2)h)((x+h)-(x-(n-1)h)$$

$$= x^{(n-1)} nh$$
$$= nh\, x^{(n-1)}$$

Again we have

$$\Delta^2 x^{(n)} = \Delta\Delta x^n$$

$$= \Delta[nhx^{n-1}]$$

$$= nh\Delta x^{n-1}$$

$$= nh[(x+h)^{n-1} - x^{n-1}]$$

$$= nh[x+h)(x+h-h)(x+h-2h)\ldots.(x+h-(n-2)h)$$

$$- x(x-h)(x-2h)\ldots.(x-(n-2)h]$$

$$= nh[(x+h)x(x-h)(x-2h)\ldots.(x-(n-3)h)$$

$$- x(x-h)(x-2h)\ldots.(x-(n-3)h)(x-(n-2)h)]$$

$$= nh\, x(x-h)(x-2h)\ldots.(x-(n-3)h)[x+h-(x-(n-2)h)]$$

$$= nh\, x^{n-2}(n-1)h$$

$$= n(n-1)h^2\; x^{n-2}$$

Proceeding in the same way, we get

$$\Delta^n x^{(n)} = n(n-1)(n-2)\ldots.1 h^n x^{(n-n)}$$

$$= n!h^n$$

Again we have

$$\Delta^{n+1}x^{(n)} = \Delta(\Delta^n x^n)$$

$$= \Delta(n!h^n)$$
$$= 0.$$

(ii) To show that $f(a+nh) = f(a) + {}^nC_1\Delta f(a) + {}^nC_2\Delta^2 f(a) + \ldots\ldots + {}^nC_n\Delta^n f(a)$

Prof. We shall prove this by the method of mathematical induction.

We have $\Delta f(a) = f(a+h) - f(a)$

$\therefore \quad f(a+h) = \Delta f(a) + f(a) = f(a) + \Delta f(a)$ [It is true for n =1]

Again we have

$$\Delta f(a+h) = f(a+2h) - f(a+h)$$

$$\therefore \quad f(a+2h) = \Delta f(a+h) + f(a+h)$$

$$= \Delta[\Delta f(a) + f(a)] + \Delta f(a) + f(a)$$

$$= f(a) + 2\Delta f(a) + \Delta^2 f(a)$$

$$f(a+2h) = f(a) + {}^2C_1\Delta f(a) + \Delta^2 f(a)$$

It is true for n =2.

Similarly, we have

$$f(a+3h) = \Delta f(a+2h) + f(a+2h)$$

$$= \Delta[f(a) + 2\Delta f(a) + \Delta^2 f(a)] + [f(a) + 2\Delta f(a) + \Delta^2 f(a)]$$

$$= f(a) + 3\Delta f(a) + 3\Delta^2 f(a) + \Delta^3 f(a)$$

$$f(a+3h)= f(a)+{}^3C_1\Delta f(a)+{}^3C_2\Delta^2 f(a)+\Delta^3 f(a)$$

It is true for n =3.

Now Assume that it is true for n = k then we have

$$f(a+kh)= f(a)+{}^kC_1\Delta f(a)+{}^kC_2 f(a)+..............+{}^kC_k\Delta^k f(a)$$

Now we shall show that this result is true for n=k +1.

Now we have

$$f(a+(k+1)h)= f(a+kh)+\Delta f(a+kh)$$

$$=[f(a)+{}^kC_1\Delta f(a)+{}^kC_2\Delta^2 f(a)+...........+{}^kC_k\Delta^k f(a)]$$

$$+\Delta[f(a)+{}^kC_1\Delta f(a)+{}^kC_2\Delta^2 f(a)+.............+{}^kC_k\Delta^k f(a)]$$

$$= f(a)+[{}^kC_1+1]\Delta f(a)+[{}^kC_2+{}^kC_1]\Delta^2 f(a)$$

$$+[{}^kC_3+{}^kC_2]\Delta f(a)+..............+\Delta^{k+1} f(a).$$

$$f(a+(k+1)h)= f(a)+{}^{k+1}C_1\Delta f(a)+{}^{k+1}C_2\Delta^2 f(a)+{}^{k+1}C_3\Delta^3 f(a)+.....+\Delta^{k+1}f(a).$$

Thus the result is true for n = k +1. $[\because\ {}^kC_r+{}^kC_{r+1}={}^{k+1}C_{r+1}]$

Hence by the principle of mathematical induction it is true for all n, we have

$$f(a+nh)= f(a)+{}^nC_1\Delta f(a)+{}^nC_2\Delta^2 f(a)+............+{}^nC_n\Delta^n f(a).$$

Examples

Example.1. Find the value of $\Delta^3(1-x)(1-2x)(1-3x), \quad where \quad h=1.$

Solution: We have

$$f(x) = (1-x)(1-2x)(1-3x)$$

$$= 1 - 6x + 11x^2 - 6x^3$$

This is the polynomial of degree 3 in *x*. Therefore we have

$$\Delta^3 f(x) = \Delta^3(1 - 6x + 11x^2 - 6x^3)$$

$$= 0 - 6.0 + 11.0 - 6.3! \qquad \left[\Delta^n x^{(n)} = n!h^n \quad \text{and} \quad \Delta^3 x^3 = 3!\right]$$

$$= -36.$$

Example.2. Using the following forward difference table, determine the value of $\Delta^4 y(1)$.

x	1	2	3	4	5
y	3	6	11	21	31

Solution: The forward difference table for the given data is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1 2	3 6	3	2		

3 4 5	11 21 31	5 10 10	5 0	3 –5	–8

From the above forward difference table we see that the value of the

$$\Delta^4 y(1) = -8.$$

Example.3. Using the backward difference table, determine the value of $\nabla^4 y(5)$ from the following data:

x	1	2	3	4	5
y	1	4	9	18	28

Solution: The forward difference table for the given data is

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1 2	1 4	3 5	2	2	

3 4 5	9 18 28	9 10	4 1	-3	-5

From the above forward difference table we see that the value of the

$$\nabla^4 y(5) = -5.$$

Example.4. Using the following forward difference table, determine the value of $\Delta^3 y(1)$.

x	1	2	3	4
y	3	8	18	45

Solution: The forward difference table for the given data is

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1 2 3 4	$f(1) = 3$ 8 18	$\Delta f(1) = 5$ 10 27	5 17	12

	45			

From the above forward difference table we see that the value of the $\Delta^3 y(1) = 12.$

Example.5. Represent the function $f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$ and its successive differences into factorial notation.

Solution: The given function is

$$x^4 - 12x^3 + 42x^2 - 30x + 9 = Ax^{(4)} + Bx^{(3)} + Cx^{(2)} + Dx^{(1)} + E$$

$$= Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + Dx + E \qquad ...(1)$$

Where A, B, C, D and E are constants. Now, we will find the value of these constants.

Putting $x = 0$ in equation (1), we get

$\Rightarrow$ E = 9

Again putting $x = 1$ in equation (1), we get

1 – 12 + 42 –30 + 9 = D + E

$\Rightarrow$ D = 1

Putting $x = 2$ in the equation (1), we get

16 – 12 × 8 + 42 ×4–30 × 2 + 9 = 2C + 2D + E

$\Rightarrow$ C = 13

Putting $x = 3$ in the equation (1), we get

81 –12 × 27 + 42 × 9 – 30 × 3 +9 = 6B + 6C + 3D + E

$\Rightarrow$ B = – 6

Equating the coefficient of x^4 on both sides, we get A =1. Putting the values of A, B, C, D, E in equation (1), we get

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9$$

$$= x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$$

Now we have

$$\Delta f(x) = 4x^{(3)} - 18x^{(2)} + 26x^{(1)} + 1$$

$$\Delta^2 f(x) = 12x^{(2)} - 36x^{(1)} + 26$$

$$\Delta^3 f(x) = 24x^{(1)} - 36$$

$$\Delta^4 f(x) = 24$$

$$\Delta^5 f(x) = 0$$

Hence $f(x) = x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9.$

Example.6. Determine the function whose first difference is e^{ax+b}.

Solution: Consider $f(x)$ is the required function.

Then we have $\Delta f(x) = e^{ax+b}$ …. (1)

Let $f(x) = Ae^{ax+b}$

Therefore we have

$$\Delta f(x) = \Delta[Ae^{ax+b}]$$

$$= A\Delta e^{ax+b}$$

$$= A[e^{a(x+1)+b} - e^{ax+b}]$$

$$= Ae^{ax+b}[e^a - 1] \qquad \text{…. (2)}$$

Comparing equations (1) and (2), we get

$$A = \frac{1}{e^a - 1}$$

Hence $f(x) = \dfrac{e^{ax+b}}{e^a - 1}$.

Example.7. Determine the function whose first difference is $9x^2 + 11x + 5$.

Solution: Consider $f(x)$ is the required function.

Then we have $\Delta f(x) = 9x^2 + 11x + 5$

Here first, we change $\Delta f(x)$ in the factorial notation.

Now we have

$$9x^2 + 11x + 5 = Ax^{(2)} + Bx^{(1)} + C$$

$$= Ax(x-1) + Bx + C \qquad \text{… (1)}$$

Putting $x = 0$ in equation (1), we get

C = 5

Putting x =1 in equation (1), we get

$$9 + 11 + 5 = B + C$$

$\Rightarrow$ B = 20

Comparing the term x^2 in equation (1), we get

$$A = 9$$

Now putting the values of A, B and C in equation (1), we get

$$\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5$$

Integrating, we get

$$f(x) = \frac{9x^{(3)}}{3} + \frac{20x^{(2)}}{2} + 5x^{(1)} + C_1.$$ where C_1 is constant of Integration.

$$= 3x(x-1)(x-2) + 10x(x-1) + 5x + C_1$$

Hence $f(x) = 3x^3 + x^2 + x + C_1.$

Example.8. Determine the lowest degree polynomial which have taken the following values:

x	**0**	**1**	**2**	**3**	**4**	**5**
***f*(*x*)**	**0**	**3**	**8**	**15**	**24**	**35**

Solution: The forward difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	0	3		
1	3	5	2	0
2	8	7	2	0
3	15	9	2	0
4	24	11	2	
5	35			

We know that

$$f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + {}^nC_3 \Delta^3 f(a) + \ldots.. \qquad \ldots(1)$$

Putting $a = 0$, $h = 1$, $n = x$ in equation (1), we get

$$f(x) = f(0) + {}^xC_1 \Delta f(0) + {}^xC_2 \Delta^2 f(0) + {}^xC_3 \Delta^3 f(0) + \ldots.. \qquad \ldots(2)$$

Now putting the value of $f(0), \Delta f(0), \Delta^2 f(0)$ and $\Delta^3 f(0)$ in the equation (2) from the above forward difference table, we get

$$f(x) = 0 + x.3 + \frac{x(x-1)}{2!}.2 + \frac{x(x-1)(x-2)}{3!}.0 + 0$$

$$f(x) = 3x + x(x-1)$$

Hence $f(x) = x^2 + 2x$.

Example.9. Evaluate $\Delta^n\left[ax^n + bx^{n\text{-}1}\right]$.

Sol. We have $\Delta^n\left[ax^n + bx^{n-1}\right] = \Delta^n\left(ax^n\right) + \Delta^n\left(bx^{n-1}\right)$

$$= a\Delta^n\left(x^n\right) + b\Delta^n\left(x^{n-1}\right)$$

$$= a\left(n!\right) + b.0$$

$$= a\left(n!\right).$$

Example.10. Given $u_0 = 1, u_1 = 11, u_2 = 21, u_3 = 28$ **and** $u_4 = 29$, **determine the value of** $\Delta^4 u_0$ **without forming difference table.**

Solution: We know that

$$\Delta^4 u_0 = (E-1)^4 u_0$$

$$= (E^4 - {}^4C_1E^3 + {}^4C_2E^2 - {}^4C_3E + 1)u_0$$

$$= E^4u_0 - 4E^3u_0 + 6E^2u_0 - 4Eu_0 + u_0$$

$$= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0$$

$$= 29 - 4\times 28 + 6\times 21 - 4\times 11 + 1$$

$$= 29 - 112 + 126 - 44 + 1$$

$$= 0.$$

Example.11. Show that $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... + \Delta^{n-1}u_{x-n} + \Delta^n u_{x-n}$.

Solution: We have

$$u_x - \Delta^n u_{x-n} = u_x - \Delta^n E^{-n} u_x$$

$$= \left(1 - \frac{\Delta^n}{E^n}\right) u_x$$

$$= \left(\frac{E^n - \Delta^n}{E^n}\right) u_x$$

$$= \frac{1}{E^n} \frac{(E-\Delta)[E^{n-1} + E^{n-2}\Delta + E^{n-3}\Delta^2 + ... + \Delta^{n-1}]u_x}{(E-\Delta)} \qquad (\because\ E = 1 + \Delta)$$

$$= (E^{-1} + \Delta E^{-2} + \Delta^2 E^{-3} + ... + \Delta^{n-1} E^{-n}) u_x$$

$$= u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... + \Delta^{n-1} u_{x-n}$$

Hence $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... + \Delta^{n-1} u_{x-n} + \Delta^n u_{x-n}$.

Example.12. Show that

(a) $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$

(b) $f(4) = f(0) + 4\Delta f(0) + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$

As for as third difference.

Solution: (a) We have

$$\Delta f(3) = f(4) - f(3)$$

or $f(4) = f(3) + \Delta f(3)$

$= f(3) + \Delta[f(2) + \Delta f(2)]$ $\quad [\because \Delta f(2) = f(3) - f(2)]$

$= f(3) + \Delta f(2) + \Delta^2 f(2)$

$= f(3) + \Delta f(2) + \Delta^2[f(1) + \Delta f(1)]$ $\quad [\because \Delta f(1) = f(2) - f(1)]$

$f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

(b) We have

$f(4) = f(-1+5)$

$= E^5 f(-1)$

$= (1+\Delta)^5 f(-1)$

$= (1 + {}^5C_1\Delta + {}^5C_2\Delta^2 + {}^5C_3\Delta^3 + {}^5C_4\Delta^4 + {}^5C_5\Delta^5) f(-1)$

$= f(-1) + 5\Delta f(-1) + 10\Delta^2 f(-1) + 10\Delta^3 f(-1)$ $\quad$ taking up to 3rd difference

$= [f(-1) + \Delta f(-1) + 4[\Delta f(-1) + \Delta^2 f(-1)] + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$

$= [f(-1) + \Delta f(-1)] + 4[f(-1) + \Delta f(-1)] + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$

$= f(0) + 4\Delta f(0) + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$

$[\because \Delta f(-1) = f(0) - f(-1)]$

Hence $f(4) = f(0) + 4\Delta f(0) + 6\Delta^2 f(-1) + 10\Delta^3 f(-1)$.

Example.13. Find the second degree polynomial which is passes through the points (0, 3), (1, 5), (2, 9) and (3, 15).

Solution: The forward difference table is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	3			
		2		
1	5		2	
		4		0
2	9		2	
		6		
3	15			

We know that

$$f(a+nh) = f(a) + {}^nC_1\Delta f(a) + {}^nC_2\Delta^2 f(a) + {}^nC_3\Delta^3 f(a) + \dots\dots\dots + {}^nC_1\Delta^n f(a) \qquad \dots(1)$$

Putting a = 0, h =1, n = x in equation (1), we get

$$f(x) = f(0) + {}^xC_1\Delta f(0) + {}^xC_2\Delta^2 f(0) + {}^xC_3\Delta^3 f(0) + \dots. \qquad \dots(2)$$

Now putting the value of $f(0), \Delta f(0), \Delta^2 f(0)$ and $\Delta^3 f(0)$ in equation (2) from the above forward difference table, we get

$$f(x) = 3 + x.2 + \frac{x(x-1)}{2!}.2 + \frac{x(x-1)(x-2)}{3!}.0$$

$$= 3 + 2x + x(x-1)$$

$$= 3 + 2x + x^2 - x$$

Hence $f(x) = x^2 + x + 3.$

Example.14. To show that $\Delta^n u_x = u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} - .... + (-1)^n u_x$

Solution: We have $u_{x+n} - {}^nC_1 u_{x+n-1} + {}^nC_2 u_{x+n-2} - .... + (-1)^n u_x$

$$= (E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - .... + (-1)^n) u_x$$

$$= (E-1)^n u_x$$

$$= \Delta^n y_x.$$

Example.15. To show that

$$u_0 + u_1 + u_2 + ... + u_n = {}^{n+1}C_1 u_0 + {}^{n+1}C_2 \Delta u_0 + {}^{n+1}C_3 \Delta^2 u_0 + ... + \Delta^n u_0.$$

Solution: We have

$$u_0 + u_1 + u_2 + ... + u_n = u_0 + Eu_0 + E^2 u_0 + ... + E^n u_0$$

$$= (1 + E + E^2 + ... + E^n) u_0$$

$$= \frac{E^{n+1} - 1}{E - 1} u_0 \qquad \text{[Using the concept of sum of } n \text{ term in G.P.]}$$

$$= \frac{(1+\Delta)^{n+1} - 1}{\Delta} u_0$$

$$= \frac{1}{\Delta} [1 + {}^{n+1}C_1 \Delta + {}^{n+1}C_2 \Delta^2 + {}^{n+1}C_3 \Delta^3 + ... + {}^{n+1}C_{n+1} \Delta^{n+1} - 1] u_0$$

$$=\frac{1}{\Delta}[{}^{n+1}C_1\Delta u_0+{}^{n+1}C_2\Delta^2 u_0+{}^{n+1}C_3\Delta^3 u_0+.....+{}^{n+1}C_{n+1}\Delta^{n+1}u_0]$$

$$={}^{n+1}C_1 u_0+{}^{n+1}C_2\Delta u_0+{}^{n+1}C_3\Delta^2 u_0+.........+\Delta^n u_0.$$

Example.16. To show that

$$u_1x+u_2x^2+u_3x^3+..........=\frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}\Delta u_1+\frac{x^3}{(1-x)^3}\Delta^2 u_1+.... \quad \text{and} \quad 0<x<1.$$

Solution: We have

$$\frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}\Delta u_1+\frac{x^3}{(1-x)^3}\Delta^2 u_1+........$$

$$=\frac{x}{1-x}u_1+\frac{x^2}{(1-x)^2}(E-1)u_1+\frac{x^3}{(1-x)^3}(E-1)^2u_1+....$$

$$=\left(\frac{x}{1-x}-\frac{x^2}{(1-x)^2}+\frac{x^3}{(1-x)^3}\right)u_1+\left(\frac{x^2}{(1-x)^2}-\frac{2x^3}{(1-x)^3}+....\right)Eu_1$$

$$+\left(\frac{x^3}{(1-x)^3}+......\right)E^2u_1+....$$

$$=\frac{x}{1-x}\left(1+\frac{x}{(1-x)}\right)^{-1}u_1+\frac{x^2}{(1-x)^2}\left(1+\frac{x}{1-x}\right)^{-2}u_2+$$

$$+\frac{x^3}{(1-x)^3}\left(1+\frac{x}{1-x}\right)^{-3}u_3+........$$

$$=\frac{x}{1-x}\left(\frac{1-x+x}{(1-x)}\right)^{-1}u_1+\frac{x^2}{(1-x)^2}\left(\frac{1-x+x}{1-x}\right)^{-2}u_2+$$

$$+\frac{x^3}{(1-x)^3}\left(\frac{1-x+x}{1-x}\right)^{-3}u_3+........$$

$$=\frac{x}{1-x}\left(\frac{1}{(1-x)}\right)^{-1}u_1+\frac{x^2}{(1-x)^2}\left(\frac{1}{1-x}\right)^{-2}u_2++\frac{x^3}{(1-x)^3}\left(\frac{1}{1-x}\right)^{-3}u_3+........$$

$$=\frac{x}{1-x}(1-x)u_1+\frac{x^2}{(1-x)^2}(1-x)^2u_2++\frac{x^3}{(1-x)^3}(1-x)^3u_3+.........$$

$$=u_1x+u_2x^2+u_3x^3+u_4x^4+................$$

Example.17. Prove that $u_0+{}^nC_1u_xx+{}^nC_2u_xx^2+.......+u_nx^n$

$$=(1+x)^nu_0+{}^nC_1(1+x)^{n-1}x\Delta u_0+{}^nC_2(1+x)^{n-2}x^2\Delta^2u_0+....+x^n\Delta^nu_0$$

Solution: We have

$$(1+x)^nu_0+{}^nC_1(1+x)^{n-1}x\Delta u_0+{}^nC_2(1+x)^{n-2}x^2\Delta^2u_0+.......+x^n\Delta^nu_0$$

$$=((1+x)+x\Delta)^nu_0$$

$$=(1+x(1+\Delta))^nu_0$$

$$=(1+xE)^nu_0$$

$$=(1+{}^nC_1xE+{}^nC_2x^2E^2+{}^nC_3x^3E^3+......+x^nE^n)u_0$$

$$=u_0+{}^nC_1u_1x+{}^nC_2x^2+{}^nC_3u_3x^3+..........+x^nu_n.$$

2.6 Summary

Let $f(x)$ be a polynomial of n^{th} degree in x, then the n^{th} difference of $f(x)$ is constant and $\Delta^{n+1} f(x) = 0$.

A product of the form $x(x-h)(x-2h)..........(x-(n-1)h)$ is known as factorial function and denoted by $x^{(n)}$.

Properties of Factorial function:

(i) $\Delta^n x^{(n)} = n!h^n$ and $\Delta^{n+1} x^{(n)} = 0$.

(ii) $f(a+nh) = f(a) + {}^nC_1 \Delta f(a) + {}^nC_2 \Delta^2 f(a) + ... + {}^nC_n \Delta^n f(a)$

2.7 Terminal Questions

Q.1. Explain the factorial function.

Q.2. State the fundamental theorem on difference calculus.

Q.3. Construct a forward difference table for the following data:

x	0	5	10	15	20	25
$f(x)$	7	11	14	18	24	32

Q.4. If $f(0) = -3,\ f(1) = 6,\ f(2) = 8,\ f(3) = 12$ prepare the forward difference table.

Q.5. Given $f(0)=3, f(10)=12, f(2)=81, f(3)=200, f(4)=100$ and $f(5)=8.$ Using the difference table and find the value of $\Delta^5 f(0)$.

Q.6. Given $u_0=3, u_1=12, u_2=81, u_3=200, u_4=100, u_5=8$, determine the value of $\Delta^5 u_0$ without forming difference table.

Q.7. If $f(0)=-3, f(1)=6, f(2)=8, f(3)=12$ and the third difference being constant, determine the value of $f(6)$.

Q.8. Represent the function $f(x)=2x^3-3x^2+3x-10$ and its successive differences into factorial notation.

Q.9. Determine the function whose first difference is $x^3+3x^2+5x+12$.

Q.10. Obtain the function whose first difference is:

(i) e^x

(ii) $x(x-1)$

(iii) a

(iv) $x^{(2)}+5x$

(v) $\sin x$

(vi) 5^x

Q.11. Prove that $u_0 + {}^xC_1\Delta u_1 + {}^xC_2\Delta^2 u_2 + \ldots.. = u_x + {}^xC_1\Delta^2 u_{x-1} + {}^xC_2\Delta^4 u_{x-2} + \ldots.$

Q.12. Evaluate the following:

(i) $\Delta^n\left(\dfrac{1}{x}\right)$

(ii) $\Delta^n[\sin(ax+b)]$

(iii) $\Delta\ (ax-1)(bx^2-1)(cx^3-1)$

(iv) $\Delta^n[ax^n+bx^{n-1}]$

Answers

5. 755

6. 755

7. 126

8. $2x^{(3)}+3x^{(2)}+2x^{(1)}-10,\ 6x^{(2)}+6x^{(1)}+2,\ 12x^{(1)}+6,\ 12.$

9. $\frac{1}{4}x^{(4)}+2x^{(3)}+\frac{9}{2}x^{(2)}+12x^{(1)}+C.$

10. (i) $\frac{e^x}{(e^h-1)}+C$ (ii) $\frac{x^{(3)}}{3}+C$

(iii) $ax+C$ (iv) $\frac{x^3}{3}+\frac{5x^2}{2}+C$

(v) $-\frac{1}{2}\sin x$ (vi) $\frac{1}{4}5^x.$

12. (i) $\frac{(-1)^n n!}{x(x+1)(x+2)\ldots.(x+n)}$

(ii) $\left(2\sin\frac{a}{2}\right)^{n}\sin\left(ax+b+n\left(\frac{a+\pi}{2}\right)\right)$

(iii) 720 *abc*

(iv) *a* (*n*!).

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

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Master of Science
PGMM -104N

Numerical Analysis

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Block

2 Interpolation

Block-2

Interpolation

Interpolation is a mathematical technique used to estimate values that fall between known, measured, or observed data points. It involves constructing a function that passes through the given data points, allowing the estimation of values at points within the range of the data. Interpolation is particularly useful when you have discrete data points and need to estimate values at other points within the dataset. Here are some key concepts and methods related to interpolation: Linear Interpolation; Polynomial Interpolation; Lagrange's Interpolation; Newton Interpolation; Spline Interpolation; Bilinear and Bicubic Interpolation. Bilinear and bicubic interpolation techniques used in image processing which estimate the pixel values between known values in an image.

Interpolation is widely used in various fields such as computer graphics, image processing, numerical analysis, and scientific computing. It provides a valuable tool for estimating values within a dataset, making it easier to analyze and visualize data. Interpolation has numerous applications across various fields: such as Computer Graphics; Geographic Information Systems; Image Processing; Numerical Analysis; Signal Processing; Finance; Cartography; Physics and Engineering; Machine Learning; Medical Imaging and Economics and Econometrics. Interpolation is applied in economic modeling to estimate values between observed economic data points. It helps in constructing economic indicators and forecasting. These applications demonstrate the versatility of interpolation in various domains, where the need to estimate values between known data points is a common requirement.

In the third unit, we shall discussed about determine one or two missing terms and Newton's forward and backward interpolation with equal intervals and in the fourth unit we deal with Gauss's and Stirling Interpolation formula for Equal Intervals. Lagrange's Interpolation Formula for Unequal Intervals is discussed in unit fifth.

UNIT- 3: NEWTON's INTERPOLATION FORMULA WITH EQUAL INTERVALS

Structure

3.1 Introduction

Interpolation is extensively used in computer graphics to generate smooth curves and surfaces. Geographic Information Systems applications often involve interpolating values between known geographic data points to generate continuous maps. In image processing, interpolation is used to estimate pixel values between known values. Bilinear and bicubic interpolation are common techniques for image resizing and enhancement. In numerical analysis, interpolation helps in estimating values at intermediate points within a set of discrete data points.

In signal processing, interpolation is used to estimate values between discrete samples of a signal. It plays a crucial role in applications like audio signal processing and telecommunications. Interpolation is used in finance for pricing financial instruments and estimating key financial metrics. It helps in modelling yield curves and estimating future cash flows. Cartographers use interpolation to generate smooth contours and surfaces on maps. Elevation data and contour lines are often interpolated to create realistic terrain representations.

Interpolation is employed in various scientific and engineering applications. It is used in finite element analysis, simulation models, and experimental data analysis. In machine learning, interpolation is used to fill in missing data points in datasets and also it can be applied to impute missing values in features or labels during data pre-processing. Interpolation is used in medical imaging to enhance image resolution and improve the quality of reconstructed images. It aids in generating smoother transitions between pixel values.

In the third unit we shall discuss the method of finding the missing one and more terms, and Newton's forward and backward interpolation with equal intervals. Suppose $y = f(x)$ be a function of x and $y_0, y_1, y_2, \ldots\ldots\ldots, y_n$ are the values of the function $f(x)$ at $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ respectively, then the method to obtaining the value of $f(x)$ at point $x = x_i$ which lie between x_0 and x_n is called interpolation. Thus, interpolation is the technique of computing the value of the function outside the given interval. If $x = x_i$ does not lie between x_0 and x_n then computing the value of $f(x)$ at this point is called the extrapolation. The study of interpolation depends on the calculus of finite difference.

3.2 Objectives

After studying this unit the learner will be able to:

- understand how to find one missing term
- understand how to find two missing terms
- understand the Newton’s forward interpolation with equal intervals
- understand the Newton’s backward interpolation with equal intervals

3.3 To find one missing term

In numerical analysis, finding a missing term typically involves identifying a pattern or relationship within a sequence of numbers. There are two method for finding one missing terms:

Method 1.

Consider one value of $f(x)$ be missing from the given set of $(n+1)$ values (*i.e.*, n values are given) of x, the values of x being equidistant. Suppose the unknown value be Y. Now construct the difference table.

We can ensure $y= f(x)$ to be a polynomial of degree $(n–1)$ in x, since n values of y are given. Therefore equating to zero the n^{th} difference to determine the value of x.

Method 2.

Consider one value of $f(x)$ be missing from the given set of $(n+1)$ values (*i.e.*, n values are given) of x, the values of x being equidistant. This means we can assume $y = f(x)$ to be a polynomial of degree $(n-1)$ in x. So we have

$$\therefore \qquad \Delta^n f(x) = 0$$

$$\text{or} \qquad (E-I)^n f(x) = 0$$

$$\text{or} \qquad \{E^n - {}^nC_1E^{n-1} + {}^nC_2 E^{n-2} - \ldots\ldots + (-1)^n I\} f(x) = 0$$

$$\text{or} \qquad E^n f(x) - {}^nC_1E^{n-1} f(x) + {}^nC_2E^{n-2} f(x) - \ldots\ldots\ldots\ldots\ldots + (-1)^n f(x) = 0$$

$$\text{or} \qquad f(x + nh) - {}^nC_1f(x+ (n-1)h) + {}^nC_2 f(x + (n-2)h) - ---- + (-1)^n f(x) = 0 \qquad \ldots(1)$$

If $x=x_0$ is the first value of x then we put $x = x_0$ in equation (1) and after solving we get the value of Y *i.e.*, missing term.

3.4 To find two missing term

To find missing terms in a sequence using a difference table, we look at the changes between the consecutive terms. The table is made by finding the differences between neighboring terms, showing patterns that help predict the missing ones.

Consider two value Y_1 and Y_2 of $f(x)$ be missing from the given set of $(n+2)$ values (*i.e.*, n values are given) of x, the values of x being equidistant. This means we can assume $y = f(x)$ be a polynomial of degree $(n-1)$ in x. Therefore we have

$$\therefore \qquad \Delta^n f(x) = 0$$

$$\text{or} \qquad f(x + nh) - {}^nC_1f(x+ (n-1) h) + {}^nC_2 f(x + (n-2)h) - ---- + (-1)^n f(x) = 0 \qquad \ldots(2)$$

If $x = x_0$ is the first value of x then we put $x = x_0$, and $x = x_1$ successively in equation (2).

Therefore we get two equation in terms of Y_1 and Y_2. After solving these two equations we get the value of Y_1 and Y_2.

Check your Progress

1. What do you mean by the concept of solving one missing term?

2. Explain the procedure to find the two missing terms in the given data.

Examples

Example.1. Find the missing term from the following data:

x	0	1	2	3	4
$y=f(x)$	$y_0 = 1$	$y_1 = 8$	?	$y_3 = 64$	$y_4 = 125$

Sol. Let y_2 be the missing term. Since there are 4 values of y are given. This means we can assume $y = f(x)$ to be a polynomial of three degree in x. So we have

$$\Delta^4 y = 0.$$

or $$(E\text{-}I)^4 y_x = 0$$

or $$\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_x = 0$$

or $$y_{x+4h} - 4y_{x+3h} + 6y_{x+2h} - 4y_{x+h} + y_x = 0 \qquad \text{…. (1)}$$

Putting $x = 0$ and $h = 1$ in equation (1), we get

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

or $$125 - 4 \times 64 + 6y_2 - 4 \times 8 + 1 = 0$$

or $6y_2 = 162$

or $y_2 = 27$.

Hence the missing term from the above data is 27.

Example.2. Given that u_0 = 230, u_1 = 210, u_2 = 120, u_3 = ----, u_4 = 110. Determine the value of u_3.

Solution. Consider the missing term u_3 = Y. The forward difference is

x	u_x	Δu_x	$\Delta^2 u_x$	$\Delta^3 u_x$	$\Delta^4 u_x$
0	230				
		-20			
1	210		-70		
		-90		Y+40	
2	120		Y-30		220 – 4Y
		Y-120		260–3Y	
3	Y		230 – 2Y		
		110-Y			
4	110				

Here four values of u_x are given. This means we can assume u_x to be a polynomial of three degree in x. So we have

$$\Delta^4 u_x = 0$$

or $220 - 4Y = 0$

or $Y = 55$.

Hence the missing term from the above data is 55.

Another method:

Since the four values of u_x are given. Therefore, we can assume u_x to be a polynomial of degree 3 in x. Therefore we have

$$\Delta^4 u_x = 0$$

or $$(E-I)^4 u_x = 0$$

or $$(E^4 - {}^4C_1 E^3 + {}^4C_2 E^2 - {}^4C_3 E + I) u_x = 0$$

or $$(E^4 u_x - 4E^3 u_x + 6E u_x - 4E u_x + u_x = 0$$

or $$u_{x+4h} - 4u_{x+3h} + 6u_{x+2h} - 4u_{x+h} + u_x = 0 \qquad \ldots(1)$$

Putting $x = 0$ and $h=1$ in equation (1), we get

$$u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 = 0$$

or $$110 - 4Y + 6\times 120 - 4\times 210 + 230 = 0$$

or $$220 - 4Y = 0$$

or $$Y = 55.$$

Example.3. Estimate the missing term in the following:

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**
y	**2**	**4**	**8**	**-**	**32**	**64**	**128**

Solution. Let Y be the missing terms. Since there are 6 values of *y* are given. Therefore, we can assume *y* to be a polynomial of degree 5 in *x*. So we have

$$\Delta^6 y = 0$$

or $$(E-I)^6 y_x = 0$$

or $$(E^6 - {}^6C_1E^5 + {}^6C_2E^4 - {}^6C_3E^3 + {}^6C_4E^2 - {}^6C_5E + I)y_x = 0$$

or $$y_{x+6h} - 6y_{x+5h} + 15y_{x+4h} - 20y_{x+3h} + 15y_{x+2h} - 6y_{x+h} + y_x = 0 \qquad \ldots(1)$$

Putting $x = 1$ and $h = 1$ in equation (1), we get

or $$128 - \times 64 + 15 \times 32 - 20Y + 15 \times 8 - 6 \times 4 + 2 = 0$$

or $$128 - 384 + 480 - 20Y + 120 - 24 + 2 = 0$$

or $$322 - 20Y = 0$$

$$Y = \frac{322}{20}$$

$$= 16.1.$$

Example.4. Find the missing terms in the following table:

x	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
***f*(*x*)**	**1**	**8**	**-**	**64**	**-**	**216**	**343**	**512**

Solution. Let Y_1 and Y_2 are the missing terms. Here six values of *f*(*x*) are given. Therefore, we can assume *f*(*x*) to be a polynomial of degree 5 in *x*.

So we have

$$\therefore \qquad \Delta^6 f(x) = 0$$

or $$(E\text{-}I)^6 f(x) = 0$$

or $$(E^6 - {}^6C_1E^5 + {}^6C_2E^4 - {}^6C_3E^3 + {}^6C_4E^2 - {}^6C_5E + I)\, f(x) = 0$$

or $$f(x+6h) - 6f(x+5h) + 15f(x+4h)$$

$$-20f(x+3h) + 15f(x+2h) - 6f(x+h) + f(x) = 0 \qquad \text{...(1)}$$

Putting *h* =1 and *x* =1, and 2 successively in equation (1), we get

$$f(7) - 6f(6) + 15f(5) - 20f(4) + 15f(3) - 6f(2) + f(1) = 0$$

and $$f(9) - 6f(7) + 15f(6) - 20f(5) + 15f(4) - 6f(3) + f(2) = 0$$

or $$343 - 6\times 216 + 15Y_2 - 20\times 64 + 15Y_1 - 6\times 8 + 1 = 0$$

and $$512 - 6\times 343 - 20Y_2 + 15\times 64 - 6Y_1 + 8 = 0$$

or $$15Y_2 + 6Y_1 = 2280 \qquad \text{....(2)}$$

and $$20Y_2 + 6Y_1 = 2662 \qquad \text{....(3)}$$

Solving equations (1) and (2), we have

$$Y_1 = 27$$

and $Y_2 = 125$

Hence the missing terms are $f(3) = 27$ and $f(5) = 125$.

Example.5. Find the first term of the series whose second and subsequent terms are 8, 3, 0, -1, 0.

Solution. The given data is

x	0	1	2	3	4	5
$f(x)$	-	8	3	0	-1	0

Let $f(0)$ be the missing terms. Here five values of $f(x)$ are given. Therefore, we can assume $f(x)$ to be a polynomial of degree 4 in x. So we have

$\therefore$ $$\Delta^5 f(x) = 0$$

or $$(E-I)^5 f(x) = 0$$

or $$(E^5 - {}^5C_1E^4 + {}^5C_2E^3 - {}^5C_3E^2 + {}^5C_4E - {}^5C_5I)\, f(x) = 0$$

or $$E^5 f(x) - 5E^4 f(x) + 10E^3 f(x) - 10E^2 f(x) + 5Ef(x) - f(x) = 0, \quad \text{Since } h=1$$

or $$f(x+5) - 5f(x+4) + 10f(x+3) - 10f(x+2) + 5f(x+1) - f(x) = 0 \qquad \ldots.(1)$$

Putting $x = 0$ in the equation (1), we get

$$f(5) - 5f(4) + 10f(3) - 10f(2) + 5f(1) - f(0) = 0$$

or $$0 - 5(-1) + 10\times 0 - 10\times 3 + 5\times 8 - f(0) = 0$$

or $\quad f(0) = 15.$

Hence the missing term is $f(0) = 15$.

Example.6. Given that $u_0+u_8 = 1.9243$, $u_1+u_7 = 1.9590$, $u_2 + u_6 = 1.9823$, and $u_3 + u_5 = 1.9956$. Determine the value of u_4.

Solution. The given values are

$u_0+u_8 = 1.9243$,

$u_1+u_7 = 1.9590$,

$u_2 + u_6 = 1.9823$,

and $u_3 + u_5 = 1.9956$.

Here 8 values of u_x are given. Therefore we have

$$\Delta^8 u_x = 0$$

or $\quad (E-I)^8 u_x = 0$

or $\quad (E^8 - {}^8C_1E^7 + {}^8C_2E^6 - {}^8C_3E^5 + {}^8C_4E^4 - {}^8C_5E^3 + {}^8C_6E^2 - {}^8C_7E + {}^8C_8I)u_x = 0$

or $\quad E^8u_x - 8E^7u_x + 28E^6u_x - 56E^5u_x + 70E^4u_x - 56E^3u_x + 28E^2u_x - 8Eu_x + u_x = 0 \quad \ldots.(1)$

Putting $x = 0$ in the equation (1), we get

or $\quad u_8 - 8u_7 + 28u_6 - 56u_5 + 70u_4 - 56u_3 + 28u_2 - 8u_1 + u_0 = 0$

or $\quad (u_8 + u_0) - 8(u_7 + u_1) + 28(u_6 + u_2) - 56(u_5 + u_3) + 70u_4 = 0$

or $\quad 1.9243 - 5 \times 1.9590 + 28 \times 1.9823 - 56 \times 1.9956 + 70u_4 = 0$

or $70u_4 = 69.9969$

or $u_4 = 0.9999$.

Hence the value of u_4 is 0.9999.

3.5 Newton's forward Interpolation with equal intervals

Newton's forward interpolation is a numerical method used to find the values of a function at points between the given data points. This method specifically applies when the intervals between the data points are equal. It is also known as Newton-Gregory's Formula for Forward Interpolation with equal intervals.

Suppose $y = f(x)$ is a function which assumes the values $f(a), f(a+h), f(a+2h), \ldots\ldots, f(a+nh)$ for $x = a, a+h, a+2h, \ldots\ldots, a+nh$ respectively where h is the difference of the arguments.

x	a	$a + h$	$a + 2h$	……..	$a + nh$
y	$f(a)$	$f(a+h)$	$f(a + 2h)$	……..	$f(a + nh)$

Consider $f(x)$ is a polynomial in x of degree n. Therefore $f(x)$ can be written as

$$f(x) = a_0 + a_1(x-a) + a_2(x-a)(x-a-h) + a_3(x-a)(x-a-h)(x-a-2h)$$

$$+ \ldots\ldots + a_n(x-a)(x-a-h)\ldots\ldots(x-a-(n-1)h) \qquad \ldots(1)$$

Where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants

Substituting the sequentially values $x = a, a+h, a+2h, \ldots\ldots, a+nh$ in equation (1), we get

$$f(a) = a_0$$

or $a_0 = f(a)$

Now we have

$$f(a + h) = a_0 + a_1 (a + h - a)$$

or $f(a + 2h) = a_0 + a_1 h$

$\Rightarrow$ $$a_1 = \frac{f(a+h) - a_0}{h}$$

$$= \frac{f(a+h) - f(a)}{h}$$

$$= \frac{\Delta f(a)}{h}$$

or $$a_1 = \frac{\Delta f(a)}{h}$$

Again we have

$$f(a+2h) = a_0 + a_1 (a+ 2h - a) + a_2 (a + 2h - a) (a + 2h - a - h)$$

$$= a_0 + a_1 2h - a_2 2h.h$$

or $2h^2 a_2 = f(a + 2h) - 2ha_1 - a_0$

or $$a_2 = \frac{f(a+2h) - 2(f(a+h) - f(a)) - f(a)}{2!h^2}$$

$$= \frac{f(a+2h) - 2f(a+h) + f(a)}{2!h^2}$$

$$= \frac{\Delta^2 f(a)}{2!h^2}$$

Proceeding in the same way, we get

$$a_3 = \frac{\Delta^3 f(a)}{3!h^3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_n = \frac{\Delta^n f(a)}{n!h^n}$$

Putting the values $a_0, a_1, a_2, a_3, \ldots\ldots\ldots, a_{n-1}, a_n$ into equation (1), we get

$$f(x) = f(a) + (x-a)\frac{\Delta f(a)}{h} + (x-a)(x-a-h)\frac{\Delta^2 f(a)}{2!h^2}$$

$$+(x-a)(x-a-h)(x-a-2h)\frac{\Delta^3 f(a)}{3!h^3} + \ldots\ldots\ldots\ldots\ldots$$

$$+(x-a)(x-a-h)(x-a-2h) + \ldots + (x-a-(n-1)h)\frac{\Delta^n f(a)}{n!h^n!} \qquad \ldots (2)$$

This is Newton-Gregory formula for forward interpolation putting $x=a+hu$ in the equation (2), we get

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots.(u-(n-1))}{n!}\Delta^n f(a)$$

This method is useful for estimating values between the given data points with equal intervals when interpolating forward.

The accuracy of the interpolation depends on the degree of the interpolating polynomial and the number of data points used in the interpolation.

3.6 Newton's backward Interpolation with equal intervals

Newton backward interpolation is a numerical method used to find the values of a function at points between the given data points, specifically when the intervals between the data points are equal. This method is an extension of Newton's forward interpolation and is useful when you need to interpolate backward from a given point. It is also known as Newton-Gregory's Formula for backward Interpolation with equal intervals.

Suppose $y = f(x)$ is a function which assumes the value $f(a), f(a+h), f(a+2h), \dots\dots\dots\dots f(a+nh)$ for $x = a, a+h, a+2h, \dots\dots\dots\dots, a+nh$, respectively where h is the difference of arguments.

x	a	$a+h$	$a+2h$	……..	$a+nh$
y	$f(a)$	$f(a+h)$	$f(a+2h)$	……..	$f(a+nh)$

Consider $f(x)$ is a polynomials in x of degree n. Therefore $f(x)$ can be written as

$$f(x) = a_0 + a_1(x-a-nh) + a_2(x-a-nh)(x-a-(n-1)h)$$

$$+ a_3(x-a-nh)(x-a-(n-1)h)(x-a-(n-2h) + \dots.$$

$$+a_n(x-a-nh)(x-a-(n-1)h)\dots\dots\dots\dots(x-a-h) \qquad \dots(3)$$

where $a_0, a_1, a_2, \dots\dots\dots, a_n$ are constants.

Substituting the sequentially values $x=a+nh$, $a+(n+1)h$, $a+(n-2)h$, $\dots\dots\dots\dots$, $a+h$ in the equation (3), we get

$$f(a+nh) = a_0$$

$$\Rightarrow \qquad a_0 = f(a+nh)$$

Now we have

$$f(a+(n-1)h) = a_0 + a_1(a+(n-1)h - a - nh)$$

or

$$a_1 = \frac{f(a+nh) - f(a+(n-1)h)}{h}$$

$$= \frac{\nabla f(a+nh)}{h}$$

or

$$a_1 = \frac{\nabla f(a+nh)}{h}$$

Again we have

$$f(a+(n-2)h) = a_0 + a_1(a+(n-2)h - a - nh) + a_2(a+(n-2)h - a - nh).(a+(n-2)h - a - (n-1)h)$$

or

$$a_2 = \frac{\nabla^2 f(a+nh)}{2!h^2}$$

Proceeding in the same way, we get

$$a_3 = \frac{\nabla^3 f(a+nh)}{3!h^3}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$a_n = \frac{\nabla^n f(a+nh)}{n!h^n}$$

Putting the values $a_0, a_1, a_2, a_3, \ldots\ldots\ldots, a_{n-1}, a_n$ into equation (3), we get

$$f(x) = f(a+nh) + (x-a-nh)\frac{\nabla f(a+nh)}{h} + (x-a-nh)(x-a-(n-1)h)\frac{\nabla^2 f(a+nh)}{2!h^2}$$

$$+.....+(x-a-nh)(x-a-(n-1)h).........(x-a-h)\frac{\nabla^n f(a+nh)}{n!h^n} \qquad \text{...(4)}$$

This is the Newton-Gregory's formula for backward interpolation putting $x=a+nh+hu$ in equation (4), we get

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+............+\frac{u(u+1)(u+2).......(u+n-1)}{n!}\nabla^n f(a+nh)$$

This method is useful for estimating values between the given data points with equal intervals when interpolating backward.

The accuracy of the interpolation depends on the degree of the interpolating polynomial and the number of data points used in the interpolation.

Examples

Example.7. Using the Newton formula to find the number of students who obtain less than 45 marks, from the following data:

Marks	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**
No. of students	**31**	**42**	**51**	**35**	**31**

Solution. The difference table of the given data is:

Marks	No. of students	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
Below 40	31				
		42			
Below 50	73		9		
		51		-25	
Below 60	124		-16		37
		35		12	
Below 70	159		-4		
		31			
Below 80	190				

Here a = 40, h = 10, x = 45.

Than we have

$$\begin{aligned} u &= \frac{x-a}{h} \\ &= \frac{45-40}{10} \\ &= 0.5. \end{aligned}$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$f(45) = f(40) + 0.5\Delta f(40) + \frac{(0.5)(0.5-1)}{2!}\Delta^2 f(40)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)}{3!}\Delta^3 f(40)$$

$$+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3}{4!}\Delta^4 f(40)$$

$$= 31 + 0.5(42) + \frac{(0.5)(-0.5)}{2}(9) + \frac{(0.5)(-0.5)(-1.5)}{6}(-25)$$

$$+ \frac{(0.5)(-0.5)(1.5)(-2.5)}{24}(37)$$

$$= 31 + 21 - 1.125 - 1.563 - 1.445$$

$f(45) = 47.867$.

Hence the approximately 47.867 students obtained less than 45 marks.

Example.8. Determine the values of $y(0.25)$, $y(0.62)$ and $y(0.46)$ from the following data:

x:	0	0.2	0.4	0.6	0.8
$y = f(x)$:	0.3989	0.3910	0.3683	0.3332	0.2897

Sol. As per discussed in the question for finding the value of y at $x = 0.25$ can be obtained by using Newton forward interpolation formula. And for $x = 0.46$ and 0.62, the values of y can be obtained by using Newton backward interpolation formula.

The forward difference table is

x	$y=f(x)$	$\Delta y=\Delta f(x)$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	0.3989				
		–0.0079			
0.2	0.3910		–0.0148		
		–0.0227		0.0024	
0.4	0.3683		–0.0124		0.0016
		–0.0351		0.0040	
0.6	0.3332		–0.0084		
		–0.0435			
0.8	0.2897				

Here a = 0, h = 0.2, x = 0.25.

Than we have

$$u = \frac{x-a}{h} = \frac{0.25-0}{0.2} = 1.25.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \quad \ldots(1)$$

From equation (1) and using above table, we have

$$y(0.25) = 0.3989 + 1.25\,(-0.0079) + \frac{1.25(1.25-1)}{2}(-0.0148)$$

$$+\frac{1.25(1.25-1)(1.25-2)}{6}(0.0024) + \frac{1.25(1.25-1)(1.25-2)(1.25-3)}{24}(0.0016)$$

$$= 0.3989 - 0.009875 - 0.00231255 - 0.00009375 + 0.000027343$$

$$= 0.386646093$$

$$\approx 0.3866\,(\text{Approximately}).$$

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots(u+n-1)}{n!}\nabla^n f(a+nh) \quad \ldots(2)$$

where $u = \dfrac{x-(a+nh)}{h}$

For $x = 0.46$, we have

$$u = \frac{0.46 - 0.8}{0.2} = -1.7 \qquad \ldots(3)$$

and for $x = 0.62$, we have

$$u = \frac{0.62 - 0.8}{0.2} = -0.9 \qquad \ldots(4)$$

Using equations (2) and (3), we have

$$f(a + nh + hu) = y(x = 0.46) = 0.2897 + (-1.7)(-0.0435)$$

$$+ \frac{(-1.7)(-1.7+1)}{2}(-0.0084) + \frac{(-1.7)(-1.7+1)(-1.7+2)}{6}(0.0040)$$

$$+ \frac{(-1.7)(-1.7+1)(-1.7+2)(-1.7+3)}{24}(0.0016)$$

$$= 0.2897 + 0.07395 - 0.004998 + 0.000238 + 0.000003094$$

$$y(0.46) = 0.358920940$$

$$\approx 0.3589 \,(\text{Approximately}).$$

Using equations (2) and (4), we have

$$y(0.62) = 0.2897 + (-0.09)(-0.0435) + \frac{(-1.9)(-0.9+1)}{2}(-0.0084)$$

$$+ \frac{(-1.9)(-0.9+1)(0.9+2)}{6}(0.0040)$$

$$+ \frac{(-1.9)(-0.9+1)(-0.9+2)(-0.9+3)}{24}(0.0016)$$

$$= 0.2897 + 0.03915 + 0.00037 - 0.000066 - 0.00001386$$

$$y(0.62) = 0.32914814$$

$$\approx 0.3291 (\text{Approximately}).$$

Hence the approximately values of $y(0.25)$, $y(0.62)$ and $y(0.46)$ are 0.3866, 0.3589 and 0.3291 respectively

Example.9. Using Newton's forward and backward interpolation formulae's, obtained the value of *f*(1.6) from the following data:

x	**1**	**1.4**	**1.8**	**2.2**
***f*(*x*)**	**3.49**	**4.82**	**5.96**	**6.5**

Solution: The forward difference table of the given data is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

Here $a = 1$, $h = 0.4$, $x = 1.6$.

Than we have $u = \frac{1.6-1}{0.4} = \frac{0.6}{0.4} = 1.5.$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$f(1.6) = f(1) + 1.5\Delta f(1) + \frac{1.5(1.5-1)}{2!}\Delta^2 f(1) + \frac{1.5(1.5-1)(15-2)}{3!}\Delta^3 f(1)$$

$$= 3.49 + 1.5 \times 1.33 + \frac{1.5 \times 0.5}{2!}(-0.19) + \frac{1.5 \times 0.5 \times (-0.5)}{3!}(-0.41)$$

=3.49+1.995-0.07125+0.025625

$f(1.6) = 5.439375.$

Now the backward difference table of the given data is:

x	$f(x)$	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+ \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots(u+n-1)}{n!}\nabla^n f(a+nh) \quad \ldots(2)$$

where $u = \dfrac{x-(a+nh)}{h}$

Here $x = 1.6$, $a + nh = 2.2$, $h = 0.4$.

Then we have

$$u = \frac{x-a+nh)}{h} = \frac{1.6-2.2}{0.4} = \frac{-0.6}{0.4} = -1.5 \quad \ldots(3)$$

Using equations (2) and (3), we have

$$f(1.6) = 6.5 + (-1.5)\times 0.54 + \frac{(-1.5)(-1.5+1)(u+1)}{2!}(-0.6)$$

$$+\frac{(-1.5)(-1.5+1)(-1.5+2)}{3!}(-0.4)$$

$= 6.5 - 0.81 - 0.225 - 0.025625$

$f(1.6) = 5.439375$.

Example.10. The population of a town in the decennial census were as under:

Year x	**1891**	**1901**	**1911**	**1921**	**1931**
Population f(x) (In thousands)	**46**	**66**	**81**	**93**	**101**

Calculate the population for the year 1895 and 1925 with the help of Newton forward as well as backward interpolation formula.

Solution: The difference table of the given data is as under:

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Here a = 1891, h =10, x = 1895.

Than we have

$$u = \frac{1895 - 1891}{10} = \frac{4}{10} = 0.4.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$f(1895) = f(1891) + (0.4)\Delta f(1891) + \frac{(0.4)(0.4-1)}{2!}\Delta^2 f(1891)$$

$$+\frac{(0.4)(0.4-1)(0.4-2)}{3!}\Delta^3 f(1891) + \frac{(0.4)(0.4-1)(0.4-2)(0.4-3)}{4!}\Delta^4 f(1891)$$

$$= 46 + (0.4)\times 20 + \frac{(0.4)(-0.6)}{2!}(-5) + \frac{(0.4)(0.6)(-1.6)}{3!}(2)$$

$$+\frac{(0.4)(-0.6)(-1.6)(-2.6)}{4!}(-3)$$

= 46+8+0.6+0.128+0.1248

$f(1895)$ =54.8528.

Now for *x*=1925. We have *a* =1891, *h* = 10, *x* = 1925.

Than we have

$$u=\frac{x-a}{h}=\frac{1925-1891}{10}=3.4. \qquad \ldots(2)$$

Using equations (1) and (2), we have

$$f(1925)=f(1891)+(3.4)\Delta f(1891)+\frac{(3.4)(3.4-1)}{2!}\Delta^2 f(1891)$$

$$+\frac{(3.4)(3.4-1)(3.4-2)}{31}\Delta^3 f(1891)$$

$$+\frac{(3.4)(3.4-1)(3.4-2)(3.4-3)}{4!}\Delta^4 f(1891)$$

$$=46+3.4\times 20+\frac{3.4\times 2.4}{2!}\times(-5)+\frac{3.4\times 2.4\times 1.4}{3!}(2)$$

$$+\frac{3.4\times 2.4\times 1.4\times 0.4}{4!}(-3)$$

$$=46+68-20.4+3.808-0.5712$$

$f(1925)=96.8368.$.

Now using the Newton Backward interpolation Formula. The backward difference table of the given data is as under

x	*y=f(x)*	$\nabla f(x)$	$\nabla^2 f(x)$	$\nabla^3 f(x)$	$\nabla^4 f(x)$
1891	46				

1901 1911 1921 1931	66 81 93 101	20 15 12 8	-5 -3 -4	2 -1	-3

Here $(a+nh) = 1931 = b$, $h = 10$, $x = 1895$.

Then we have

$$u = \frac{1895 - 1931}{10}$$

$$= \frac{-36}{10}$$

$$= -3.6 \qquad \ldots(3)$$

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots.(u+n-1)}{n!}\nabla^n f(a+nh) \qquad \ldots(4)$$

From equation (4) and using above table, we have

$$f(1895) = f(1931) + (-3.6)\times\nabla f(1931) + \frac{(-3.6)(-3.6+1)}{2!}\nabla^2 f(1931)$$

$$+ \frac{(-3.6)(-3.6+1)(-3.6+2)}{3!}\nabla^3 f(1931)$$

$$+ \frac{(-3.6)(-3.6+1)(-3.6+2)(-3.6+3)}{4!}\nabla^4 f(1931)$$

$$=101+(-3.6)\times 8+\frac{(-3.6)\times(2.6)}{2!}(-4)+\frac{(-3.6)(2.6)(-1.6)}{3!}(-1)$$

$$+\frac{(-3.6)(-2.6)(-1.6)(-0.6)}{4!}(-3)$$

$$=101-28.8-18.72+2.496-1.1232$$

$$f(1895)=54.8528..$$

Now for x=1925.

We have $a+nh$ =1931, h = 10, x = 1925.

Than we have

$$u=\frac{x-(a+nh)}{h} \qquad \ldots(5)$$

$$=\frac{1925-1931}{10}=\frac{-6}{10}=-0.6$$

Using equations (4) and (5), we have

$$f(1925)= f(1931)+(-0.6)\nabla f(1931)+\frac{(-0.6)(-0.6+1)}{2!}\nabla^3 f(1931)$$

$$+\frac{(-0.6)(-0.6+1)(-0.6+2)}{3!}\nabla^3 f(1931)$$

$$+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{4!}\nabla^4 f(1931)$$

$$=101+(-0.6)\times 8+\frac{(-0.6)\times(0.4)}{2!}(-4)+\frac{(-0.6)(0.4)(1.4)}{}(-1)$$

$$+\frac{(0.6)(0.4)(1.4)(2.4)}{4!}\times(-3)$$

=101 - 4.8 + 0.48 + 0.056+ 0.1008

$f(1925) = 96.8368.$

Example.11. From the following table, determine the form of the function *f*(*x*):

x	**0**	**1**	**2**	**3**	**4**
f(x)	**13**	**19**	**28**	**40**	**55**

Solution: The forward difference table of the given data is:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^3 f(x)$
0	13				
		6			
1	19		3		
		9		0	
2	28		3		0
		12		0	
3	40		3		
		15			
4	55				

Here a = 0, h =1.

Than we have

$$u = \frac{x-0}{1} = x.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \quad \ldots(1)$$

From equation (1) and using above table, we have

$$f(x) = f(0) + x\Delta f(0) + \frac{x(x-1)}{2!}\Delta^2 f(0) + \frac{x(x-1)(x-2)}{3!}\Delta^3 f(0) + \ldots.$$

$$= 13 + x(6) + \frac{x(x-1)}{2!}(3)$$

$$= 13 + 6x + \frac{3}{2}x^2 - \frac{3}{2}x$$

Hence the function $f(x)$ is $\frac{3}{2}x^2 + \frac{9}{2}x + 13$.

3.7 Summary

In numerical analysis, discovering a missing term usually entails recognizing a pattern or relationship within a sequence of numbers.

Consider one value of $f(x)$ be missing from the given set of $(n+1)$ values (*i.e.*, n values are given) of x, the values of x being equidistant. This means we can assume $y = f(x)$ to be a polynomial of degree $(n-1)$ in x. Therefore, we have

$\Delta^n f(x) = 0$ or $(E-I)^n f(x) = 0$.

Newton forward interpolation formula is

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots$$

where $u = \frac{x-a}{h}$.

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+ \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots.(u+n-1)}{n!}\nabla^n f(a+nh)$$

where $u = \dfrac{x-(a+nh)}{h}$.

3.8 Terminal Questions

Q.1. Write the Newton Forward and backward difference formula.

Q.2. From the following table to determine the values of $f(0.2),\ f(1.1),\ f(1.9), f(2.1), f(2.9),\ f(3.8)$; suggest which method (either Newton forward interpolation formula or Newton backward interpolation formula) is appropriate for finding the $f(0.2),\ f(1.1),\ f(1.9), f(2.1), f(2.9),\ f(3.8)$ values.

x	0	1	2	3	4
f(x)	2	5	15	22	42

Q.3. Determine the missing term in the following table:

x	0	1	2	3	4
f(x)	1	3	9	–	81

Q.4. Determine the missing terms in the following table:

x	2	2.1	2.2	2.3	2.4	2.5	2.6
$f(x)$	0.135	–	0.111	0.100	–	0.082	0.024

Q.5. Determine the value of the area of the circle of diameter 82 from the following data:

D(Diameter)	80	85	90	95	100
A(Area)	5026	5674	6362	7088	7854

Q.6. From the following table, obtain the form of the function $f(x)$:

x	3	5	7	9	11
$f(x)$	6	24	58	108	174

Q.7. Obtain the values of $f(1.5)$ and $f(7.5)$ from the following data:

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Answer

3. 31.

4. 0.123 and 0.090.

5. 5280.

6. $2x^2 + 7x + 9$

7. 3.375, 421.87.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-4: Gauss' and Stirling Interpolation Formula with Equal Intervals

Structure

4.1 Introduction

Gauss' Interpolation Formula is a mathematical technique used to estimate values between known data points in a sequence. It helps in predicting intermediate values within a set of given data points. This method is especially useful when you want to approximate values between two known points in a sequence or dataset. Gauss' Forward Interpolation Formula is a numerical method used for estimating values between known data points in a sequence or dataset. Specifically, it is employed for forward interpolation, helping approximate values that come after the known data points. This formula is particularly useful when the intervals between data points are equal. In numerical analysis, Gauss' Forward Interpolation Formula serves as a mathematical tool for making predictions or approximations in situations where only specific data points are available.

The Stirling interpolation Formula is a method used in numerical analysis for interpolation. It allows for the estimation of values between known data points in a sequence or dataset. In essence, this formula helps fill in the gaps between existing data points, providing a way to make predictions or approximations within a given set of numerical values.

In this unit we shall discuss about the Gauss' Forward interpolation formula with equal intervals, Gauss' Backward interpolation formula with equal intervals and Stirling difference formula.

4.2 Objectives

After reading this unit the learner should be able to understand about:

- Gauss' Forward interpolation formula with equal intervals
- Gauss' Backward interpolation formula with equal intervals
- Stirling Difference Formula

4.3 Gauss' Forward Interpolation Formula with Equal Intervals

Gauss' Forward Interpolation formula is a mathematical method for estimating intermediate values within a set of known data points. It is particularly useful for approximating values between two known points in a sequence.

The Gauss's forward interpolation formula is

$$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots$$

4.4 Gauss' Backward Interpolation Formula with Equal Intervals

Gauss' Backward Interpolation Formula is a numerical technique used to estimate values between known data points in a sequence or dataset. It is particularly useful for interpolating values when the intervals between data points are not equal. This formula allows for backward interpolation, meaning it helps approximate values that precede the known data points in the sequence.

In numerical analysis, Gauss' Backward Interpolation Formula provides a mathematical tool for making predictions or approximations in situations where only specific data points are available. The Gauss's backward interpolation formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+\frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots$$

4.5 Stirling Interpolation Formula

Stirling interpolation Formula is a mathematical tool in numerical analysis used for interpolation. Specifically, it's employed to estimate values between known data points in a sequence or dataset. This formula is part of the broader set of techniques for interpolating values and finding intermediate points within a given dataset.

Stirling's interpolation Formula is employed for both forward and backward interpolation, allowing for the approximation of values that precede or succeed the available data points. In essence, this formula is a mathematical tool that aids in making predictions or approximations in situations where only specific data points are known and a continuous estimate is needed. The mean of Gauss's forward interpolation formula and Gauss's backward interpolation formula gives Stirling's interpolation formula. The Gauss's forward interpolation formula is

$$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots \quad \ldots.(1)$$

and the Gauss's backward interpolation formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots \quad \ldots.(2)$$

The mean of equation (1) and (2) is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3} + \ldots\ldots \quad \ldots.(3)$$

The equation (3) is known as the Stirling's interpolation formula.

Check your Progress

1. Write the formula for Gauss's forward and backward interpolation formula?

2. Write the formula for Stirling interpolation formula.

Examples

Example.1. Use Gauss's Forward formula to find the value of *f*(25), from the following data:

x	**20**	**24**	**28**	**32**
$y = f(x)$	**14**	**32**	**35**	**40**

Solution. Here h =4. Taking 24 as origin, *i.e.* a = 24. Now we have

$$u = \frac{x-a}{h} = \frac{25-24}{4} = \frac{1}{4} = 0.25.$$

The difference table is:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	–1	14			
			18		
24	0	32		–15	
			3		

28	1	35	5	2	17
32	2	40			

The Gauss's forward interpolation formula is

$$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots \quad ....(1)$$

From equation (1) and using above table, we have

$$=32+(0.25)\times 3+\frac{(0.25)(0.25\text{-}1)}{2!}(\text{-}15)+\frac{(0.25+1)(0.25)(0.25\text{-}1)}{3!}(17)$$

$$=32+(0.25)\times 3+\frac{(0.25)(0.25\text{-}1)}{2}(\text{-}15)+\frac{(0.25+1)(0.25)(0.25\text{-}1)}{6}(17)$$

$$=32+0.75+(\text{-}0.09375)(\text{-}15)+(\text{-}0.0390625)(17)$$

$$=32+0.75+1.40625\text{-}0.6640625$$

=33.4921875.

$\approx$ 33.49.

Hence the value of *f*(25) is 33.49.

Example.2. Use Gauss's Backward formula to show that the value of $\sqrt{12516}$ =111.8749301**, from the following data:**

x	**12500**	**12510**	**12520**	**12530**
$y = f(x) = \sqrt{x}$	**111.803399**	**111.848111**	**111.892806**	**111.937483**

Solution. Here $h = 10$. Taking 12520 as origin, *i.e.* $a = 12520$. Now we have

$$u = \frac{x-a}{h} = \frac{12516-12520}{10} = \frac{-4}{10} = -0.4.$$

The difference table is:

x	u	$10^6 f(x)$	$10^6 \Delta f(x)$	$10^6 \Delta^2 f(x)$	$10^6 \Delta^3 f(x)$
12500	–2	111803399			
			44712		
12510	-1	111848111		–17	
			44695		-1
12520	0	111892806		-18	
			44677		
12530	1	111937483			

The Gauss's backward interpolation formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots \quad \text{....(1)}$$

From equation (1) and using above table, we have

$$10^6 f(-0.4) = 111892806 + (-0.4)\times(44695) + \frac{(-0.4)(-0.4+1)}{2!}(-18) + \frac{(-0.4+1)(-0.4)(-0.4-1)}{3!}(-1)$$

$$= 111892806 - 17878 + \frac{(0.4)(0.6)}{2}(18) - \frac{(0.6)(0.4)(1.4)}{6}(1)$$

$$= 111892806 - 17878 + 2.16 - 0.056$$

$$= 111874930.104$$

$$f(-0.4) = 111.874930104$$

Hence the value of $\sqrt{12516}$ is 111.874930104.

Example.3. Use Stirling's formula to obtain y_{32} given that

x :	20	25	30	35	40	45
$y = f(x)$:	14.035	13.674	13.257	12.734	12089	11.309

Solution. Here h =5. Taking 30 as origin, *i.e.* a = 30. Now we have

$$u = \frac{x-a}{h} = \frac{32-30}{5} = 0.4.$$

The difference table is:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
20	–2	14.035	–0.361				

25	–1	13.674	–0.399	–0.038	– 0.104		
30	0	13.275	– 0.541	–0.142	– 0.038	0.142	–0.211
35	1	12.734	–0.645	–0.104	– 0.031	–0.069	
40	2	12.089	–0.780	–0.135			
45	3	11.309					

We know that the Stirling's interpolation formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots \quad \ldots.(1)$$

From equation (1) and using above table, we have

$$=13.275+(0.4)\frac{(-0.399-0.541)}{2}+\frac{(0.4)^2}{2!}(-0.142)$$

$$+\frac{(0.4)((0.4)^2-1)}{3!}\left(\frac{0.038-0.104}{2}\right)+\frac{(0.4)^2((0.4)^2-1)}{4!}(0.142)$$

=13.275 – 0.188 – 0.01136 + 0.001848 – 0.0007952

=13.07669.

$\approx$13.077.

Hence the value of y_{32} is 13.077.

Example.4. Use Stirling formula to obtain $y_{12.2}$ from the following data:

x	**10**	**11**	**12**	**13**	**14**
$10^5 y_x$	**23967**	**28060**	**31788**	**35209**	**38368**

Solution. Here h = 1. Taking 12 as origin, *i.e.* a = 12.

Now we have

$$u = \frac{x-a}{h} = \frac{12.2-12}{1} = 0.2.$$

The difference table is:

x	***u***	**$10^5 y_u$**	**$10^5 \Delta y_u$**	**$10^5 \Delta^2 y_u$**	**$10^5 \Delta^3 y_u$**	**$10^5 \Delta^4 y_u$**
10	–2	23967				
			4093			
11	– 1	28060		– 365		
			3728		58	
12	0	31788		– 307		–13
			3421		45	
13	1	35209		– 262		
			3159			
14	2	38368				

We know that the Stirling's interpolation formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + ... \qquad ....(1)$$

From equation (1) and using above table, we have

$$= 31788 + (0.2)\frac{(3421+3728)}{2} + \frac{(0.2)^2}{2!}(-307)$$

$$+ \frac{(0.2)((0.2)^2-1}{3!}(45+58) + \frac{(0.2)^2((0.2)^2\text{-}1)}{4!}(\text{-}13)$$

$= 31788 + 714.9 – 6.14 – 3.296 + 0.0208$

$= 32493.4848$

$\approx 32493.49.$

Hence the value of $y_{12.2}$ is 32493.49.

Example.5. Use Stirling formula, to determine *log* 337.5 from the following data:

x	**310**	**320**	**330**	**340**	**350**	**360**
log₁₀x	**2.4913617**	**2.5051500**	**2.5185139**	**2.5314789**	**2.5440680**	**2.5563025**

Solution: Here *h* =10. Taking 330 as origin, *i.e.* *a* = 330.

Now we have

$$u = \frac{x-a}{h} = \frac{337.5-330}{10} = 0.75.$$

The difference table is:

x	u	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$	$\Delta^4 y_u$	$\Delta^5 y_u$
310	– 2	2.4913617					
			0.0137883				
320	– 1	2.5051500		– 0.0004244			
			0.0133639		0.0000255		
330	0	2.5185139		– 0.0003989		–0.0000025	
			0.0129650		0.0000230		0.0000008
340	1	2.5314789		– 0.0003759		–0.0000017	
			0.0128891		0.0000213		
350	2	2.5440680		– 0.0003546			
			0.0122345				
360	3	2.5563025					

We know that the Stirling's interpolation formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \dots\dots\dots \quad \text{....(1)}$$

From equation (1) and using above table, we have

$$= 2.5185139 + (0.75)\frac{(0.0133639 + 0.0129650)}{2} + \frac{(0.75)^2}{2!}(-0.0003989)$$

$$+\frac{(0.75)((0.75)^2-1)}{3!}+\frac{(0.0000255+0.0000230)}{2}$$

$$+\frac{(0.75)^2((0.75)^2-1)}{4!}(0.0000025)$$

=2.52827374

$\approx$ 2.52827374.

Hence the value of $log_{337.5}$ is 2.5282737.

Example.6. For the following data:

$x:$	0.10	0.15	0.20	0.25	0.30
$y=f(x):$	0.1003	0.1511	0.2026	0.2554	0.3093

Use suitable interpolation formula to calculate the values of *y* for:

(i) *x* = 0.14 (ii) *x* = 0.21 (iii) *x* = 0.28.

Sol. The difference table is

x	***y* = *f*(*x*)**	**Δy**	**$\Delta^2 y$**	**$\Delta^3 y$**	**$\Delta^4 y$**
0.10	0.1003				
0.15	0.1511	0.0508	0.0007		
0.20	0.2026	0.0515	0.0013	0.0006	-0.0008
0.25	0.2554	0.0528	0.0011	-0.0002	
0.30	0.3093	0.0539			

(i) For x = 0.14, to determine the value of y(0.14) we use Newton Forward interpolation formula.

Here a = 0.10, h = 0.05, x = 0.14.

Than we have

$$u = \frac{x-a}{h} = \frac{0.14-0.10}{0.05} = \frac{0.04}{0.05} = 0.8.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$f(0.14) = 0.1003 + 0.8(0.0508) + \frac{0.8\times(0.8-1)}{2}(0.0007)$$

$$+\frac{0.8(0.8-1)(0.8-2)}{3!}\times 0.0006 + \frac{0.8(0.8-1)(0.8-2)(0.8-3)}{4!}\times(-0.0008)$$

$$= 0.1003 + 0.04064 + \frac{0.8\times(-0.2)}{2}(0.0007)$$

$$+\frac{0.8\times(-0.2)\times(-1.2)}{6}\times 0.0006 + \frac{0.8\times(-0.2)\times(-1.2)\times(-2.2)}{24}\times(-0.0008)$$

$$= 0.1003 + 0.04064 - \frac{0.8\times(0.2)}{2}(0.0007)$$

$$+\frac{0.8\times(0.2)\times(1.2)}{6}\times 0.0006 + \frac{0.8\times(0.2)\times(1.2)\times(2.2)}{24}\times(0.0008)$$

= 0.1003+0.04064-0.00056+0.0000192+0.00001408

= 0.100333328

(ii) Now to determine the value of *y* = (0.21), we use Stirling's interpolation formula.

The difference table is:

x	*u*	***y* = *f*(*x*)**	**Δ*y***	**Δ²*y***	**Δ³*y***	**Δ⁴*y***
0.10	– 2	0.1003				
0.15	– 1	0.1511	0.0508	0.0007		
0.20	0	0.2026	0.0515	0.0013	0.0006	-0.0008
0.25	1	0.2554	0.0528	0.0011	-0.0002	
0.30	2	0.3093	0.0539			

Here h =0.05. Taking 0.20 as origin, *i.e.* a = 0.20.

Now we have

$$u = \frac{x-a}{h} = \frac{0.21-0.20}{0.05} = \frac{0.01}{0.05} = 0.2.$$

The Stirling's difference formula is

$$y_u = y_0 + u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1}$$

$$+\frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \text{..........} \qquad \text{....(2)}$$

From equation (2) and using above table, we have

$$= 0.2026 + (0.2)\left(\frac{0.0528 + 0.0515}{2}\right) + \frac{(0.2)^2}{2} \times (0.0013)$$

$$+ \frac{(0.2)\left((0.2)^2 - 1\right)}{3!}\left(\frac{-0.0002 + 0.0006}{2}\right)$$
$$+ \frac{(0.2)^2\left((0.2)^2 - 1\right)}{4} \times (-0.0008)$$

$$= 0.2026 + (0.2)(0.05215) + \frac{0.04}{2} \times (0.0013)$$

$$+ \frac{(0.2)(0.04 - 1)}{6}(0.0002) + \frac{(0.04)(0.04 - 1)}{4} \times (-0.0008)$$

$$= 0.2026 + 0.01043 + (0.02) \times 0.0013$$
$$+ (-0.032) \times (0.0002) + (-0.0096) \times (-0.0008)$$

$$= 0.2026 + 0.01043 + 0.000026 - 0.0000064 + 0.00000768$$

$$= 0.21305728.$$

(iii) For x = 0.28, to determine the value of $y(0.28)$ we use Newton backward interpolation formula.

Here $(a + hn) = 0.30 = b$, $h = 0.05$, $x = 0.28$.

Then we have

$$u = \frac{0.28 - 0.30}{0.05} = \frac{-0.02}{0.05} = -0.4. \qquad \ldots(3)$$

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+ \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots(u+n-1)}{n!}\nabla^n f(a+nh) \qquad \ldots(4)$$

From equations (3), (4) and above table, we have

$$= 0.3093 + (-0.4)\times 0.0539 + \frac{(-0.4)(-0.4+1)}{2!}\times(0.0011)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)}{3!}\times(-0.0002)$$

$$+\frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!}\times(-0.0008)$$

$$= 0.3093 - 0.02156 - \frac{(0.4)(0.6)}{2}\times 0.0011$$

$$+\frac{(0.4)(0.6)(1.6)}{6}\times(0.0002) - \frac{(0.4)(0.6)(1.6)(2.6)}{24}\times(-0.0008)$$

$$= 0.3093 - 0.02156 - 0.12\times 0.0011$$

$$+(0.64)\times(0.0002) + (0.0416)\times(0.0008)$$

$= 0.3093 – 0.02156 - 0.000132 + 0.0000128 + 0.00003328$

$= 0.28765408.$

Example.7. Given the following data:

θ	0°	5°	10°	15°	20°	25°	30°
***f* = tanθ**	**0.00**	**0.0875**	**0.1763**	**0.2679**	**0.3640**	**0.4663**	**0.5774**

Determine the value of tan 3°, tan16°, tan28° stating the appropriate formula used.

Sol. The difference table for given data is:

θ°	tanθ	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$	$\Delta^6 f$
0°	0						
		0.0875					
5°	0.0875		0.0013				
		0.0888		0.0015			
10°	0.1763		0.0028		0.0002		
		0.0916		0.0017		–0.0002	
15°	0.2679		0.0045		0		0.0011
		0.0961		0.0017		0.0009	
20°	0.364		0.0062		0.0009		
		0.1023		0.0026			
25°	0.4663		0.0088				
		0.1111					
30°	0.5774						

For $\theta = 3^0$, to determine the value of $f(\theta) = \tan 3°$, we use Newton Forward interpolation formula.

Here $a = 0$, $h = 5$, $x = 3$.

Than we have

$$u = \frac{x-a}{h} = \frac{3-0}{5} = \frac{3}{5} = 0.6.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$\tan 3^\circ = 0 + (0.6)(0.0875) + \frac{(0.6)(0.6-1)}{2!}(0.0013) + \frac{(0.6)(0.6-1)(0.6-2)}{3!}(0.0015)$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)(0.6-3)}{4!}(0.0002)$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)(0.6-3)(0.6-4)}{5!}(-0.0002)$$

$$+ \frac{(0.6)(0.6-1)(0.6-2)(0.6-3)(0.6-4)(0.6-5)}{6!} \times (0.0011)$$

$$= 0 + 0.0525 + \frac{(0.6)(-0.4)}{2}(0.0013) + \frac{(0.6)(-0.4)(-1.4)}{6}(0.0015)$$

$$+ \frac{(0.6)(-0.4)(-1.4)(-2.4)}{24}(0.0002)$$

$$+ \frac{(0.6)(-0.4)(-1.4)(-2.4)(-3.4)}{120}(-0.0002)$$

$$+ \frac{(0.6)(-0.4)(-1.4)(-2.4)(-3.4)(-4.4)}{720} \times (0.0011)$$

$$= 0 + 0.0525 - (0.12)(0.0013) + (0.056)(0.0015)$$

$$- (0.0336)\,(0.0002) - (0.022848)\,(0.0002) - (0.0167552) \times (0.0011)$$

$$= 0 + 0.0525 - 0.00156 + 0.000084 - 0.00000672 - 0.0000045696 - 0.00001843072$$

$$= 0.05099427968.$$

$$\approx 0.0510.$$

Now to determine the value of of $f(\theta) = \tan 16°$, we use Stirling's interpolation formula.

The difference table is:

θ°	*u*	**tanθ**	**Δf**	**Δ²f**	**Δ³f**	**Δ⁴f**	**Δ⁵f**	**Δ⁶f**
0°	-3	0						
			0.0875					
5°	– 2	0.0875		0.0013				
			0.0888		0.0015			
10°	– 1	0.1763		0.0028		0.0002		
			0.0916		0.0017		–0.0002	
15°	0	0.2679		0.0045		0		0.0011
			0.0961		0.0017		0.0009	
20°	1	0.364		0.0062		0.0009		
			0.1023		0.0026			
25°	2	0.4663		0.0088				
			0.1111					
30°	3	0.5774						

Here $h = 5$. Taking 15 as origin, *i.e.* $a = 15$.

Now we have

$$u = \frac{x-a}{h}$$

$$= \frac{16-15}{5}$$

$$= \frac{1}{5} = 0.2.$$

The Stirling's interpolation formula is

$$y_u = y_0 + u\,\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1}$$
$$+ \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3} + \ldots\ldots. \qquad \ldots.(2)$$

From equation (2) and using above table, we have

$$\tan\ 16° = 0.2679 + (\ 0.2\ )\left[\frac{0.0961+0.0916}{2}\right] + \frac{(0.2)^2}{2!}(0.0045)$$

$$+ \frac{(0.2)\left[(0.2)^2-1\right]}{3!} \times \left[\frac{0.0017+0.0017}{2}\right] + \frac{(0.2)^2\left[(0.2)^2-1\right]}{4!} \times 0$$

$$+ \frac{(0.2)^2\left[(0.2)^2-1\right]\left[(0.2)^2-2^2\right]}{5!}\left[\frac{-0.0002+0.0009}{2}\right]$$

$$+ \frac{(0.2)^2\left[(0.2)^2-1\right]\left[(0.2)^2-2^2\right]}{6!} \times (0.0011)$$

$$= 0.2679 + (\ 0.2\ )(0.09385) + (\ 0.02\ )(0.0045) + (-0.032) \times (0.0017) + 0$$

$$+ (0.006336)(0.00035) + (0.0002112)(0.0011)$$

$$= 0.2679 + 0.01877 + 0.00009 - 0.0000544 + 0.0000022176 + 0.00000023232$$

$$= 0.2867080499.$$

$$\approx 0.2867.$$

Now to determine the value of of $f(\theta) = \tan 28°$, we use Newton backward interpolation formula.

Here $(a + hn) = 30 = b$, $h = 5$, $x = 28$.

Then we have

$$u = \frac{28-30}{5} = \frac{-2}{5} = -0.4. \qquad \ldots(3)$$

Newton backward interpolation formula is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh) + \ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots\ldots(u+n-1)}{n!}\nabla^n f(a+nh) \qquad \ldots(4)$$

From equations (3), (4) and above table, we have

$$\tan 28° = 0.5774 + (-0.4)(0.1111) + \frac{(-0.4)(-0.4+1)}{2!}(0.0088)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)}{3!} \times 0.0026$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)}{4!} \times 0.0009$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)}{5!} \times (0.0009)$$

$$+ \frac{(-0.4)(-0.4+1)(-0.4+2)(-0.4+3)(-0.4+4)(-0.4+5)}{6!} \times (0.0011)$$

$= 0.5324$

Example.8. Use the following data to calculate the values of *y* when *x*=1.1, *x*=1.7, *x*=1.9.

$x:$	1.00	1.2	1.4	1.6	1.8	2.0
$y = f(x):$	0.1120	0.1125	0.1243	0.1475	0.1623	0.1824

Sol. The forward difference table is

x	*y*	Δ**y**	Δ^2**y**	Δ^3**y**	Δ^4**y**	Δ^5**y**
1.0	0.1120					
		0.0005				
1.2	0.1125		0.0113			
		0.0118		0.0001		
1.4	0.1243		0.0114		–0.0199	
		0.0232		–0.0198		0.0534
1.6	0.1475		–0.0084		0.0335	
		0.0148		0.0137		
1.8	0.1623		0.0053			
		0.0201				
2.0	0.1824					

For $x = 1.1$, to determine the value of $y(1.1)$ we use Newton Forward interpolation formula.

Here $a = 1$, $h = 0.2$, $x = 1.1$.

Than we have

$$u = \frac{x-a}{h} = \frac{1.1-1}{0.2} = \frac{0.1}{0.2} = 0.5.$$

Using Newton forward interpolation formula, we have

$$f(x) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a) + \ldots\ldots \qquad \ldots(1)$$

From equation (1) and using above table, we have

$$y(1.1) = 0.1120 + (0.5)\,(0.0005) + \frac{(0.5)\,(0.5-1)}{2!}(0.0113)$$

$$+ \frac{(0.5)(-0.5)\,(-1.5)}{3!}\,(0.0001) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{4!}(-0.0199)$$

$$+ \frac{(0.5)(-0.5)(-1.5)(-2.5)(-3.5)}{5!}\,(0.0534)$$

$$y(1.1) = 0.1120 + (0.5)\,(0.0005) + \frac{(0.5)\,(0.5-1)}{2}(0.0113)$$

$$+ \frac{(0.5)(-0.5)\,(-1.5)}{6}\,(0.0001) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}(-0.0199)$$

$$+ \frac{(-0.25)\,(-1.5)(-2.5)(-3.5)}{120}\,(0.0534)$$

$$= 0.1120 + 0.00025 - (0.125)(0.0113) + (0.0625)\,(0.0001)$$

$$+ (0.0390625)(0.0199) + (0.02734375)\,(0.0534)$$

$$= 0.1120 + 0.00025 - 0.0014125 + 0.00000625$$

$$+ 0.00077734375 + 0.00146015625$$

= 0.1120 + (–14125×10–3) + 0.0000025+0.00077734+0.0014601

= 0.11308125.

Now to determine the value of y = (1.7), we use Stirling's interpolation formula.

Here h =0.2. Taking 1.6 as origin, *i.e.* a = 1.6.

Now we have

$$u = \frac{x-a}{h} = \frac{1.7-1.6}{0.2} = \frac{0.1}{0.2} = 0.5.$$

The difference table is:

x	y	u	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1.0	0.1120	– 3					
1.2	0.1125	– 2	0.0005	0.0113			
1.4	0.1243	– 1	0.0118	0.0114	0.0001	–0.0199	
1.6	0.1475	0	0.0232	–0.0084	–0.0198	0.0335	0.0534
1.8	0.1623	1	0.0148	0.0053	0.0137		
2.0	0.1824	2	0.0201				

The Stirling's difference formula is

$$\begin{aligned} y_u = y_0 &+ u\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} \\ &+ \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} \\ &+ \frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3} + \ldots\ldots \end{aligned}$$

....(2)

From equation (2) and using above table, we have

$$y(1.7) = 0.1475 + (0.5)\left[\frac{0.0148+0.0232}{2}\right]$$

$$+\frac{(0.5)^2}{2!}(-0.0084)+\frac{(0.5)(0.25-1)}{3!}\left[\frac{0.0137+(-0.0198)}{2}\right]$$

$$+\frac{(0.5)^2(0.25-1)}{4!}(0.0335)$$

$$= 0.1475 + (0.5)(0.019)-(0.125)(0.0084)$$

$$+(0.0625)(0.00305)+(0.0078125)(0.0335)$$

$$= 0.1475 + 0.0095-0.00105+0.000190625+0.00026171875$$

$$= 0.15640234375$$
$$\approx 0.1564.$$

For $x = 1.9$, to determine the value of $y(1.9)$ we use Newton backward interpolation formula.

Here $(a+hn) = 2.0= b$, $h = 0.2$, $x = 1.9$.

Then we have

$$u=\frac{1.9-2.0}{0.2}=\frac{-0.1}{0.2}=-0.5. \quad \ldots(3)$$

Newton backward interpolation formula is

$$f(a+nh+hu)= f(a+nh)+u\nabla f(a+nh)+\frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$
$$+\ldots\ldots\ldots\ldots+\frac{u(u+1)(u+2)\ldots\ldots.(u+n-1)}{n!}\nabla^n f(a+nh) \quad \ldots(4)$$

From equations (3), (4) and above table, we have

$$y(1.9) = 0.1824 + (-0.5)(0.0201) + \frac{(-0.5)(0.5)}{2!}(-0.0053)$$

$$+ \frac{(-0.5)(0.5)(1.5)}{3!}(0.0137) + \frac{(-0.5)(0.5)(1.5)(2.5)}{4!}(0.0335)$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(3.5)}{5!}(0.0534)$$

$$= 0.1824 - (0.5)(0.0201) + \frac{(0.5)(0.5)}{2}(0.0053)$$

$$- \frac{(0.5)(0.5)(1.5)}{6}(0.0137) - \frac{(0.5)(0.5)(1.5)(2.5)}{24}(0.0335)$$

$$- \frac{(0.5)(0.5)(1.5)(2.5)(3.5)}{120}(0.0534)$$

$$= 0.1824 - 0.01005 + 0.0006625$$

$$- 0.00085625 - 0.00130859375 - 0.00146015625$$

$$= 0.1693875$$
$$\approx 0.1694.$$

4.6 Summary

The Gauss's forward interpolation formula is

$$y_u = y_0 + u\,\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1}$$
$$+ \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots$$

and the Gauss's backward interpolation formula is

$$y_u = y_0 + u\,\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \frac{(u+2)(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots$$

The mean of Gauss's forward interpolation formula and Gauss's backward interpolation formula gives Stirling's interpolation formula. The Stirling's interpolation formula is

$$y_u = y_0 + u\,\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3} + \ldots\ldots.$$

4.7 Terminal Questions

Q.1. Write the Gauss's Interpolation Formula.

Q.2. What do you mean by Stirling Interpolation Formula.

Q.3. Use Gauss Forward Interpolation formula to find the value of $f(3.75)$, from the following data:

x	2.5	3	3.5	4	4.5	5
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

Q.4.. Use Gauss Forward Interpolation formula to obtain a polynomial of degree four which takes the following values of the function $f(x)$:

x	1	2	3	4	5
$y = f(x)$	1	-1	1	-1	1

Q.5. Use Gauss Backward Interpolation formula to obtain the values of the function $f(x)$ at x=5.8:

x	4	5	6	7
$y = f(x)$	270	648	1330	2448

Q.6. Use Stirling Interpolation formula to obtain the values of the function $f(x)$ at x = 0.41:

x	0.30	0.35	0.40	0.45	0.50
$y = f(x)$	0.1179	0.1368	0.1554	0.1736	0.1915

Answer

3. 19.407426.

4. $f(x) = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31.$

5. 1162.944.

6. 0.15907168.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-5: Lagrange's Interpolation Formula for Unequal Intervals

Structure

5.1 Introduction

Lagrange's Interpolation Formula for Unequal Intervals is a mathematical method used to estimate the value of a function between known data points when the intervals between these points are not equal. This interpolation formula is based on the Lagrange polynomial, which is a polynomial that passes through given data points. This formula allows for the estimation of the function's value at any point within the given range, even if the intervals between the known data points are irregular. It is a powerful tool in numerical analysis for approximating values in situations where the data is unevenly distributed.

Lagrange's Interpolation Formula can be applied when the intervals between the known data points are not uniform or equal. This makes it suitable for scenarios where data points are irregularly spaced. The formula is used to construct a polynomial (Lagrange polynomial) that passes through the given data points. This polynomial can then be used to approximate the function's values at points within the range.

Lagrange's method is effective for estimating function values at points within the range covered by the known data points. In this unit we shall discuss about Lagrange’s interpolation formyla for unequal length and inverse lagrange’s interpolation formula for unequal length.

5.2 Objectives

After reading this unit the learner should be able to understand about

- the Lagrange’s Interpolation Formula for unequal intervals
- the Inverse Lagrange’s Interpolation Formula for unequal intervals

5.3 Lagrange's Interpolation Formula for Unequal Intervals

Lagrange's Interpolation Formula is used when you have a set of data points with associated function values, and estimate the value of the function at a point that falls within the range of those data points. Lagrange's Interpolation Formula is beneficial in the some special situations such as unequal intervals, polynomial interpolation and Interpolation within a range.

The general form of the Lagrange's Interpolation Formula is versatile and can be applied to various contexts, providing a flexible tool for numerical analysis and approximation. Let us consider $y_0, y_1, y_2, \ldots\ldots\ldots\ldots, y_n$ be the values of function $y=f(x)$ corresponding to the arguments $x_0, x_1, x_2, \ldots\ldots\ldots, x_n$ not necessarily equally spaced.

If the $(n+1)$ values of the function $f(x)$ are given then the $(n+1)^{th}$ difference is zero. Thus $f(x)$ is supposed to be polynomial in x of degree n.

We have

$$y(x) = f(x) = a_0(x-x_1)(x-x_2)(x-x_3)\ldots\ldots\ldots\ldots(x-x_n)$$

$$+a_1(x-x_0)(x-x_2)(x-x_3)\ldots\ldots\ldots\ldots(x-x_n)$$

$$+a_2(x-x_0)(x-x_1)(x-x_3)\ldots\ldots\ldots\ldots(x-x_n)$$

$$+a_3(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots\ldots(x-x_n)$$

$$+\ldots\ldots\ldots\ldots+a_n(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots\ldots(x-x_{n-1}) \quad \ldots(1)$$

Where $a_0, a_1, a_2, \ldots.., a_n$ all are constants.

To determine the value of a_0,

Put $x = x_0$ and $y = y_0$ in the equation (1), we get

$$y_0 = a_0(x_0 - x_1)\ (x_0 - x_2)(x_0 - x_3)..................(x_0 - x_n)$$

$$\Rightarrow \qquad a_0 = \frac{y_0}{(x_0 - x_1)\ (x_0 - x_2)(x_0 - x_3)..................(x_0 - x_n)}$$

Similarly to determine the value of a_1,

Put $x = x_1$ and $y = y_1$ in the equation (1), we get

$$y_1 = a_1(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)..............(x_1 - x_n)$$

$$\Rightarrow \qquad a_1 = \frac{y_1}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)..........(x_1 - x_n)}$$

Put $x = x_2$ and $y = y_2$ in the equation (1), we get

$$y_2 = a_1(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)................(x_2 - x_n)$$

$$\Rightarrow \qquad a_2 = \frac{y_2}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)................(x_2 - x_n)}$$

Put $x = x_3$ and $y = y_3$ in the equation (1), we get

$$y_3 = a_1(x_3 - x_0)(x_3 - x_1)(x_3 - x_2).................(x_3 - x_n)$$

$$\Rightarrow \qquad a_3 = \frac{y_3}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)..................(x_3 - x_n)}$$

Proceeding in this way, we get

$$a_n = \frac{y_n}{(x_n - x_0)(x_n - x_1)(x_n - x_2)..............(x_n - x_{n-1})}$$

Putting these values of a_1, a_2, ……………, a_n in the equation (1), we get

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).................(x_0-x_n)} y_0$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3).............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3).............(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3).........(x_2-x_n)} y_2$$

$$+.......+ \frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})} y_n \qquad \ldots(2)$$

This equation (2) is known as the Lagrange's interpolation formula for unequal intervals.

5.4 Inverse Lagrange's Interpolation Formula for Unequal Intervals

The Inverse Lagrange's Interpolation formula for unequal intervals is

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3).............(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)..........(y_0-y_n)} x_0$$

$$+ \frac{(y-y_0)(y-y_2)(y-y_3).............(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)..........(y_1-y_n)} x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)............(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)..........(y_2-y_n)} x_2$$

$$+............+ \frac{(y-y_0)(y-y_1)(y-y_2)............(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)..........(y_n-y_{n-1})} x_n$$

Check your Progress

1. What do you mean by Lagrange interpolation formula for unequal interval?

2. Write the inverse Lagrange interpolation formula for unequal interval.

Examples

Example.1. Using Lagrange interpolation formula to find a cubic polynomial that approximation the data given below

x :	0	1	4	5
y :	4	3	24	39

Also determine the value of (i) $y(3)$ (ii) $y(6)$ and (iii) $y(12)$.

Sol. It is given that $x_0 = 0$, $x_1 = 1$, $x_2 = 4$ and $x_3 = 5$. Also $y_0 = 4$, $y_1 = 3$, $y_2 = 24$ and $y_3 = 39$.

We know that the Lagrange's Interpolation formula is

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)} y_1$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})} y_n \qquad ...(1)$$

or $$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$=\frac{(x-1)(x-4)(x-5)}{(0-1)(0-4)(0-5)}\times(4)+\frac{(x-0)(x-4)(x-5)}{(1-0)(1-4)(1-5)}\times(3)$$

$$+\frac{(x-0)(x-1)(x-5)}{(4-0)(4-1)(4-5)}\times 24+\frac{(x-0)(x-1)(x-4)}{(5-0)(5-1)(5-4)}\times 39$$

$$=\frac{40x^2-60x+80}{20}$$

$$y(x)=2x^2-3x+4 \qquad \ldots(2)$$

(i) Put x =3 in the equation (2), we get

$$y(3)=13.$$

(ii) Put x = 6 in the equation (2), we get

$$y(6)=58.$$

(iii) Put x =12 in the equation (2), we get

$$y(12)=256.$$

Hence the values of $y(3)$, $y(6)$ and $y(12)$ are 13, 58 and 256 respectively.

Example.2. Use Lagrange's interpolation formula to find *f*(10), from the following data:

$x:$	5	6	9	11
$y=f(x)$	12	13	14	16

Sol. It is given that $x_0 = 5$, $x_1 = 6$, $x_2 = 9$ and $x_3 = 11$. Also $y_0 = 12$, $y_1 = 13$, $y_2 = 14$ and $y_3 = 16$.

We know that the Lagrange's Interpolation formula is

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)}y_1$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})}y_n \qquad ...(1)$$

or $$f(x) = \frac{(x-x_1)(x-x_2).....(x-x_n)}{(x_0-x_1)(x_0-x_2)-(x_0-x_n)}f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)-(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}f(x_1)+....$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)..........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2).......(x_n-x_{n-1})}f(x_n)$$

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times 13$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16$$

$$=\frac{(4)(1)(-1)}{(-1)(-4)(-6)}\times 12 + \frac{(5)(1)(-1)}{(1)(-3)(-5)}\times 13$$

$$+\frac{(5)\,(4)(-1)}{(4)(3)(-2)}\times 14+\frac{(5)\,(4)\,(1)}{(6)\,(5)(2)}\times 16$$

$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}$$

= 2 – 4.3333 + 11.6667+ 5.3333

Hence the value of $f(10)$ is 14.6667.

Example.3. Use Lagrage's formula inversely to find upto the two decimal places, the value of *x*, when $y=19$, given the following

x:	**0**	**1**	**2**
y:	**0**	**1**	**20**

Sol. Using the Inverse Lagrange's interpolation formula for unequal intervals is

$$x(y)=\frac{(y-y_1)(y-y_2)(y-y_3)............(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)..........(y_0-y_n)}x_0$$
$$+\frac{(y-y_0)(y-y_2)(y-y_3)............(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)..........(y_1-y_n)}x_1$$
$$+\frac{(y-y_0)(y-y_1)(y-y_3)............(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)..........(y_2-y_n)}x_2$$
$$+...........+\frac{(y-y_0)(y-y_1)(y-y_2)............(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)..........(y_n-y_{n-1})}x_n$$

…(1)

It is given that y_0 = 0, y_1 = 1, and y_2 = 20. Also x_0 = 0, x_1 = 1 and x_2 = 2.

Put the above values in the equation (1), we get

$$x(y)=\frac{(19-1)(19-20)}{(0-1)(0-20)}\times 0+\frac{(19-0)(19-20)}{(1-0)(1-20)}\times 1+\frac{(19-0)(19-1)}{(20-0)(20-1)}\times 2$$

$$=0+\frac{19\times(-1)\times 1}{1\times(-19)}+\frac{19\times 18}{20\times 19}\times 2$$

$$=1+1.8$$

$$=2.8.$$

Hence the value of $x(19)$ is 2.8.

Example.4. Use Lagrange's interpolation formula to find the value of *y* when *x* = 2, from the following table:

x	**0**	**1**	**3**	**4**
y	**5**	**6**	**50**	**105**

Solution: Here $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$

and $y_0 = 5$, $y_1 = 6$, $y_2 = 50$, $y_3 = 105$.

We know that the Lagrange's Interpolation formula is

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)....................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)}y_1$$

$$+\ldots\ldots+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})}y_n \qquad \ldots(1)$$

or $$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Putting $x = 2$ in above expression, we get

$$y(2)=\frac{(2-1)(2-3)(2-4)}{(0-1)(0-3)(0-4)}\times 5+\frac{(2-0)(2-3)(2-4)}{(1-0)(1-3)(1-4)}\times 6+\frac{(2-0)(2-1)(2-4)}{(3-0)(3-1)(3-4)}\times 50$$

$$+\frac{(2-0)(2-1)(2-3)}{(4-0)(4-1)(4-3)}\times 105$$

$$=\frac{1\times(-1)\times(-2)}{(-1)\times(-3)\times(-4)}\times 5+\frac{2\times(-1)\times(-2)}{1\times(-2)\times(-3)}\times 6+\frac{2\times 1\times(-2)}{3\times 2\times(-1)}\times 50+\frac{2\times 1\times(-1)}{4\times 3\times 1}\times 105$$

$$=-\frac{10}{12}+4+\frac{100}{3}-\frac{105}{6}$$

$$=\frac{-10+48+400-210}{12}$$

$$=19.$$

Hence the value of $y(2)$ is 19.

Example.5. The value of *x* and *y* are given as below:

x	**0**	**1**	**2**	**5**
y	**2**	**5**	**7**	**8**

Find the value of y when $x = 4$.

Solution: Here $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 5$

and $y_0 = 2$, $y_1 = 5$, $y_2 = 7$, $y_3 = 8$.

We know that the Lagrange's Interpolation formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots\ldots\ldots\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots\ldots\ldots\ldots(x_0-x_n)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots\ldots\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots\ldots(x_1-x_n)} y_1$$

$$+\ldots\ldots+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots(x_n-x_{n-1})} y_n \qquad \ldots(1)$$

or $$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Putting $x = 4$ in above expression, we get

$$y(x) = \frac{(4-1)(4-2)(4-5)}{(0-1)(0-2)(0-5)} \times 2 + \frac{(4-0)(4-2)(4-5)}{(1-0)(1-2)(1-5)} \times 5$$

$$+\frac{(4-0)(4-1)(4-5)}{(2-0)(2-1)(2-5)} \times 7 + \frac{(4-0)(4-1)(4-2)}{(5-0)(5-1)(5-2)} \times 8$$

$$= \frac{3\times 2\times(-1)}{(-1)\times(-2)\times(-5)}\times 2+\frac{4\times 2\times(-1)}{1\times(-1)\times(-4)}\times 5+\frac{4\times 3\times(-1)}{2\times 1\times(-3)}\times 7+\frac{4\times 3\times 2}{5\times 4\times 3}\times 8$$

$$= \frac{6}{10}+(-10)+14+\frac{16}{5}$$

$$= \frac{3+20+16}{5}$$

$$= \frac{3.9}{5}$$

$$y(4)=7.8.$$

Hence the value of $y(4)$ is 7.8.

Example.6. Find the value of *y* at *x* = 5 from the following data:

x	**1**	**3**	**4**	**8**	**10**
***y* = *f*(*x*)**	**8**	**15**	**19**	**32**	**40**

Solution: Here $x_0 = 1$, $x_1 = 3$, $x_2 = 4$, $x_3 = 8$, $x_4 = 10$

and $y_0 = 8$, $y_1 = 15$, $y_2 = 19$, $y_3 = 32$, $y_4 = 40$.

We know that the Lagrange's Interpolation formula is

$$y(x)=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)}y_1$$

$$+\ldots\ldots+\frac{(x-x_0)(x-x_1)(x-x_2)\ldots\ldots\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\ldots\ldots\ldots(x_n-x_{n-1})}y_n \qquad \ldots(1)$$

or $$y(x)=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Putting x = 5 in above expression, we get

$$y(5)=\frac{(5-3)(5-4)(5-8)(5-10)}{(1-3)(1-4)(1-8)(1-10)}\times 8+\frac{(5-1)(5-4)(5-8)(5-10)}{(3-1)(3-4)(3-8)(3-10)}\times 15$$

$$+\frac{(5-1(5-3)(5-8)(5-10)}{(4-1)(4-3)(4-8)(4-10)}\times 19+\frac{(5-1)(5-3)(5-4)(5-10)}{(8-1)(8-3)(8-4)(8-10)}\times 32$$

$$+\frac{(5-1)(5-3)(5-4)(5-8)}{(10-1)(10-3)(10-4)(10-8)}\times 40$$

$$=\frac{2\times 1\times(-3)\times(-5)}{(-2)\times(-3)\times(-7)\times(-9)}\times 8+\frac{4\times 1\times(-3)\times(-5)}{2\times(-1)\times(-5)\times(-7)}\times 15$$

$$+\frac{4\times 2\times(-3)\times(-5)}{3\times 1\times(-4)\times(-6)}\times 19+\frac{4\times 2\times 1\times(-5)}{7\times 5\times 4\times(-2)}\times 32+\frac{4\times 2\times 1\times(-3)}{9\times 7\times 6\times 2}\times 40$$

$$=\frac{40}{63}-\frac{90}{7}+\frac{95}{3}+\frac{32}{7}-\frac{80}{63}$$

$$=\frac{40-9\times 90+95\times 21+32\times 9-80}{63}$$

$$=\frac{40-810+1995+288-80}{63}$$

$$=\frac{-930+2323}{63}$$

$$=\frac{1433}{63}$$

$$=22.746031746031$$

$y(5)=22.75.$

Hence the value of $y(5)$ is 22.75.

Example.7. Apply Lagrange's formula to find the cubic polynomial which includes the following values of *x* and *y*.

x	**0**	**1**	**4**	**6**
$y(x)=f(x)$	**1**	**-1**	**1**	**-1**

Solution: Here $x_0 = 0$, $x_1 = 1$, $x_2 = 4$, $x_3 = 6$, and $y_0 = 1$, $y_1 = -1$, $y_2 = 1$, $y_3 = -1$.

We know that the Lagrange's Interpolation formula is

$$y(x)=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3).............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)}y_1$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})}y_n \qquad ...(1)$$

or $$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$f(x)=\frac{(x-1)(x-4)(x-6)}{(0-1)(0-4)(0-6)}\times 1+\frac{(x-0)(x-4)(x-6)}{(1-0)(1-4)(1-6)}\times(-1)$$
$$+\frac{(x-0)(x-1)(x-6)}{(4-0)(4-1)(4-6)}\times 1+\frac{(x-0)(x-1)(x-4)}{(6-0)(6-1)(6-4)}\times(-1)$$

$$=-\frac{1}{24}[x^3-11x^2+34x-24]-\frac{1}{15}[x^3-10x^2+24x]$$

$$-\frac{1}{24}[x^3-7x^2+6x]-\frac{1}{60}[x^3-5x^2+4x]$$

$$f(x)=-\frac{1}{6}x^3+\frac{3}{2}x^2-\frac{10}{3}x+1.$$

Hence the cubic polynomial is $f(x)=-\frac{1}{6}x^3+\frac{3}{2}x^2-\frac{10}{3}x+1.$

Example.8. Using Lagrange's method to show that

$$y_3=0.05(y_0+y_6)-0.3(y_1+y_5)+0.75(y_2+y_4)$$

Solution: Here $x_0 = 0,\ x_1 = 1,\ x_2 = 2, x_3 = 4, x_4 = 5, x_5 = 6$ and their corresponding values of function are given by y_0, y_1, y_2, y_4, y_5 and y_6.

We know that the Lagrange's Interpolation formula is

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)......................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3).................(x_0-x_n)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3).............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)} y_1$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})} y_n \qquad \ldots(1)$$

or $$y_x = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)} \times y_0 + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)} y_1$$

$$+\frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)} \times y_2 + \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)} y_4$$

$$+\frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(5-0)(5-1)(5-2)(5-4)(5-6)} \times y_5 + \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)} y_6$$

To determine the value of y_3, put $x = 3$ in above expression, we get

$$y_3 = \frac{12}{240} y_0 - \frac{18}{60} y_1 + \frac{36}{48} y_4 - \frac{18}{60} y_5 + \frac{12}{240} y_6$$

$$= \frac{1}{20}(y_0 + y_6) - \frac{3}{10}(y_1 + y_5) + \frac{3}{4}(y_2 + y_4)$$

$$= 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(y_2 + y_4).$$

Example.9. Using Inverse Lagrange interpolation formula to find a cubic polynomial that approximation the following data:

$x:$	0	1	3	6
$y:$	1	2	4	9

Sol. It is given that $x_0 = 0$, $x_1 = 1$, $x_2 = 3$ and $x_3 = 6$. Also $y_0 = 1$, $y_1 = 2$, $y_2 = 4$ and y3 = 9. Using the Inverse Lagrange's interpolation formula for unequal intervals is

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)............(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)..........(y_0-y_n)}x_0$$

$$+\frac{(y-y_0)(y-y_2)(y-y_3)............(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)..........(y_1-y_n)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)............(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)..........(y_2-y_n)}x_2$$

$$+............+\frac{(y-y_0)(y-y_1)(y-y_2)............(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)..........(y_n-y_{n-1})}x_n$$

…(1)

From equation (1), we get

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

or

$$x(y) = \frac{(y-2)(y-4)(y-9)}{(1-2)(1-4)(1-9)}\times 0 + \frac{(y-1)(y-4)(y-9)}{(2-1)(2-4)(2-9)}\times 1$$

$$+\frac{(y-1)(y-2)(y-9)}{(4-1)(4-2)(4-9)}\times 3+\frac{(y-1)(y-2)(y-4)}{(9-1)(9-2)(9-4)}\times 6$$

$$x(y)=0+\frac{(y-1)(y-4)(y-9)}{1(-2)(-7)}\times 1+\frac{(y-1)(y-2)(y-9)}{(3)(2)(-5)}\times 3+\frac{(y-1)(y-2)(y-4)}{(8)(7)(5)}\times 6$$

$$=\frac{(y-1)(y-4)(y-9)}{14}-\frac{(y-1)(y-2)(y-9)}{10}+\frac{3(y-1)(y-2)(y-4)}{140}$$

$$=(y-1)\left\{\frac{(y-4)(y-9)}{14}-\frac{(y-2)(y-9)}{10}+\frac{3(y-2)(y-4)}{140}\right\}$$

$$=(y-1)\left\{\frac{y^2-13y+36}{14}-\frac{y^2-11y+18}{10}+\frac{3y^2-18y+24}{140}\right\}$$

$$=(y-1)\left\{\frac{10y^2-130y+360}{140}-\frac{14y^2-154y+252}{140}+\frac{3y^2-18y+24}{140}\right\}$$

$$=(y-1)\left\{\frac{-y^2+6y+132}{140}\right\}$$

$$x(y)=\frac{-y^3+7y^2+126y-132}{140}$$

Hence the cubic polynomial is $x(y)=\dfrac{-y^3+7y^2+126y-132}{140}$.

5.9 Summary

Lagrange's Interpolation Formula is beneficial in the some special situations such as unequal intervals, polynomial interpolation and interpolation within a range, and extrapolation.

The Lagrange's Interpolation formula is

$$y(x) = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)....................(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)................(x_0-x_n)} y_0$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)............(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3).........(x_1-x_n)} y_1$$

$$+.......+\frac{(x-x_0)(x-x_1)(x-x_2)...........(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)........(x_n-x_{n-1})} y_n$$

The Inverse Lagrange's Interpolation formula for unequal intervals is

$$x(y) = \frac{(y-y_1)(y-y_2)(y-y_3).............(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)..........(y_0-y_n)} x_0$$

$$+\frac{(y-y_0)(y-y_2)(y-y_3).............(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)..........(y_1-y_n)} x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)............(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)..........(y_2-y_n)} x_2$$

$$+............+\frac{(y-y_0)(y-y_1)(y-y_2)............(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)..........(y_n-y_{n-1})} x_n$$

5.10 Terminal Questions

Q.1. Write the Lagrange's Interpolation Formula for unequal intervals.

Q.2. Write the Inverse Lagrange's Interpolation Formula for unequal intervals.

Q.3. Use Lagrange's interpolation formula to find $f(35)$, from the following data:

$x:$	25	30	40	50
$y = f(x)$	52	67.3	84.1	94.4

Q.4. Find the value of y at $x = 7$ given that:

x	0	2	5	8	10	12
$y = f(x)$	7.5	10.25	15	16	18	21

Q.5. Using the Lagrange's formula to find the polynomial which includes the following values of x and y.

x	0	1	2	3	4
$y = f(x)$	3	6	11	18	27

Q.6. Using the Lagrange's interpolation formula to determine the value $f(5)$, from the following data:

x	1	2	3	4	7
$y = f(x)$	2	4	8	16	128

Answer

3. 77.41

4. 15.7

5. $y = f(x) = x^2 + 2x + 3.$

6. 32.93.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

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Block

3 Solution of Linear Simultaneous Equations

Block-3

Solution of inear Simultaneous Equations-I

Simultaneous linear equations are fundamental in various fields, including physics, engineering, economics, and many other scientific disciplines. The choice of method for solving such systems depends on the characteristics of the system, the size of the problem, and the efficiency of the chosen algorithm. Simultaneous linear equations are of significant importance in various areas of mathematics, science, engineering, and other fields. Simultaneous linear equations refer to a system of multiple equations, each of which is a linear equation, all involving the same set of variables. These systems are commonly represented in matrix form as AX=B, where A is the coefficient matrix; X is the column vector of variables and B is the column vector of constants. There are various methods to solve simultaneous linear equations, including: Matrix Methods (Gauss Elination and LU Decomposition); Iterative methods (Jacobi Method and Gauss Seidel Method); and Determinants and Cramer's rule.

Simultaneous linear equations are mathematical model to describe the physical systems in various disciplines such as physics and engineering. These equations provide a mathematical representation of relationships between different variables in a system. In engineering, simultaneous linear equations are crucial for solving problems related to circuit analysis, structural analysis, control systems, and optimization. They help engineers analyze and design systems efficiently. Economic models often involve systems of linear equations to represent relationships among economic variables such as supply, demand, production, and consumption. In finance, these equations are used for portfolio optimization and risk management.

In the sixth unit, we shall discussed about the simultaneous linear equations by Gauss Elimination method and Gauss seidel Method. And in unit seventh we solved the simultaneous linear equations by LU Decomposition method, Crout's method and Choleski's method.

UNIT-6: Solution of Linear Simultaneous Equations-I

Structure

6.1 Introduction

Simultaneous linear equations occur in the field of science and engineering like as analysis of a network under sinusoidal steady-state condition, determination of the output of a chemical plant and finding the cost of reaction, the analysis of electronic circuits having a number of invariant element etc. Gaussian Elimination is a fundamental method in solving linear systems and is used in various applications, including solving systems of equations in engineering, physics, computer science, and more. The Gauss-Seidel method is generally faster than the Gauss elimination method for solving large systems of linear equations, especially when the coefficient matrix is sparse.

However, it may not converge for all systems, and the convergence rate can be influenced by the properties of the coefficient matrix. Some systems may require preconditioning or other methods to improve convergence. We can solved the system of simultaneous linear equations by matrix method or by Cramer's rule. But for large system, these methods are failed.

In this unit we shall discuss some direct and iterative method of solutions: Gauss Elimination Method and Gauss Seidel Method. The resulting matrix in row-echelon form will have a triangular shape, and the solutions can be easily obtained through back substitution. If the system is consistent and has a unique solution, the matrix will be in reduced row-echelon form, and each variable will have a unique value.

6.2 Objectives

After reading this unit the learner should be able to understand about:

- the linear equations and their structure in matrix form
- the Gauss Elimination Method with their solution procedure
- the Gauss Seidel Method with their solution procedure

6.3 Linear Equations

Consider the m first degree equations in n variables

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m \end{array}\right\}$$

The above system of equations can be written in the matrix form:

$$\begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e., AX = B

where A = $[a_{ij}]$, X = $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Homogeneous System:

The above system of equations is said to be homogeneous if all the b_i (i=1, 2, ……., m) are zero.

Non-homogeneous System:

If atleast one b_i (i=1, 2, ……., m) is not equal to zero then the above system of equations is called as non-homogeneous system.

The solution of such types of equations can be obtained by

1. Determinant method
2. Matrix inversion method
3. Direct methods

   (i) Gauss elimination method

   (ii) Gauss-Jordan method

   (iii) Triangularization method.

4. Indirect methods

   (i) Tacobi iterative method

   (ii) Gauss-Seidel iterative method

   (iii) Relaxation method.

Here in this unit, we shall discuss only two important methods Gauss elimination method and Gauss-Seidel method.

6.4 Gauss Elimination Method

Gaussian Elimination is a method used in linear algebra to solve systems of linear equations by transforming the augmented matrix of the system reduced into row-echelon form. This process simplifies the system and makes it easier to find the solutions.

In this method, the unknowns from the system of equations are eliminated successively such that system of equation is reduced to an upper triangular system from which the unknowns are determined by back substitution. We proceed a step-by-step explanation of the Gaussian Elimination method as follows:

Consider the given system of equations are

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n &= b_m \end{aligned}\right\} \qquad \ldots(1)$$

Step-I. First we eliminating x_1 from the second, third, fourth, ……….. nth equation. Consider $a_{11} \neq 0$. The variable x_1 eliminated from the second equation by subtracting $\frac{a_{21}}{a_{11}}$ times the first equation from the second equation, similarly we eliminate x_1 from third equation by subtracting $\frac{a_{31}}{a_{11}}$ times the first equation from the third equation, etc. Then we get the new system of equation

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n &= b_1 \\ b_{22}x_2 + \ldots\ldots + b_{2n}x_n &= b'_2 \\ b_{32}x_2 + \ldots\ldots + b_{3n}x_n &= b'_3 \\ \vdots \qquad\quad \vdots \qquad &\quad \vdots \\ b_{m2}x_2 + \ldots\ldots + b_{mn}x_n &= b'_m \end{aligned}\right\} \qquad \ldots(2)$$

Step-II. To eliminate x_2 from the third, fourth ….. nth equation. Consider $b_{22} \neq 0$. The variable x_2 eliminated from the third equation by subtracting $\frac{b_{32}}{b_{22}}$ times the second equation from the third equation, similarly we eliminate x_2 from fourth equation by subtracting $\frac{b_{42}}{b_{22}}$ times the second equation from the fourth equation, etc. Then we get the new system of equation as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots\ldots + a_{1n}x_n &= b_1 \\ b_{22}x_2 + b_{23}x_3 + \ldots\ldots + b_{2n}x_n &= b'_2 \\ c_{33}x_3 + \ldots\ldots + c_{3n}x_n &= b''_3 \\ \vdots \qquad\quad \vdots \qquad &\quad \vdots \\ c_{m3}x_3 + \ldots\ldots + c_{mn}x_n &= b_m^{(m-1)'} \end{aligned}\right\} \qquad \ldots(3)$$

Step-III. Proceeding in the same ways, eliminated x_3 in third step, and eliminate x_4 in fourth step and so on.

Therefore we get new the system of equation as

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \ldots.. + a_{1n}x_n &= b_1 \\ b_{22}x_2 + b_{23}x_3 + \ldots\ldots + b_{2n}x_n &= b_2' \\ c_{33}x_3 + \ldots\ldots + c_{3n}x_n &= b_3'' \\ \vdots \qquad & \quad \vdots \\ d_{mn}x = b_m^{(m-1)'} & \end{aligned}\right\} \qquad \ldots(4)$$

To determine the value of unknown

Hence the value of x_1 x_2………… x_n are given by the system of equations (4) by back substitution.

Check your Progress

1. What do you mean by Linear system of equation?

2. Explain the homogeneous and non-homogeneous system.

Examples

Example.1. Solve the following system of equations using Gauss's elimination method:

$$2x+3y-z=5,\ 4x+4y-3z=3,\ 2x-3y+2z=2$$

Solution: The give system of equations are

$$2x+3y-z=5 \qquad \ldots(1)$$

$$4x+4y-3z=3 \qquad \ldots(2)$$

$$2x-3y+2z=2 \qquad \ldots(3)$$

First eliminating x from the equations (2) and (3), by subtracting 2 and 1 times of equation (1) respectively, we get

$$2x+3y-z=5 \qquad \ldots(4)$$

$$2y+z=7 \qquad \ldots(5)$$

$$6y-3z=3 \qquad \ldots(6)$$

Again eliminating y from the equation (6) with the help of equation (5). Equation (5) is subtracted after multiplies by 3 from the equation (6), we get

$$2x+3y-z=5 \qquad \ldots(7)$$

$$2y+z=7 \qquad \ldots(8)$$

$$-6z=-18 \quad \text{or} \quad z=3 \qquad \ldots(9)$$

Put the value of z into equation (8), we get

$$y=2$$

Now put the values of y and z into equation (7), we get

$$x=1$$

Hence the solutions of the given system of equations are

$$x_1=1,\ x_2=2,\ x_3=3.$$

Example.2. Solve the following system of equations using Gauss's elimination method:

$$x_1+x_2+2x_3=4, \quad 3x_1+x_2-3x_3=-4, \quad 2x_1-3x_2-5x_3=-5.$$

Solution: The give system of equations are

$$x_1 + x_2 + 2x_3 = 4 \qquad \ldots(1)$$

$$3x_1 + x_2 - 3x_3 = -4 \qquad \ldots(2)$$

$$2x_1 - 3x_2 - 5x_3 = -5 \qquad \ldots(3)$$

First eliminating x_1 from the equations (2) and (3), by subtracting 3 and 2 times of equation (1) respectively, we get

$$x_1 + x_2 + 2x_3 = 4 \qquad \ldots(4)$$

$$2x_2 + 9x_3 = 16 \qquad \ldots(5)$$

$$5x_2 + 9x_3 = 13 \qquad \ldots(6)$$

Again eliminating x_2 from the equation (6) with the help of equation (5). Divided the equation (5) by 2 and then this equation is subtracted after multiplies by 5 from equation (6), we get

$$x_1 + x_2 + 2x_3 = 4 \qquad \ldots(7)$$

$$x_2 + \frac{9}{2}x_3 = 8 \qquad \ldots(8)$$

$$-\frac{27}{2}x_3 = -27 \qquad \text{or} \qquad x_3 = 2 \qquad \ldots(9)$$

Put the value of x_3 into equation (8), we get

$$x_2 = 8 - \frac{9}{2} \times 2 = -1$$

Now put the values of x_2 and x_3 into equation (7), we get

$$x_1 = 4 + 1 - 4 = 1$$

Hence the solutions of the given system of equations are

$$x_1 = 1,\ x_2 = -1,\ x_3 = 2.$$

Example.3. Solve the following simultaneous linear equations:

$$2x_1 + 4x_2 + x_3 = 3,\ 3x_1 + 2x_2 - 2x_3 = -2,\ x_1 - x_2 + x_3 = 6.$$

using Gauss's elimination method.

Solution: The give system of equations can be written in the following order:

$$x_1 - x_2 + x_3 = 6 \qquad \ldots(1)$$

$$2x_1 + 4x_2 + x_3 = 3 \qquad \ldots(2)$$

$$3x_1 + 2x_2 - 2x_3 = -2 \qquad \ldots(3)$$

First eliminating x_1 from equations (2) and (3) by subtracting 2 and 3 times of equation (1) respectively, we get

$$x_1 - x_2 + x_3 = 6 \qquad \ldots(4)$$

$$6x_2 - x_3 = -9 \qquad \ldots(5)$$

$$5x_2 - 5x_3 = -20 \qquad \ldots(6)$$

Again eliminating x_2 from equation (6) with the help of equation (5). Divided equation (5) by 6 and then this equation is subtracted after multiplies by 5 from equation (6), we get.

$$x_1 - x_2 + x_3 = 6 \qquad \ldots(7)$$

$$6x_2 - x_3 = -9 \qquad \ldots(8)$$

$$x_3 = 3 \qquad \ldots(9)$$

Put the value of x_3 into equation (8), we get

$$x_2 = \frac{-9+3}{6} = -1$$

Now put the values of x_2 and x_3 into equation (7), we get

$$x_1 = 6 - 1 + 3 = 2$$

Hence the solutions of the given system of equations are

$$x_1 = 2, \; x_2 = -1, \; x_3 = 3.$$

Example.4. Solve the following system of simultaneous linear equations:

$$6x + 3y + 2z = 6, \; 6x + 4y + 3z = 0, \; 20x + 15y + 12z = 0.$$

by Gauss's elimination method.

Solution: The give system of equations are

$$6x + 3y + 2z = 6 \qquad \ldots(1)$$

$$6x + 4y + 3z = 0 \qquad \ldots(2)$$

$$20x + 15y + 12z = 0 \qquad \ldots(3)$$

First, divide the equation (1) by 6, we get

$$x + \frac{1}{2}y + \frac{1}{3}z = 1 \qquad \ldots(4)$$

$$6x + 4y + 3z = 0 \qquad \ldots(5)$$

$$20x + 15y + 12z = 0 \qquad \ldots(6)$$

Now eliminating x from (5) and (6) equations by subtracting 6 and 20 times of equation (1) respectively, we get

$$x+\frac{1}{2}y+\frac{1}{3}z=1 \qquad \ldots(7)$$

$$y+z=-6 \qquad \ldots(8)$$

$$5y+\frac{16}{3}z=-20 \qquad \ldots(9)$$

Now eliminating y from equation (9) by subtracting 5 times of equation (8), we get

$$x+\frac{1}{2}y+\frac{1}{3}z=1 \qquad \ldots(10)$$

$$y+z=-6 \qquad \ldots(11)$$

$$\frac{1}{3}z=10 \quad \text{or} \quad z=30 \qquad \ldots(12)$$

Substitute the values of z into equation (11), we get

$$y=-6-30=-36$$

And again substitute the values of y and z into equation (10), we get

$$x=1-\frac{1}{2}(-36)-\frac{1}{3}(80)$$

or $$x=9$$

Hence the solutions of the given system of equations are

$$x=9,\ y=-36,\ z=30.$$

6.5 Gauss Seidel Method

The Gauss-Seidel method is an iterative numerical technique used to solve a system of linear equations. It is named after the mathematicians Carl Friedrich Gauss and Philipp Ludwig von Seidel.

This method is particularly useful for solving large systems of linear equations efficiently.

Consider a system of *n* equation in *n* variables in which $a_{ii} \neq 0$

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + \ldots\ldots + a_{3n}x_n = b_3 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots(1)$$

The above system of equation can be written as

$$\left.\begin{array}{l} x_1 = \dfrac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3 \ldots\ldots\ldots\ldots\ldots\ldots a_{1n}x_n\right] \\ x_2 = \dfrac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3 \ldots\ldots\ldots\ldots\ldots\ldots a_{2n}x_n\right] \\ x_3 = \dfrac{1}{a_{nn}}\left[b_3 - a_{31}x_1 - a_{32}x_2 \ldots\ldots\ldots\ldots\ldots\ldots a_{3n}x_n\right] \\ \qquad\qquad \vdots \\ x_n = \dfrac{1}{a_{33}}\left[b_n - a_{n1}x_1 - a_{n2}x_2 \ldots\ldots\ldots a_{n(n-1)}x_{n-1}\right] \end{array}\right\} \quad \ldots(2)$$

First we put the first approximations $x_1^{(1)}, x_2^{(1)}, \ldots\ldots\ldots\ldots, x_n^{(1)}$ in the right hand side of first equation of (2), we get

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)} \ldots\ldots\ldots\ldots\ldots\ldots a_{1n}x_n^{(1)}\right]$$

Now we put $x_1^{(2)}, x_2^{(1)}, x_3^{(1)}, ...x_n^{(1)}$ in the right hand side of second equation of (2), so we get

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(2)} - a_{23}x_3^{(1)}................a_{2n}x_n^{(1)}\right]$$

Now again we put $x_1^{(2)}, x_2^{(2)}, x_3^{(1)}, ..., x_n^{(1)}$ in the right hand side of third equation of (2), so we get.

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(2)} - a_{32}x_2^{(2)}................ a_{3n}x_n^{(1)}\right]$$

Proceeding in the same way we put $x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, ..., x_n^{(1)}$ in the last equation of (2), so we get

$$x_n^{(2)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(2)} - a_{n2}x_2^{(2)}............a_{n(n-1)}x_{n-1}^{(2)}\right]$$

Here the first stage of iteration is completed.

The whole process is repeated untill the values of x_1, x_2, …………, x_n are obtained upto the desired accuracy level. Gauss-Seidel method is also known as a method of successive displacement.

Examples

Example.5. Solve the following system of equations by Gauss-Seidel iteration method:

$$83x + 11y - 4z = 95, \quad 7x + 52y + 13z = 104, \; 3x + 8y + 29z = 71.$$

Solution. The given system of equations are

$$83x + 11y - 4z = 95 \qquad \text{…(1)}$$

$$7x+52y+13z=104 \qquad \ldots(2)$$

$$3x+8y+29z=71 \qquad \ldots(3)$$

The above given equation can be written as in the iteration form:

$$x=\frac{1}{83}(95-11y+4z) \qquad \ldots(4)$$

$$y=\frac{1}{52}(104-7x-13z) \qquad \ldots(5)$$

$$z=\frac{1}{29}(71-3x-8y) \qquad \ldots(6)$$

Here first we taking the initial solution $x^{(1)}=0, y^{(1)}=0, z^{(1)}=0$ and put these values in the equation (4), we get

$$x^{(2)}=\frac{1}{83}(95-11y^{(1)}-4z^{(1)})$$

$$=\frac{1}{83}(95-11\times 0+4\times 0)$$

$$=\frac{95}{83}$$

$$=1.14$$

Now put $x^{(2)}=1.14, y^{(1)}=0, z^{(1)}=0,$ in the equation (5), we get

$$y^{(2)}=\frac{1}{52}(104-7x^{(2)}-13z^{(1)}$$
$$=\frac{1}{52}(104-7\times 1.14-13\times 0)$$

$$= \frac{96.02}{52}$$

$$= 1.85$$

Now put $x^{(2)} = 1.14, y^{(2)} = 1.85, z^{(1)} = 0$ in the equation (6), we get

$$z^{(2)} = \frac{1}{29}(71 - 3x^{(2)} - 8y^{(2)})$$

$$= \frac{1}{29}(71 - 3\times1.14 - 8\times1.85)$$

$$= \frac{52.78}{29}$$
$$= 1.82$$

Now put $x^{(2)} = 1.14, y^{(2)} = 1.85, z^{(2)} = 1.82$ in the equation (4), we get

$$x^{(3)} = \frac{1}{83}(95 - 11y^{(2)} + 4z^{(2)})$$

$$= \frac{1}{83}(95 - 11\times1.85 + 4\times1.82)$$

$$= \frac{81.93}{83}$$
$$= 0.99$$

Now put $x^{(3)} = 0.99, y^{(2)} = 1.85, z^{(2)} = 1.82$ in the equation (5), we get

$$y^{(3)} = \frac{1}{52}(104 - 7x^{(3)} - 13z^{(3)}$$
$$= \frac{1}{52}(104 - 7\times0.99 - 13\times1.82)$$

$$= \frac{73.41}{52}$$
$$= 1.41$$

Now put $x^{(3)} = 0.99, y^{(3)} = 1.41, z^{(2)} = 1.82$ in the equation (6), we get

$$z^{(3)} = \frac{1}{29}(71 - 3x^{(3)} - 8y^{(3)})$$
$$= \frac{1}{29}(71 - 3\times 0.99 - 8\times 1.41)$$

$$= \frac{56.75}{29}$$
$$= 1.95$$

Now put $x^{(3)} = 0.99, y^{(3)} = 1.44, z^{(3)} = 1.95$ in the equation (5), we get

$$x^{(4)} = \frac{1}{83}(95 - 11y^{(3)} + 4z^{(3)})$$

$$= \frac{1}{83}(95 - 11\times 1.41 + 4\times 1.95)$$

$$= \frac{87.29}{83}$$
$$= 1.05$$

Now put $x^{(4)} = 1.05, y^{(3)} = 1.41, z^{(3)} = 1.95$ in the equation (5), we get

$$y^{(4)} = \frac{1}{52}(104 - 7x^{(3)} - 13z^{(3)})$$

$$= \frac{1}{52}(104 - 7\times 1.05 - 13\times 1.95)$$

$$= \frac{71.3}{52}$$
$$= 1.37$$

Now put $x^{(4)} = 1.05, y^{(4)} = 1.37, z^{(3)} = 1.95$ in the equation (6), we get

$$z^{(4)} = \frac{1}{29}(71 - 3x^{(4)} - 8y^{(4)})$$

$$= \frac{1}{29}(71 - 3\times1.05 - 8\times1.37)$$

$$= \frac{56.89}{29}$$

$$= 1.96$$

Here $x^{(4)} = 1.05,\ y^{(4)} = 1.37,\ z^{(4)} = 1.96$. These values are sufficiently close to the above $x^{(3)}$, $y^{(3)}$, $z^{(3)}$ respectively. Hence the solutions of the given system of equations are

$$x = 1.05,\ y = 1.37,\ \ z = 1.96.$$

Example.6. Solve the following system of equation by Gauss-Seidel iteration method

$$10x + 2y + z = 9,\ 2x + 20y - 2z = -44,\ -2x + 3y + 10z = 22$$

Solution: The give system of equations are

$$10x + 2y + z = 9 \qquad \ldots(1)$$

$$2x + 20y - 2z = -44 \qquad \ldots(2)$$

$$-2x + 3y + 10z = 22 \qquad \ldots(3)$$

The above given equation can be written as in the iteration form:

$$x = \frac{1}{10}(9 - 2y - z) \qquad \ldots(4)$$

$$y = \frac{1}{20}(-44 - 2x + 2z) \qquad \ldots(5)$$

$$z = \frac{1}{10}(22 + 2x - 3y) \qquad \ldots(6)$$

Here first we taking the initial solution $x^{(1)} = 0, y^{(1)} = 0, z^{(1)} = 0$ and put these values in the equation (4), we get

$$x^{(2)} = \frac{1}{10}(9 - 2y^{(1)} - z^{(1)})$$

$$= \frac{1}{10}(9 - 2\times 0 - 0)$$
$$= \frac{9}{10}$$
$$= 0.9$$

Now put $x^{(2)} = 0.9, y^{(1)} = 0, z^{(1)} = 0$ in the equation (5), we get

$$y^{(2)} = \frac{1}{20}(-44 - 2x^{(2)} + 2z^{(1)})$$
$$= \frac{1}{20}(-44 - 2\times 0.9 + 2\times 0)$$

$$= -\frac{45.8}{20}$$
$$= -2.29$$

Now put $x^{(2)} = 0.9, y^{(2)} = -2.29, z^{(1)} = 0$ in the equation (6), we get

$$z^{(2)} = \frac{1}{10}(22 + 2x^{(2)} - 3y^{(2)})$$
$$= \frac{1}{10}(22 + 2\times 0.9 - 3\times(-2.29))$$

$$= \frac{1}{10}((30.67)$$
$$= 3.067$$

Now put $x^{(2)} = 0.9, y^{(2)} = -2.29, z^{(2)} = 3.067$ in the equation (4), we get

$$x^{(3)} = \frac{1}{10}(9 - 2y^{(2)} - z^{(2)})$$
$$= \frac{1}{10}(9 - 2\times(-2.29) - 3.067)$$

$$= \frac{10.513}{10}$$
$$= 1.051$$

Now put $x^{(3)} = 1.051, y^{(2)} = -2.29,\ z^{(2)} = 3.067$ in the equation (5), we get

$$y^{(3)} = \frac{1}{20}(-44 - 2\,x^{(3)} + 2z^{(2)})$$
$$= \frac{1}{20}(-44 - 2\times 1.051 + 2\times 3.067)$$

$$= \frac{1}{20}(-39.968)$$
$$= -1.99$$

Now put $x^{(3)} = 1.051, y^{(3)} = -1.99,\ z^{(2)} = 3.067$ in the equation (6), we get

$$z^{(3)} = \frac{1}{10}(22 + 2x^{(3)} - 3y^{(3)})$$
$$= \frac{1}{10}(22 + 2\times 1.051 - 3\times(-1.99))$$

$$= \frac{1}{10}(30.078)$$
$$= 3.007$$

Now Put $x^{(3)} = 1.051, y^{(3)} = -1.99,\ z^{(3)} = 3.007$ in the equation (4), we get

$$x^{(4)} = \frac{1}{10}(9 - 2y^{(3)} - z^{(3)})$$

$$= \frac{1}{10}(9 - 2\times(-1.99) - 3.007)$$

$$= \frac{9.973}{10}$$

$$= 0.997$$

Now Put $x^{(4)} = 0.997, y^{(3)} = -1.99,\ z^{(3)} = 3.007$ in the equation (5), we get

$$y^{(4)} = \frac{1}{20}(-44 - 2x^{(4)} + 2z^{(3)})$$

$$= \frac{1}{20}(-44 - 2\times 0.997 + 2\times 3.007)$$

$$= \frac{1}{20}(39.98)$$

$$= -1.99$$

Now Put $x^{(4)} = 0.997, y^{(4)} = -1.99,\ z^{(3)} = 3.007$ in the equation (6), we get

$$z^{(4)} = \frac{1}{10}(22 + 2x - 3y)$$

$$= \frac{1}{10}(22 + 2\times 0.997 - 3\times(-1.99))$$

$$= \frac{1}{10}(29.964)$$

$$= 2.99$$

Here $x^{(4)} = 0.997, y^{(4)} = -1.99,\ z^{(4)} = 2.99$. These values are sufficiently close to the above $x^{(3)}$, $y^{(3)}$, $z^{(3)}$ respectively. Hence the solutions of the given system of equations are

$$x = 0.997 \approx 1,\ y = -1.99 \approx -2,\ z = 2.99 \approx 3.$$

6.6 Summary

Simultaneous linear equations provide a powerful and versatile mathematical framework for modeling, analyzing, and solving a wide range of real-world problems in diverse fields. Their importance lies in their applicability to understanding and optimizing complex systems and phenomena. Gaussian Elimination is a method in which the unknowns from the system of equations are eliminated successively such that system of equation is reduced to an upper triangular system from which the unknowns are determined by back substitution. Gauss-Seidel method is an iterative numerical technique used to solve a system of linear equations. This method is particularly useful for solving large systems of linear equations efficiently.

6.7 Terminal Questions

Q.1 Write the procedure for solving simultaneous linear equations by Gauss's elimination method.

Q.2. Explain the Gauss' Seidel Method.

Q.3. Solve the following system of equation by Gauss elimination method:

$$2x + y + z = 10,\ 3x + 2y + 3z = 18,\ x + 4y + 9z = 16$$

Q.4. Using Gauss elimination method to solve the following simultaneous equations:

$$x_1 + x_2 + x_3 = 10,\ 2x_1 + x_2 + 2x_3 = 17,\ 3x_1 + 2x_2 + x_3 = 17$$

Q.5. Solve the following system of equation by Gauss Seidel method:

$$27x + 6y - z = 85,\ 6x + 15y + 2z = 72,\ x + y + 54z = 110$$

Q.6. Using Gauss Seidel method to solve the following simultaneous equations:

$$20x + y - 2z = 17\ ,\ 3x + 20y - z = -18\ ,\ 2x - 3y + 20z = 25$$

Q.7. Solve the following system of equations by Gauss-Seidel iteration method:

$$2x + 4y + z = 3,\ 3x + 2y - 2z = -2\ ,\ x - y + z = 6$$

Answer

3. $x = 7, y = -9, z = 5$

4. $x_1 = 2, x_2 = 3, x_3 = 5$

5. $x = 2.43, y = 3.57, z = 1.92$

6. $x = 1, y = -1, z = 1$.

7. $x_1 = 2,\ x_2 = -1,\ x_3 = 3.$

Suggested Further Readings:

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1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

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UNIT-7: Solution of Linear Simultaneous Equations-II

Structure

7.1 Introduction

Numerical methods play a crucial role in various applications, particularly in determining finite differences, employing finite element techniques, and modeling equations and differential equations. Matrix algorithms, in particular, have garnered significant attention for solving engineering and industrial problems. Matrix computations serve as essential and versatile tools in a wide range of engineering applications, including image processing, control theory, network analysis, queuing theory, telecommunication, machine learning, data mining, data science, computational finance, and bioinformatics.

In the realm of matrix calculations, methods such as LU factorization, Eigen decomposition, and Crout's method are commonly employed for solving linear systems of equations and finding eigenvalues. Choleski's method, on the other hand, is utilized for determining the inverse of a matrix. It is worth noting that Gauss elimination remains a fundamental technique for solving systems of linear equations. LU decomposition, or LU factorization, is a numerical method used to factorize a square matrix into the product of a lower triangular matrix (L) and an upper triangular matrix (U). This factorization is particularly useful for solving linear systems of equations and finding the inverse of a matrix.

Crout's method is an iterative numerical technique used for solving systems of linear equations, and it is often applied in the context of LU decomposition. Crout's method is one of the variations of LU decomposition, and its implementation involves solving a system of linear equations using forward and backward substitutions. Like other LU decomposition methods, Crout's method is valuable for efficiently solving systems of equations and finding inverses of matrices in numerical computations.

Choleski's method is widely used in various fields, including numerical analysis, statistics, and optimization, where symmetric positive definite matrices are prevalent. Its efficiency and stability make it a preferred choice for solving systems of linear equations involving such matrices.

7.2 Objectives

After reading this unit the learner should be able to understand about:

- the LU Decomposition method or Triangular method
- Procedure for Solving System of Equations by LU Decomposition Method
- Crout's method
- Choleski's method

7.3 LU Decomposition Method or Triangular Method

The LU decomposition is often employed to simplify the process of solving systems of linear equations and to enhance computational efficiency. LU decomposition is widely used in various numerical algorithms and applications, including solving linear systems, computing matrix inverses, and finding determinants. It is particularly advantageous for systems with multiple right-hand sides, as the LU decomposition can be reused for each system.

LU decomposition is a foundational concept in numerical linear algebra, contributing to the development of efficient algorithms for solving matrix-related problems.

According to LU decomposition method, every square matrix A can be expressed in the form of LU, where L is a lower triangular matrix and U is an upper triangular matrix, such that A = LU and provided that all the principal minors of A are non-singular

i.e.,

$$|a_{11}| \neq 0,$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0,$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0, \text{ and so on.}$$

7.4 Procedure for Solving System of Equations by LU Decomposition Method

The LU decomposition method is better than the Gaussian elimination method, and it's procedure is similar to Gaussian elimination. However, LU decomposition works well only when the coefficient matrix can be broken down into a product of lower and upper triangular matrices. Let us consider the system of equations:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above given system of equations can be written as in matrix form:

$$AX = B \qquad .....(1)$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Suppose $$A = LU \qquad ....(2)$$

Where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

From the equation (2), we have

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

or $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Now comparing the above two matrices for determining the values of the elements of L and U, so we have

(i) $u_{11} = a_{11}, \quad u_{12} = a_{12}, \quad u_{13} = a_{13}.$

(ii) $l_{21}u_{11} = a_{21} \quad \Rightarrow \quad l_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{a_{21}}{a_{11}}.$

(iii) $l_{31}u_{11} = a_{31} \quad \Rightarrow \quad l_{31} = \dfrac{a_{31}}{u_{11}} = \dfrac{a_{31}}{a_{11}}.$

(iv) $l_{21}u_{12} + u_{22} = a_{22} \quad \Rightarrow \quad l_{21} = \dfrac{a_{22} - u_{22}}{u_{12}}$

$$\Rightarrow \quad u_{22} = a_{22} - l_{21}u_{12} = a_{22} - \frac{a_{21}}{a_{11}}a_{12}.$$

$$(v)\quad l_{21}u_{13}+u_{23}=a_{23} \quad\Rightarrow\quad u_{23}=a_{23}-l_{21}u_{13}$$

$$\Rightarrow \quad u_{23}=a_{23}-\frac{a_{21}}{a_{11}}a_{13}.$$

$$(vi)\quad l_{31}u_{12}+l_{32}u_{22}=a_{32} \quad\Rightarrow\quad l_{32}=\frac{1}{u_{22}}\left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right)$$

$$\Rightarrow \quad l_{32}=\frac{\left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right)}{\left(a_{22}-\frac{a_{21}}{a_{11}}a_{12}\right)}.$$

$$(vii)\quad l_{31}u_{13}+l_{32}u_{23}+u_{33}=a_{33} \quad\Rightarrow\quad u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}$$

$$\Rightarrow \quad u_{33}=a_{33}-l_{31}u_{13}-l_{32}u_{23}$$

$$\Rightarrow \quad u_{33}=a_{33}-\frac{a_{31}}{a_{11}}a_{13}-\frac{\left(a_{32}-\frac{a_{31}}{a_{11}}a_{12}\right)}{\left(a_{22}-\frac{a_{21}}{a_{11}}a_{12}\right)}\left(a_{23}-\frac{a_{21}}{a_{11}}a_{13}\right).$$

Now put $A=LU$ in the equation (1), so we have

$$AX=B \quad\Rightarrow\quad LUX=B \qquad ....(3)$$

Assume $UX=Y,\quad \text{where}\quad Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ ....(4)

Now from the equations (3) and (4), we have

$$LY = B \qquad ....(5)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

From the above expression, we get

$$\left.\begin{aligned} y_1 &= b_1 \\ y_2 &= b_2 - l_{21} b_1 \\ y_3 &= b_3 - l_{31} b_1 - l_{32} y_2 \end{aligned}\right\} \qquad ....(6)$$

From the equations (4) and (6), we have

$$UX = Y \qquad ....(7)$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From the above expression, we get

$$\left.\begin{aligned} x_3 &= \frac{y_3}{u_{33}} \\ x_2 &= \frac{y_2 - u_{23} x_3}{u_{22}} \\ x_3 &= \frac{y_1 - u_{12} x_2 - u_{13} x_3}{u_{11}} \end{aligned}\right\} \qquad ....(8)$$

Hence LU decomposition can offer computational advantages over Gaussian elimination in certain scenarios. Once the LU decomposition is computed, it can be reused for solving multiple linear systems with different right-hand sides efficiently. This can result in significant time savings, especially when dealing with systems of equations with the same coefficient matrix and different constant vectors.

LU decomposition is particularly relevant and beneficial when the coefficient matrix can be expressed as the product of lower and upper triangular matrices (*i.e.,* A=LU) This condition is met in many practical applications, and when applicable, LU decomposition becomes a powerful tool for solving linear systems and related problems.

Examples

Example.1. Solve the following system of equations by LU decomposition method:

$$4x_1 + 6x_2 + 2x_3 = 18,\ x_1 + 2x_2 + 3x_3 = 6,\ 3x_1 + x_2 + 2x_3 = 8.$$

Solution: Given that the system of equations are

$$4x_1 + 6x_2 + 2x_3 = 18,$$

$$x_1 + 2x_2 + 3x_3 = 6,$$

$$3x_1 + x_2 + 2x_3 = 8.$$

The given system of equations can be written as in matrix form:

$$AX = B \qquad ....(1)$$

Where $A=\begin{bmatrix}4 & 6 & 2\\1 & 2 & 3\\3 & 1 & 2\end{bmatrix}, \quad X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \quad B=\begin{bmatrix}18\\6\\8\end{bmatrix}$

Consider $$A=LU \qquad ....(2)$$

Where $L=\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}$ and $U=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}$

From equation (2), we have

$$\begin{bmatrix}4 & 6 & 2\\1 & 2 & 3\\3 & 1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\l_{21} & 1 & 0\\l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}u_{11} & u_{12} & u_{13}\\0 & u_{22} & u_{23}\\0 & 0 & u_{33}\end{bmatrix}$$

$$\begin{bmatrix}4 & 6 & 2\\1 & 2 & 3\\3 & 1 & 2\end{bmatrix}=\begin{bmatrix}u_{11} & u_{12} & u_{13}\\l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23}\\l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33}\end{bmatrix}$$

For determining the values of elements of L and U, we comparing the above two matrices. So we have

$(i) \quad u_{11}=4, \quad u_{12}=6, \quad u_{13}=2.$

$(ii) \quad l_{21}u_{11}=1 \quad \Rightarrow \quad l_{21}=\dfrac{1}{u_{11}}=\dfrac{1}{4}.$

$(iii) \quad l_{31}u_{11}=3 \quad \Rightarrow \quad l_{31}=\dfrac{3}{u_{11}}=\dfrac{3}{4}.$

$$(iv) \quad l_{21}u_{12} + u_{22} = 2 \quad \Rightarrow \quad u_{22} = 2 - l_{21}u_{12}$$

$$\Rightarrow \quad u_{22} = 2 - 6l_{21} = 2 - 6\frac{1}{6} = \frac{1}{2}.$$

$$(v) \quad l_{21}u_{13} + u_{23} = 3 \quad \Rightarrow \quad u_{23} = 3 - l_{21}u_{13}$$

$$\Rightarrow \quad u_{23} = 3 - \frac{1}{4}(2) = \frac{5}{2}.$$

$$(vi) \quad l_{31}u_{12} + l_{32}u_{22} = 1 \quad \Rightarrow \quad l_{32}u_{22} = 1 - l_{31}u_{12}$$

$$\Rightarrow \quad l_{32} = \frac{1}{1/2}\left(1 - \frac{3}{4}.6\right) \quad \Rightarrow \quad l_{32} = -7.$$

$$(vii) \quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \quad \Rightarrow \quad u_{33} = 2 - l_{31}u_{13} - l_{32}u_{23}$$

$$\Rightarrow \quad u_{33} = 2 - \frac{3}{4}(2) - (-7)\frac{5}{2}$$

$$\Rightarrow \quad u_{33} = 18.$$

Thus the L and U are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & -7 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 6 & 2 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix}$$

Now putting $A = LU$ in equation (1), we have

$$AX = B \quad \Rightarrow \quad LUX = B \qquad \text{....(3)}$$

Consider

$$UX = Y, \quad \text{where} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad \text{....(4)}$$

Now the equations (3) and (4), we have

$$LY = B \qquad \text{....(5)}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & -7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ 8 \end{bmatrix}$$

From the above expression, we obtain

$$\left. \begin{array}{r} y_1 = 18 \\ \frac{1}{4} y_1 + y_2 = 6 \\ \frac{3}{4} y_1 - 7 y_2 + y_3 = 8 \end{array} \right\} \qquad \text{....(6)}$$

Solving above exprression (6), we get

$$y_1 = 18, \quad y_2 = 3/2, \quad y_3 = 5.$$

From the equations (4) and (6), we have

$$U\,X = Y \qquad \text{....(7)}$$

$$\begin{bmatrix} 4 & 6 & 2 \\ 0 & 1/2 & 5/2 \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

From the above we get

$$\left.\begin{array}{c} 4x_1 + 6x_2 + 2x_3 = 18 \\ \frac{1}{2}x_2 + \frac{5}{2}x_3 = \frac{3}{2} \\ 18x_3 = 5 \end{array}\right\} \quad \text{....(8)}$$

Solving above expression (8), we obtain

$$x_1 = \frac{35}{18}, \quad x_2 = \frac{29}{18}, \quad x_3 = \frac{5}{18}.$$

7.5 Crout's Method

Crout's method is an iterative numerical technique used for solving systems of linear equations, and it is often applied in the context of LU decomposition. Named after the mathematician Roland E. Crout, this method is a variation of LU decomposition that aims to factorize a square matrix A into the product of a lower triangular matrix L and an upper triangular matrix U.

Crout's method is similar to Gauss elimination method and LU decomposition method. Here we explain the Court's method by assuming three system of equation:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

The above system of equation can be written as in matrix form:

$$AX = B \qquad \qquad ....(1)$$

Where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

The augmented matrix $[A \vdots B]$ of equation (1) is

$$[A \vdots B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \qquad \qquad .....(2)$$

Thus the augmented derived matrix $[A' \vdots B']$ is given as

$$\begin{bmatrix} a_{11}' & a_{12}' & a_{13}' & b_1' \\ a_{21}' & a_{22}' & a_{23}' & b_2' \\ a_{31}' & a_{32}' & a_{33}' & b_3' \end{bmatrix} \qquad \qquad .....(3)$$

For determining a_{ij}' and b_i' from equations (2) and (3), we have

$$\therefore \quad a_{11}' = a_{11}, \quad a_{21}' = a_{21}, \quad a_{31}' = a_{31}.$$

$$a_{12}' = \frac{a_{12}}{a_{11}}, \quad a_{13}' = \frac{a_{13}}{a_{11}}, \quad b_1' = \frac{b_1}{a_{11}}.$$

$$a_{22}' = a_{22} - a_{21}' a_{12}'.$$

$$a_{32}' = a_{32} - a_{31}' a_{12}'.$$

$$a_{23}' = \frac{a_{23} - a_{21}' a_{13}'}{a_{22}'}.$$

$$b_2' = \frac{b_2 - a_{21}' b_1'}{a_{22}'}.$$

$$a_{33}' = a_{33} - a_{31}' a_{13}' - a_{32}' a_{23}'.$$

$$b_3' = \frac{b_3 - a_{31}' b_1' - a_{32}' b_2'}{a_{33}'}.$$

Hence the solution of the above system of equation is

$$x_3 = b_3'$$

$$x_2 = b_2' - a_{23}' x_3$$

$$x_1 = b_1' - a_{12}' x_2 - a_{13}' x_3.$$

Check your Progress

1. What do you mean by LU Decomposition method?

2. Explain the crout's method.

Examples

Example.2. Apply Court's method to solve

$$x_1 - 3x_2 + 4x_3 = 12,\ 5x_1 + 4x_2 - 3x_3 = 2, 3x_1 + x_2 + 2x_3 = 16.$$

Solution: Given that the system of equations are

$$x_1 - 3x_2 + 4x_3 = 12,$$

$$5x_1 + 4x_2 - 3x_3 = 2,$$

$$3x_1 + x_2 + 2x_3 = 16.$$

The given system of equations can be written as in matrix form:

$$AX = B \qquad .....(1)$$

Where $A = \begin{bmatrix} 1 & -3 & 4 \\ 5 & 4 & -3 \\ 3 & 1 & 2 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 12 \\ 2 \\ 16 \end{bmatrix}$

The augmented matrix $[A/B]$ is given as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 & 12 \\ 5 & 4 & -3 & 2 \\ 3 & 1 & 2 & 16 \end{bmatrix} \qquad \ldots(2)$$

Thus the augmented derived matrix $[A'/B']$ is given as

$$\begin{bmatrix} a_{11}' & a_{12}' & a_{13}' & b_1' \\ a_{21}' & a_{22}' & a_{23}' & b_2' \\ a_{31}' & a_{32}' & a_{33}' & b_3' \end{bmatrix} \qquad \ldots(3)$$

For determining a_{ij}' and b_i' from equations (2) and (3), we have

$$\therefore \quad a_{11}' = a_{11} = 1, \quad a_{21}' = a_{21} = 5, \quad a_{31}' = a_{31} = 3.$$

$$a_{12}' = \frac{a_{12}}{a_{11}} = -\frac{3}{1} = -3.$$

$$a_{13}' = \frac{a_{13}}{a_{11}} = \frac{4}{1} = 4.$$

$$b_1' = \frac{b_1}{a_{11}} = \frac{12}{1} = 12.$$

$$\begin{aligned} a_{22}' &= a_{22} - a_{21}'a_{12}' \\ &= 4 - 5(-3) \\ &= 19. \end{aligned}$$

$$\begin{aligned} a_{32}' &= a_{32} - a_{31}'a_{12}' \\ &= 1 - 3(-3) \\ &= 10. \end{aligned}$$

$$a_{23}' = \frac{a_{23} - a_{21}'a_{13}'}{a_{22}'}$$

$$= \frac{-3-5(4)}{19}$$

$$= -\frac{23}{19}.$$

$$b_2' = \frac{b_2 - a_{21}'b_1'}{a_{22}'}$$

$$= \frac{2-5(12)}{19}$$

$$= -\frac{58}{19}.$$

$$a_{33}' = a_{33} - a_{31}'a_{13}' - a_{32}'a_{23}'$$

$$= 2 - 3(4) - 10\left(-\frac{23}{19}\right)$$

$$= \frac{40}{19}.$$

$$b_3' = \frac{b_3 - a_{31}'b_1' - a_{32}'b_2'}{a_{33}'}$$

$$= \frac{16 - 3(12) - 10\left(-\frac{58}{19}\right)}{40/19}$$

$$= 5.$$

Hence the solution of the above system of equation is

$$x_3 = b_3' = 5$$

$$x_2 = b_2' - a_{23}'x_3$$

$$= -\frac{58}{19} + \frac{23}{19}(5)$$
$$= 3$$

$$x_1 = b_1' - a_{12}'x_2 - a_{13}'x_3$$

$$= 12 - (-3)(3) - 4(5)$$
$$= 1.$$

7.6 Choleski's Method

Choleski's method, named after the mathematician André-Louis Cholesky, is an algorithm used for the decomposition of a positive definite matrix into the product of a lower triangular matrix L and its transpose L^T. This decomposition is a specialized form of the more general LU decomposition, tailored for symmetric and positive definite matrices. The Choleski decomposition is particularly useful for solving linear systems, calculating determinants, and generating samples from multivariate normal distributions.

Choleski's method is used to determine the inverse of a matrix provided that matrix is symmetric i.e., $A' = A$.

Let us assume $A = LL'$

or
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \quad \ldots(1)$$

Where L is a lower triangular matrix and L' is transpose of L (an upper triangular matrix).

The inverse of A is

$$A^{-1} = (LL')^{-1} = (L')^{-1}L^{-1} = (L^{-1})'L^{-1}$$ … (2)

To determine the inverse of A first we comparing both sides of equation (1) to finding the values of $l_{11}, l_{21}, l_{22}, \ldots\ldots..$ Putting these value of $l_{11}, l_{21}, l_{22}, \ldots\ldots..$ in L .

Now to find L^{-1} we have

$$L^{-1} = X \qquad \Rightarrow \quad LX = I$$ …(3)

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving the above system we get the matrix L^{-1} and using equation (2), we get the inverse of the given matrix A.

Examples

Example.3. Determine the inverse of the following matrix using Choleski's method:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}.$$

Solution: Given that the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix}$ is symmetric so we use Choleski's method to find inverse of the matrix.

Let us assume $A = L\,L'$

or $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \qquad ....(1)$$

Comparing both sides, we get

$$l_{11}^2 = 1 \quad \Rightarrow \quad l_{11} = 1.$$

$$l_{11} l_{21} = 1 \quad \Rightarrow \quad l_{21} = 1.$$

$$l_{11} l_{31} = 1 \quad \Rightarrow \quad l_{31} = 1.$$

$$l_{21}^2 + l_{22}^2 = 2 \quad \Rightarrow \quad l_{22} = 1.$$

$$l_{21} l_{31} + l_{22} l_{32} = 3 \quad \Rightarrow \quad l_{32} = 2.$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 6 \quad \Rightarrow \quad l_{33} = 1.$$

So we get the matrix L is

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Now to find L^{-1} we have

$$L^{-1} = X \quad \Rightarrow \quad LX = I \qquad ....(2)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving the above system we get

$$x_{11} = 1.$$

$$x_{11} + x_{21} = 0 \quad \Rightarrow \quad x_{21} = -1.$$

$$x_{22} = 1.$$

$$x_{11} + 2x_{21} + x_{31} = 0 \quad \Rightarrow \quad x_{31} = 1.$$

$$2x_{22} + x_{32} = 0 \quad \Rightarrow \quad x_{32} = -2.$$

$$x_{33} = 1.$$

Therefore the matrix L^{-1} is

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

The transpose of matrix L^{-1} is

$$\left(L^{-1}\right)' = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

The inverse of A is

$$A^{-1} = \left(L\,L'\right)^{-1} = \left(\,L'\,\right)^{-1} L^{-1} = \left(\,L^{-1}\right)' L^{-1}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 1 \\ -3 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}.$$

7.7 Summary

The superiority of the LU decomposition method over the Gaussian elimination method is evident, and there exists a resemblance between LU decomposition and Gaussian elimination. The effectiveness of the LU decomposition method is contingent upon the coefficient matrix being representable as the product of lower and upper triangular matrices. LU decomposition offers advantages in terms of efficiency and reusability over Gaussian elimination, especially when dealing with multiple linear systems. Its relevance is contingent upon the factorization conditions A=LU which is satisfied in many real-world scenarios. Crout's method is valuable for efficiently solving systems of equations and finding inverses of matrices in numerical computations. The Choleski decomposition is particularly useful for solving linear systems, calculating determinants, and generating samples from multivariate normal distributions.Choleski's method is computationally efficient, requiring only half as many calculations as the LU decomposition since it exploits the symmetry of positive definite matrices.

7.8 Terminal Questions

Q.1. What do you mean by LU Decomposition method.

Q.2. Explain Choleski' method.

Q.3. Solve the following system of equations using LU decomposition method:

$$2x_1 + x_2 + x_3 = 2,\ x_1 + 3x_2 + 2x_3 = 2, 3x_1 + x_2 + 2x_3 = 2.$$

Q.4. Solve the following system of equations using LU decomposition method:

$$x_1 + 2x_2 + 3x_3 = 14,\ 2x_1 + 5x_2 + 2x_3 = 18, 3x_1 + x_2 + 5x_3 = 20.$$

Q.5. Apply Crout's method to solve

$$x_1 + 2x_2 + 3x_3 = 6,\ 2x_1 + 3x_2 + x_3 = 9, 3x_1 + x_2 + 2x_3 = 8.$$

Q.6. Apply Crout's method to solve

$$5x_1 + 2x_2 + x_3 = -12,\ -x_1 + 4x_2 + 2x_3 = 20, 2x_1 - 3x_2 + 10x_3 = 3.$$

Q.7. Applying Choleski's method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$

Q.8. Applying Choleski's method to find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 6 \\ 2 & 5 & 15 \\ 6 & 15 & 46 \end{bmatrix}.$$

ANSWERS

Q.3. $x_1 = 1, \quad x_2 = 1, \quad x_3 = -1.$

Q.4. $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3.$

Q.5. $x_1 = \frac{35}{18}, \quad x_2 = \frac{29}{18}, \quad x_3 = \frac{5}{18}.$

Q.6. $x_1 = -4, \quad x_2 = 3, \quad x_3 = 2.$

Q.7. $\begin{bmatrix} -23/13 & -3/13 & 14/13 \\ -3/13 & 3/13 & -1/13 \\ 14/13 & -1/13 & -4/13 \end{bmatrix}$

Q.8. $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 10 & -3 \\ 0 & -3 & 1 \end{bmatrix}$

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

Master of Science
PGMM -104N
Numerical Analysis

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Block

4 Solving Algebraic and Transcendental Equations

Block-4

Solving Algebraic and Transcendental Equations

Many challenges in science and engineering can be expressed through equations, making the solution of equations a pivotal aspect in scientific and engineering domains. The significance of solving equations extends to various mathematical problems as well. In earlier units, we extensively explored the concept of numerical methods and the associated operators. In this block, our focus shifts to the task of determining solutions for algebraic and transcendental equations, commonly referred to as finding the roots of an equation. There are so many numerical methods for solving algebraic and transcendental equations. Some important methods are following: Bisection Method; Newton-Raphson Method; Regula Falsi Method and Secant Method.

While mathematical methods readily handle linear, quadratic, cubic, and biquadratic equations, transcendental equations and those of higher degrees pose a more intricate challenge. The conventional mathematical approaches may not be as effective for these types of equations. Consequently, numerical methods such as the Bisection method, Regula falsi method, Secant method, and Newton Raphson's method come into play to address and solve transcendental equations and equations of higher degrees.

Numerical methods provide effective techniques for solving algebraic and transcendental equations when analytical solutions are challenging or impossible to obtain. Here, we'll explore some commonly used numerical methods for solving such equations: Bisection method, Regula Falsi method, Secant method and Newton-Raphson method. These numerical methods are essential for solving algebraic and transcendental equations encountered in various scientific, engineering, and mathematical applications. The choice of method depends on the specific characteristics of the equation and the desired level of precision.

In the eighth unit, we shall discussed about the Bisection Method and Newton Raphson method and in the ninth unit we deal with Regula falsi method and secant method.

UNIT-8: Numerical Method for solving Algebraic and Transcendental Equations-I

Structure

8.1 Introduction

Numerical methods offer powerful approaches to solve algebraic and transcendental equations, especially in cases where obtaining analytical solutions proves difficult or impractical. These techniques play a crucial role in addressing a wide range of scientific, engineering, and mathematical problems where precise solutions are essential but challenging to derive through traditional analytical methods. The Bisection Method is a simple yet effective numerical technique for finding the root of a real-valued function within a specified interval. This method is particularly useful when dealing with continuous functions where the root exists and changes sign over the chosen interval.

The Newton-Raphson Method is an iterative numerical technique for finding the roots of a real-valued function. Named after Sir Isaac Newton and Joseph Raphson, this method is particularly efficient for obtaining accurate approximations of roots, especially when the initial guess is close to the actual root.

8.2 Objectives

After reading this unit the learner should be able to understand about:

- The Polynomial
- Algebraic and Transcendental Equations
- Bisection method and procedure to find the real root by Bisection method
- Newton method and procedure to find the real root by Newton method

8.3 Polynomial

An expression of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n x^n$$

where all *a's* are constant provided that $a_n \neq 0$ and *n* is a positive integer, known as a polynomial in *x* of degree *n*.

8.4 Algebraic Equations

An expression of the form

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots.. + a_{n-1} x + a_n = 0, \; a_0 \neq 0$$

where all $a_0, a_1, \ldots\ldots., a_n$ are constants and *n* is a positive integer, known as an algebraic equation of degree *n*, in terms of *x*.

Here $4x^6 + 3x^5 + 9x^4 + x^3 + 3x - 6 = 0,\; x^3 - x - 1 = 0$ are the examples of algebraic equation.

8.5 Transcendental Equations

If $f(x)$ is an expression involving some functions such as trigonometric, exponential, logarithmic etc., then the equation

$$f(x) = 0 \qquad \ldots. (1)$$

is known as transcendental equation.

Here $\cos x + 4\sin x + 1 = 0;\quad x^2 \log_{10} x - x + 1 = 0;\quad xe^{\sqrt{x}} = \sin x$ all are transcendental equations.

8.6 Root of the Equation

The value of x which satisfying the equation $f(x) = 0$ is known as the root of the equation. The roots of the linear, quadratic, cubic, or biquadratic equations are obtained by available methods, but for transcendental equation or higher degree equation cannot solved by these methods easily. So those types of equation can be solved by numerical methods such as Bisection, Secant, Newton-Raphson, Regula-Falsi method etc.

Note: Every algebraic equation of degree n has only n roots real as well as imaginary.

8.7 Bisection Method

The Bisection Method is a fundamental tool in numerical analysis and serves as the basis for more advanced root-finding algorithms. Bisection method is used to find the root of an equation $f(x) = 0$ to the desired degree of accuracy.

According to this method to find root of an equation first we check the given function $f(x)$ is continuous in a closed interval $[a, b]$ or not. If $f(x)$ is not continuous in a closed interval $[a, b]$ then Bisection method is failed. Also if $f(x)$ is continuous in a closed interval $[a, b]$ and does not cut the x-axis, then $f(x)$ does not have a real root.

8.8 Procedure to find the real root by Bisection Method

The procedure to determine the real root by Bisection Method are as follows:

Consider the given equation is

$$f(x) = 0. \qquad \ldots(1)$$

Step-1: First determine a closed interval $[a, b]$ in which the function $f(x)$ is continuous and the values $f(a)$ and $f(b)$ are opposite sign.

Step-2: If $f(a) < 0$ and $f(b) > 0$, then there exist one real root of the given equation (1) between $a < x < b$.

Step-3: For the first approximation to get the root by bisecting the interval (a, b); we have

$$x_1 = \frac{a+b}{2}.$$

Step-4: Suppose if $f(x_1) = 0$, then x_1 is a required root of the equation (1) *i.e.*, $f(x) = 0$.

Otherwise the root will be either in the interval (a, x_1) or in the interval (x_1, b) according as $f(x)$ is positive or negative.

Step-5: Now bisect the interval as before and continue this process until we get the root of the given equation $f(x) = 0$ is found to a desired degree of accuracy.

Case-1: If $f(x_1) > 0$, so that the root lies between a and x_1; then for second approximation we bisect the interval (a, x_1);

$$i.e., x_2 = \frac{a + x_1}{2}.$$

Case-2: If $f(x_2) < 0$, so that the root lies between x_1 and x_2; then for third approximation we bisect the interval (x_1, x_2); *i.e.*, $x_3 = \dfrac{x_1 + x_2}{2}$.

Step-6: Continue this process, until we get the root of the given equation (1) to the desired accuracy.

Check your Progress

1. What do you mean by algebraic equation?

2. What is transcendental equations.

3. How to find the root of equation by Bisection method.

Examples

Example.1. Determine the real root of the equation $x^3 - 4x + 1 = 0$ by Bisection method.

Sol. It is given that $f(x) = x^3 - 4x + 1 = 0$.

Here $f(1) = (1)^3 - 4 + 1 = -2 < 0$

and $f(2) = (2)^3 - 4(2) + 1 = 1 > 0$.

Therefore $f(1) < 0$ and $f(2) > 0$ so at least one root of the given equation lies between 1 and 2.

Using Bisection method, the first approximation is

$$x_1 = \frac{1+2}{2} = 1.5.$$

Now we see that

$$f(1.5) = (1.5)^3 - 4(1.5) + 1 = -1.625 < 0$$

Therefore $f(1.5) < 0$ and $f(2) > 0$ so the root lies between 1.5 and 2.

For the second approximation is

$$x_2 = \frac{1.5+2}{2} = 1.75.$$

Now we see that

$$f(1.75) = (1.75)^3 – 4(1.75)+1= –0.640625 < 0$$

Therefore the root lies between 1.75 and 2.

For the third approximation is

$$x_3 = \frac{1.75+2}{2} = 1.875.$$

Now we see that

$$f(1.875) = (1.875)^3 – 4(1.875)+1 = 0.091796875 > 0$$

Therefore the root lies between 1.75 and 1.875.

For the fourth approximation is

$$x_4 = \frac{1.75+1.875}{2} = 1.8125.$$

Now we see that

$$f(1.8125) = (1.8125)^3 – 4(1.8125)+1= –0.295654296875 < 0$$

Therefore the root lies between 1.8125 and 1.875.

For the fifth approximation is

$$x_5 = \frac{1.8125+1.875}{2} = 1.84375.$$

Now we see that

$$f(1.84375) = (1.84375)^3 – 4(1.84375)+1= –0.107330322265 < 0$$

Therefore the root lies between 1.84375 and 1.875.

For the sixth approximation is

$$x_6 = \frac{1.84375 + 1.875}{2} = 1.859375.$$

Now we see that

$$f(1.859375) = (1.859375)^3 – 4(1.859375) + 1 = –0.009128570556 < 0$$

Therefore the root lies between 1.859375 and 1.875.

For the seventh approximation is

$$x_7 = \frac{1.859375 + 1.875}{2} = 1.8671875.$$

Now we see that

$$f(1.8671875) = (1.8671875)^3 – 4(1.8671875) + 1 = 0.0409922599792 > 0$$

Therefore the root lies between 1.859375 and 1.8671875.

For the eight approximation is

$$x_8 = \frac{1.859375 + 1.8671875}{2} = 1.86328125.$$

Now we see that from above iterations,

x_1=1.5, x_2=1.75, x_3=1.875,

x_4=1.8125, x_5=1.84375, x_6=1.859375,

x_7=1.8671875, x_8=1.86328125.

The root of the given equation $f(x) = x^3–4x+1= 0$ up-to two decimal places is 1.86, which is of desired accuracy.

Example.2. Using Bisection method to determine the real root of the equation $x^3 - 5x + 1 = 0.$

Solution: It is given that $f(x) = x^3 - 5x + 1 = 0.$.

Here $f(2) = 2^3 - 5 \times 2 + 1 = -1$

and $f(3) = 3^3 - 5 \times 3 + 1 = 13$

Therefore $f(2) < 0$ and $f(3) > 0$ so at least one root of the given equation lies between 2 and 3.

Using Bisection method, the first approximation is

$$x_1 = \frac{2+3}{2} = 2.5.$$

Now we see that

$$f(2.5) = (2.5)^3 - 5(2.5) + 1 = 4.125 > 0$$

Therefore $f(2) < 0$ and $f(2.5) > 0$ so the root lies between 2 and 2.5.

For the second approximation is

$$x_2 = \frac{2+2.5}{2} = 2.25.$$

Now we see that

$$f(2.25) = (2.25)^3 - 5(2.25) + 1 = 1.140625 > 0$$

Therefore the root lies between 2 and 2.25.

For the third approximation is

$$x_3 = \frac{2 + 2.25}{2} = 2.125.$$

Now we see that

$f(2.125) = (2.125)^3 – 5(2.125)+1 = -0.029296875 < 0$

Therefore the root lies between 2.125 and 2.25.

For the fourth approximation is

$$x_4 = \frac{2.125 + 2.25}{2} = 2.1875$$

Now we see that

$f(2.1875) = (2.1875)^3 – 5(2.1875)+1 = 0.530029296875 > 0$

Therefore the root lies between 2.125 and 2.1875.

For the fifth approximation is

$$x_5 = \frac{2.125 + 2.1875}{2} = 2.15625.$$

Now we see that

$f(2.15625) = (2.15625)^3 – 5(2.15625)+1 = 0.2440490722656 > 0$

Therefore the root lies between 2.125 and 2.15625.

For the sixth approximation is

$$x_6 = \frac{2.125 + 2.15625}{2} = 2.140625.$$

Now we see that

$$f(2.140625) = (2.140625)^3 – 5(2.140625)+1 = 0.105808280566 > 0$$

Therefore the root lies between 2.125 and 2.140625.

For the seventh approximation is

$$x_7 = \frac{2.125 + 2.140625}{2} = 2.1328.$$

Now we see that

$$f(2.1328) = (2.1328)^3 – 5(2.1328)+1 = 0.037757079552 > 0$$

Therefore the root lies between 2.125 and 2.1328.

For the eight approximation is

$$x_8 = \frac{2.125 + 2.1328}{2} = 2.1289.$$

Now we see that

$$f(2.1289) = (2.1289)^3 – 5(2.1289)+1 = 0.004132960569 > 0$$

Therefore the root lies between 2.125 and 2.1289.

For the ninth approximation is

$$x_9 = \frac{2.125 + 2.1298}{2} = 2.12695.$$

Now we see that from above iterations,

x_1=2.5, x_2=2.25, x_3=2.125, x_4=2.1875,

x_5=2.15625, x_6=2.140625, x_7=2.1328, x_8=2.1289,

x_9=2.12695.

The root of the given equation $f(x) = x^3 - 5x + 1 = 0$ up-to two decimal places is 2.12, which is of desired accuracy.

8.9 Newton-Raphson Method

The Newton-Raphson Method is widely used in various scientific and engineering applications for solving nonlinear equations and finding roots due to its speed of convergence when applicable conditions are met. The Newton-Raphson Method is an iterative numerical technique designed for approximating the roots of a real-valued function. This method, named after Sir Isaac Newton and Joseph Raphson, is particularly effective when seeking accurate solutions to Algebraic and Trancedental equations.

Newton-Raphson Method is known for its rapid convergence, especially when the initial guess is close to the root and the function behaves well. However, it may exhibit convergence issues if the initial guess is far from the root or if the function possesses certain characteristics. It is a powerful tool widely utilized in scientific and engineering applications for solving nonlinear equations.

8.10 Procedure to find a real root by Newton-Raphson Method

Let us consider $x = x_0$ be an approximate value of the roots of the equation $f(x) = 0$ which may be algebraic or transcendental and let $x_0 + h$ be the correct vale of the corresponding root where h be a real number sufficiently small. Then we have

$$f(x_0 + h) = 0 \qquad \ldots..(1)$$

The above equation (1) expanding by Taylor's theorem, we have

$$f(x_0 + h) = f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \ldots.. = 0$$

Since h is very small, so neglecting the second and higher order terms, so we get

$$f(x_0) + h\, f'(x_0) = 0$$

$$h = -\frac{f(x_0)}{f'(x_0)} \text{ also } f'(x_0) \neq 0 \qquad \text{.....(2)}$$

Thus from equations (1) and (2), the first approximation of the root is given by

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly, we taking x_1 as initial approximation, to be the better approximation of the root x_2 is obtained as

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \qquad f'(x_0) \neq 0$$

Proceeding in the same way, we get the better approximation of the root is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, 3....$$

This is known as the Newton-Raphson formula which is very important for solving the algebraic equations and transcendental equations.

Examples

Example.3. Determine the real root of the equation $x^2 - 4x + 2 = 0$ using Newton-Raphson's method.

Solution: The given equation is $x^2 - 4x + 2 = 0$.

Consider $\qquad f(x) = x^2 - 4x + 2 = 0$

and $\qquad f'(x) = 2x - 4$

Now we have

$$f(3) = 3^2 - 4\times 3 + 2 = -1.$$

and $$f(4) = 4^2 - 4\times 4 + 2 = 2.$$

Therefore, one real root of the given equation is lies between 3 and 4.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^2 - 4x_n + 2}{2x_n - 4}$$

$$= \frac{2x_n^2 - 4x_n - x_n^2 + 4x_n - 2}{2x_n - 4}$$

$$x_{n+1} = \frac{x_n^2 - 2}{2x_n - 4} \qquad \text{where } n = 0, 1, 2, 3, \ldots.. \qquad \ldots(1)$$

First we take $x_0 = 3$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{x_0^2 - 2}{2x_0 - 4}$$

$$= \frac{3^2 - 2}{2\times 3 - 4}$$

$$= \frac{9 - 2}{6 - 4}$$

$$= \frac{7}{2}$$

$$= 3.5$$

Again, putting n =1 in the equation (1), so we get second approximation

$$x_2 = \frac{x_1^2 - 2}{2x_1 - 4}$$

$$= \frac{(3.5)^2 - 2}{2 \times 3.5 - 4}$$

$$= \frac{12.25 - 2}{7 - 4}$$

$$= \frac{10.25}{3}$$

$$= 3.41667$$

Putting n =2 in the equation (1), so we get the third approximation

$$x_3 = \frac{x_2^2 - 2}{2x_2 - 4}$$

$$= \frac{(3.41667)^2 - 2}{2 \times (3.41667) - 4}$$

$$= \frac{11.67363 - 2}{6.83334 - 4}$$

$$= \frac{9.67363}{2.83334}$$

$$= 3.41421$$

Putting n = 3 in the equation (1), so we get the fourth approximation

$$x_4 = \frac{x_3^3 - 2}{2x_3 - 4}$$

$$= \frac{(3.41421)^2 - 2}{2 \times 3.41421 - 4}$$

$$= \frac{11.65683 - 2}{6.82842 - 4}$$

$$= \frac{9.65683}{2.82842}$$

$$= 3.41421$$

Here we see that $x_3 = x_4$. Hence the root of the given equation $x^2 - 4x + 2 = 0$ is 3.41421.

Example.4. Using Newton-Raphson's method to determine the real root of the equation $x^3 - 5x + 1 = 0$.

Solution: The given equation is $x^3 - 5x + 1 = 0$.

Consider $\quad f(x) = x^3 - 5x + 1 = 0$

and $\quad f'(x) = 3x^2 - 5$

Now we have

$$f(2) = 2^3 - 5 \times 2 + 1 = -1$$

and $\quad f(3) = 3^3 - 5 \times 3 + 1 = 13$

Therefore, one real root of the given equation is lies between 2 and 3.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5}$$

$$= \frac{3x_n^3 - 5x_n - x_n^3 + 5x_n + 1}{3x_n^2 - 5}$$

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 - 5} \quad \text{where } n = 0, 1, 2, 3, \ldots \qquad ..(1)$$

First we take $x_0 = 2$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{2x_0^3 + 1}{3x_0^2 - 5}$$

$$= \frac{2(2)^3 + 1}{3(2)^2 - 5}$$

$$= \frac{17}{7}$$

$$= 2.42857.$$

Again, putting $n = 1$ in the equation (1), so we get second approximation

$$x_2 = \frac{2x_1^3 + 1}{3x_1^2 - 5}$$

$$= \frac{2(2.42857)^3 + 1}{3(2.42857)^2 - 5}$$

$$= \frac{29.64718}{12.69386}$$

$$= 2.33555$$

Putting n =2 in the equation (1), so we get the third approximation

$$x_3 = \frac{2x_2^3 + 1}{3x_2^2 - 5}$$

$$= \frac{2(2.33555)^3 + 1}{3(2.33555)^2 - 5}$$

$$= \frac{26.47989}{11.36438}$$

$$= 2.33008$$

Putting n = 3 in the equation (1), so we get the fourth approximation

$$x_4 = \frac{2x_3^3 + 1}{3x_3^2 - 5}$$

$$= \frac{2(2.33008)^3 + 1}{3(2.33008)^2 - 5}$$

$$= \frac{26.30128}{11.28782}$$

$$= 2.33006$$

Here we see that $x_3 = x_4$. Hence the root of the given equation $x^3 - 5x + 1 = 0$ is 2.3300.

Example.5. Determine the cube root of 10 correct to six decimal places with the help of Newton-Raphson's method.

Solution: The given equation is $x^3 - 10 = 0$.

Consider $f(x) = x^3 - 10 = 0$

and $f'(x) = 3x^2$

Now we have

$$f(1) = 1^3 - 10 = -9$$

$$f(2) = 2^3 - 10 = -2$$

and $f(3) = 3^3 - 10 = 17$

Therefore, one real root of the given equation is lies between 2 and 3.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 10}{3x_n^2}$$

$$= \frac{3x_n^3 - x_n^3 + 10}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + 10}{3x_n^2} \quad \text{where } n = 0, 1, 2, 3\ldots. \qquad \ldots(1)$$

First we take $x_0 = 2$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{2x_0^3 + 10}{3x_0^2}$$

$$= \frac{2(2)^3 + 10}{3(2)^2}$$

$$= \frac{26}{12}$$

$$= 2.16667$$

Again, putting *n* =1 in the equation (1), so we get the second approximation

$$x_2 = \frac{2x_1^3 + 10}{3x_1^2}$$

$$= \frac{2(2.16667)^3 + 10}{3 \times (2.16667)^2}$$

$$= \frac{30.342686}{14.083377}$$

$$= 2.154504$$

Putting *n* =2 in the equation (1), so we get the third approximation

$$x_3 = \frac{2x_2^3 + 10}{3x_2^2}$$

$$= \frac{2.(2.154504)^3 + 10}{3.(2.154504)^2}$$

$$= \frac{30.001930}{13.925662}$$

$$= 2.154435$$

Putting n = 3 in the equation (1), so we get the fourth approximation

$$x_4 = \frac{2x_3^3 + 10}{3x_3^2}$$

$$= \frac{2(2.154435)^3 + 10}{3(2.154435)^2}$$

$$= \frac{30.000009}{13.924771}$$

$$= 2.154435$$

Here we see that $x_3 = x_4$. Hence the root of the given equation $x^3 - 10 = 0$ is 2.154435.

Example.6. Determine the real root of tan x = 4x by using Newton-Raphson's method.

Solution. The given equation is $\tan x - 4x = 0$.

Consider $\quad f(x) = \tan x - 4x = 0$

and $\quad f'(x) = \sec^2 x - 4$

Now we have

$$f(0) = 0$$

and $\quad f(1) = -3.982$

Therefore, one real root of the given equation is 0.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{\tan x_n - 4x_n}{\sec^2 x_n - 4}$$

$$= \frac{x_n \sec^2 x_n - \tan x_n}{\sec^2 x_n - 4}, \quad \text{where } n = 0, 1, 2, 3, \ldots \qquad \ldots(1)$$

First we take $x_0 = 2$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{x_0 \sec^2 x_0 - \tan x_0}{\sec^2 x_0 - 4}$$

$$= \frac{0.1 - 0}{1 - 4}$$

$$= 0$$

Hence the root of the given equation tan $x - 4x = 0$ is 0.

Example.7. Using Newton-Raphson's method to determine the root of $e^x = 3x$ upto correct to four decimal places.

Solution: The given equation is $e^x = 3x$.

Consider $f(x) = e^x - 3x = 0$

and $f'(x) = e^x - 3$

Now we have

$$f(0) = 1$$

and $f(1) = -0.28$

Therefore, one real root of the given equation is lies between 0 and 1.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3}$$

$$= \frac{x_n e^{x_n} - 3x_n - e^{x_n} + 3x_n}{e^{x_n} - 3}$$

$$x_{n+1} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 3}, \text{ where } n = 0, 1, 2, 3 \ldots.. \qquad ..(1)$$

First we take $x_0 = 0$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{e^{x_0}(x_0 - 1)}{e^{x_0} - 3}$$

$$= \frac{e^0(0-1)}{e^0 - 3}$$

$$= \frac{-1}{-2}$$

$$= 0.5$$

Again, putting $n = 1$ in the equation (1), so we get the second approximation

$$x_2 = \frac{e^{x_1}(x_1 - 1)}{e^{x_1} - 3}$$

$$= \frac{e^{0.5}(0.5-1)}{e^{0.5} - 3}$$

$$= \frac{-0.82436}{-1.35127}$$

$$= 0.61006$$

Putting n =2 in the equation (1), so we get the third approximation

$$x_3 = \frac{e^{x_2}(x_2 - 1)}{e^{x_2} - 3}$$

$$= \frac{e^{0.61006}(0.61006 - 1)}{e^{0.61006} - 3}$$

$$= \frac{-0.71770}{-1.15945}$$

$$= 0.61900$$

Putting n =3 in the equation (1), so we get the fourth approximation

$$x_4 = \frac{e^{x_3}(x_3 - 1)}{e^{x_3} - 3}$$

$$= \frac{e^{0.61900}(0.61900 - 1)}{e^{0.61900} - 3}$$

$$= \frac{-0.71770}{-1.14293}$$

$$= 0.61905$$

Here we see that $x_3 = x_4$. Hence the root of the given equation $e^x = 3x$ is 0.6190.

Example.8. Determine the real root of the equation $3x = \cos x + 1$ by Newton-Raphson's method.

Solution: The given equation is $3x = \cos x + 1$.

Consider $f(x) = 3x - \cos x - 1 = 0$

and $f'(x) = 3 + \sin x$

Now we have

$$f(0) = -2$$

and $f(1) = 3 - \cos(1) - 1 = 2 - 0.5403 = 1.4597$

Therefore, one real root of the given equation is lies between 0 and 1.

Using Newton-Raphson's formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{3x_n - \cos x_n - 1}{3 + \sin x_n}$$

$$= \frac{3x_n + x_n \sin x_n - 3x_n + \cos x_n + 1}{3 + \sin x_n}$$

$$= \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n} \qquad \ldots(1)$$

First we take $x_0 = 0$

Putting $n = 0$ in the equation (1), so we get the first approximation

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$$= \frac{0 \sin 0 + \cos 0 + 1}{3 + \sin 0}$$

$$= \frac{2}{3}$$

$$= 0.6666$$

Putting n = 1 in the equation (1), so we get the second approximation

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1}$$

$$= \frac{(0.6666)\sin(0.6666)\cos(0.6666) + 1}{3 + \sin(0.6666)}$$

$$= \frac{0.6666 \times 0.6183 + 0.7859 + 1}{3 + 0.6183}$$

$$= \frac{0.4122 + 0.7859 + 1}{3.6183}$$

$$= \frac{2.1481}{3.6183}$$

$$= 0.6074$$

Putting n =2 in the equation (1), so we get the third approximation

$$x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2}$$

$$= \frac{(0.6074)\sin(0.6074) + \cos(0.6074) + 1}{3 + \sin(0.6074)}$$

$$= \frac{0.6074 \times 0.5707 + 0.8211 + 1}{3 + 0.5707}$$

$$= \frac{0.3466 + 0.8211 + 1}{3.5707}$$

$$= \frac{2.1677}{3.5707}$$

$$= 0.6071$$

Putting n =3 in the equation (1), so we get the fourth approximation

$$x_4 = \frac{x_3 \sin x_3 + \cos x_3 + 1}{3 + \sin x_3}$$

$$= \frac{(0.6071)\sin(0.6071) + \cos(0.6071) + 1}{3 + \sin(0.6071)}$$

$$= \frac{0.6071 \times 0.5704 + 0.8213 + 1}{3 + 0.5704}$$

$$= \frac{0.3463 + 0.8213 + 1}{3.5704}$$

$$= \frac{2.1676}{3.5704}$$

$$= 0.6071$$

Here we see that $x_3 = x_4$. Hence the root of the given equation $3x - \cos x - 1 = 0$ is 0.6071.

8.11 Summary

Bisection Method, another essential tool in numerical analysis, serves as a foundational technique for more advanced root-finding algorithms. Primarily used to locate the root of an equation $f(x) = 0$ with a desired degree of accuracy, the Bisection Method is a reliable approach applicable in various mathematical and computational contexts.

Newton-Raphson Method stands out as a widely employed and powerful numerical technique in the realms of science and engineering. Specifically designed for approximating roots of real-valued functions, this iterative method, named after Sir Isaac Newton and Joseph Raphson, is acclaimed for its rapid convergence under favorable conditions. It is a preferred choice in applications requiring accurate solutions to nonlinear equations.

8.12 Terminal Questions

Q.1. Explain the procedure for solving algebraic equation by Newton-Raphson's method.

Q.2. Write the procedure for solving algebraic equation by Bisection method.

Q.3. Using Bisection method to find a real root of the equation $f(x) = 3x - \sqrt{1+\sin x} = 0.$

Q.4. By using Newton-Raphson's method, find the real root of the equation $x^4 - x - 13 = 0$.

Q.5. Using Newton-Raphson's method. Find the square root of 12 correct to three places of decimal.

Answer

3. 0.39

4. 1.967

5. 3.4641.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.
2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.
3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.
4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.
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UNIT-9: Numerical Method for solving Algebraic and Transcendental Equations-II

Structure

9.1 Introduction

Numerical methods are instrumental in addressing the challenges posed by both algebraic and transcendental equations when analytical solutions become difficult or impracticable. This collective set of methods constitutes a versatile toolkit, catering to a broad spectrum of algebraic and transcendental equations prevalent in scientific, engineering, and mathematical contexts. The selection of a specific method hinges on the distinctive features of the problem at hand, balancing the trade-off between precision and computational efficiency based on the desired outcome. Secant Method provides an effective approach for finding roots of real-valued functions, offering a good compromise between simplicity and convergence speed, especially in situations where derivatives are not readily available.

The Regula-Falsi method is a numerical technique that merges the simplicity of the Bisection method with faster convergence, leading to quicker results in many cases. This method proves advantageous in approximating roots of real-valued functions within a specified interval. However, it is essential to note that the Regula-Falsi method may face convergence challenges under certain circumstances. Convergence issues may arise if the function being analyzed possesses specific characteristics or if the initial interval chosen for the method is not well-suited to the nature of the function. Careful consideration of these factors is crucial for the successful application of the Regula-Falsi method in root-finding problems.

9.2 Objectives

After reading this unit the learner should be able to understand about:

- The Regula-Falsi Method with their solution procedure
- The Secant Method and their problem

9.3 Regula Falsi Method

The Regula-Falsi method, also known as the False Position method, is an iterative numerical technique used for finding the root of a real-valued function within a given interval. The Regula-Falsi method combines the simplicity of the Bisection method with faster convergence, often providing quicker results. However, it may encounter convergence issues if the function has certain characteristics or if the initial interval is poorly chosen. This method is the oldest method for finding the real root of the equation $f(x) = 0$.

In this method we take two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs *i.e.*, $f(x_0)f(x_1) < 0.$ The root must lie in between x_0 and x_1 since the graph $y = f(x)$ crosses the x-axis between these two points.

Now the equation of the chord joining the two points $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \text{....(1)}$$

In this method the curve between the point $A[x_0, f(x_0)]$ and $B[x_1, f(x_1)]$ is replaced by the chord AB by joining the points A and B taking the point of intersection of the chord with the x-axis as an approximation to the root which is given by putting $y = 0$ in the equation (1). Thus, we have

$$x_2 = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. Then replace the part of curve between the points $A(x_0, f(x_0))$ and $C(x_2, f(x_2))$ by the chord joining these points and this chord intersect the x-axis then we get second approximation to the root which is given by

$$x_3 = x_0 - \frac{(x_2 - x_0)}{f(x_2) - f(x_0)} f(x_0)$$

The procedure is repeated till the root is found to desired accuracy. The Regula-Falsi method is a valuable tool for approximating roots of real-valued functions, striking a balance between computational efficiency and simplicity.

Examples

Example.1. Use the method of false position, find the real root of the equation $x^3 - 2x - 5 = 0$**.**

Solution: The given equation is $x^3 - 2x - 5 = 0$.

Consider $\quad f(x) = x^3 - 2x - 5 = 0$

Now we have

$$f(2) = 2^3 - 2\times 2 - 5 = -1$$

and $\quad f(3) = 3^3 - 2\times 3 - 5 = 16$

Therefore, one real root of the given equation is lies between 2 and 3.

Taking $x_0 = 2$ and $x_1 = 3$

$\Rightarrow \quad f(x_0) = f(2) = -1$

and $\quad f(x_1) = f(3) = 16$

Using Regula-Falsi method, we have

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3-2}{16-(-1)}(-1)$$

$$= 2 + \frac{1}{17}$$

$$= \frac{35}{17}$$

$$= 2.0588$$

Now we have

$$f(x_2) = f(2.0588)$$

$$= (2.0588)^3 - 2 \times (2.0588) - 5$$

$$= -0.3910$$

Therefore, one real root of the given equation is lies between 2.0588 and 3.

Now we take $x_0 = 2.0588$, $x_1 = 3$

$\Rightarrow$ $f(x_0) = f(2.0588) = -0.3910$

and $f(x_1) = f(3) = 16$

Using Regula-Falsi method, we have

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0588 - \frac{3 - 2.0588}{16 - (-0.3910)} \times (-0.3910)$$

$$= 2.0588 + \frac{0.9412}{16.391} \times (0.3910)$$

$$= 2.0588 + 0.02245$$

$$= 2.0812$$

Now, we have

$$f(x_3) = f(2.0812)$$

$$= (2.0812)^3 - 2 \times (2.0812) - 5$$

$$= -0.1479$$

Therefore, one real root of the given equation is lies between 2.0812 and 3.

Now we take $x_0 = 2.0812$, $x_1 = 3$

$\Rightarrow$ $$f(x_0) = f(2.0812) = -0.1479$$

and $$f(x_1) = f(3) = 16$$

Using Regula-Falsi method, we have

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0812 - \frac{3 - 2.0812}{16 - (-0.1479)} \times (-0.1479)$$

$$= 2.0812 + \frac{0.9188}{16.1479} \times (0.1479)$$

$$= 2.0812 + 0.0084$$

$$= 2.0896$$

Now, we have

$$f(x_4) = f(2.0896)$$

$$= (2.0896)^3 2 \times (2.0896) - 5$$

$$= -0.0551$$

Therefore, one real root of the given equation is lies between 2.0896 and 3.

Now we take $x_0 = 2.0896,\ x_1 = 3$

$$f(x_0) = f(2.0896) = -0.0551$$

and $$f(x_1) = f(3) = 16$$

Using Regula-Falsi method, we have

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0896 - \frac{3 - 2.0896}{16 - (-0.0551)}(-0.0551)$$

$$= 2.0896 + \frac{0.9404}{16.0551} \times 0.0551$$

$$= 2.0896 + 0.0031$$

$$= 2.0927$$

Now we have

$$f(x_5) = f(2.0927)$$

$$= (2.0927)^3 - 2 \times (2.0927) - 5$$

$$= -0.0206$$

Therefore, one real root of the given equation is lies between 2.0927 and 3.

Now we take $x_0 = 2.0927, x_1 = 3$

$$f(x_0) = f(2.0927) = -0.0206$$

and $$f(x_1) = f(3) = 16$$

Using Regula-Falsi method, we have

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0927 - \frac{3 - 2.0927}{16 - (-0.0206)} \times (-0.0206)$$

$$= 2.0927 + \frac{0.9073}{16.02206} \times 0.0206$$

$$= 2.0927 + 0.0011$$

$$= 2.0938$$

Now, we have

$$f(x_6) = f(2.0938)$$

$$= (2.0938)^3 - 2 \times (2.0938) - 5$$

$$= -0.0083$$

Therefore, one real root of the given equation is lies between 2.0938 and 3.

Now we take $x_0 = 2.0938$, $x_1 = 3$

$$f(x_0) = f(2.0938) = -0.0083$$

and $f(x_1) = f(3) = 16$

Using Regula-Falsi method, we have

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$$

$$= 2.0938 - \frac{3 - 2.0938}{16 - (-0.0083)} \times (-0.0083)$$

$$= 2.0938 + \frac{0.9062}{16.0083} \times 0.0083$$

$$= 2.0938 + 0.00046$$

$$= 2.0942$$

Now, we have

$$f(x_7) = f(2.0942)$$

$$= (2.0942)^3 - 2 \times (2.0942) - 5$$

$$= -0.0030$$

Therefore, one real root of the given equation is lies between 2.0942 and 3.

Now we take $x_0 = 2.0942$, $x_1 = 3$

$\Rightarrow$ $f(x_0) = f(2.0942) = -0.0030$

and $f(x_1) = f(3) = 16$

Using Regula-Falsi method, we have

$$x_8 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0942 - \frac{3 - 2.0942}{16 - (-0.0030)} \times (-0.0030)$$

$$= 2.0942 + \frac{0.9052}{16.0030} \times 0.0030$$

$$= 2.0942 + 0.00016$$

$$= 2.0943$$

Here $x_7 = x_8$.

Hence the root of the given equation is 2.094 which is correct to three decimal places.

Example.2. Find the real root of the equation $xe^x = 2$ correct upto three decimal places using Regula-Falsi method.

Solution. The given equation is $xe^x = 2$.

Consider $f(x) = xe^x - 2 = 0$

Now we have

$$f(0) = 0 - 2 = -2$$

$$f(0.5) = 0.5e^{0.5} - 2 = -1.1756$$

$$f(0.7) = -0.5903$$

$$f(0.9) = 0.2136$$

Therefore, one real root of the given equation is lies between 0.7 and 0.9.

Taking $x_0 = 0.7, x_1 = 0.9$

$\Rightarrow \quad f(x_0) = f(0.7) = -0.5903$

and $\quad f(x_1) = f(0.9) = 0.2136$

Using Regula-Falsi method, we have

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$$

$$= 0.7 - \frac{0.9 - 0.7}{0.2136 - (-0.5903)} \times (-0.5903)$$

$$= 0.7 + \frac{0.2}{0.8039} \times (0.5903)$$

$$= 0.8469$$

Now, we have

$$f(x_2) = f(0.8469)$$

$$= (0.8469)\,e^{0.8469} - 2$$

$$= -0.0247$$

Therefore, one real root of the given equation is lies between 0.8469 and 0.9.

Now we take x_0 = 0.8469, x_1 = 0.9

$\Rightarrow \quad f(x_0) = f(0.8469) = -0.0247$

and $\quad f(x_1) = f(0.9) = 0.2136$

Using Regula-Falsi method, we have

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$$

$$= 0.07 - \frac{0.9 - 0.8469}{0.2136 - (-0.0247)} \times (0.0247)$$

$$= 0.7 + \frac{0.0531}{0.2383} \times 0.0247$$

$$= 0.8524$$

Now, we have

$$f(x_3) = f(0.8524)$$

$$= (0.8524)e^{0.8524} - 2$$

$$= -0.00089$$

Therefore, one real root of the given equation is lies between 0.8524 and 0.9.

Now we take $x_0 = 0.8524$, $x_1 = 0.9$

$\Rightarrow$ $$f(x_0) = f(0.8524) = -0.00089$$

and $$f(x_1) = f(0.9) = 0.2136$$

Using Regula-Falsi method, we have

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)}$$

$$= 0.8524 - \frac{0.9 - 0.8524}{0.2136 - (-0.00089)} \times (-0.00089)$$

$$= 0.8524 + \frac{0.0476}{0.21449} \times (0.00089)$$

$$= 0.8526.$$

9.4 Secant Method

The Secant Method is an iterative numerical technique utilized for approximating the root of a real-valued function. It is particularly useful for finding solutions to equations when the derivative of the function is either unknown or difficult to compute. The Secant Method shares similarities with the Newton-Raphson Method but does not require the computation of the derivative. While it may not converge as rapidly as Newton's method, it is versatile and applicable in cases where derivatives are challenging to obtain.

Secant method is an improvement/extension of the Regula-Falsi method and according to this method no requirement of the condition $f(x_0).\ f(x_1)<0$; *i.e.,* no condition for the interval (x_0, x_1) must contains the root.

In this method the function $y= f(x)$ graph is near to secant line in each iteration of the method. If any stage of iteration the value of $f(x_n)= f(x_{n-1})$, then the secant method is fail.

The secant formula for *n*th approximation is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n); \qquad n \geq 1$$

Note: Secant method is not necessarily for $f(x_0)$ and $f(x_1)$ are of opposite sign. Also it is not necessary for the interval (x_0, x_1) must contain the root.

Check your Progress

1. What is the difference between regula falsi method and secant method..
2. Which one is better in between regula falsi method and secant method

Examples

Example.3. Find the real root of the equation x-e^{-x}=0 by Secant Method, up-to three places of decimal.

Sol. The given equation is $x - e^{-x} = 0$.

Consider $f(x) = x - e^{-x} = 0$

Now we have

$f(0) = -1 < 0$

and $f(1) = 1 - e^{-1} = 0.6321 > 0$

Therefore, atleast one root of the given equation is lies between 0 and 1.

Taking $x_0 = 0$, $x_1 = 1$, $f(x_0) = -1$ and $f(x_1) = 0.6321$.

Using Secant method, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{0.6321 - (-1)}(0.6321)$$

$$= 1 - \frac{0.6321}{1.6321}$$

$$= 0.6127.$$

Now, we have

$f(x_2) = f(0.6127)$

$= (0.6127) - e^{-0.6127}$

$= 0.6127 - 0.5419$

$= 0.0708.$

Taking $x_1 = 1$, $x_2 = 0.6127$, $f(x_1) = 0.6321$ and $f(x_2) = 0.0708$.

Using Secant method, we have

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.6127 - \frac{0.6127 - 1}{0.0708 - 0.6321}(0.0708)$$

$$= 0.6127 - 0.0488$$

$$= 0.5639.$$

Now we have

$f(x_3) = f(0.5639)$

$= (0.5639)\text{-}e^{-0.5639}$

$= 0.5639\text{-}0.5690$

$= -0.0051.$

Now taking x_2 =0.6127, x_3 =0.5639, $f(x_2)$ =0.0708 and $f(x_3)$ = –0.0051.

Using Secant method, we have

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.5639 - \frac{0.5639 - 0.6127}{-0.0051 - 0.0708}(-0.0051)$$

$$= 0.5639 + 0.0033$$

$$= 0.5672.$$

Now we have

$f(x_4) = f(0.5672)$

$= (0.5672)\text{-}e^{-0.5672}$

$= 0.5639\text{-}0.5671$

$= 0.0001.$

Now taking x_3 =0.5639, x_4 =0.5672, $f(x_3)$ =-0.0051 and $f(x_4)$ = 0.0001.

Using Secant method, we have

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5672 - \frac{0.5672 - 0.5639}{0.0001 - (-0.0051)}(0.0001)$$

$$= 0.5672 - 0.00063$$

$$= 0.5670.$$

Hence the root of the given equation $f(x) = x - e^{-x} = 0$ up-to three decimal places is 0.567, which is of desired accuracy.

Example.4. Find the real root of the equation $\cos x - xe^x = 0$ by Secant Method.

Sol. The given equation is $\cos x - xe^x = 0$.

Consider $f(x) = \cos x - xe^x = 0$

Now we have

$$f(x_0) = f(0) = \cos(0) - 0.e^0 = 1 > 0$$

and $f(x_1) = f(1) = \cos(1) - 1.e^1 = -2.1780 < 0$

Therefore, atleast one root of the given equation is lies between 0 and 1.

Taking $x_0 = 0$, $x_1 =1$, $f(x_0) = 1$ and $f(x_1) = -2.1780$.

Using Secant method, we have

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$= 1 - \frac{1-0}{-2.1780-1}(-2.1780)$$

$$= 1 - \frac{2.1780}{3.1780}$$

$$= 0.3147.$$

Now, we have

$$f(x_2) = f(0.3147)$$

$$= \cos(0.3147) - (0.3147)\, e^{0.3147}$$

$$= 0.9509 - 0.4311$$

$$= 0.5198\,.$$

Taking $x_1 = 1$, $x_2 = 0.3147$, $f(x_1) = -2.1780$ and $f(x_2) = 0.5198$.

Using Secant method, we have

$$x_3 = x_2 - \frac{(x_2 - x_1)}{f(x_2) - f(x_1)} f(x_2)$$

$$= 0.3147 - \frac{0.3147 - 1}{0.5198 - (-2.1780)}(0.5198)$$

$$= 0.3147 + \frac{0.3562}{2.6978}$$

$$= 0.4467.$$

Now we have

$f(x_3) = f(0.4467)$

$$= \cos(0.4467) - (0.4467)\, e^{0.4467}$$

$$= 0.9019 - 0.6983$$

$$= 0.2036.$$

Now taking x_2 =0.3147, x_3 =0.4467, $f(x_2)$ =0.5198 and $f(x_3)$ = 0.2036.

Using Secant method, we have

$$x_4 = x_3 - \frac{(x_3 - x_2)}{f(x_3) - f(x_2)} f(x_3)$$

$$= 0.4467 - \frac{0.4467 - 0.3147}{-0.2036 - 0.5198}(0.2036)$$

$$= 0.4467 + \frac{0.0269}{0.3162}$$

$$= 0.5318.$$

Now we have

$f(x_4) = f(0.5318)$

$$= \cos(0.5318) - (0.5318)\, e^{0.5318}$$

$$= 0.8619 - 0.9051$$

$$= -0.0432.$$

Now taking x_3 =0.4467, x_4 =0.5318, $f(x_3)$ =0.2036 and $f(x_4)$ = –0.0432.

Using Secant method, we have

$$x_5 = x_4 - \frac{(x_4 - x_3)}{f(x_4) - f(x_3)} f(x_4)$$

$$= 0.5318 - \frac{0.5318 - 0.4467}{-0.0432 - 0.2036}(-0.0432)$$

$$= 0.5318 - \frac{0.0037}{0.2468}$$

$$= 0.5168.$$

Now we have

$f(x_5) = f(0.5168)$

$$= \cos(0.5168) - (0.5168)\, e^{0.5168}$$

$$= 0.8694 - 0.8665$$

$$= 0.0029.$$

Now taking $x_4 = 0.5318$, $x_5 = 0.5168$, $f(x_4) = -0.0432$ and $f(x_5) = 0.0029$.

Using Secant method, we have

$$x_6 = x_5 - \frac{(x_5 - x_4)}{f(x_5) - f(x_4)} f(x_5)$$

$$= 0.5168 - \frac{0.5168 - 0.5318}{0.0029 - (-0.0432)}(0.0029)$$

$$= 0.5168 + \frac{0.0000435}{0.0461}$$

$$= 0.5177.$$

Now we have

$f(x_6) = f(0.5177)$

$$= \cos(0.5177) - (0.5177)\, e^{0.5177}$$

$$= 0.8690 - 0.8688$$

$$= 0.0002.$$

Now taking x_5 =0.5168, x_6 =0.5177, $f(x_5)$ =0.0029 and $f(x_6)$ = 0.0002.

Using Secant method, we have

$$x_7 = x_6 - \frac{(x_6 - x_5)}{f(x_6) - f(x_5)} f(x_6)$$

$$= 0.5177 - \frac{0.5177 - 0.5168}{0.0002 - 0.0029}(0.0002)$$

$$= 0.5177 + 0.00006$$

$$= 0.5178.$$

Hence the root of the given equation $f(x) = \cos x - x e^x = 0$ up-to three decimal places is 0.517, which is of desired accuracy.

9.5 Summary

The Regula-Falsi method, also recognized as the False Position method, is an iterative numerical approach utilized to determine the root of a real-valued function within a specified interval. Combining the simplicity of the Bisection method with accelerated convergence, the Regula-Falsi method typically yields faster results. The Secant method represents an enhancement or extension of the Regula-Falsi method.

Unlike the Regula-Falsi method, the Secant method eliminates the need for the condition $f(x_0)$ $f(x_1) < 0$, meaning there is no requirement for the interval (x_0, x_1) to necessarily contain the root. It's worth noting that the Regula-Falsi method, being the oldest technique in the pursuit of real roots for the equation $f(x) = 0$, serves as the foundation for the Secant method.

Secant method is not necessarily for $f(x_0)$ and $f(x_1)$ are of opposite sign. Also it is not necessary for the interval (x_0, x_1) must contain the root. The secant formula for nth approximation is given by

$$x_{n+1} = x_n - \frac{(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} f(x_n); \qquad n \geq 1$$

9.6 Terminal Questions

Q.1. Explain the Regula-Falsi Method.

Q.2. Write the formula of Secant Method.

Q.3. Find a real root of the equation $x^2 - \log_e x - 12 = 0,$ using Regula-Falsi method correct to three decimal place.

Q.4. Find real root of the following equation by using Regula-Falsi method: $xe^x - 3 = 0$.

Q.5. Using Secant method to find the real root of the equation $f(x) = \cos x - xe^x = 0$ up-to four decimal places.

Answer

3. 3.6461 4. 1.046 5. 0.5177

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.
2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.
3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.
4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.
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Master of Science
PGMM -104N
Numerical Analysis

Block

5 Numerical Differentiation and Integration

Block-5

Numerical Differentiation And Integration

Numerical differentiation is a computational approach used to estimate the derivative of a function at a given point, especially when an analytical expression for the derivative is not readily available or when dealing with discrete data points. There are several numerical methods for approximating derivatives, each with its own set of advantages and limitations. For higher-order derivatives or more complex cases, numerical differentiation can be extended using methods such as Richardson extrapolation or finite difference formulas. Numerical differentiation is particularly valuable in situations where obtaining the analytical derivative is challenging, and it finds applications in various fields such as physics, engineering, and data analysis.

Numerical differentiation is a method used to estimate the derivative of a function at a specific point or over a range of values when an analytical expression for the derivative is not readily available. This approach is particularly useful in scenarios involving discrete data points or functions defined by complex algorithms. The choice between these methods often depends on the specific requirements of the problem and the trade-off between accuracy and computational efficiency. Derivatives using the forward and backward difference formulas are numerical methods to estimate the first derivative of a function at a specific point. These methods are particularly useful when an analytical expression for the derivative is not available or when dealing with discrete data points.

In the 10^{th} unit, we shall discuss about derivatives using forward difference formula, derivatives using backward difference formula. In unit eleventh we deals with derivatives using Stirling difference formula, derivatives using Newton's divided difference formula. In unit twelveth we shall discuss about general quadrature formula for equally spaced arguments, Trapezoidal rule, Simpson's 1/3 rule, Simpson's 3/8 rule. Numerical solution of ordinary differential equation with Euler's method, Euler's modified method, Taylor Series method discussed in unit thirteen. In unit fourtheen we shall discussed the Picard's method. Runge-Kutta method for fourth order, Milne's predictor-corrector method.

UNIT-10: Numerical Differentiation-I

Structure

10.1 Introduction

Numerical differentiation is the process of obtaining the values of the derivative of a function from a set of numerical values of that function. Two common numerical differentiation methods are the Forward Difference Method and the Central Difference Method. The forward difference formula approximates the derivative by considering the difference between function values at a given point and a slightly displaced point in the positive x-direction. Similar to the forward difference formula, the backward difference formula estimates the derivative by computing the difference between function values at a given point and a slightly displaced point in the negative x -direction.

If the arguments are uniformly spaced, the preferred choice is the Newton-Gregory forward formula when aiming to find the derivative of a function near the beginning. Conversely, the Newton-Gregory backward formula is employed when seeking the derivative at a point near the end. When the derivative at a point is situated close to the middle of the table, the Stirling difference formula is applied. For unevenly spaced data, go for Newton's divided difference formula.In the tenth unit, we will explore the topic of derivatives, specifically delving into the application of the forward difference formula and the backward difference formula.

10.2 Objectives

After reading this unit the learner should be able to understand about:

- the derivatives using forward difference formula
- the derivatives using backward difference formula

10.3 Derivatives Using Forward Difference Formula

The Newton-Gregory formula for numerical differentiation is employed to estimate the derivative of a function at a given point using equally spaced data points. It relies on interpolating a polynomial through these data points and then differentiating the polynomial to approximate the derivative. First differentiate the interpolating polynomial with respect to *x* to find the derivative approximation at the desired point. This will yield an expression for the derivative in terms of the function values and their respective forward differences.

The Newton-Gregory formula for forward interpolation is

$$f(a+hu) = f(a) + u\Delta f(a) + \frac{u(u-1)}{2!}\Delta^2 f(a) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(a)$$

$$+ \ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots.(u-(n-1))}{n!}\Delta^n f(a)$$

or

$$y = u_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots\ldots$$

$$+ \ldots\ldots\ldots\ldots + \frac{u(u-1)(u-2)\ldots.(u-(n-1))}{n!}\Delta^n y_0 \qquad \ldots\ldots (1)$$

where $$u = \frac{x-a}{h} \qquad \ldots\ldots (2)$$

Differentiating both sides of equation (1) with respect to *x*, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\begin{array}{l} y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \\ + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots\ldots\ldots \end{array}\right]$$

$$= \left[\begin{array}{l} \Delta y_0 + \frac{(2u-1)}{2!} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!} \Delta^3 y_0 \\ + \frac{4u^3 - 18u^2 + 22u - 6}{4!} \Delta^4 y_0 \ldots\ldots\ldots\ldots \end{array} \right] \frac{du}{dx} \qquad \ldots\ldots (3)$$

From equation (2), we get

$$\frac{du}{dx} = \frac{1}{h} \qquad \ldots\ldots (4)$$

Using equations (3) and (4), we get

$$\frac{dy}{dx} = \frac{1}{h} \left[\begin{array}{l} \Delta y_0 + \frac{(2u-1)}{2!} \Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!} \Delta^3 y_0 \\ + \frac{4u^3 - 18u^2 + 22u - 6}{4!} \Delta^4 y_0 \ldots\ldots\ldots\ldots \end{array} \right] \qquad \ldots\ldots (5)$$

Put $x = a$ in equation (2), we get $u = 0$.

At $x = a$ in the equation (5), we get

$$\left(\frac{dy}{dx} \right)_{x=a} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 + \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \ldots\ldots\ldots \right]$$

Now differentiating both sides of the equation (5) with respect to x, we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h} \left[\Delta^2 y_0 + \frac{6u-6}{3!} \Delta^3 y_0 + \frac{12u^2 - 36u + 22}{4!} \Delta^4 y_0 + \ldots\ldots\ldots \right] \frac{du}{dx}$$

$$= \frac{1}{h^2} \left[\Delta^2 y_0 + \frac{6u-6}{3!} \Delta^3 y_0 + \frac{12u^2 - 36u + 22}{4!} \Delta^4 y_0 + \ldots\ldots\ldots \right] \qquad \ldots\ldots (6)$$

At $x = a$, in the equation (6), we get

$$\left(\frac{d^2 y}{dx^2} \right)_{x=a} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 \ldots\ldots\ldots\ldots \right]$$

Proceeding in the same way, we get the third differentiations at the required points as

$$\left(\frac{d^2y}{dx^3}\right)_{x=a} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\ldots\ldots\ldots\ldots\ldots\right]$$

10.4 Derivatives Using Backward Difference Formula

The Newton-Gregory formula for backward interpolation is

$$f(a+nh+hu) = f(a+nh) + u\nabla f(a+nh) + \frac{u(u+1)}{2!}\nabla^2 f(a+nh)$$

$$+\ldots\ldots\ldots\ldots + \frac{u(u+1)(u+2)\ldots\ldots.(u+n-1)}{n!}\nabla^n f(a+nh)$$

or

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+\frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \ldots\ldots\ldots. \qquad \ldots\ldots (1)$$

Where $u = \frac{x - x_n}{h}$ …… (2)

Differentiating both sides of equation (1) with respect to *x*, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\begin{array}{l} y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n \\ + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right]$$

$$= \left[\begin{array}{l} \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \\ + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right] \frac{du}{dx} \qquad \ldots\ldots (3)$$

From equation (2), we get

$$\frac{du}{dx} = \frac{1}{h} \qquad \ldots\ldots (4)$$

Using equations (3) and (4), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\begin{array}{l} \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \\ + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right] \qquad \ldots\ldots (5)$$

Put $x = x_n$ in equation (2), we get $u = 0$.

At $x = x_n$ in the equation (5), we get

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\ldots\ldots\ldots \right]$$

Now differentiating both sides of the equation (5) with respect to *x*, we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h}\left[\nabla^2 y_n + \frac{6u+6}{3!}\nabla^3 y_n + \frac{12u^2+36u+22}{4!}\nabla^4 y_n + \ldots\ldots\ldots\ldots \right]\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{3!}\nabla^3 y_n + \frac{12u^2+36u+22}{4!}\nabla^4 y_n + \ldots\ldots\ldots \right] \qquad \ldots\ldots (6)$$

At $x = x_n$ in the equation (6), we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\ldots\ldots \right]$$

Proceeding in the same way, we get the third differentiation at the required point.

$$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right].$$

Check your Progress

1. What do you mean by derivatives using forward difference method?

2. Write the derivatives using backward difference formula.

Examples

Example.1. Determine the $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y at $x = 50$ **from the following table:**

x	50	51	52	53	54	55	56
$y = f(x)$	**3.6840**	**3.7084**	**3.7325**	**3.7563**	**3.7798**	**3.8030**	**3.8259**

Solution: In this problem we find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y at $x = 50$. Here we see that x = 50 lies near the starting of the table therefore in this case we shall use Newton's forward interpolation formula for derivtives. The Newton-Gregory forward interpolation formula for derivatives, we have

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + ...\right] \quad ....(1)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \ldots\ldots\ldots\ldots\ldots\ldots\right] \qquad \ldots.(2)$$

The difference table is as below:

x	$y = x^{1/3}$	Δy	$\Delta^2 y$	$\Delta^3 y$
50	3.6840			
		0.0244		
51	3.7084		-0.0003	
		0.0241		0
52	3.7325		-0.0003	
		0.0238		0
53	3.7563		-0.0003	
		0.0235		0
54	3.7798		-0.0003	
		0.0232		0
55	3.8030		-0.0003	
		0.0229		
56	3.8259			

Here a = 50, h =1 then from equation (1), we get

$$\left(\frac{dy}{dx}\right)_{x=50} = \frac{1}{1}\left[0.0244 - \frac{1}{2}(-0.0003) + \frac{1}{3}(0)\right]$$

$$= 0.0244 + 0.00015$$

$$= 0.02455.$$

and also put a = 50, h =1 in the equation (2), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \ldots\ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = \frac{1}{1^2}(-0.0003)$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=50} = -0.0003.$$

Hence the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of y at $x = 50$ are 0.02455 and -0.0003 respectivey.

Example.2. Determine the first and second derivatives of the functions *f*(*x*) at *x* = 1.1 from the following data:

x	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**
***f*(*x*)**	**0**	**0.1280**	**0.5440**	**1.2960**	**2.4320**	**4.0000**

Solution: In this problem we find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $f(x)$ at $x = 1.1.$ Here we see that x = 1.1 lies near the starting of the table therefore in this case we shall use Newton's forward interpolation formula for derivtives. The Newton-Gregory forward interpolation formula is

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0) + \ldots\ldots \quad \ldots (1)$$

where $u = \frac{x - x_0}{h} = \frac{x-1}{0.2} = 5(x-1)$ … (2)

The difference table is as below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0				
		0.1280			
1.2	0.1280		0.2880		
		0.4160		0.0480	
1.4	0.5440		0.3360		0
		0.7520		0.480	
1.6	1.2960		0.3840		0
		1.1360		0.480	
1.8	2.4320		0.4320		
		1.5680			
2.0	4.0000				

Differentiating both sides of equation (1) with respect to *x*, we get

$$f'(x) = \left[\Delta f(x_0) + \frac{2u-1}{2!}\Delta^2 f(x_0) + \frac{3u^2-6u+2}{3!}\Delta^3 f(x_0) + ... \right] \frac{du}{dx} \quad \text{… (3)}$$

From equation (2), we get

$$\frac{du}{dx} = 5 \quad \text{… (4)}$$

$$f'(x) = 5\left[\Delta f(x_0) + \frac{2u-1}{2!}\Delta^2 f(x_0) + \frac{3u^2-6u+2}{3!}\Delta^3 f(x_0) + ... \right] \quad \text{… (5)}$$

Put $x = 1.1$ in the equation (2), we get $u = 5(1.1 - 1) = 0.5$.

Put u=0.5 then from equation (5), we get

$$f'(x) = 5\left[0.128 + \frac{2(0.5)-1}{2}(0.2880) + \frac{3(0.5)^2 - 6(0.5)+2}{6}(0.048)\right]$$

$$= 5[0.128 + 0 - 0.002]$$

$$= 0.63$$

$$f'(1.1) = 0.63$$

Now differentiating both sides of equation (3) with respect to x, we get

$$f''(x) = 5\left[\Delta^2 f(x_0) + \frac{6u-6}{3!}\Delta^3 f(x_0)......\right]\frac{du}{dx}$$

$$= 5\left(\Delta^2 f(x_0) + (u-1)\Delta^3 f(x_0) + ...\right)(5) \qquad \left(\because \frac{du}{dx} = 5\right)$$

$$= 25\left(\Delta^2 f(x_0) + (u-1)\Delta^3 f(x_0) + ...\right) \qquad \ldots (6)$$

Put u =0.5 then from equation (6), we get

$$f''(1.1) = 25[0.2880 + (0.5-1)(0.0480)]$$

$$= 25[0.2880 - 0.024]$$

$$= 6.6$$

$$f''(1.1) = 6.6$$

Hence the value of the first and second derivatives of the functions $f(x)$ at x = 1.1 are 0.63 and 6.6 respectively.

Example.3. Determine the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 2.03$ with the help of the following data:

x	**1.96**	**1.98**	**2.00**	**2.02**	**2.04**
y	**0.7825**	**0.7739**	**0.7651**	**0.7563**	**0.7473**

Solution: In this problem we find the value of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 2.03$. Here we see that x = 2.03 lies near the last of the table therefore in this case we shall use Newton's backward interpolation formula for derivtives. The difference table is as below:

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.96	0.7825				
		- 0.0086			
			-0.0002		
1.98	0.7739				
				0.0002	
		- 0.0088			
					- 0.0004
			0		
2.00	0.7651				
				- 0.0002	
		- 0.0088			
			-0.0002		
2.02	0.7563				
		- 0.0090			
2.04	0.7473				

Here $x_n = 2.04$, $h = 0.02$ and $x = 2.03$ then we have

$$u = \frac{x - x_n}{h}$$

$$= \frac{2.03 - 2.04}{0.02}$$

$$= -\frac{1}{2} = \text{-0.5} \qquad \text{….(1)}$$

Now by the Newton's backward interpolation formula for derivative, we have

$$\frac{dy}{dx} = \frac{1}{h}\begin{bmatrix} \nabla y_n + \frac{2u+1}{2!}\nabla^2 y_n + \frac{3u^2+6u+2}{3!}\nabla^3 y_n \\ + \frac{4u^3+18u^2+22u+6}{4!}\nabla^4 y_n + \text{..............................} \end{bmatrix} \qquad \text{….(2)}$$

Put u = -0.5 and h = 0.02 in the equation (2), we get

$$\left(\frac{dy}{dx}\right)_{x=2.03} = \frac{1}{0.02}\left[-0.0090 + \frac{2\left(-\frac{1}{2}\right)+1}{2!}(-0.0002) + \frac{3\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right)+2}{4!}(-0.0002) \right.$$

$$\left. + \frac{4\left(-\frac{1}{2}\right)^3 + 18\left(-\frac{1}{2}\right)^2 + 22\left(-\frac{1}{2}\right)+6}{4!}(-0.0004) \right]$$

$$= \frac{1}{0.02}\left[-0.0009 + 0 + 0.000008 + 0.000017\right]$$

$$= -0.44875$$

Again differentiating both sides of equation (2) with respect to x, we get

$$\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{3!}\nabla^3 y_n + \frac{12u^2+36u+22}{4!}\nabla^4 y_n \text{...........}\right] \qquad \text{….(3)}$$

Put u = -0.5 and h = 0.02 in the equation (3), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=2.03} = \frac{1}{(0.02)^2}\left[-0.0002 + \frac{6\left(-\frac{1}{2}\right)+1}{6!}(-0.0002)\right.$$

$$\left. + \frac{12\left(-\frac{1}{2}\right)^2 + 36\left(-\frac{1}{2}\right) + 22}{24} \times (-0.0004)\right]$$

$$= \frac{1}{0.0004}[-0.0002 - 0.0001 - 0.00012]$$

= -1.05

$$\left(\frac{d^2y}{dx^2}\right)_{x=2.03} = -1.05$$

Hence the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of $f(x)$ at $x = 2.03$ are -0.44875 and -1.05 respectivey.

Example.4. Find the value of $f'(1.5)$ and $f''(1.5)$ from the following table:

x	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
***f*(*x*)**	**3.375**	**7.000**	**13.625**	**24.000**	**38.875**	**59.000**

Solution: : In this problem we find the value of $f'(1.5)$ and $f''(1.5)$. Here we see that x = 1.5 lies near the starting of the table therefore in this case we shall use Newton's forward interpolation formula for derivtives. The difference table is as below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3.0	24.000		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4.0	59.000				

The Newton-Gregory forward interpolation formula for derivatives, we have

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\ldots\ldots.\right] \qquad \ldots.(1)$$

$$\left(\frac{d^2 y}{dx^2}\right)_{x=a} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 \ldots\ldots\ldots\ldots\ldots\ldots.\right] \qquad \ldots.(2)$$

Here a = 1.5 and h =0.5 then from equation (1), we get

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}\times 3.000 + \frac{1}{3}\times(0.75) - 0\ldots\ldots\right]$$

or $f'(1.5) = \dfrac{1}{0.5}[3.625 - 1.5 + 0.25]$

$$= \frac{1}{0.5} \times 2.375$$

$$= 4.75$$

Put a = 1.5 and h = 0.5 in the equation (2), we get

$$\left(\frac{d^2 y}{dx^2} \right)_{x=a} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 .... \right]$$

$$\left(\frac{d^2 y}{dx^2} \right)_{x=1.5} = \frac{1}{(0.5)^2} \left[3.000 - 0.75 + \frac{11}{12} .0 .... \right]$$

or $f''(1.5) = \frac{1}{(0.5)^2} [3.000 - 0.75 + \frac{11}{25} .0]$

$$f''(1.5) = \frac{2.25}{0.25}$$

$$= 9$$

Hence the values of $f'(1.5)$ and $f''(1.5)$ are 4.75 and 9 respectivey.

Example.5. Given the following data:

θ°	**0**	**10**	**20**	**30**	**40**
***sin* θ°**	**0.000**	**0.1736**	**0.3420**	**0.5000**	**0.6428**

Find the value of cos θ° by numerical method when θ = 10^0.

Solution. In this problem we find the value of cos θ° when θ = 10^0. Here we see that θ = 10^0 lies near the starting of the table therefore in this case we shall use Newton's forward interpolation formula for derivtives. The difference table is as below:

θ°	sin θ°	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	0.000				
		0.1736			
10	0.1736		- 0.0052		
		0.1684		- 0.0052	
20	0.3420		- 0.0104		0.0004
		0.1580		- 0.0048	
30	0.5000		- 0.0152		
		0.1428			
40	0.6428				

The Newton-Gregory forward interpolation formula for derivatives, we have

$$\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots\dots\dots\right] \qquad \dots.(1)$$

Here $f(x) = \sin\theta,\ a = 10,\ h = 10^\circ = 0.1745$ radian then from equation (1), we get

$$\left(\frac{dy}{d\theta}\right)_{\theta=10^\circ} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 \dots\dots\dots\right]$$

$$(\cos\theta)_{\theta=10^\circ} = \frac{1}{10}\left[0.1684 + \frac{1}{2}(0.0104) + \frac{1}{3}(-0.0048)\right]$$

$$= \frac{1}{0.1745} \times 0.1720$$

$$= 0.9856.$$

Hence the values of cos θ° at $\theta = 10^{0}$ is 0.9856.

10.5 Summary

Numerical differentiation methods play a crucial role in situations where analytical derivatives are challenging to obtain, making them valuable tools in fields such as physics, engineering, and data analysis. The choice between forward and backward difference often depends on the direction of the data or the nature of the problem. While these methods are straightforward and easy to implement, it's essential to choose an appropriate step size h to balance accuracy and numerical stability.

In practice, the central difference method (using both forward and backward differences) is also commonly employed to improve accuracy by considering function values on both sides of the point of interest.

10.6 Terminal Questions

Q.1. Explain the numerical differentiation method for equal intervals.

Q.2. Write the procedure for determine the Derivatives Using the Newton's Backward Difference Formula.

Q.3. Find the first, second and third derivatives of the function $f(x)$ tabulated below, at the point $x = 1.5$.

x	1.5	2.0	2.5	3.0	3.5	4.0
$f(x)$	3.375	7.000	13.625	24.000	38.875	59.000

Q.4. Find $y'(0)$ and $y''(0)$ from the following table:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Q.5. Find the derivative of $f(x)$ at $x = 0.4$ from the following table:

x	0.1	0.2	0.3	0.4
$y=f(x)$	1.10517	1.22140	1.34986	1.49182

Q.6. Find $f'(1.1)$ and $f''(1.1)$ from the following table:

x	1	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

Answer

3. 4.75, 9, 6.

4. – 27.9 and 117.67.

5. 1.49133.

6. 3.9435 and – 3.545.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-11: Numerical Differentiation-II

Structure

11.1 Introduction

Numerical differentiation serves as a method to approximate the derivative of a function either at a specific point or across a range of values. This approach proves valuable when an analytical expression for the derivative is not readily accessible or when working with discrete data points. The main objective is to provide an estimation of how quickly a function changes concerning its independent variable. The selection of a suitable method is contingent on considerations such as the desired level of accuracy, computational efficiency, and the inherent characteristics of the problem at hand. This technique finds widespread application in diverse scientific and engineering contexts, particularly when addressing experimental or discrete data scenarios.

Newton's Divided Difference Formula is a mathematical method for constructing an interpolating polynomial for a set of given data points. It is particularly useful for approximating a function when only discrete data points are known. The formula is named after Sir Isaac Newton, who developed this method. Newton's Divided Difference Formula is a fundamental tool in numerical analysis, especially for constructing interpolating polynomials, and it is widely used in various applications, including numerical integration and differentiation. Moving on to the eleventh unit, our focus will shift towards derivatives, examining their computation through the Stirling difference formula and Newton's divided difference formula.

11.2 Objectives

After reading this unit the learner should be able to understand about:

- the derivatives using Stirling Difference formula
- the derivatives using Newton’s divided difference formula

11.3 Derivatives Using Stirling Difference Formula

The Stirling's finite difference formula is a method used for numerical differentiation, which approximates the derivative of a function using its function values at evenly spaced points. The formula can be derived from the finite difference approximation, and it's particularly useful when a high degree of accuracy is required. The Stirling's difference formula is

$$y_u = y_0 + u\,\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u^2}{2!}\Delta^2 y_{-1}$$
$$+\frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2}+\frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}+..........$$
$$+\frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3}+\Delta^5 y_{-2})}{2}+\frac{u^2(u^2-1)(u^2-2^2)}{6!}\Delta^6 y_{-3}+....... \quad ....\ (1)$$

Where $u=\dfrac{x-x_0}{h}$ or $\dfrac{x-a}{h}$ .... (2)

Differentiating both sides of equation (1) with respect to *x*, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left[\begin{array}{l} y_0+u\left(\dfrac{\Delta y_0+\Delta y_{-1}}{2}\right)+\dfrac{u^2}{2!}\Delta^2 y_{-1}+\dfrac{u\,(u^2-1)}{3!}\left(\dfrac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) \\ +\dfrac{u^2(u^2-1)}{4!}\Delta^4 y_{-2}+................ \end{array}\right]$$

$$=\left[\begin{array}{l} \left(\dfrac{\Delta y_0+\Delta y_{-1}}{2}\right)+u\Delta^2 y_{-1}+\dfrac{3u^2-1}{3!}\left(\dfrac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) \\ +\left(\dfrac{4u^3-2u}{4!}\right)\Delta^4 y_{-2}+........... \end{array}\right]\frac{du}{dx} \quad ....\ (3)$$

From equation (2), we get

$$\frac{du}{dx} = \frac{1}{h} \qquad \text{.... (4)}$$

Now from equation (2) and (4), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\begin{array}{l}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\ + \left(\frac{4u^3 - 2u}{4!}\right)\Delta^4 y_{-2} + ...\end{array}\right] \qquad \text{.... (5)}$$

At $x = x_0$, $u = 0$ then from equation (5), we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\left(\frac{\Delta^2 y_{-1} + \Delta^3 y_{-2}}{2}\right) + ...\right]$$

Again differentiating equation (5) with respect to *x*, we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h}\left[\Delta^2 y_{-1} + \frac{6u}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{12u^2 - 2}{4!}\Delta^4 y_{-2} + ...\right]\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\begin{array}{l}\Delta^2 y_{-1} + \frac{6u}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\ + \left(\frac{12u^2 - 2}{4!}\right)\Delta^4 y_{-2} + ...\end{array}\right] \qquad \left[\because \frac{du}{dx} = \frac{1}{h}\right]$$

$$= \frac{1}{h^2}\left[\begin{array}{l}\Delta^2 y_{-1} + \frac{6u}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\ + \left(\frac{12u^2 - 2}{4!}\right)\Delta^4 y_{-2} + ...\end{array}\right] \qquad \text{.....(6)}$$

At $x = x_0$, $u = 0$ then from equation (6), we get

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots\ldots\ldots\right]$$

Proceeding in the same way, we get the successive differentiation at the required point.

11.4 Derivatives Using Newton's Divided Difference Formula

Newton's divided difference formula is another method for numerical differentiation, which provides a polynomial approximation to the derivative of a function using its function values at distinct points. The formula is based on the concept of divided differences, which are a sequence of coefficients computed from the function values.

The Newton's divided difference formula is

$$y = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots..(1)$$

Differentiating equation (1) with respect to x , we get

$$\frac{dy}{dx} = \Delta f(x_0) + \left[(x-x_1) + (x-x_0)\right]\Delta^2 f(x_0) +$$

$$[(x-x_1)(x-x_2) + (x-x_0)(x-x_2) + (x-x_0)(x-x_1)]\Delta^3 f(x_0) + \ldots\ldots\ldots\ldots\ldots \qquad \ldots..(2)$$

Put $x = a$ in the equation (2) and we get the value of first derivative at $x = a$.

Again differentiating equation (1) with respect to x , we get

$$\frac{d^2y}{dx^2} = 2\Delta^2 f(x_0) + [2(x-x_0) + 2(x-x_1) + 2(x-x_2)]\Delta^3 f(x_0) + \ldots\ldots\ldots\ldots$$

Put $x = a$ in the equation (2) and we get the value of second derivative at $x = a$.

Note:

1. To determine the value of the derivatives of a function near the beginning of the arguments, Newton's forward formula is employed.

2. For derivatives required near the end of the arguments, then we use Newton's backward formula.

3. When the derivative is needed at the middle of the given arguments, the central difference formula is applied.

4. Newton's divided difference formula is employed when the arguments are not equally spaced.

Check your Progress

1. What do you mean by derivatives using stirling difference formula?

2. Write the derivatives using newton's divided difference formula.

Examples

Example.1. Determine the value of $f'(93)$ from the following table:

x	**60**	**75**	**90**	**105**	**120**
***f*(*x*)**	**28.2**	**38.2**	**43.2**	**40.9**	**37.2**

Solution: Since 93 lies near the central point of the table therefore in this case we shall use Stirling difference formula for derivatives.

The difference table is given by

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
- 2	60	28.2				
			10.0			
-1	75	38.2		-5		
			5		**-2.3**	
0	90	**43.2**		**-7.3**		**8.2**
			-2.3		**5.9**	
1	105	40.9		-1.4		
			-3.7			
2	120	37.2				

The Stirling's difference formula for derivatives is

$$\frac{dy}{dx} = \frac{1}{h}\left[\begin{aligned}&\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)\\ &+\left(\frac{4u^3 - 2u}{4!}\right)\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\end{aligned}\right] \quad \ldots. (1)$$

Here $x_0 = 90,\ \ x = 93,\ h = 15$.

Then we have

$$u = \frac{93-90}{15}$$

$$= \frac{3}{15}$$

$$= 0.2$$

Putting these values in Stirling formula for first derivative (1), we get.

$$\left(\frac{dy}{dx}\right)_{x=93} = \frac{1}{15}\left(\frac{5+(-2.3)}{2} + (0.2)\times(-7.3) + \frac{3(0.2)^2-1}{3!}\left(\frac{(-2.3)+(5.9)}{2}\right)\right.$$

$$\left. + \frac{4(0.2)^2 - 2(0.2)}{4!} 8.2\right)$$

$$f'(93) = \frac{1}{15}\left[\frac{2.7}{2} + (-1.46) - \frac{3.168}{3!\times 2} - \frac{3.0716}{4!}\right]$$

$$= \frac{1}{15}\left[\frac{2.7}{2} + (-1.46) - \frac{3.168}{12} - \frac{3.0716}{24}\right]$$

$$= \frac{1}{15}(1.35 - 1.46 - 0.26400 - 0.1257)$$

$$f'(93) = -0.3331.$$

Hence the value of $f'(93)$ is -0.3331.

Example.2. Determine the value of $f'(0.6)$ and $f''(0.6)$ from the following table:

x	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**
$f(x)$	**1.5836**	**1.7974**	**2.0442**	**2.3275**	**2.6510**

Solution: Since 0.6 lies near the central point of the table therefore in this case we shall use Stirling difference formula for derivatives. The difference table is given by

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
- 2	0.4	1.5836				
			0.2138			
-1	0.5	1.7974		0.0330		
			0.2468		**0.0035**	
0	0.6	2.0442		**0.0365**		**0.0002**
			0.2833		**0.0037**	
1	0.7	2.3275		0.0402		
			0.3235			
2	0.8	2.6510				

The Stirling's difference formula for derivatives is

$$\frac{dy}{dx} = \frac{1}{h}\left[\begin{array}{l}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \frac{3u^2 - 1}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\ + \left(\frac{4u^3 - 2u}{4!}\right)\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots \end{array}\right] \qquad \text{.... (1)}$$

Here $x_0 = 0.6,\ \ x = 0.6,\ \ h = 0.1.$

Then we have

$$u = \frac{0.6 - 0.6}{0.1}$$

$$= \frac{0}{0.1}.$$

$$= 0.$$

Putting these values in Stirling formula for first derivative (1), we get.

$$\left(\frac{dy}{dx}\right)_{x=0.6} = \frac{1}{(0.1)}\left[\frac{0.2468 + 0.2833}{2} + 0 + \frac{3(0)^2 - 1}{3!}\left(\frac{0.0035 + 0.0037}{2}\right) + 0\right]$$

$$f'(0.6) = \frac{1}{(0.1)}\left[\frac{0.2468 + 0.2833}{2} - \frac{1}{6}\left(\frac{0.0035 + 0.0037}{2}\right)\right]$$

$$= 10(0.26505 - 0.0006)$$

$$= 2.6445$$

$$f'(0.6) = 2.6445.$$

Again differentiating equation (1) with respect to *x*, we get

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\begin{array}{l}\Delta^2 y_{-1} + \dfrac{6u}{3!}\left(\dfrac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) \\ + \left(\dfrac{12u^2 - 2}{4!}\right)\Delta^4 y_{-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right] \quad \ldots..(2)$$

$$f''(0.6) = \frac{1}{(0.1)^2}\left[0.0365 + 0 - \frac{1}{12}(0.0002)\right]$$

$$= 100(0.0365 - 0.000016)$$

$$= 3.6484$$

$$f''(0.6) = 3.6484.$$

Hence the value of $f'(0.6)$ and $f''(0.6)$ are 2.6445 and 3.6484.

Example.3. Determine the value of $f'(6)$ from the following table:

x	**0**	**1**	**3**	**4**	**5**	**7**	**9**
***f*(*x*)**	**150**	**108**	**0**	**-54**	**-100**	**-144**	**-84**

Solution: In this case the values of the arguments are not equally spaced. So we will use the Newton's divided difference formula.

The Newton's divided difference formula is

$$y = f(x_0) + (x - x_0)\Delta f(x_0) + (x - x_0)(x - x_1)\Delta^2 f(x_0)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\Delta^3 f(x_0) + ... \qquad .....(1)$$

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = \Delta f(x_0) + \left[(x - x_1) + (x - x_0)\right]\Delta^2 f(x_0) +$$

$$[(x-x_1)(x-x_2)+(x-x_0)(x-x_2)+(x-x_0)(x-x_1)]\Delta^3 f(x_0)+.............\qquad .....(2)$$

Here $x=6, x_0=0,\ x_1=1,\ x_2=3,\ x_3=4,\ x_4=5,\ x_5=7,\ x_6=9$.

The divided difference table is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	150				
		$\frac{108-150}{1-0}=-42$			
1	108				
			$\frac{-54+42}{3-0}=-4$		
				$\frac{0+4}{4-0}=1$	
		$\frac{0-108}{3-1}=-54$			
3	0				
			$\frac{-54+54}{4-1}=0$		
					$\frac{1-1}{5-0}=0$
				$\frac{4-0}{5-1}=1$	
		$\frac{-54-0}{4-3}=-54$			
4	-54				
					$\frac{1-1}{7-1}=0$
			$\frac{-46+54}{5-3}=4$		
				$\frac{8-4}{7-3}=1$	
		$\frac{-100+54}{5-4}=-46$			
					$\frac{1-1}{9-3}=0$
5	-100				
			$\frac{-22+46}{7-4}=8$		
				$\frac{13-8}{9-4}=1$	
		$\frac{-144+100}{7-5}=-22$			
7	-144				
			$\frac{30+22}{9-5}=13$		
		$\frac{-84+144}{9-7}=30$			
9	-84				

Putting above these values in the equation (2), we get

$$f'(6) = -42 + \{(6-1)+(6-0)\}(-4) + \{(6-1)+(6-3)+(6-0)(6-3)+(6-0)(6-1)\}1$$

$$= -42 + 11 \times (-4) + (15+18+30) \times 1$$

$$= -42 - 44 + 63 = -23$$

$$f'(6) = -23.$$

Hence the value of $f'(6)$ is -23.

Example.4. From the following table, determine the value of $f'(10)$:

x	**3**	**5**	**11**	**27**	**34**
***f*(*x*)**	**-13**	**23**	**899**	**17315**	**35606**

Solution: In this case the values of the arguments are not equally spaced. So we willl use here the Newton's divided difference formula.

The Newton's divided difference formula is

$$y = f(x_0) + (x-x_0)\Delta f(x_0) + (x-x_0)(x-x_1)\Delta^2 f(x_0)$$

$$+ (x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0) + \ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots..(1)$$

Differentiating equation (1) with respect to x, we get

$$\frac{dy}{dx} = \Delta f(x_0) + \left[(x - x_1) + (x - x_0)\right] \Delta^2 f(x_0) +$$

$$[(x - x_1)(x - x_2) + (x - x_0)(x - x_2) + (x - x_0)(x - x_1)] \Delta^3 f(x_0) + \ldots\ldots\ldots\ldots \qquad \ldots..(2)$$

Here $x = 10, x_0 = 3, \; x_1 = 5, \; x_2 = 11, \; x_3 = 27, \; x_4 = 34$.

The divided difference table is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
3	-13				
		$\frac{23-(-13)}{5-3} = 18$			
5	23		$\frac{146-18}{11-3} = 16$		
		$\frac{899-23}{11-5} = 146$		$\frac{40-16}{27-3} = 1$	
11	899		$\frac{1026-146}{27-5} = 40$		$\frac{1-1}{34-3} = 0$
		$\frac{17315-899}{27-11} = 1026$		$\frac{69-40}{34-5} = 1$	
27	17315		$\frac{2613-1026}{34-11} = 69$		
		$\frac{35606-17315}{34-27} = 2613$			
34	35606				

Putting above these values in the equation (2), we get

$$f'(10) = 18 + \left\{(10 - 5) + (10 - 3)\right\}(16)$$

$$+[(10-5)(10-11)+(10-3)(10-11)+(10-3)(10-5)]\times 1$$

$$=18+12\times 16+[(-5)+(-7)+35]\times 1$$

$$=18+192+23$$

$$f'(10)=233.$$

Hence the value of $f'(10)$ is 233.

Example.5. From the following table, determine the value of $f'(2.5)$:

x	**1.5**	**1.9**	**2.5**	**3.2**	**4.3**	**5.9**
***f*(*x*)**	**3.375**	**6.059**	**13.625**	**29.368**	**73.907**	**196.579**

Solution: In this case the values of the arguments are not equally spaced. So we will use here the Newton's divided difference formula.

The Newton's divided difference formula is

$$y= f(x_0)+(x-x_0)\Delta f(x_0)+(x-x_0)(x-x_1)\Delta^2 f(x_0)$$

$$+(x-x_0)(x-x_1)(x-x_2)\Delta^3 f(x_0)+.................. \qquad .....(1)$$

Differentiating equation (1) with respect to x , we get

$$\frac{dy}{dx}=\Delta f(x_0)+\left[(x-x_1)+(x-x_0)\right]\Delta^2 f(x_0)+$$

$$[(x-x_1)(x-x_2)+(x-x_0)(x-x_2)+(x-x_0)(x-x_1)]\Delta^3 f(x_0)+............. \qquad .....(2)$$

Here $x = 2.5, x_0 = 1.5,\ x_1 = 1.9,\ \ x_2 = 2.5,\ \ x_3 = 3.2,\ \ x_4 = 4.3,\ \ x_5 = 5.9.$

The divided difference table is given below:

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375				
		$\frac{6.059-3.375}{1.9-1.5} = 6.71$			
1.9	6.059		$\frac{12.61-6.71}{2.5-1.5} = 5.9$		
		$\frac{13.625-6.059}{2.5-1.9} = 12.61$		$\frac{7.6-5.9}{3.2-1.5} = 1$	
2.5	13.625		$\frac{22.49-12.61}{3.2-1.9} = 7.6$		$\frac{1-1}{4.3-1.5} = 0$
		$\frac{29.368-13.625}{3.2-2.5} = 22.49$		$\frac{10-7.6}{4.3-1.9} = 1$	
3.2	29.368		$\frac{40.49-22.49}{4.3-2.5} = 10$		$\frac{1-1}{5.9-1.9} = 0$
		$\frac{73.907-29.368}{4.3-3.2} = 40.49$		$\frac{13.4-10}{5.9-2.5} = 1$	
4.3	73.907		$\frac{76.67-40.49}{5.9-3.2} = 13.4$		
		$\frac{196.579-73.907}{5.9-4.3} = 76.67$			
5.9	196.579				

Putting above these values in the equation (2), we get

$$f'(2.5) = 6.7 + \left\{(2.5-1.9) + (2.5-1.5)\right\}(5.90)$$

$$+[(2.5-1.9)(2.5-2.5) + (2.5-1.5)(2.5-1.9) + (2.5-1.5)(2.5-2.5)] \times (1)$$

$= 6.71 + 9.44 + 0.6$

$= 16.75$

$f'(2.5) = 16.75.$

Hence the value of $f'(2.5)$ is 16.75.

11.5 Summary

Newton's Divided Difference Formula holds a fundamental role in numerical analysis, primarily employed for the construction of interpolating polynomials. This method is extensively utilized in various applications, spanning numerical integration and differentiation. The essence of Newton's Divided Difference Formula lies in its mathematical approach to creating an interpolating polynomial based on a set of provided data points. This technique proves particularly valuable when tasked with approximating a function in situations where only discrete data points are available.

If your data points are evenly spaced, use the Newton-Gregory forward formula for derivatives near the beginning and the backward formula for derivatives near the end. If the derivative is around the middle of your data, use the Stirling difference formula. For unevenly spaced data, go with Newton's divided difference formula.

11.6 Terminal Questions

Q.1. Explain the Newton’s divided difference formula.

Q.2. When we use Stirling difference formula for derivaties.

Q.3. Find $f'(5)$ from the following table:

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

Q.4. From $f'(7.50)$ from the following table:

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
$y=f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Q.5. Find $f'(0.6)$ and $f''(0.6)$ from the following table:

x	0.4	0.5	0.6	0.7	0.8
$f(x)$	1.5836	1.7974	2.0442	2.3275	2.6510

Q.6. Find $f'(0.8)$ from the following table:

x	6	7	9	12
$f(x)$	1.556	1.690	1.908	2.158

Q.7. Find $f'(1)$ for $f(x) = \dfrac{1}{1+x^2}$ from the following table:

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	0.5	0.4524	0.4098	0.3717	0.3378

Answer

3. 84856

4. 0.233

5. 2.6445 and 3.64833

6. 0.10848

7. -0.5031.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.
2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.
3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.
4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.
5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-12: Numerical Integration

Structure

12.1 Introduction

Numerical integration is a technique used to approximate the definite integral of a function when an analytical solution is challenging or impossible to obtain. It involves dividing the integration interval into smaller subintervals and approximating the area under the curve within each subinterval. Various methods exist for numerical integration, and the choice often depends on the nature of the function and the desired level of accuracy.

Numerical integration is valuable in cases where the antiderivative of a function is difficult to determine or when dealing with numerical data. It finds applications in various fields, including physics, engineering, finance, and computer science. The method chosen depends on factors such as the complexity of the function, the desired precision, and the computational resources available. Numerical integration is the process of obtaining the value of a definite integral from a set of numerical values of the integrand.

The process to finding the value of the define integral I = $\int_a^b f(x)dx$ of a function of a single variable, is called as numerical quadrature. If we apply this for function of two variables it is called mechanical cubature.

12.2 Objectives

After reading this unit the learner should be able to understand about:

- the Quadrature formula
- the trapezoidal rule and their problems
- Simpson's 1/3 and 3/8 Rule for solving integration

12.3 Quadrature Formula

A quadrature formula is a method for numerical integration, also known as quadrature, where we approximate the integral of a function by summing weighted function values at discrete points within the interval of integration. There are various types of quadrature formulas, such as the midpoint rule, trapezoidal rule, Simpson's rule, and Gaussian quadrature, among others. The problem of numerical integration is solved by first approximating the function *f(x)* by a interpolating polynomial an then integrating it between the desired limit.

Thus $f(x) = P_n(x)$

$$\int_a^b f(x)\,dx = \int_a^b P_n(x)dx.$$

Further suppose $y = f(x)$.

Let us consider the values $y_0,\ y_1,\ y_2,\ y_3,\ \ldots\ldots\ldots\ldots\ldots,\ y_{n-1},\ y_n$ for $x_0,\ x_0+h,\ x_0+2h,\ x_0+3h, \ldots\ldots\ldots, x_0+nh.$

Let us consider, divide the interval (*a*, *b*) into *n* sub-intervals of equal width *h i.e.,*

$$\frac{b-a}{n} = h$$

Let $x_0 = a, x_1 = x_0 + h = a + h,\ x_2 = x_0 + 2h = a + 2h,\ \ldots\ldots\ldots,\ x_n = x_0 + nh = a + nh = b$

Then we have

$$\text{I} = \int_a^b f(x)dx = \int_{x_0}^{x_0+nh} f(x)dx$$

Using the Newton's forward interpolation formula is

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

…..(1)

Where $u = \dfrac{x - x_0}{h}$

Then we have $du = \dfrac{dx}{h} \quad \Rightarrow \quad dx = h\ du$

Putting the above values in the equation (1), we get

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots\ldots \right] du$$

$$= h\left[\begin{array}{l} ny_0 + \dfrac{n^2}{2}\Delta y_0 + \left(\dfrac{n^3}{3} - \dfrac{n^2}{2}\right)\dfrac{\Delta^2 y_0}{2!} \\ + \left(\dfrac{n^4}{4} - n^3 + n^2\right)\dfrac{\Delta^3 y_0}{3!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + \text{up to} (\text{n} + 1) \text{ terms} \end{array} \right]$$

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[\begin{array}{l} ny_0 + \dfrac{n^2}{2}\Delta y_0 + \left(\dfrac{n^3}{3} - \dfrac{n^2}{2}\right)\dfrac{\Delta^2 y_0}{2!} \\ + \left(\dfrac{n^4}{4} - n^3 + n^2\right)\dfrac{\Delta^3 y_0}{3!} + \ldots\ldots\ldots\ldots + \text{up to} (\text{n} + 1) \text{ terms} \end{array} \right] \quad \ldots..(2)$$

This is the general quadrature formula.

12.4 Trapezoidal Rule

Trapezoidal Rule is a numerical method used for approximating definite integrals, especially when an analytical solution is challenging to obtain. It is based on dividing the integration interval into smaller subintervals and approximating the area under the curve using trapezoids.

The Trapezoidal Rule is a straightforward method that provides reasonable accuracy, especially for functions with varying slopes. However, it may require more subintervals to achieve high precision compared to some other methods. The general quadrature formula is

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[\begin{array}{l} ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} \\ + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \text{.............} + \text{up to} (\text{n}+1) \text{ terms} \end{array}\right] \quad \text{.....(1)}$$

Putting $n = 1$ in the general quadrature formula (1) and neglecting all the differences higher terms than first, we get

$$\int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

$$= h\frac{(y_0 + y_1)}{2}$$

Similarly, we get

$$\int_{x_0+h}^{x_0+2h} f(x)dx = h\frac{(y_1 + y_2)}{2}$$

$$\vdots \qquad \qquad \vdots$$

$$\vdots \qquad \qquad \vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = h\frac{(y_{n-1} + y_n)}{2}$$

Adding all these above *n* integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[\frac{1}{2}(y_0 + y_n) + (y_1 + y_2 + y_3 + \text{................} + y_{n-1})\right]$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ... + y_{n-1})\right] \qquad ....(2)$$

This formula (2) is known as Trapezoidal rule. Trapezoidal Rule is a practical and widely used numerical technique for solving the integration. It is suitable for approximating the definite integral of a function when analytical methods are impractical.

Examples

Example.1. Use Trapezoidal rule to evaluate $\int_0^1 x^3\, dx$ considering the five sub-intervals.

Solution: Divide the interval [0, 1] into five sub parts in which each of width

$$h = \frac{1-0}{5} = 0.2$$

For computing the value of the given function $y = x^3$ at each points of sub-interval, are as following:

x	0	0.2	0.4	0.6	0.8	1
$y = x^3$	0	0.008	0.064	0.216	0.512	1

Using Trapezoidal rule, we have

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ... + y_{n-1})\right]$$

or

$$\int_0^1 x^3 dx = \frac{h}{2}\left[\left(y_0 + y_5\right) + 2\left(y_1 + y_2 + y_3 + y_4\right)\right]$$

$$=\frac{0.2}{2}\left[(0+1)+2(0.008+0.064+0.216+0.512)\right]$$

$$=\frac{0.2}{2}[2\times 0.8+1]$$

$$=\frac{0.2}{2}(2.6)$$

$$=0.26.$$

12.5 Simpson's 1/3 Rule

Simpson's Rule is a numerical method for approximating definite integrals, particularly useful when an analytical solution is challenging to obtain. This rule is an improvement over the Trapezoidal Rule and provides a more accurate estimation of the area under a curve. Simpson's Rule generally provides a more accurate result than the Trapezoidal Rule, and the error decreases significantly with each doubling of the number of subintervals.

Simpson's Rule is more accurate for smooth and well-behaved functions, especially those that can be approximated well by quadratic polynomials. However, it requires an even number of subintervals, and the accuracy improvement may be modest for functions with rapidly changing slopes.

The general quadrature formula is

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[\begin{array}{l} ny_0+\frac{n^2}{2}\Delta y_0+\left(\frac{n^3}{3}-\frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} \\ +\left(\frac{n^4}{4}-n^3+n^2\right)\frac{\Delta^3 y_0}{3!}+.............+\text{up to}\,(\text{n}+1)\text{ terms}\end{array}\right] \quad .....(1)$$

Putting $n = 2$ in the general quadrature formula (1) and neglecting all the differences higher terms then second, we get

$$\int_{x_0}^{x_0+2h} f(x)dx = h\left[2y_0 + \frac{2^2}{2}\Delta y_0 + \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\frac{\Delta^2 y_0}{2!}\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= h\left[2y_0 + 2y_1 - 2y_0 + \frac{y_2}{3} - \frac{2}{3}y_1 + \frac{y_0}{3}\right]$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Similarly

$$\int_{x_0+2h}^{x_0+4h} f(x)dx = \frac{h}{3}[y_2 + 4y_3 + y_4]$$

$$\vdots \qquad\qquad \vdots$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these above *n* integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[\begin{array}{l} y_0 + 2\left(y_2 + y_4 + y_6 + ... + y_{n-2}\right) \\ +4(y_1 + y_3 + y_5 + ... + y_{n-1}) + y_n \end{array}\right]$$

$$= \frac{h}{3}\left[\begin{array}{l} \left(y_0 + y_n\right) + 2\left(y_2 + y_4 + y_6 + ... + y_{n-2}\right) \\ +4(y_1 + y_3 + y_5 + ... + y_{n-1}) \end{array}\right] \qquad \ldots.. (2)$$

This formula (2) is known as Simpson's $\frac{1}{3}$ rule.

Check your Progress

1. What do you mean by trapezoidal rule?

2. Write the simpson's 1/3 formula.

Examples

Example.2. Find the value of $\log_e 2$ **from** $\int_0^1 \frac{x^2}{1+x^3}\,dx$, **using Simpson's one-third rule, by dividing the range into four equal parts. Also solve the given integral with usual method and find the error.**

Solution: Divide the interval [0, 1] into four equal parts in which each of width

$$h = \frac{1-0}{4} = 0.25$$

For computing the value of the given function $y = f(x) = \frac{x^2}{1+x^3}$ at each points of sub-interval, are as following:

x	x^2	x^3	$1+x^3$	$y = f(x) = \frac{x^2}{1+x^3}$
0	0	0	1	0
0.25	0.0625	0.015625	1.015625	0.061538
0.50	0.2500	0.12500	1.12500	0.222222
0.75	0.5625	0.421875	1.421875	0.395604
1	1	1	2	0.500000

Using Simpson's one-third rule, we have

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[\begin{array}{l}(y_0 + y_n) + 2(y_2 + y_4 + y_6 + ... + y_{n-2}) \\ +4(y_1 + y_3 + y_5 + ... + y_{n-1})\end{array}\right]$$

or

$$\int_0^1 \frac{x^2}{1+x^3}\, dx = \frac{h}{3}\left[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)\right]$$

$$= \frac{0.25}{3}[(0+0.5)+2(0.222222)+4(0.061538+0.395604)]$$

$$= \frac{0.25}{3}(2.773015)$$

$$= 0.231084\,.$$

Again we have

$$\int_0^1 \frac{x^2}{1+x^3}\, dx = \frac{1}{3}\int_0^1 \frac{3x^2}{1+x^3}\, dx$$

$$= \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\log 2$$

$$= \frac{1}{3}\times(0.693147)$$

$$= 0.231049$$

Therefore the error between usual method and numerical method is

$= 0.231084 - 0.231049$

$= 0.000034.$

Example.3. Use Simpson's 1/3 rule, dividing the range into ten equal parts, to prove that

$$\int_0^1 \frac{\log(1+x^3)}{1+x^2} = 0.1730.$$

Solution: Divide the interval [0, 1] into ten equal parts in which each of width $h = \frac{1-0}{10} = 0.1.$

For computing the value of the given function $y = f(x) = \frac{\log(1+x^3)}{1+x^2}$ at each points of sub-interval, are as following:

x	x^2	$1+x^2$	$\log(1+x^2)$	$\frac{\log(1+x^2)}{1+x^2}$
0	0	1.0	0	0
0.1	0.01	1.01	0.009950	0.009851
0.2	0.04	1.04	0.039220	0.037712
0.3	0.09	1.09	0.086177	0.079062
0.4	0.16	1.16	0.14842	0.127948
0.5	0.25	1.25	0.223143	0.178514
0.6	0.36	1.36	0.307484	0.226091
0.7	0.49	1.49	0.398776	0.267634

0.8	0.64	1.64	0.494696	0.301644
0.9	0.81	1.81	0.593326	0.327804
1.0	1.0	2.0	0.693147	0.346573

Using Simpson's one-third rule, we have

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}dx = \frac{h}{3}[(y_0 + y_{10}) + 2(y_2 + y_4 + y_6 + y_8) + 4(y_1 + y_3 + y_5 + y_7 + y_9)]$$

$$= \frac{0.1}{3}[(0+0.346573) + 2(0.037712 + 0.127948 + 0.226091 + 0.301644)$$

$$+4(0.009851 + 0.079062 + 0.178514 + 0.26634 + 0.327804)]$$

$$= \frac{0.1}{3}(5.184839)$$

$$= 0.17282793.$$

Example.4. Evaluate the integral $\int_{0.5}^{0.7} x^{1/2}e^{-x}dx$ using the Trapezoidal rule and Simpson's 1/3 rule.

Solution: Divide the interval [0.5, 0.7] into four equal parts in which each of width

$$h = \frac{0.7 - 0.5}{4}$$
$$= 0.05$$

For computing the value of the given function $y = f(x) = x^{1/2}e^{-x}$ at each points of sub-interval, are as following:

x	$x^{1/2}$	e^{-x}	$f(x) = x^{1/2}e^{-x}$
0.50	0.707106	0.606530	0.428881
0.55	0.741619	0.576949	0.427876
0.60	0.774596	0.548811	0.425107
0.65	0.806225	0.522045	0.420886
0.70	0.836660	0.496858	0.415473

Using Trapezoidal rule, we have

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + ... + y_{n-1})\right]$$

or

$$\int_{0.5}^{0.7} x^{1/2}e^{-x}dx = \frac{h}{2}[(y_0 + y_4) + 2(y_1 + y_2 + y_3)]$$

$$= \frac{0.05}{2}[(0.428881 + 0.415473) + 2(0.427876 + 0.425107 + 0.420886)]$$

$$= \frac{0.05}{2}(3.392092)$$

$$= 0.0848023$$

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_{0.5}^{0.7} x^{1/2}e^{-x}dx = \frac{h}{3}[(y_0 + y_4) + 2y_2 + 4(y_1 + y_3)]$$

$$= \frac{0.05}{3}[(0.428881 + 0.415473) + 2(425107) + 4(0.427876 + 0.420886)]$$

$$= \frac{0.05}{3}(5.089616)$$

$$= 0.0848269\,.$$

12.5 Simpson's 3/8 Rule

Simpson's Rule is a valuable method for numerical integration which is offering improved accuracy over simpler techniques, especially for functions with relatively smooth behavior.

The general quadrature formula is

$$\int_{x_0}^{x_0+nh} f(x)dx = h\left[\begin{array}{l} ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} \\ + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{3!} + \ldots\ldots\ldots\ldots + \text{up to}\,(\text{n}+1)\ \text{terms} \end{array}\right] \quad \ldots..(1)$$

Putting n =3 in the general quadrature formula (1) and neglecting all differences higher then third, we get.

$$\int_{x_0}^{x_0+3h} f(x)dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{3^3}{3} - \frac{3^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{3^4}{4} - 3^3 + 3^2\right)\frac{\Delta^3 y_0}{3!}\right]$$

$$= h\left[3y_0 + \frac{9}{2}(y_1 - y_0) + \frac{9}{4}(y_2 - 2y + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3]$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6]$$
$$\vdots \qquad\qquad \vdots$$
$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)dx = \frac{3h}{8}[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n]$$

Adding all these above n integrals, we get

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\begin{bmatrix}(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + y_{10} + y_{11} + \ldots\ldots\ldots\ldots + y_{n-1}) \\ +2(y_3 + y_6 + y_9 + y_{12} + \ldots\ldots\ldots\ldots + y_{n-3})\end{bmatrix}$$

….. (2)

This formula (2) is known as Simpson's three-eight's rule.

Examples

Example.5. Evaluate $\int_0^{\pi/2} e^{\sin x}\, dx$ correct to four decimal places by Simpson's one-third and three-eighth rule, dividing the interval $\left(0, \frac{\pi}{2}\right)$ into three equal parts.

Solution: Divide the interval $\left(0, \frac{\pi}{2}\right)$ into three equal parts in which each of width

$$h = \frac{(\pi/2) - 0}{3}$$
$$= \frac{\pi}{6}$$

For computing the value of the given function $y = e^{\sin x}$ at each points of sub-interval, are as following:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = e^{\sin x}$	1	1.64872	2.45960	2.71828

Using Simpson's one-third rule, we have

$$\int_0^{\pi/2} e^{\sin x}\, dx = \frac{h}{3}[(y_0 + y_3) + 2(y_2) + 4(y_1)]$$

$$= \frac{\pi/6}{3}[(1+2.71828) + 2(2.45960) + 4(1.64872)]$$

$$= \frac{\pi}{18}(15.23236)$$

$$= 2.6596.$$

Using Simpson's three-eighth rule, we have

$$\int_0^{\pi/2} e^{\sin x}\, dx = \frac{3h}{8}[y_0 + 3(y_1 + y_2) + y_3]$$

$$= \frac{3\left(\frac{\pi}{6}\right)}{8}\left[(1+2.71828) + 3(1.65872 + 2.45960)\right]$$

$$= \frac{3\pi}{48}(16.04324)$$

$= 3.1513$.

Example.6. Evaluate the integral $\int_0^1 \frac{dx}{1+2x}$, using Simpson's 1/3 rule and Simpson's 3/8 rule.

Sol. : Divide the interval [0, 1] into three equal parts in which each of width

$$h = \frac{1-0}{6}$$
$$= \frac{1}{6}$$

For computing the value of the given function $\int_0^1 \frac{dx}{1+2x}$, at each points of sub-interval, are as following:

x	0	1/6	2/6	3/6	4/6	5/6	6/6=1
$f(x)$	1	0.75	0.6	0.5	0.4285714	0.375	0.333333

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 f(x)dx = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{1}{18}[(1+0.333333) + 2(0.6+0.4285714) + 4(0.75+0.5+0.375)]$$

$$= \frac{1}{18}[1.33333 + 6.5 + 2.0571428]$$

$= 0.5494708.$

Using Simpson's $\frac{3}{8}$ rule, we have

$$\int_0^1 f(x)\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 2y_3 + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3}{48}\left[(1+0.333333)+2(0.5) + 3(0.75+0.6+0.4285714 +0.375)\right]$$

$$=\frac{1}{16}\left[1.333333 +1+6.4607142\right]$$

$$=\frac{1}{16}(8.7940475)$$

= 0.5496279.

Example.7. Evaluate the integral $\int_0^6 \frac{dx}{1+x^2}$ by using Trapezoidal, Simpson's one-third and three-eighth rule.

Solution. Divide the interval [0, 6] into six equal parts in which each of width

$$h = \frac{6-0}{6}$$

$$= 1$$

For computing the value of the given function $y = \frac{1}{1+x^2}$ at each points of sub-interval, are as following:

x	$y=\dfrac{1}{1+x^2}$
0	1
1	0.5
2	0.2
3	0.1
4	0.0588
5	0.0385
6	0.027

Using Trapezoidal rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$=\frac{1}{2}[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)]$$

$$=\frac{1}{2}[2.8216]$$

$$=1.4108$$

Using Simpson's one-third rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)]$$

$$= \frac{1}{3}[(1+0.027)+2(0.2+0.0588)+4(0.5+0.1+0.0385)]$$

$$= \frac{1}{3}[1.027+2(0.2588)+4(0.6385)]$$

$$= \frac{1}{3}[4.0986]$$

$$= 1.3662 .$$

Using Simpson's three-eight rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)]$$

$$= \frac{3}{8}[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)]$$

$$= \frac{3}{8}(3.6189)$$

$$= 1.3570 .$$

Example.8. Calculate the approximate value of $\int_{-3}^{3} x^4 dx$ by using Trapezoidal rule, Simpson's one-third and three eight rule, by dividing the range in six equal parts.

Solution: Divide the interval [-3, 3] into six equal parts in which each of width

$$h = \frac{3-(-3)}{6}$$

$$= 1$$

For computing the value of the given function $y = x^4$ at each points of sub-interval, are as following:

x	-3	-2	-1	0	1	2	3
$y=x^4$	81	16	1	0	1	16	81

Using Trapezoidal rule, we have

$$\int_{-3}^{3} x^4 dx = \frac{h}{2}[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}[(81+81) + 2(16+1+0+1+16)]$$

$$= \frac{1}{2}[162 + 68]$$

$$= 115$$

Using Simpson’s one-third rule, we have

$$\int_{-3}^{3} x^4 dx = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_2 + y_3 + y_5)]$$

$$= \frac{1}{3}[(81+81) + 2(1+1) + 4(16+0+16)]$$

$$= \frac{1}{3}[162 + 2(2) + 4(32)]$$

$$= \frac{1}{3}(294)$$

$$= 98$$

Using Simpson's three-eight rule, we have

$$\int_{-3}^{3} x^4 dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3}{8}[(81+81) + 2(0) + 3(16+1+1+16)]$$

$$= \frac{3}{8}[162 + 3\times 34]$$

$$= 99$$

The exact value of

$$\int_{-3}^{3} x^4 dx = 2\int_{0}^{3} x^4 dx$$

$$= 2\left[\frac{x^5}{5}\right]_0^3$$

$$= \frac{2}{5}(3)^5$$

$$= \frac{2}{5} \times 243$$

$$= 97.2 .$$

Example.9. Evaluate $\int_{4}^{5.2} \log_e x\, dx$ by Simpson's one-third and three-eighth rule.

Solution: Divide the interval [4, 5.2] into six equal parts in which each of width

$$h = \frac{5.2 - 4}{6}$$

$$= 0.2$$

For computing the value of the given function $f(x) = \log_e x$ at each points of sub-interval, are as following:

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$f(x)$	1.386294	1.435084	1.481604	1.526056	1.568615	1.609437	1.648658

Using Simpson's one-third rule, we have

$$\int_4^{5.2} \log_e x\,dx = \frac{h}{3}[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_3 + y_5)]$$

$$= \frac{0.2}{3}[(1.386294 + 1.648658) + 2(1.481604 + 1.568615)$$

$$+4(1.435284 + 1.526056 + 1.609437)]$$

$$= \frac{0.2}{3}(27.417698)$$

$$= 1.827847$$

Using Simpson's three-eighth rule, we have

$$\int_4^{5.2} \log_e x\,dx = \frac{3h}{8}[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)]$$

$$= \frac{3(0.2)}{8}[(1.386294 + 1.648658) + 2(1.526056)$$

$$+3(1.435284 + 1.481604 + 1.568615 + 1.609437)]$$

$$= \frac{0.6}{8} \times (24.371294)$$

$$= 1.827847.$$

Example.10. Evaluate the integral $\int_0^1 \frac{1}{1+x^2}\,dx$ using using Trapezoidal rule, Simpson's one-third and three eight rule, by dividing the range in six equal parts. Hence obtain the value of π in each case.

Solution. For applying the trapezoidal rule, the interval must be divided into number of intervals to multiple of 1, for Simpson's 1/3 rule, in number of multiple of 2, for Simpson 3/8 rule in number of multiple of 3. So when applying all the rules then number of intervals must be divided by 1, 2, 3 and 6. Let n=6 intervals then we have

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

x	$x_0 = 0$	$x_1 = \frac{1}{6}$	$x_2 = \frac{2}{6}$	$x_3 = \frac{3}{6}$	$x_4 = \frac{4}{6}$	$x_5 = \frac{5}{6}$	$x_6 = 1$
y	$\frac{1}{1+0} = 1$ y_0	$\frac{1}{1+\frac{1}{36}} = 0.97297$ y_1	$\frac{1}{1+\frac{1}{9}} = 0.9$ y_2	$\frac{1}{1+\frac{1}{4}} = 0.8$ y_3	$\frac{1}{1+\frac{4}{9}} = 0.0692307$ y_4	$\frac{1}{1+\frac{25}{36}} = 0.59016$ y_5	$\frac{1}{1+1} = 0.5$ y_6

Using Trapezoidal rule, we have

$$\int_{x_0}^{x_0+nh} f(x)\, dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \dots\dots\dots + y_{n-1})\right]$$

$$\int_0^1 \frac{1}{(1+x^2)} dx = \frac{1}{12}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{12}\left[(1 + 0.5) + 2(0.97297 + 0.9 + 0.8) + (0.692307 + 0.59016)\right]$$

$$= 0.7842395$$

Using Simpson's one-third rule, we have

$$\int_0^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0 + y_n) + 2(y_3 + y_4 + \dots.) + 4(y_1 + y_3 + \dots\dots)\right]$$

$$\int_0^{x_0+nh} \frac{1}{(1+x^2)} dx = \frac{h}{3}\left[(y_0 + y_6) + 2(y_3 + y_4) + 4(y_1 + y_3 + y_5)\right]$$

$$= \frac{1}{18}\left[(1+0.5) + 2(0.9 + 0.692307) + 4(0.97297 + 0.8 + 0.59016)\right]$$

$$= 0.78539633$$

Using Simpson's three-eight rule, we have

$$\int_0^{x_0+nh} f(x)dx = \frac{3}{8}h\left[(y_0 + y_n) + 2(y_3 + y_6 + y_9 + \dots) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 \dots\dots)\right]$$

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{3}{48}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{16}\left[(1+0.5) + 2(0.8) + 3(0.97297 + 0.9 + 0.59016 + 0.692307 + 0.59016)\right]$$

$$= 0.78539437$$

Using integration, we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}$$

or $$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{\pi}{4}$$

or $$\pi = \int_0^1 \frac{1}{1+x^2}\,dx.$$

Using Trapezoidal rule, π = 4[0.7842395]=3.136958.

Using Simpson's 1/3 rule, we have π=4[0.785396333]=3.141585332.

Using Simpson's 3/8 rule, we have π=4[0.78394437]=3.14157748.

12 Summary

The Trapezoidal rule is

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}[(y_0+y_n)+2(y_1+y_2+..........+y_{n-1})]$$

The Simpson's 1/3 rule is

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}[(y_0+y_n)+2(y_2+y_4+y_6+........+y_{n-2})+4(y_1+y_3+y_5+..........+y_{n-1})]$$

The Simpson's 3/8 rule is

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + ... + y_{n-3}) + 3(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1})]$$

12 Terminal Questions

Q.1. Write the formula for Trapezoidal rule.

Q.2. Which method give the more approximate result in the following method:

(i) Trapezoidal rule (ii) Simpson's one third rule (iii) Simpson's three-eighth rule.

Q.3. Explain the Simpson's 3/8 rule.

Q.4. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's one-third and three-eighth rule. Hence obtain the approximate value of π in each case.

Q.5. Evaluate $\int_2^{10} \frac{dx}{1+x}$ by dividing the range into eight equal parts by Simpson's one-third rule.

Q.6. Calculate an approximate value of the integral $\int_0^{x/2} \sin x\, dx$ by (i) Trapezoidal rule (ii) Simpson's one third rule (iii) Simpson's three-eighth rule.

Answer

4. 0.785397 and 0.785395, π = 3.141588

5. 1.29962

6. 0.99795, 1.0006, 1.1003.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.

2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.

3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.

4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.

5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-13: Numerical Solution of Ordinary Differential Equations-I

Structure

13.1 Introduction

The numerical solution of ordinary differential equations (ODEs) involves using computational methods to approximate the solutions of differential equations when explicit analytical solutions are not readily available or feasible. ODEs describe how a function changes with respect to an independent variable and find applications in various fields such as physics, engineering, biology, and economics.

In this unit, we will discuss the important methods of solving ordinary differential equation of first order having numerical coefficients and given boundary or initial conditions $\left(i.e., \ \frac{dy}{dx} = f(x, y) \text{ given } y(x_0) = y_0 \right)$ numerically. These methods also useful to solve those types of problem related to first order differential equations which cannot be integrated analytically. For example, $\frac{dy}{dx} = x^2 + y^2 - c^2$.

Some important numerical methods for solving ordinary differential equations are Euler's method; Euler's Modified method; Taylor's Series Method; Picard's method of successive approximation; Runge-Kutta method and Milne's predictor-corrector method. Here in this unit we discuss only Euler's method; Euler's Modified method and Taylor's Series Method.

13.2 Objectives

After reading this unit the learner should be able to understand about:

- The Euler's Method
- The Euler's modified Method
- The Taylor Series method

13.3 Euler's Method

Euler's Method is a simple and straightforward numerical technique used for approximating the solution of ordinary differential equations (ODEs) when an explicit analytical solution is either challenging or impossible to obtain. Developed by Leonhard Euler, this method is particularly suitable for introductory purposes and provides a basic understanding of numerical integration. Euler's Method is based on the idea of approximating the solution of an ODE by taking small steps along the curve, using the slope at each point to predict the next value.

The method is easy to implement, making it a suitable choice for introductory courses in numerical methods. However, it may not provide accurate results for certain types of differential equations, especially those with rapid changes. This is simplest and oldest method was devised by Euler. It illustrates, the basic idea of those numerical methods which seeks to determine the change Δy in y corresponding to a small increase in the arguments x.

Euler's Method serves as a foundation for more advanced numerical methods and is a valuable tool for gaining insight into the numerical solution of ordinary differential equations.

Consider the differential equation

$$y' = \frac{dy}{dx} = f(x, y) \qquad \text{....(1)}$$

with initial condition $y = y_0$ when $x = x_0$, *i.e.*, $y(x_0) = y_0$

We wish to solve the equation (1), for the values of y at $x = x_i$

Where $x_i = x_0 + ih,\ i = 1, 2, 3, 4, ...$

Now integrate the equation (1), we have

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y)\, dx$$

Let $f(x,y) = f(x_0, y_0)$ where $x_0 \le x \le x_1$

Now we have

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0, y_0)\, dx$$

$$= y_0 + (x_1 - x_0) f(x_0, y_0) \qquad [\because\ h = x_1 - x_0]$$

Similarly for $x_1 \le x \le x_2$, we have

$$y_2 = y_1 + h\ f(x_1, y_1)$$

Proceeding in the same way, we have finally

$$y_{n+1} = y_n + h\ f(x_n, y_n)$$

Thus, starting from x_0 when $y = y_0$ we can construct a table of y for given steps of h in x.

Euler's method is a simple numerical technique for solving ordinary differential equations (ODEs) that can be computationally inefficient for certain problems, especially when higher accuracy is required.

Euler's method approximates the solution of an initial value problem by advancing the solution in small steps of size h along the direction of the derivative.

Examples

Example.1. Given $\frac{dy}{dx} = \frac{y - x}{y + x}$ with the initial condition y =1 when x = 0 find y for x = 0.1 in four steps by Euler's method.

Solution: It is given that

$$\frac{dy}{dx} = \frac{y-x}{y+x} = f(x, y),\ x_0 = 0,\ y_0 = 1.$$

We have $h = \frac{0.1}{4} = 0.025$

The Euler's formula is

$$y_{n+1} = y_n + h\, f(x_n, y_n) \qquad \text{....(1)}$$

Putting n = 0, 1, 2, 3, …… in equation (1), we get

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$= y_0 + h\frac{y_0 - x_0}{y_0 + x_0}$$

$$= 1 + 0.025\left(\frac{1-0}{1+0}\right)$$
$$= 1.025$$

$$y_1 = 1.025$$

Again we have

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$= y_1 + h\frac{y_1 - x_1}{y_1 + x_1}$$

$$= 1.025 + 0.025\left(\frac{1.025 - 0.025}{1.025 + 0.025}\right)$$

$$= 1.025 + 0.025 \times \frac{1}{1.05}$$

$$y_2 = 1.0488$$

Now we have

$$y_3 = y_2 + h\,f(x_2, y_2)$$

$$= y_2 + h\,\frac{y_2 - x_2}{y_2 + x_2}$$

$$= 1.0488 + 0.025\left(\frac{1.0488 + 0.05}{1.0488 + 0.05}\right)$$

$$= 1.0488 + 0.025 \times \frac{1.0438}{1.0988}$$

$$y_3 = 1.07152$$

Now we have

$$y_4 = y_3 + h\,f(x_3, y_3)$$

$$= y_3 + h\,\frac{y_3 - x_3}{y_3 + x_3}$$

$$= 1.07152 + 0.025\left(\frac{1.07152 - 0.075}{1.07152 + 0.075}\right)$$

$$y_4 = 1.09324$$

Hence the value of y at $x = 0.1$ for the differential equation $\frac{dy}{dx} = \frac{y - x}{y + x}$ is 1.09324.

Example 2. Using the Euler's method to compute the y(0.5) for the differential equation $\frac{dy}{dx} = y^2 - x^2$ with $y = 1$ when $x = 0$.

Solution: It is given that

$$\frac{dy}{dx} = y^2 - x^2 = f(x, y), \ \ x_0 = 0, \ y_0 = 1.$$

And we have $h = \dfrac{0.5}{5} = 0.1$

The Euler's formula is

$$y_{n+1} = y_n + h\, f(x_n, y_n) \qquad \text{....(1)}$$

Putting n = 0, 1, 2, 3, …… in equation (1), we get

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_1 = y_0 + h(y_0^2 - x_0^2)$$

$$= 1 + (0.1)(1^2 - 0^2)$$

$$= 1 + (0.1)1$$

$$= 1.1$$

Again we have

$$y_2 = y_1 + h\, f(x_1, y_1)$$

$$y_2 = y_1 + h(y_1^2 - x_1^2)$$

$$= 1.1 + (0.1)[(1.1)^2 - (0.1)^2]$$

$$= 1.1 + (0.1)(1.21 - 0.01)$$

$$= 1.220 \qquad [\because \ x_1 = x_0 + h]$$

Now we have

$$y_3 = y_2 + h\, f(x_2, y_2)$$

$$= y_2 + h(y_2^2 - x_2^2)$$

$$= 1.22 + (0.1)[(1.22)^2 - (0.2)^2] \qquad [\because \; x_2 = x_1 + h]$$

$$= 1.22 + (0.1)(1.4484)$$

$$= 1.36484$$

Now we have $y_4 = y_3 + h\, f(x_3, y_3)$

$$y_4 = y_3 + h(y_3^2 - x_3^2)$$

$$= 1.36484 + (0.1)[(1.36484)^2 - (0.3)^2]$$

$$= 1.36484 + (0.1)(1.7728)$$

$$= 1.54212$$

Now we have $y_5 = y_4 + h\, f(x_4, y_4)$

$$= y_4 + h(y_4^2 - x_4^2)$$

$$= 1.54212 + (0.1)[(1.54212)^2 - (0.4)^2]$$

$$= 1.7639$$

Hence the value of y at x = 0.5 for the given differential equation $\dfrac{dy}{dx} = y^2 - x^2$ is 1.7639.

13.4 Euler's Modified Method

Euler's Modified Method, also known as the Improved Euler Method or Heun's Method, is an enhancement of the basic Euler's Method for approximating the solution of ordinary differential equations (ODEs). This modification seeks to improve the accuracy of the solution by incorporating a more sophisticated approach to predicting the next values. Similar to Euler's Method, Euler's Modified Method is based on the idea of approximating the solution of an ODE by taking small steps along the curve. However, it employs a more refined prediction step. While Euler's Modified Method requires an additional evaluation of the function to improve accuracy, it is still relatively straightforward to implement. It strikes a balance between simplicity and accuracy.

The Euler Modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})] \quad (n = 0, 1, 2....)$$

This is the nth approximation of y_1. For determining the initial value $y_1^{(0)}$ we use the Euler's method. $y_1^{(0)} = y_0 + h f(x_0, y_0)$.

Check your Progress

1. What do you mean by Euler's method?

2. Write the Euler's modified method formula.

Examples

Example.3. Solve the differential equation $\frac{dy}{dx} = x + \sqrt{|y|} \quad with \; y(0) = 1 \; for \; 0 \le x \le 0.6$ **in the steps of 0.2; using Euler's modified method.**

Solution: It is given that

$$f(x, y) = x + \sqrt{|y|}, x_0 = 0, \quad y_0 = 1 \;\; and \;\; h = 0.2 \qquad .....(1)$$

Using Euler's method, we have

$$y_1 = y_0 + h f(x_0, y_0)$$

$$= 1 + (0.2)(0 + \sqrt{1})$$

$$= 1.2$$

Hence $y_1 = y_1^0 = 1.2$.

The value of y_1, thus determined is improved by Euler's modified method. The Euler's modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(n)})] \qquad \text{.....(2)}$$

Put n = 0 in the equation (2), we get

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 1 + \frac{0.2}{2}[(0 + \sqrt{1}) + (0.2 + \sqrt{1.2})]$$

$$= 1 + 0.2295 = 1.2295$$

Put n = 1 in the equation (2), we get

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 1 + \frac{0.2}{2}\left[(0 + \sqrt{1}) + (0.2 + \sqrt{1.2295})\right]$$

$$=1+0.2309$$

$$=1.2309$$

Put n = 2 in the equation (2), we get

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 1 + \frac{0.2}{2}[(0 + \sqrt{1}) + (0.2 + \sqrt{1.2309})]$$

$$= 1 + 0.2309$$

$$= 1.2309$$

Hence we take $y_1 = 1.2309$ at x = 0.2.

Now, we proceed to obtain y at x = 0.4.

Using Euler's method, we have

$$y_2 = y_1 + h\,f(x_1, y_1)$$

$$= y_1 + h(x_1 + \sqrt{|y_1|}$$

$$= 1.2309 + 0.2(0.2 + \sqrt{1.2309})$$

$$= 1.2309 + 0.2(1.30945)$$

$$= 1.49279$$

The value of y_2, thus determined is improved by Euler's modified method. The Euler's modified formula is

$$y_2^{(n+1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^n)] \qquad \text{…..(3)}$$

Put n = 0 in the equation (3), we get

$$y_2^{(1)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$= 1.2309 + \frac{0.2}{2}\left[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.49279})\right]$$

$$= 1.52402$$

Put n =1 in the equation (3), we get

$$y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$= 1.2309 + \frac{0.2}{2}\left[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.52402})\right]$$

$$= 1.525297$$

Put n =2 in the equation (3), we get

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$= 1.2309 + \frac{0.2}{2}\left[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.525297})\right]$$

$$= 1.52535$$

Put n = 3 in the equation (3), we get

$$y_2^{(4)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(3)})]$$

$$= 1.2309 + \frac{0.2}{2}\left[(0.2 + \sqrt{1.2309}) + (0.4 + \sqrt{1.52535})\right]$$

$$= 1.52535$$

Thus $y_2^3 = y_2^{(4)}$

Here we take $y_2 = 1.52535$ at $x = 0.4.$

Now we proceed to obtain *y* at $x = 0.6$.

Using Euler's method, we have

$$y_3 = y_2 + h f(x_2, y_2)$$

$$= y_2 + h(x_2 + \sqrt{|y_2|})$$

$$= 1.52535 + 0.2(0.4 + \sqrt{1.52535})$$

$$= 1.85236.$$

The value of y_3, thus determined is improved by Euler's modified method. The Euler's modified formula is

$$y_3^{(n+1)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(n)})] \qquad \ldots.(4)$$

Put *n* = 0 in the equation (4), we get

$$y_3^{(1)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(0)})]$$

$$= 1.52535 + \frac{0.2}{2}\left[90.4 + \sqrt{1.52535}) + (0.6 + \sqrt{1.85236})\right]$$

$$= 1.88496$$

Put *n* = 1 in the equation (4), we get

$$y_3^{(2)} = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3^{(1)})]$$

$$=1.52535+\frac{0.2}{2}\left[(0.4+\sqrt{1.52535})+(0.6+\sqrt{1.88496})\right]$$

$$=1.88615$$

Put n =2 in the equation (4), we get

$$y_3^{(3)}=y_2+\frac{h}{2}[f(x_2,y_2)+f(x_3,y_3^{(3)})]$$

$$=1.52535+\frac{0.2}{2}\left[(0.4+\sqrt{1.52535})+(0.6+\sqrt{1.88615})\right]$$

$$=1.88619$$

Put n = 3 in the equation (4), we get

$$y_3^{(4)}=y_2+\frac{h}{2}[f(x_2,y_2)+f(x_3,y_3^{(3)})]$$

$$=1.52535+\frac{0.2}{2}\left[(0.4+\sqrt{1.52535})+(0.6+\sqrt{1.88619})\right]$$

$$=1.88619$$

Here $y_3^{(3)}=y_3^{(4)}$.

Thus we get $y_3=1.88619$ at $x=0.6$.

Hence the value of $y(0.2)=1.2309,\ y(0.4)=1.52535,\ y(0.6)=1.88619.$

Example.4. Using the Euler's modified method, compute y(0.1) correct to six decimal figures, where $\frac{dy}{dx}=x^2+y$ with $y(0)=0.94.$

Solution: It is given that

$$f(x,y)=x^2+y,\ x_0=0,\ y_0=0.94,\ h=0.1. \qquad \ldots..(1)$$

Using Euler's method, we have

$$y_1 = y_0 + h\,f(x_0, y_0)$$

$$= y_0 + h[x_0^2 + y_0]$$

$$= 0.94 + (0.1)[0 + 0.94]$$

$$= 1.034$$

Hence $y_1^0 = 1.034.$

The value of y_1, thus determined is improved by Euler's modified method. The Euler's modified formula is

$$y_1^{(n+1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(n)})] \qquad \text{.....(2)}$$

Put *n* = 0 in the equation (2), we get

$$y_1^{(1)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + ((0.1)^2 + 1.034)\right]$$

$$= 1.0392$$

Put *n* = 1 in the equation (2), we get

$$y_1^{(2)} = y_0 + \frac{h}{2}[\,f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + ((0.1)^2 + (1.0392))\right]$$

$$= 1.03946$$

Put n = 2 in the equation (2), we get

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + ((0.1)^2 + (1.03946))\right]$$

$$= 1.039473$$

Put n = 3 in the equation (2), we get

$$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})]$$

$$= 0.94 + \frac{0.1}{2}\left[(0 + 0.94) + ((0.1)^2 + (1.039473))\right]$$

$$= 1.039473$$

Here $y_1^{(3)} = y_1^{(4)}$. Therefore $y_1 = 1.039473.$

Hence the value of $y(0.1) = 1.039473.$

13.5 Taylor's Series Method

Taylor's Series Method is a numerical technique used for approximating the solution of ordinary differential equations (ODEs) by representing the solution as a Taylor series expansion. This method provides a systematic way to obtain accurate numerical solutions by considering higher-order derivatives of the function. Taylor's Series Method expands the solution of an ODE into a Taylor series around a given point. The series includes terms involving the function's values and its derivatives.

The method requires the calculation of higher-order derivatives of the function at the chosen point a. These derivatives contribute terms to the series expansion. The accuracy of the approximation depends on the order of the Taylor series used. Higher-order series provide more accurate results, but they require more derivatives to be computed.

Consider the differential equation

$$y' = \frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0.$$

The Taylor's series method is

$$y(x) = y_0 + (x - x_0)\, y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + \ldots\ldots..$$

Putting the values of x, x_0, y_0, y_0', y_0'', ………. in above equation and we get value of $y(x)$. Taylor's series method is derived in any order and values of $y(x)$ are easily obtained. But this method take long time in computing higher derivatives.

Taylor's Series Method is often implemented using computer software due to the need for multiple derivative evaluations. The series expansion is truncated at a certain order, and the terms are used to iteratively update the solution.

Examples

Example.5. Solve the differential equation $\frac{dy}{dx} = x + y$ with $y(0)=1$, by Taylor's series method to compute y for $x=0.1$.

Solution: It is given that

$$y' = \frac{dy}{dx} = x + y, \quad x_0 = 0, \quad y_0 = 1. \qquad \ldots(1)$$

Here we find some derivatives and their values at $x_0=0$, $y_0=1$ are

$y' = x + y$	$y'_0 = 1$
$y'' = 1 + y'$	$y''_0 = 2$
$y''' = 0 + y''$	$y'''_0 = 2$
$y'''' = y'''$	$y''''_0 = 2$

The Taylor's series method is

$$y(x) = y_0 + (x - x_0)\, y'_0 + \frac{(x - x_0)^2}{2!}\, y''_0 + \frac{(x - x_0)^3}{3!}\, y'''_0 + \frac{(x - x_0)^4}{4!}\, y''''_0 + \ldots\ldots..$$

Putting the value of x_0, y_0, $y'_0, y''_0, y'''_0, y''''_0, \ldots\ldots..$, we get

$$y(x) = 1 + (x - 0).(1) + \frac{(x - 0)^2}{2!}(2) + \frac{(x - 0)^3}{3!}(2) + \frac{(x - 0)^4}{4!}(2) + \ldots\ldots..$$

$$= 1 + x + \frac{2x^2}{2!} + \frac{2x^3}{3!} + \frac{2x^4}{4!} + \ldots\ldots..$$

Now put x=0.1 and taking up to fourth terms, we get

$$y(0.1) = 1 + 0.1 + (0.1)^2 + \frac{2}{6}(0.1)^3 + \frac{2}{24}(0.1)^4$$

$$= 1 + 0.1 + (0.1)^2 + \frac{1}{3}(0.1)^3 + \frac{1}{12}(0.1)^4$$

$$=1+0.1+0.01+\frac{1}{3}(0.001)+\frac{1}{12}(0.0001)$$

$$=1+0.1+0.01+0.00033+0.0000083$$

$$=1.1103383.$$

13 Summary

Numerical solutions of ODEs play a crucial role in simulating dynamic systems and understanding their behavior. The choice of method depends on factors such as the nature of the problem, desired accuracy, stability, and computational efficiency. Euler's method is a simple and intuitive approach for solving ODEs, its efficiency and accuracy might be limited, especially for complex problems. For higher accuracy and faster convergence, more sophisticated numerical methods should be considered.

The Euler's method is

$$y_{n+1}=y_n+h\,f(x_n,y_n)$$

The Euler's modified method is

$$y_1^{(n+1)}=y_0+\frac{h}{2}[f(x_0,y_0)+f(x_1,y_1^{(n)})]\quad (n=0,1,2....)$$

The Taylor Series method is

$$y(x)=y_0+(x-x_0)\,y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+........$$

13 Terminal Questions

Q.1. Explain the Euler's method.

Q.2. Which method give the amore appropriate result in Euler's method and Euler's modified method.

Q.3. Write the Taylor Series formula.

Q.4. Use Euler's method compute the value of $y(0.04)$ for the differential equation $\frac{dy}{dx} = -y$ with $y = 1$ at $x = 0$.

Q.5. Using Euler's modified method, compute $y(2)$ in steps of 0.2 given that $\frac{dy}{dx} = 2 + \sqrt{xy}$ with $y(1) = 1$.

Q.6. Using Taylor's series method to compute $y(2.1)$ correct to 5 decimal places, where $x\frac{dy}{dx} = x - y$ with $y(2)=2$.

Answer

4. -0.6705 5. $y(2) = 5.0516$ 6. 2.00238125.

Suggested Further Readings:

1. Atkinson, K. and Han, W. Theoretical Numerical Analysis, Springer Science & Business Media, 2010.
2. Jain, M.K., Iyengar, S.R.K and Jain, R.K.: Numerical Methods for Scientific and Engineering Computations, NewAge International (P) Ltd. New Delhi, 2014.
3. Sastry, S.S.: Introductory Methods of Numerical Analysis, UBS Publishers, 2012.
4. Bradie, B. A friendly introduction to Numerical Analysis. Pearson Education, 2007.
5. Gupta. R. S., Elements of Numerical Analysis, 2nd Edition, Cambridge University Press, 2015.

UNIT-14: Numerical Solution of Ordinary Differential Equations-II

Structure

14.1 Introduction

Numerical solutions of ordinary differential equations (ODEs) are essential for simulating dynamic systems and gaining insights into their behavior. The selection of a specific numerical method is influenced by various factors, each playing a crucial role in determining the most suitable approach for a given problem. Picard's Method, also known as the Picard Iteration or the Method of Successive Approximations, is an iterative numerical technique used for solving ordinary differential equations (ODEs), particularly initial value problems. Named after the French mathematician Émile Picard, this method involves constructing a sequence of successive approximations to converge towards the true solution. Indeed, the Runge-Kutta methods, and particularly the fourth-order Runge-Kutta method (RK4), are widely regarded as some of the most commonly used and versatile numerical techniques for solving ordinary differential equations (ODEs).

The popularity of RK4 is attributed to its balance between accuracy and simplicity, making it suitable for a broad range of applications. Milne's Predictor-Corrector Method is a numerical technique used for solving ordinary differential equations (ODEs), particularly for initial value problems. It belongs to the family of predictor-corrector methods, where an initial prediction of the solution is refined iteratively to improve accuracy.

Picard's method of successive approximation; Runge-Kutta method and Milne's predictor-corrector methods are discussed in this unit.

14.2 Objectives

After reading this unit the learner should be able to understand about:

- the Picard Method
- the Runge-Kutta Method for fourth order
- the Milne's Predictor-Corrector Method

14.3 Picard's Method

Picard's Method is based on the concept of iterative refinement. It begins with an initial guess for the solution and successively refines the approximation through a series of iterations. The method is implemented by performing iterative calculations using the recurrence relation until the desired level of accuracy is achieved. It often requires the solution of integral equations. Let us consider the differential equation

$$y' = \frac{dy}{dx} = f(x, y) \qquad \text{....(1)}$$

with the initial condition $y = y_0$ for $x = x_0$ i.e., $y(x_0) = y_0$.

Integrating the differential equation (1), we get

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx \qquad \text{...(2)}$$

Equation (2) in which the unknown function y appears under the integral sign, is called an integral equation. In this method, the first approximation $y^{(1)}$ is obtained by replacing y by y_0 in $f(x, y)$ in right hand side of (2) and integrating with respect to x, we get

i.e., $$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx \qquad \text{...(3)}$$

The second approximation $y^{(2)}$ is determined by replacing y by $y^{(1)}$ in $f(x, y)$ in right hand side of (2) and integrating with respect to x, we get

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx \qquad \text{....(4)}$$

The third approximation $y^{(3)}$ is determined by replacing y by $y^{(2)}$ in $f(x, y)$ in right hand side of (3) and integrating with respect to x, we get

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})dx \qquad \text{....(5)}$$

Proceeding in the same way we obtain $y^{(3)}, y^{(4)}, \ldots\ldots\ldots\ldots, y^{n-1}$ and y^n where

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})dx$$

With $\quad y^{(0)} = y_0$

Repeat this steps till whenever upto the two value of y becomes same to the desired degree of accuracy.

Examples

Example.1. Use Picard's method to solve $\frac{dy}{dx} = x - y$ for $x = 0.1$ **and 0.2 given that** ***y*****=1 when** ***x*** **= 0.**

Solution: It is given that

$$f(x, y) = x - y \quad \text{and} \quad x_0 = 0, y_0 = 1 \qquad \ldots(1)$$

The first approximation is

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= y_0 + \int_{x_0}^{x} (x - y_0)dx$$

$$= 1 + \int_{0}^{x} (x-1)\,dx$$

$$= 1 + \left[\frac{x^2}{2} - x\right]_0^x$$

$$= \frac{x^2}{2} - x + 1$$

The second approximation is

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})dx$$

$$= y_0 + \int_{x_0}^{x} (x - y^{(1)})dx$$

$$= 1 + \int_0^x \left[x - \left(\frac{x^2}{2} - x + 1 \right) \right] dx$$

$$= 1 + \int_0^x \left(2x - \frac{x^2}{2} - 1 \right) dx$$

$$= 1 + \left[x^2 - \frac{x^3}{6} - x \right]_0^x$$

$$= 1 + x^2 - \frac{x^3}{6} - x$$

$$= -\frac{x^3}{6} + x^2 - x + 1$$

The third approximation is

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)})dx$$

$$= y_0 + \int_{x_0}^{x} (x - y^{(2)})dx$$

$$= 1 + \int_0^x \left(x + \frac{x^3}{6} - x^2 + x - 1 \right) dx$$

$$= 1 + \left(x^2 + \frac{x^4}{24} - \frac{x^3}{3} - x \right)_0^x$$

$$= \frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1$$

The fourth approximation is

$$y^{(4)} = y_0 + \int_{x_0}^{x} f(x, y^{(3)}) dx$$

$$= y_0 + \int_{x_0}^{x} (x - y^{(3)}) dx$$

$$= 1 + \int_0^x \left(x - \left(\frac{x^4}{24} - \frac{x^3}{3} + x^2 - x + 1 \right) \right) dx$$

$$= 1 + \int_0^x \left(2x - \frac{x^4}{24} + \frac{x^3}{3} - x^2 - 1 \right) dx$$

$$= 1 + \left[x^2 - \frac{x^5}{120} + \frac{x^4}{12} - \frac{x^3}{3} - x \right]_0^x$$

$$= -\frac{x^5}{120} + \frac{x^4}{12} - \frac{x^3}{3} + x^2 - x + 1$$

The fifth approximation is

$$y^{(5)} = y_0 + \int_{x_0}^{x} f(x, y^{(4)}) dx$$

$$= y_0 + \int_{x_0}^{x} (x - y^{(4)}) dx$$

$$= 1 + \int_0^x \left(x - \left(-\frac{x^5}{120} + \frac{x^4}{12} - \frac{x^3}{3} + x^2 - x + 1 \right) \right) dx$$

$$=1+\int_0^x\left(2x+\left(\frac{x^5}{120}-\frac{x^4}{12}+\frac{x^3}{3}-x^2-1\right)\right)dx$$

$$=1+\left(x^2+\frac{x^6}{720}-\frac{x^5}{60}+\frac{x^4}{12}-\frac{x^3}{3}-x\right)_0^x$$

$$=\frac{x^6}{720}-\frac{x^5}{60}+\frac{x^4}{12}-\frac{x^3}{3}+x^2-x+1$$

When x =0.1, we have

$$y_0=1,\ y^{(1)}=0.905,\ y^{(2)}=0.9098,\ y^{(3)}=0.90967,\ y^{(4)}=0.90967.$$

Hence $y=0.90967$ at $x=0.1.$

When x = 0.2, we have

$$y_0=1,\ y^{(1)}=0.82,\ y^{(2)}=0.83867,\ y^{(3)}=0.83740,\ y^{(4)}=0.83746,\ \ y^{(5)}=0.83746\ .$$

Hence $y=0.83746$ at $x=0.2.$

Example.2. Apply Picard's method to solve $\frac{dy}{dx}=x+y^2$ **given that when** $x_0=0,\ y_0=0$ **up to third order of approximation.**

Solution. It is given that

$$\frac{dy}{dx}=f(x,y)=x+y^2 \text{ and } x_0=0, y_0=0 \qquad \ldots(1)$$

The first approximation is

$$y^{(1)}=y_0+\int_{x_0}^{x}f(x,\ y_0)dx$$

$$= y_0 + \int_{x_0}^{x} (x + y_0^2) dx$$

$$= 0 + \int_{0}^{x} (x + 0) dx$$

$$= \left[\frac{x^2}{2} \right]_0^x$$

$$= \frac{x^2}{2}$$

The second approximation is

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)}) dx$$

$$= y_0 + \int_{x_0}^{x} (x + (y^{(1)})^2) dx$$

$$= 0 + \int_{x_0}^{x} \left(x + \left(\frac{x^2}{2} \right)^2 \right) dx$$

$$= \int_{0}^{x} \left(x + \frac{x^4}{4} \right) dx$$

$$= \left(\frac{x^2}{2} + \frac{x^5}{20} \right)_0^x$$

$$= \frac{x^2}{2} + \frac{x^5}{20}$$

The third approximation is

$$y^{(3)} = y_0 + \int_{x_0}^{x} f(x, y^{(2)}) dx$$

$$= y_0 + \int_{x_0}^{x} [x + y^{(2)})^2]dx$$

$$= 0 + \int_0^x \left(x + \left(\frac{x^2}{2} + \frac{x^5}{20} \right)^2 \right) dx$$

$$= \int_0^x \left(x + \frac{x^4}{4} + \frac{x^{10}}{400} + \frac{x^7}{20} \right) dx$$

$$= \left(\frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \right)_0^x$$

$$= \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

14.4 Runge-Kutta Method For Fourth Order

Runge-Kutta methods are a family of numerical techniques commonly used for solving ordinary differential equations (ODEs). These methods provide a systematic way to approximate the solution of ODEs with improved accuracy compared to simple methods like Euler's Method. The most widely used among them is the fourth-order Runge-Kutta method (RK4). This method is most commonly used method and most suitable when computation of higher derivatives is complicated. Runge-Kutta methods involve using weighted averages of function values at different points within each step to obtain more accurate approximations of the solution.

Consider the following differential equation.

$$\frac{dy}{dx} = f(x, y) \text{ with } y(x_0) = y_0$$

Runge-Kutta method of order four is given by

$$y_{n+1} = y_n + k \ \ for \ \ x = x_0 + h$$

Where $k = \frac{h}{6}\left[k_1 + \frac{4[k_2 + k_3]}{2} + k_4\right]$

$$= \frac{h}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

Where $k_1 = f(x_0, y_0)$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + k_1 \frac{h}{2}\right)$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + k_2 \frac{h}{2}\right)$$

$$k_4 = f\left(x_0 + h, y_0 + k_3 h\right)$$

Runge-Kutta methods, particularly fourth-order Runge-Kutta method (RK4), are powerful tools in the numerical solver toolkit. Their combination of accuracy, versatility, and ease of implementation makes them a preferred choice for many applications where the computation of higher derivatives might be challenging or impractical.

Check your Progress

1. What do you mean by Picard's method?

2. Write the Runge Kutta method formula for fourth order.

Examples

Example.3. Use Runge-Kutta method to solve $\frac{dy}{dx} = x$ for $x = 1.2, 1.4,$ initially $x = 1, y = 2.$

Solution: It is given that

$$f(x,y)=\frac{dy}{dx}=xy, \quad x_0=1,\ y_0=2.$$

Then we have

$$f(x_0,y_0)=1\times 2$$

$$=2$$

Assume $h=0.2$. Then we have

$$k_1=f(x_0,y_0)$$

$$=2$$

$$k_2=f\left(x_0+\frac{h}{2},y_0+k_1\frac{h}{2}\right)$$

$$=\left(x_0+\frac{h}{2}\right)\left(y_0+k_1\frac{h}{2}\right)$$

$$=\left(1+\frac{0.2}{2}\right)\left(2+2\times\frac{0.2}{2}\right)$$

$$=(1.1)(2.2)$$

$$=2.42$$

$$k_3=f\left(x_0+\frac{h}{2},y_0+k_2\frac{h}{2}\right)$$

$$=\left(x_0+\frac{h}{2}\right)\left(y_0+k_2\frac{h}{2}\right)$$

$$= \left(1 + \frac{0.2}{2}\right)\left(2 + 2.42 \times \frac{0.2}{2}\right)$$

$$= (1.1)(2.242)$$

$$= 2.4662$$

$$k_2 = f(x_0 + h, y_0 + k_3 h)$$

$$= (x_0 + h)(y_0 + k_3 h)$$

$$= (1 + 0.2)(2 + 2.4662 \times 0.2)$$

$$= (1.2)(2.49324)$$

$$= 2.9918$$

Now we have

$$k = \frac{h}{6}[k_1 + 2(k_2 + k_3) + k_4]$$

$$= \frac{0.2}{6}[2 + 2(2.42 + 2.4662) + 2.9918]$$

$$k = 0.49214$$

Therefore we have

$$x_1 = x_0 + h$$

$$= 1 + 0.2$$
$$= 1.2$$

and $y_1 = y_0 + k$

$$= 2 + 0.49214$$

$$= 2.4921$$

Hence $y(1.2) = 2.4921.$

Now for the second interval, we have

$$x_1 = 1.2,\ y_1 = 2.4921,\ f(x, y) = xy.$$

Now we have

$$f(x_1, y_1) = x_1 y_1$$

$$= 1.2 \times 2.4921$$

$$= 2.99052$$

Assume $h = 0.2.$ Then we have

$$k_1 = f(x_1, y_1)$$

$$= 2.99052$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + k_1 \frac{h}{2}\right)$$

$$= \left(x_1 + \frac{h}{2}\right)\left(y_1 + k_1 \frac{h}{2}\right)$$

$$= (1.2 + 0.1)\left(2.4921 + 2.9905 \times \frac{0.2}{2}\right)$$

$$= (1.3)(2.79105)$$

$$= 3.6283$$

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + k_2 \frac{h}{2}\right)$$

$$= \left(x_1 + \frac{h}{2}\right)\left(y_1 + k_2 \frac{h}{2}\right)$$

$$= \left(1.2 + \frac{0.2}{2}\right)\left(2.4921 + 3.6283 \times \frac{0.2}{2}\right)$$

$$= (1.3)(2.8548)$$

$$= 3.71143$$

$$k_4 = f(x_1 + h, y_1 + k_3 h)$$

$$= (x_1 + h)(y_1 + k_3 h)$$

$$= (1.2 + 0.2)(2.4921 + 3.71128 \times 0.2)$$

$$= (1.4)(3.2343)$$

$$= 4.5281$$

Now we have

$$k = \frac{h}{6}[(k_1 + 2(k_2 + k_3) + k_4]$$

$$= \frac{0.2}{6}[2.9905 + 2(3.6283 + 3.7114) + 4.5281]$$

$$= 0.73992$$

Therefore we have

$$x_2 = x_1 + h$$

$$= 1.2 + 0.2$$

$$= 1.4$$

and $y_2 = y_1 + k$

$$= 2.4921 + 0.73992$$

$$= 3.2330$$

Hence $y(1.4) = 3.2321$.

Example.4. Solve the equation $\frac{dy}{dx} = -2xy^2$ with initial condition *y*(0) = 1 by Runge-Kutta's method for *x* = 0.2 and 0.4 with *h* = 0.2.

Solution: It is given that

$$\frac{dy}{dx} = -2xy^2$$

Then we have

$$f(x_0, y_0) = -2x_0 y_0^2$$

$$= -2(0)(1)^2$$

$$= 0$$

Assume $h = 0.2$. Then we have

$$k_1 = f(x_0, y_0)$$

$$= 0.$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + k_1 \frac{h}{2}\right)$$

$$= -2\left(x_0 + \frac{h}{2}\right)\left(y_0 + k_1 \frac{h}{2}\right)^2$$

$$= -2\left(0 + \frac{0.2}{2}\right)\left(1 + 0 \times \frac{0.2}{2}\right)^2$$

$$= -2(0.1)(1)$$

$$= -0.2$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + k_2\frac{h}{2}\right)$$

$$= -2\left(x_0 + \frac{h}{2}\right)\left(y_0 + k_2\frac{h}{2}\right)^2$$

$$= -2\left(0 + \frac{0.2}{2}\right)\left(1 + (-0.2) \times \frac{0.2}{2}\right)^2$$

$$= -2(0.1)(0.98)^2$$

$$= -0.1920$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$= -2(x_0 + h)(y_0 + k_3 h)^2$$

$$k_4 = f(x_0 + h, y_0 + k_3 h)$$

$$= -2(x_0 + h)(y_0 + k_3 h)^2$$

$$= -2(0 + 0.2)[1 + (-0.1920)(0.2)]^2$$

$$= -0.36986$$

Now we have

$$k = \frac{h}{6}[(k_1 + 2(k_2 + k_3) + k_4]$$

$$= \frac{0.2}{6}[0 + 2((-0.2) + (-0.1920)) + (-0.36986)]$$

$$= \frac{0.2}{6}(-1.15386)$$

$$= -0.03846$$

Therefore we have

$$x_1 = x_0 + h$$

$$= 0 + 0.2$$
$$= 0.2$$

and $y_1 = y_0 + k$

$$= 1 + (-0.03846)$$

$$= 0.96154$$

Hence $y(0.2) = 0.96154.$

Now for the second interval, we have

$$x_1 = 0.2,\ y_1 = 0.9615,\ f(x, y) = -2xy^2$$

Now we have

$$f(x_1, y_1) = -2x_1 y_1^2$$

$$= -2 \times (0.2) \times (0.9615)^2$$

$$= -0.3697929.$$

Assume $h = 0.2.$ Then we have

$$k_1 = f(x_1, y_1)$$

$$= -0.3697929.$$

$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + k_2\frac{h}{2}\right)$$

$$= -2\left(x_1 + \frac{h}{2}\right)\left(y_1 + k_1\frac{h}{2}\right)^2$$

$$= -2\left(0.2 + \frac{0.2}{2}\right)\left(0.9615 + \left(-0.36979 \times \frac{0.2}{2}\right)\right)^2$$

$$= -2(0.3)(0.85473)$$

$$= -0.51284$$

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + k_2\frac{h}{2}\right)$$

$$= -2\left(x_1 + \frac{h}{2}\right)\left(y_1 + k_2\frac{h}{2}\right)^2$$

$$= -2\left(0.2 + \frac{0.2}{2}\right)\left(0.9615 + \left(-0.51284 \times \frac{0.2}{2}\right)\right)^2$$

$$= -2(0.3)(0.82849)$$

$$= -0.49709$$

$$k_4 = f(x_1 + h, y_1 + k_3 h)$$

$$= -2(x_1 + h)(y_1 + k_3 h)^2$$

$$= -2(0.2 + 0.2)(0.9615 + (-0.49709) \times 0.2)^2$$

$= -2(0.4)(0.7431)$

$= -0.59454$

Now we have

$$k = \frac{h}{6}[(k_1 + 2(k_2 + k_3) + k_4]$$

$$= \frac{0.2}{6}(-0.36979 + 2(-0.51284 - 0.49709) - 0.59454)$$

$$= -0.099473$$

Therefore we have

$$x_2 = x_1 + h$$

$$= 0.2 + 0.2$$

$$= 0.4$$

and $y_2 = y_1 + k$

$$= 0.9615 - 0.99473$$

$$= 0.86202$$

Hence $y(0.4) = 0.86202$.

14.5 Milne’s Predictor-Corrector Method

Milne's method combines both prediction and correction steps to iteratively refine the solution of an ODE. It uses a third-order Adams-Bashforth predictor and a fourth-order Adams-Moulton corrector. Milne's Predictor-Corrector Method is a combination of third and fourth-order methods, providing higher accuracy compared to some lower-order methods. This contributes to improved numerical stability. If we solve the differential equation $\frac{dy}{dx} = f(x, y)$ with $y(x_0) = y_0$ by this method, we first obtain the approximate value of y_{n+1} by predictor formula and then improve this value by means of a corrector formula.

The predictor formula is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n)$$

The Corrector formula is

$$y_{n+1}^{(1)} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

which improve that predicted value.

The method requires the calculation of predictor and corrector values at each step. While more involved than simpler methods, it is still relatively straightforward to implement.

Examples

Example.5. Compute $y(2)$, if $y(x)$ is the solution of $\frac{dy}{dx} = \frac{1}{2}(x + y)$ assuming *y*(0) = 2, *y*(0.5) = 2.636, *y*(1) = 3.595, *y*(1.5)= 4.968.

Solution. It is given that

$$f(x, y) = \frac{dy}{dx} = \frac{1}{2}(x + y)$$

and the values assuming $y(0) = 2$, $y(0.5) = 2.636$, $y(1) = 3.595$, $y(1.5) = 4.968$.

$x_0 = 0$	$y_0 = 2$	$y'_0 = \frac{1}{2}(0+2) = 1$
$x_1 = 0.5$	$y_1 = 2.636$	$y'_1 = \frac{1}{2}(0.5+2.636) = 1.568$
$x_2 = 1$	$y_2 = 3.595$	$y'_2 = \frac{1}{2}(1.5+3.595) = 2.297$
$x_3 = 1.5$	$y_3 = 4.968$	$y'_3 = \frac{1}{2}(1.5+4.968) = 3.234$

Using the predictor formula, we have

$$y_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 2 + \frac{4}{3}(0.5)[2 \times 1.568 - 2.2975 + 2 \times 3.234]$$

$$= 2 + \frac{2}{3}[7.3065)$$

$$= 6.871$$

Now we have

$$y'_4 = \frac{1}{2}(x_4 + y_4)$$

$$= \frac{1}{2}(2 + 6.871)$$

$$= 4.4355$$

Now using the corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 3.595 + \frac{0.5}{3}[2.2975 + 4 \times 3.234 + 4.4355]$$

$$= 3.595 + \frac{0.5}{3}[19.669]$$

$$= 6.873166 \approx 6.8732$$

Now we have

$$y_4' = \frac{1}{2}(x_4 + y_4)$$

$$= \frac{1}{2}(2 + 6.8732)$$

$$= 4.4366$$

Again using the Corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 3.595 + \frac{0.5}{3}[2.2975 + 4 \times 3.234 + 4.4366]$$

$$= 3.595 + \frac{0.5}{3}[19.6701]$$

$= 6.87335 \approx 6.8734$

Hence the value of $y(2) = 6.8734$.

Example.6. Solve initial value problem $\frac{dy}{dx} = 1 + xy^2, y(0) = 1, h = 0.1$ for $x = 0.4$ **by using Milne's method when it is given.**

x	**0.1**	**0.2**	**0.3**
y	**1.105**	**1.223**	**1.355**

Solution:

It is given that

$$\frac{dy}{dx} = 1 + xy^2, y(0) = 1, h = 0.1 \text{ for } x = 0.4.$$

$x_0 = 0$	$y_0 = 1$	$y_0' = 1 + 0 \times 1^2 = 1$
$x_1 = 0.1$	$y_1 = 1.105$	$y_1' = 1 + (0.1)(1.105)^2 = 1.1221$
$x_2 = 0.2$	$y_2 = 1.223$	$y_2' = 1 + (0.2)(1.223)^2 = 1.2991$
$x_3 = 0.3$	$y_3 = 1.355$	$y_3' = 1 + (0.3)(1.355)^2 = 1.5508$

Using the predictor formula, we have

$$y_4 = y_0 + \frac{4h}{3}[2y_1' - y_2' + 2y_3']$$

$$= 1 + \frac{4\times(0.1)}{3}[2\times1.1221 - 1.2991 + 2\times1.5508]$$

$$= 1 + \frac{0.4}{3}[4.0467]$$

$$= 1.53956 \approx 1.539$$

Now we have

$$y_4' = 1 + x_4 y_4^2$$

$$= 1 + (0.4)(1.539)^2$$

$$= 1.9474$$

Now using the corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}[y_2' + 4y_3' + y_4']$$

$$= 1.223 + \frac{0.1}{3}[1.2991 + 4\times1.5508 + 1.9474]$$

$$= 1.223 + \frac{0.1}{3}[9.4497]$$

$$= 1.53799$$

Now we have

$$y_4' = 1 + x_4 y_4^2$$

$$=1+(0.4)(1.538)^2$$

$$=1.9461$$

Again using the Corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$=1.223+\frac{0.1}{3}[1.2991+4\times 1.5508+1.9461]$$

$$=1.223+\frac{0.1}{3}[9.4497]$$

$$=1.53799 \cong 1.538$$

Hence the value of $y(0.4)=1.538.$

14.6 Summary

Numerical solution of ODEs involves a thoughtful consideration of the specific problem's characteristics, accuracy requirements, stability constraints, and computational efficiency. The choice of method should align with the unique features of the ODE and the goals of the simulation or analysis.

The Picard's Method is

$$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{n-1})dx \quad \text{with} \quad y^{(0)} = y_0 .$$

The Runge-Kutta's Method for fourth order is

$$y_{n+1} = y_n + k \quad \text{for} \quad x = x_0 + h$$

Then $k = \frac{h}{6}\left[k_1 + \frac{4[k_2 + k_3]}{2} + k_4\right] = \frac{h}{6}[k_1 + 2(k_2 + k_3) + k_4]$

Where $k_1 = f(x_0, y_0)$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + k_1\frac{h}{2}\right)$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + k_2\frac{h}{2}\right)$$

$$k_4 = f\left(x_0 + h, y_0 + k_3 h\right)$$

The Milne's Predictor-Corrector Method is

$$y_{n+1} = y_{n-3} + \frac{4h}{3}(2y'_{n-2} - y'_{n-1} + 2y'_n) \text{ and } y^{(1)}_{n+1} = y_{n-1} + \frac{h}{3}(y'_{n-1} + 4y'_n + y'_{n+1})$$

14.7 Terminal Questions

Q.1. Write the solution procedure of Picard's method for solving ordinary differential equation.

Q.2. Explain the Runge-Kutta's method for fourth order.

Q.3. What do you mean by Milne's Predictor-Corrector Method.

Q.4. Use Picard's method to solve $\frac{dy}{dx} = y - x$ with y = 2 when x = 0 up to third order of approximation.

Q.5. Solve the equation $\frac{dy}{dx} = x + y$ with initial condition y(0) =1 by Runge – Kutta's rule from x = 0 to x = 0.4 with h = 0.1.

Q.6. Use Milne's method to solve $\frac{dy}{dx} = x + y$ with initial condition y(0) =1 from x = 0.20 to x = 0.30.

Answer

4. $y^{(3)} = -\frac{x^4}{24} - \frac{x^3}{6} + \frac{x^2}{2} + 2x + 2$

5. $y(0.1) = 1.1103, y(0.2) = 1.2428, y(0.3) = 1.3997, y(0.4) = 1.5836.$

6. $(y)_{x=0.20} = 1.2428$ and $(y)_{x=0.30} = 1.3997.$

Suggested Further Readings:

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