Uttar Pradesh Rajarshi Tandon Open University

Master of Business Administration

BBA-117

Operations Research

॥ सरस्वती नः सुभगा मयस्करत् ॥

Uttar Pradesh Rajarshi Tandon
Open University

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BLOCK

1

INTRODUCTION TO OPERATION RESEARCH

UNIT-1

Operation Research : An Overview

UNIT-2

Review of Probability and Statistics

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BLOCK INTRODUCTION

In block 1 you will learn about the introductory part of operations research. This block explains the meaning of operations research, its historical background, features and scope etc. In this block, the role of probability and statistics in operation research is also explained.

There are two units in this block.

Unit-1- It discusses about meaning of operations research, history of OR, phases of OR, and its scope.

Unit-2- It explains the meaning of statistics and basic statistical tools used in decision making. Unit also discusses about the concepts of probability, different theorems of probability and role of probability and statistics in OR.

UNIT 1 OPERTAION RESEARCH : AN OVERVIEW

Structure

1.1 OBJECTIVES

After completing this unit, you will be able to:

- To explain the meaning of Operations Research
- To understand historic development of OR
- To Describe the techniques of O.R
- To explain the Applications of Operations Research
- To describe the Limitations of Operation Research
- To understand the OR specialist and Manager relationship

1.2 INTRODUCTION

Operation Research (OR) also called as 'Management Science' or 'Decision Science' is the science that deals with decision making. It uses mathematical models and statistics to help in decision making.

Traditionally, people considered decision making to be an art that is learned over a period of time with experience. But due to the complex and changing environment, there was a need to supplement the art of decision making with scientific and systematic approach. Businesses today have to take complicated decisions everyday and cannot use only common sense or thumb rule for taking these decisions as the cost of making error may be too high. OR allows decision makers to base their decisions on data driven and scientific methods.

OR is a area of study that developed after World War II, when the failures of military operations were very high. At that time, Scientists formed a team to study the problems arising due to difficult situations and developed solutions to these problems. OR helps in determining the most efficient way to solve a problem. It is used to analyze complex real life problems with the objective of optimizing performance. The tools of operations research have come from various disciplines such as statistics, mathematics, economics, psychology, engineering etc. Today OR is a separate discipline that deals with the various optimization techniques to solve a problem. The purpose of OR is to provide a rational basis for making decisions. It can be seen as a science to describe, understand, and predict the behaviour of systems.

1.3 HISTORY OF OPERATIONS RESEARCH

The history of operations research can be traced back since the time of industrial revolution but OR in its present form, was developed during the second world war when the British military management formed a team of scientists to examine the strategies of military operations so that allocation of scarce resources may be done effectively. During this period, many new scientific techniques were developed for military operations ans therefore the name 'operations research' was given to this area.

The effectiveness of OR techniques in military operations attracted the attention of other government departments and industries towards this field. Today almost all organizations make use of OR techniques for decision making at all levels. The growth of operations research was not limited to the U.S.A. and U.K., today it has reached many countries of the world.

Most of the OR models require complex computation, large data handling and computation time to provide solutions, therefore until the year 1980, OR remained the domain of mathematicians and statisticians, Practicing managers did not have much confidence in solutions due to the lack of understanding. Only after the development of high-speed digital computers, managers were able to apply OR techniques to business problems. The advent of digital computers contributed much to the growth of OR.

Many Indian organizations also started using OR techniques. In India, Regional Research Laboratory located at Hyderabad was the first Operations Research unit established during 1949. At the same time another unit was set up in Defense Science Laboratory to solve the Stores, Purchase and Planning Problems. In 1953, Operations Research unit was established in Indian Statistical Institute, Calcutta, with the objective of using Operations Research methods in

National Planning and Survey. Today, Railways, Defence, Indian Airline, Fertilizer Corporation of India, Delhi Cloth Mills, Tata Steel and many other companies are using techniques of OR for decision making.

1.4 MEANING AND FEATURES OF OPERATIONS RESEARCH

In general, Operations Research can be defined as the application of scientific methods to decision making. It provides a systematic approach to the decision making problems. This approach requires quantification of parameters and objectives, economic and statistical modeling, quantitative and logical formulation of model, mathematical methods to olve formulated problems and provide quantitative answers for decision making. It generally involves optimum utilization of resources.

Churchman et al. gave the classical definition of OR as, "Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of systems so as to provide those in control of operations with optimum solutions to the problems."

Some other definitions of OR given by different authors are mentioned below:-

1. According to the Operational Research Society of Great Britain (OPERATIONAL RESEARCH QUARTERLY, l3 (3):282, l962), Operational Research is the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defense. Its distinctive approach is to develop a scientific model of the system, incorporating measurements of factors such as change and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically.

2. Randy Robinson defined OR as the application of scientific methods to improve the effectiveness of operations, decisions and management. By means such as analyzing data, creating mathematical models and proposing innovative approaches, Operations Research professionals develop scientifically based information that gives insight and guides decision-making.

3. According to Morse and Kimball, "O.R. is a quantitative approach and described it as a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control".

4. According to Saaty, "O.R. is a tool of improving quality of answers. O.R. is the art of giving bad answers to problems which otherwise have worse answers".

5. Miller and Starr defined OR as the applied decision theory, which uses any scientific, mathematical or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a thorough-going rationality in dealing with his decision problem”.

6. Pocock says that O.R. is an applied Science. He states “O.R. is scientific methodology (analytical, mathematical, and quantitative) which by assessing the overall implication of various alternative courses of action in a management system provides an improved basis for management decisions”

1.4.1 FEATURES OF OPERATIONS RESEARCH

The significant features of operations research are explained below:-

1. **Problem Solving-** Operations Research is mainly provides solution to various decision making problems. It follows a general systematic process of decision making in which first the problem is defined and objectives are established, second the various alternative courses of action are considered, next the model for decision making is developed and values for parameters of the process are determined, finally alternatives are evaluated and the optimal one is chosen as the final solution of the problem.

2. **Scientific Approach-** Operations Research uses a scientific approach for solving problems. It does not use intuitions, and whims rather it follows a formalized process of logic and reasoning to solve the problems. Following steps are used to solve any problem:

   a. Problem is clearly defined and various environmental conditions are determined.

   b. Observations take place under various environmental conditions to determine behaviour of system.

   c. A hypothesis is formulated on the basis of observations.

   d. Hypothesis formulated is tested using an experiment.

   e. Results of experiments are analysed and hypothesis is accepted/ rejected.

3. **Objective approach-** OR uses objective methods for finding the solutions. It first help in defining the objectives and then alternative courses of action are identified on the basis of goals, various measures are used to compare these alternatives and finally the best alternative is selected.

4. **Inter- Disciplinary-** Operations Research is inter-disciplinary in nature and requires a team effort to find the solution of a problem. Various problems may be related to economic, physical, psychological, biological, sociological and engineering areas. It requires blend of people with

expertise in various areas of mathematics, statistics, management, economics, computer science etc.

5. **Use of Computers-** In the current ecenario, use of digital computers has become integral part of operations research. Computers are required due to complexity of model, volume of data etc.

1.5 RELATIONSHIP BETWEEN MANAGER AND OR SPECIALIST

The key responsibility of managers is to take decisions. OR specialists help managers in taking the best decision. A manager recognizes the symptoms that indicate the existence of an y problem. OR specialist along with the manager identifies the relevant key variables to be studied in order to solve the problem. They also help in converting the problem in form of quantitative relationships among variables. Further it is the task of OR specialist to investigate methods for solving the problems and determine appropriate quantitative tools to be used. An OR specialist develops alternative solutions to the problems; finds various solutions; state assumptions underlying these solutions and tests alternative solutions. The manager determines which solution is most effective because of practical constraints within the organization and decides what the solution means for the organization. Ultimately the best solution is chosen by the manager. Thus both manager and OR specialist work together for solving problems.

1.6 OR MEHODOLOGY

The important feature of OR is that it uses mathematical models to analyse problems. This feature requires a scientific methodology to translate a real problem into a mathematical model which can be further solved to find the solution. The OR practioners have found that a number of problems reappear in different circumstances. However, their essential nature and features remain the same. Therefore, a common analytical methodology can be used to find the solution of problems belonging to same general category. The OR methodology consists of the following steps:-

a. Formulate the decision making problem.

b. Determine the assumptions and formulate a mathematical model for the problem.

c. Get the input data.

d. Solve the model formulated and interpret the results.

e. Validate the model.

f. Implement the solution obtained.

The steps may be overlapping also. The details of each step are as follows:-

a. Formulating the decision making problem- The first step is to define the problem clearly. The objectives and limitations of the decision making problems are also defined.

b. Determine the assumptions and formulate a mathematical model for the problem- In this step, the environment is analysed and the assumptions behind the problem are determined. A model is developed to represent the decision making problem. This is a mathematical model that represents system, process or environment in the form of mathematical equations. The proposed model is finally tested and modified to work under stated environmental conditions.

c. Get the input data- Or models need right kind of data in order to be tested. In this step, the internal-external data and facts are collected using computers and the input is provided to models to operate.

d. Solve the model formulated and interpret the results- In this step, the solution of problem is obtained with the help of model and input data.

e. Validate the model- This solution is further tested for optimality. If the solution is not optimal or the model is not working properly, it is revised and the end result is obtained which is desirable and supports the objectives of the organization.

f. Implement the solution obtained- This is the last step of OR methodology. In this step, the decision is implemented however the implementation involves many behavioural issues which are to be resolved by the implementation authority. A properly implemented solution obtained through OR techniques results in improved working conditions.

1.7 OR TECHNIQUES

Operations Research makes use of various techniques to solve any decision making problem. It not only uses mathematical procedures and cost analysis but also uses other techniques such as linear programming, game theory, decision theory, queuing theory, inventory models and simulation. Some other tools such as non-linear programming, integer programming, dynamic programming, sequencing theory, Markov process, network scheduling (PERT/CPM) etc.

Some of the important OR techniques are explained as below-

a. Linear Programming- is the most powerful tool of OR. Linear programming models are used to deal with the problem of making optimal allocation of scarce resources to complete production objective, determining transportation schedules, selecting plant location, assigning tasks to various personnel and machines, selection of media investment portfolio selection etc.

b. Decision Models- are related to decision making under uncertainty and risk. In business, a lot of uncertainty is involved, decision models make use of probability theory to calculate the probabilities of occurrence of various events which are further used to arrive at a suitable course of action.

c. Network theory- is related to the complexities involved in various projects. It helps managers in planning, analyzing and scheduling the network activites involved in a project. CPM/ PERT are network techniques that help in minimizing the cost and completion time of a project.

d. Inventory Control Models- are used to determine optimal timing and quantities for ordering items and the buffer stock of various items. These models help in minimizing ordering costs and holding costs of inventory.

e. Queuing Theory- is related to the management of waiting lines to determine the no. of service units required so as to minimize the waiting lines. It has wide use in solving problems of traffic congestion, job scheduling, hospital operations, bank operations etc.

f. Game Theory- is the modeling technique used for assessing the impact of competitor's decision on the pay-off. It deals with the situations where two or more persons are making decisions with conflicting interests. It is mainly used in decision making in competitive situations.

g. Simulation- is a widely used technique of OR in which the model is built to represent real situation and then observing the behavior. This technique allows to examine the consequences of decisions without risk of real life experimentation. If the result of simulation model indicate modification or improvement , the manager can confidently decide to implement such change in the real system.

h. Sequencing- help in solving the problems where sequence in which jobs should be performed with minimization of total efforts is to be determined.

i. Markov Process- are the models used in the situation where status of the system (states) can be defined by some descriptive measure of numerical value and where system moves from one state to another on a probability basis. These models are applied to analyse consumer buying patterns, to forecast bad debts, planning personnel needs etc.

j. Non-linear Programming- is used in problems where some or all variables are non linear. Most of the real life problems are non-linear.

k. Integer Programming- is used when the values of decision variables are restricted to integers such as in case of financial management.

l. Dynamic Programming- is used in the systems where optimal solutions to the problem may involve highly complex interrelations.

m. Goal Programming- is related to the problems having multiple objectives. It is similar to linear programming.

1.8 APPLICATIONS OF OPERATIONS RESEARCH

Operations Research has wide scope in various industries. In general we can say that wherever there is a problem of decision making, OR may help. Originally the field of OR was developed to be used in military but now OR is used in many organizations including business and industry. OR is widely applied in following fields-

n. In Defence- OR was initially used during world war II where OR teams of Britain and America developed techniques and strategies to win the war. OR helps the military executives and leaders to select the best strategies to win the battle.

o. In industry- Post world war II, industry adapted OR techniques for managing different resources to achieve success. With the increasing size of companies, it became difficult for leaders to manage them, therefore jobs were divided into parts and assigned to various persons/ machines. OR helps in managing various resources to achieve the profit objective.

p. In production – OR is useful in production of goods and services to minimize the cost of production. OR helps in designing and selecting locations, scheduling and sequencing production, allocation of machinery, and determining optimum product mix.

q. In Marketing- OR is useful in marketing to maximize the sales and minimize cost of sales. OR helps in determining the frequency of purchase, finding minimum sale price per unit, understanding customer's choice about colour, packing and size etc.

r. In field of finance- OR is useful in the field of finance to minimize the cost of capital. It helps in finding out the term capital requirements, determining a profit plan and optimum replacement policies.

s. In Agriculture- OR is useful in the field of agriculture for increasing agricultural output of a country. It helps in determining optimal distribution of water resources, maximum sales price and agricultural policies.

t. In planning- OR helps government in planning. It is useful for making policies economic planning and development such as optimum size of caravelle fleet of Indian Airlines.

u. In Research and Development- OR is useful in the field of R & D for determining areas of concentration for R &D, control of development projects. Determination of time and cost requirements etc.

Thus it can be concluded that OR is widely used in taking various types of managerial decisions. It is also used in Railways for resolving waiting problems of passengers on ticket windows, and in public transport to determine a suitable fare structure etc.

1.9 LIMITATIONS OF OPERATIONS RESEARCH

OR has certain limitations. These are explained below-

a. Problem of computation- OR attempts to find the optimal solution taking all factors in account. In real life problems, there may be enormous factors involved and it may not be possible to express them all in quantities.

b. Applied on quantitative variables- OR provides solution when all variables in a problem can be quantified. It does not take into account qualitative factors.

c. Requirement of mathematical knowledge- OR techniques require good knowledge of mathematical and statistical tools. Thus the business executives are required to have this background apart from business expertise. Since most of business executives do not possess these mathematical skills therefore, it is required to hire external OR experts.

d. Money and Time cost- Application of OR techniques is a time taking and costly activity.

e. Difficulty in implementation- Implement OR decisions is a complicated task as it does not take into account the complexities of human behavior and relations. So there can be resistance to implement the decisions.

1.10 SUMMARY

This unit explains the meaning, concept and techniques of Operations Research. The unit discusses about meaning and features of OR. In this unit, a brief history of OR and its use in various fields is discussed. The methodology of OR and OR techniques are briefly explained. The unit also covers the applications and limitations of OR.

1.11 SELF ASSESSMENT QUESTIONS

1. What is Operations Research?
2. Define OR and explain its various features.
3. How does a manager relate to an OR specialist?
4. Explain the methodology of operations research.
5. What are various OR techniques? Explain in brief.
6. Briefly explain the applications of OR.
7. What are various limitations of OR?
8. Discuss the essential features of operations research.

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UNIT 2 REVIEW OF PROBABILITY AND STATISTICS

Structure

2.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the fundamentals of probability and statistics and its use in OR.
- Understand the amount of uncertainty that is involved in decision making.
- Understand the fundamental of probability and various probability rules that help in calculating probability.
- Perform several analyses that help in decision making involving uncertainty.

2.2 INTRODUCTION

In our day to day life, we usually listen and use statements like: "the average price of the apples was Rs. 100 per kg", or "he travelled at an average speed of 50km/hr", or "the average return on investment A is 10%". In all these statements, the term average is used, which refers to the very important statistical tool. Therefore, it is very important to have a basic understanding of statistics to take our day to day decisions.

Similarly we also use the statements like: "Probably it will rain today", "it is likely that Mr. X will not come for this party", "Team A seems to have more chances to win this game", "It is possible that he will join us at 2'O'clock."All these words probably, likely, chances, possible sounds same and having similar meaning. In all these statements an element of uncertainty is associated. In layman's language the word 'Probability' thus connotes that there is uncertainty about the happening of the event.

In this unit, the review of basics of probability and statistics is given so that the other concepts of operations research may be clearly understood.

2.3 PROBABILITY DEFINED

The probability of a given event is an expression of likelihood or chance of occurrence of an event. A probability ranges from zero to one. Zero means event will not occur whereas one means event will definitely occur. Thus, the theory of probability provides a numerical measure of the element of uncertainty. It enables us to take decision under conditions of uncertainty with a calculated risk.

2.4 TERMINOLOGY USED IN PROBABILITY THEORY

Before starting the calculation of probability, it is important to understand few terms:

2.4.1 EVENT AND EXPERIMENT

The term experiment means an act which can be repeated under some given conditions. The term Random Experiment means, an experiment if conducted repeatedly under homogeneous conditions, does not give the same result. If an unbiased dice is thrown, any of six numbers on the dice can come up. Here, throwing the dice is an experiment.

The outcome of any random experiment is called an event. For exp- in throwing a dice, appearance of a number i.e. 1, 2, 3, 4, 5, or 6 is an event.

An event could be either simple or compound. If the possible outcome corresponding to an event is simple, it is known as simple event otherwise it is known as compound event. For example- in throwing a dice, probability of getting 4 is a simple event as there is only outcome 4 is on the dice, but getting an even no. as an outcome is a compound event as there are three possible even numbers (2, 4, 6) on a dice that can be the outcome.

2.4.2 EXHAUSTIVE CASES

In a random experiment, all possible outcomes are known as exhaustive cases. For example- In tossing a coin, total number of possible outcomes are two (Head or Tail). Thus exhaustive cases are two. Similarly in throwing a dice, total number of possible outcomes is six, so the exhaustive cases are six. However, if two dices are thrown simultaneously, total number of possible outcomes will be 6 *6 = 36.

2.4.3 FAVORABLE EVENT

In any random experiment, the total number of desired outcomes is known as favorable event. For example- in a game of cards, if one card is chosen at random from a deck of cards, the probability of getting a diamond card will be 13/52 as there are 13 diamond cards in one set. Hence total number of favorable events will be 13.

2.4.4 EQUALLY LIKELY EVENT

Events are said to be equally likely if the possibility of their happening is equal. For example- if we toss a coin, we can get either head or tail, occurrence of these two events is equally likely.

2.4.5 INDEPENDENT AND DEPENDENT EVENTS

If the happening of one event is totally free from the happening of other event, the events are called as independent events. For example- in tossing a coin, possibility of getting a head is independent of getting a tail. Similarly in playing cards, possibility of getting a king is equal in every experiment (i.e.4/52 as there are 4 kings in one set), if the cards are not replaced (total cards are 52 every time) after any experiment. But if the card is replaced after every experiment, then the possibility of getting a king is 3/51 as one king is replaced after an experiment so remaining kings in the set are three and total cards in set are 51 only. This event of getting a king in the second experiment is known as dependent event.

2.5 TYPES OF PROBABILITY

Probability can be defined in three ways-

2.5.1 Classical or a priori Probability

2.5.2 Relative frequency theory of probability

2.5.3 Subjective theory of probability

2.5.1 CLASSICAL OR A PRIORI PROBABILITY

This is the oldest and simplest way of defining probability. This approach is given by French Mathematician 'Laplace' in eighteenth century. According to this theory, probability is the ratio of favorable events to the possible events. Thus probability of occurring an event A is given as:

$$p(A) = a/n$$

Where 'a' is the no. of favorable events and n is the total no. of possible events. Therefore,

$$\text{Probability of an event} = \frac{\text{No. of favorable outcomes}}{\text{Total no. of possible outcomes}}$$

For example- In a random throw of a dice, the probability of getting a number 2 on the face of dice is - p (2) = 1/6 = 0.166 , as 1 is the no. of favorable outcome in one throw and total no. of possible outcomes in one throw are 6.

Thus, for calculating probability under this approach we must calculate:

(i) No. of favorable outcomes

(ii) Total number of possible outcomes

Another example of tossing a coin can be taken, in this case the probability of getting a Head is

$$p(H) = 1/2 = 0.5$$

Similarly In a random selection of a card from a deck of playing cards, the probability of getting a specific card (Ace of spade) is p (A) = 1/52

Classical probability is also called as priori probability because it assumes that various outcomes of an event are equally likely and the device with which the experiment is performed is a fair device, and so the probability of their happening is also equal. Thus, the theory determines the probability of an event in advance i.e. before or without an actual experiment is being performed.

Since this theory assumes all event are equally likely, hence this approach is applicable in situations like throwing a dice, tossing a coin, playing card selection etc. but in real life there are many conditions where occurrence of events are not equally likely. For example- in a person is suffering with a very serious disease, his probability of survival will not be 50 % since survival and death, i.e. the two mutually exclusive and exhaustive events are not equally likely.

Mathematically, it can be said that if there are 'a' possible outcomes are favorable and 'b' no of events are unfavorable, then probability of occurring favorable events will be:

$$\mathbf{p = a/(a + b)}$$

And, probability of non- occurrence will be:

$$\mathbf{q = b/(a + b)}$$

Here it should be noted that,

$$\mathbf{p + q = 1}$$

$$\text{or,} \quad \mathbf{p = 1 - q}$$

2.5.2 RELATIVE FREQUENCY THEORY OF PROBABILITY

In eighteenth century, British statisticians try to calculate risk of losses in life insurance & other commercial insurances. They defined probability from statistical data collection process of birth and death. This theoretical foundation of probability is known as relative frequency theory of probability.

This approach of probability is impossible to implement until you actually perform the experiment once. Secondly, this approach may not be able to explain certain cases.

For example- if a coin is tossed 10 times, it is possible that you get 7 heads & 3 tails. The probability of head is thud 0.7 and that of a tail is 0.3. However, if the same experiment is continued up to 100 or 1000 times, we should expect approximately equal no. of heads and tails. Thus as n (total trials) increases or tends towards ∞, we find that probability of all the events is equal. The probability of an event can thus be defined as the relative frequency with which it occurs indefinitely large no. of trials.

Thus, the probability of an event can be defined as the relative frequency with which it occurs in an indefinitely large no. of trials.

Thus for an event A,

$$p(A) = \text{Lim}_{n\to\infty} (a/n)$$

Where 'a' is the number of time an event occurred out of n trials, where n is large or approaches to infinity.

Hence,

$$\text{Probability} = \frac{\text{Relative frequency}}{\text{Total no. of trials}}$$

For example- If XYZ firm produces 1000 items per month and 10 items out of that are defective, then the probability of being an item defective would be 10/1000 i.e. 0.01.

2.5.3 SUBJECTIVE THEORY OF PROBABILITY

This theory of probability is subjective in nature as it depends on the belief of decision maker. The subjective probability is defined as the probability assigned to an event by any decision maker depending upon his/her knowledge or evidence with him.

For example- we have data related to share prices of XYZ Company for the last 1 year. Out of these 100 values share price of this company rise 40 times. According to classical theory the probability of price rise of this Company will be 40/100 = 0.4. Now this is an individual decision whether he/she will purchase share of this company or not. Many will think a rise in price where as many will be at a safer side and will not purchase share of this company. Since the decision taken by any individual reflects his/her personality, hence this theory is also known as Personality theory of probability.

Since most of the time top-level management decisions are concerned with specific conditions, rather than what happened in the past therefore decision makers at this level make considerable use of subjective probability.

2.5.4 ILLUSTRATION BASED ON PROBABILITY THEORY

Example 1- A bag containing 20 green and 50 yellow balls, a ball is drawn at random. What is the probability that it is a green ball?

Solution-

Total number if balls in the bag = 20 + 50 =70

Number of green ball in the bag = 20

Probability of getting a green ball

$$P(G) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

=20/70

= 2/7

Example 2- One card is drawn from a standard pack of 52. What is the probability that it is a king?

Solution-

Total number if cards in a standard pack = 52

Number of King in a standard pack = 4

Probability that the card drawn is a king

$$P(K) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

=4/52

= 1/13

Example 3- What is the probability of getting a head in tossing a coin?

Solution-

Total number of possible events = 2 (either a head or tail)

Number of favorable event = 1 (getting a head)

Probability of getting a head

$$P(H) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 1/2

Example 4- A dice is thrown, what is the probability of getting Number Six?

Solution-

Total number of possible events = 6 (since, a dice is having 1, 2, 3, 4, 5 & 6 coded on it)

Number of favorable event = 1(getting a no.6 only)

Probability of getting six

$$P(6) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 1/6

Example 5- One card is drawn from a standard pack of 52. What is the probability that it is a card of diamond?

Solution-

Total number if cards in a standard pack = 52

Number of diamond cards in a standard pack = 13

Probability that the card drawn is a diamond card

$$P(K) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

=13/52

= 1/4

2.6 THEOREMS OF PROBABILITIES

There are two important theorems of probabilities, namely:

- The addition theorem, and
- The multiplication theorem

2.6.1 THE ADDITION THEOREM

According to this theorem if two events A and B are mutually exclusive the probability of the occurrence of either A or B is the sum of the individual probability of A and B.

Mathematically,

$$P\ (A \text{ or } B) = P(A) + P(B)$$

Justification of theorem- Suppose in an experiment of n trials, A can happen in a_1 ways and B can happen in a_2 ways, then total number of ways in which either of the event can happen can be represents as $a_1 + a_2$. Then by definition, Probability of A or B can be calculated as

$$P\ (A \text{ or } B) = \frac{a_1+a_2}{n}$$

$$= \frac{a_1}{n} + \frac{a_2}{n}$$

Thus,

$$\mathbf{P(A \text{ or } B) = P(A) + P(B)}$$

The theorem can be extended to three or more mutually exclusive events.

Thus,

$$\mathbf{P\ (A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)}$$

2.6.2 ILLUSTRATION BASED ON ADDITION THEOREM

Example 6- One card is drawn from a standard pack of 52. What is the probability that it is either a queen or a king?

Solution-

Total number if cards in a standard pack = 52

Number of King in a standard pack = 4

Number of Queen in a standard pack = 4

Probability that the card drawn is a King

$$P(K) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 4/52

= 1/13

Probability that the card drawn is a Queen

$$P(Q) = \frac{\text{Number of favorable cases}}{\text{Total number of equally likely cases}}$$

= 4/52

= 1/13

Since, the event are mutually exclusive, the probability that the card drawn is either a king or a queen

P(K or Q) = P(K) + P(Q)

= 1/13 + 1/13

= 2/13

Modification of addition rule- In many cases it is possible that events are not mutually exclusive. It means that it is possible that both the events will occur. For example if we want to draw a card from a standard pack of cards and that should be a queen or a card of diamond , then it is possible that you will get a queen of diamond , means both the favorable event happened simultaneously. Thus in this case addition theorem can be stated as,

P (A or B) = P(A) + P(B) – P(A and B)

Here P(A and B) means probability of happening A & B togther.

Thus in the above example,

P (Queen or Diamond) = P(Queen) + P(Diamond) – P(queen and Diamond)

= 4/52+ 13/52- 1/52

= 16/52

= 4/13

Similarly, in case of three events, theorem can be stated as,

P(A or B or C)= P(A) + P(B) + P(C) - P(A and B) - P(B and C) - P(C and A) + P(ABC)

Example 7- XYZ ltd wants to recruit an HR manager. Five candidates apply for that a brief profile of all five candidates are as follows:

Male with working experience of 11years

Female with working experience of 12years

Female with working experience of 09years

Female with working experience of o8years

Female with working experience of 05years

If, the company want to select a female of experience more than 10years. Then what would be the probability that she would be either female or having experience more than 10 years?

Solution- P(female or experience more than 10 years)= P(female)+P(experience more than 10 years)-P(female and experience more than 10 years)

=4/5 + 2/5 – 1/5

= 5/5

= 1

Example 8- Find the probability of getting

a) King or jack.

b) Any card of spade, and

c) Jack of diamond. From a standard pack of playing cards.

Solution –

a) No. of king in a standard pack of 52 cards = 4

No. of jack in a standard pack of 52 cards = 4

Thus, Probability of getting a king or jack = (4+4)/52 = 8/52= 2/13

b) No. of spade cards in a standard pack of 52 cards = 13

Thus, probability of getting a spade card = 13/52 = ¼

c) No of jack in a standard pack of 52 cards = 4

No. of diamond in a standard pack of 52 cards = 13

In a standard pack of 52, there is also a jack of diamond,

Thus, P(jack or diamond) = P(jack)+P(diamond)-P(jack and diamond)

= 4/52 + 13/52 – 1/52

= 16/52 = 4/13

Example 9- A bag containing 50 balls numbering from 1 to 50. One ball is picked randomly, what is the probability that the ball picked is a multiple of

a) 7 or 9

b) 6 or 7

Solution-

a) Probability of getting a number multiple of 7is

P (7, 14, 21, 28, 35, 42, 49) = 7/50

Probability of getting a number multiple of 9is

P (9, 18, 27, 36, 45) = 5/50

Since events are mutually exclusive,

Hence P (multiple of 7 or 9) = 7/50 + 5/50

= 12/50 = 6/25

b) Probability of getting a number multiple of 6 is

P (6, 12, 18, 24, 30, 36, 42, 48) = 8/50

Probability of getting a number multiple of 7is

P (7, 14, 21, 28, 35, 42, 49) = 7/50

Since 42 is a multiple of 6 as well as 7, thus getting a ball of no. 42 is common for the events hence the probability of getting a multiple of 6 & 7 is

P(6 or 7) = 8/50 + 7/50 – 1/50

= 14/50

= 7/25

2.6.3 THE MULTIPLICATION THEOREM

According to this theorem, multiplication of probabilities of two independent events A and B is equal to probability they both will occur. Symbolically:

P (A and B) = P(A) x P(B)

Justification of theorem- Suppose in an experiment event A can happen successfully a_1 times out of n_1 ways and event B can happen successfully a_2 times out of n_2 ways. Then the total number of successful events in this case is $a_1 \times a_2$. Similarly, the total number of possible cases is $n_1 \times n_2$.

Then by definition the probability of the occurrence of both events is,

$$\frac{a_1 \times a_2}{n_1 \times n_2} = \frac{a_1}{n_1} \times \frac{a_2}{n_2}$$

P (A and B) = P(A) x P(B)

In the similar way the theorem can be extended to three or more events.

P (A, B and C) = P(A) x P(B) x P(C)

2.6.4 ILLUSTRATION BASED ON MULTIPLICATION THEOREM

Example 10- A bag contains 5 white and 7 black balls. A ball is drawn out of it and replaced in the bag. Then the ball is drawn again. What is the probability that (i) both the balls drawn were white, (ii) both were black, (iii) the first ball was white and the second black, (iv) the first balls was black and the second white, (v) both were of the same colour, (vi) both were different colours.

Solution-

(i) Probability of first ball being white P(A) = 5/12

Probability of second ball also being white P(B) = 5/12

Since, both the events are independent. Hence the probability of both balls being white:

P (A & B) = P(A) x P(B)

= 5/12 x 5/12

= 25/144

(ii) Probability of first ball being black P(A) = 7/12

Probability of second ball also being black P(B) = 7/12

Since, both the events are independent. Hence the probability of both balls being black:

P (A & B) = P(A) x P(B)

= 7/12 x 7/12

= 49/144

(iii) Probability of first ball being white P(A) = 5/12

Probability of second ball being black P(B) = 7/12

Since, both the events are independent. Hence the probability of first ball being white and second being black:

P (A & B) = P(A) x P(B)

= 5/12 x 7/12

= 35/144

(iv) Probability of first ball being black P(A) = 7/12

Probability of second ball being white P(B) = 5/12

Since, both the events are independent. Hence the probability of first ball being black and second being white:

P (A & B) = P(A) x P(B)

= 7/12 x 5/12

= 35/144

(v) Bothe of same colour means either both are white or both are black:

Probability of both being white = 25/144

Probability of both being black = 49/144

Both the events are mutually exclusive. Hence, the probability of both balls of same colour = 25/144 + 49/144

= 74/144

= 37/72

(vi) Both balls of different colours means :

Either first ball black and second white or first ball white and second black:

= (7/12 x 5/12) + (5/12 x 7/12)

= 35/144 + 35/144

= 70/144

= 35/72

Example 11- Find the probability in the following cases:

1. Two cards are drawn from a pack of cards in succession with replacement. Find the probability that both are aces.
2. From a pack of cards four cards are drawn with replacement. What is the probability that they all will be (a) Aces, (b) Club, (c) Red colour.

Solution-

(1) Probability of first card being a ace P(A) = 4/52

Probability of second card also being a ace P(B) = 4/52

Since, both the events are independent, i.e.

P (A x B)= P(A) x P(B)

= 4/52 x 4/52

= 16/2701 = 1/169

(2) Total number of events in each drawn = 52 (since the cards are being replaced)

(a) Probability of getting a ace = 4/52 = 1/13

Hence the probability of getting all the four cards of ace = 1/13 x 1/13 x 1/13 x 1/13 = 1/28,561

(b) Probability of getting the card of club in each drawn = 13/52 = ¼

Hence, probability of getting all the four cards of club:

= 1/4 x 1/4 x 1/4x 1/4x 1/4

= 1/256

(c) Probability of getting the cards of red colour in each draw = 26/52 = 1/2

Hence, probability of getting all the four cards of red colour = 1/2x 1/2x 1/2x ½ = 1/16

2.7 RANDOM VARIABLE FUNCTION

The outcome of any random experiment can be termed in the form of an variable, such variable are known as random variable. For example- if two dices are thrown simultaneously, the 36 outcomes are possible whose value can vary from 2 to 12, if we represent all these values by a variable X, and then X will be called as random variable. Since the value of such variable is totally depends upon chance hence it is also known as chance or stochastic variable.

A random variable can be of two types

- Discrete random variable
- Continuous random variable

If the value of any random variable is not following any pattern or a fix interval it is called as discrete random variable. For example- number of patients a doctor is treated in a particular month, number of runs per match a batsman is scoring in a particular series.

On the other hand variable is called continuous random variable if the value of that variable is within a certain interval.

Symbolically, if $X_1, X_2, X_3, \ldots X_k$ are the set of values of a discrete variable X with their respective probabilities $p_1, p_2, p_3, \ldots p_k$, where $p_1+p_2+p_3+\ldots p_k = 1$we can say that the function P(X) which has respective values $p_1, p_2, p_3, \ldots p_k$ for X= X1, X2, X3,……….X_k is called the probability function or frequency function of X.

It should be noted that a probability distribution is analogous to relative frequency distribution with probability replacing relative frequencies. Thus we can think of probability distribution as theoretical or ideal limiting forms of relative frequency distribution when the number of observation is made very large. For this reason, we can think of probabilities distribution as being distribution for populations, whereas relative frequency distributions are distribution drawn from this population.

Example 12- A vendor estimating from his past experiences the probability of his selling breads in a day. These are as follows:

No. of breads sold in a day:	0	1	2	3	4	5
Probability:	0.02	0.20	0.30	0.35	0.10	0.03

Find the mean number of breads sold in a day.

Solution-

Mean no of bread sold= 0.02x0+0.20x1+0.30x2+0.35x3+0.10x4+ 0.03x5

= 0+0.20+0.60+1.05+0.40+0.15

=2.4

Hence, mean number of bread sold in a day is 3.

Example 13- A book seller sells weekly magazine, which he purchases at a wholesale price of Rs. 35 each; and sells it a retail price of Rs. 40 each. An unsold copy after a week is a clear loss to the seller. The seller estimated the following probabilities for the number of copies demanded.

No. of copies:	50	51	52	53	54	55
Probabilities:	0.07	0.1	0.28	0.30	0.22	0.03

How many copies should he ordered, so that his expected profit will be maximum?

Solution-

Given that:

Cost per magazine = Rs. 35

Selling price per magazine = Rs. 40

Profit per magazine = 40-35 = Rs. 5

Expected profit= no. of copies sold x probability x profit per copy

Computation of expected profit

No. of copies	Probabilities	Profit per copy	Expected profit
50	0.07	5	17.5
51	0.1	5	25.5
52	0.28	5	72.8
53	0.30	5	79.5
54	0.22	5	59.4
55	0.03	5	8.25

53 Copies will give the maximum expected profit of Rs. 79.50.

2.8 STATISTICS DEFINED

Whenever numbers are collected and compiled, regardless of what they represent, they become statistics. In other words, the term statistics is considered synonymous with ways and means of presenting and handling data, making inferences logically and drawing relevant conclusions.

Statistics may be defined as the collection, presentation, analysis and interpretation of numerical data." This definition clearly points out four stages in a statistical investigation, namely : (i) collection of data, (ii) presentation of data, (iii) analysis of data, and (iv) interpretation of data.

The following are the important functions of the science of Statistics:

a. It presents facts in a definite form.

b. It simplifies mass of figures.

c. It facilitates comparison.

d. It helps in formulating and testing hypothesis.

e. It helps in prediction.

f. It helps in the formulation of suitable policies.

2.9 BASIC STATISTICAL TOOLS

Generally, it is difficult for a human mind to remember huge and unwieldy set of numeric values; again we will not be able to draw some conclusion with that vague data. Thus for that purpose we use some statistical tool. These tools not only make the data simple, precise and more understandable but also help us in many further statistical analyses.

Some of the basic statistical tools are: mean, mode, median, standard deviation, correlations, regression, sampling, hypothesis testing and probability distributions.

2.10 USE OF PROBABILITY AND STATISTICS IN OR

Operations research is a discipline that takes tools from different discipline such as mathematics, economics, psychology and statistics. Probability and statistics are two important disciplines that help in different OR models for the purpose of decision making. The probability theory is very much helpful in making prediction. Statistical methods help in estimating and predicting the future values of decision variables. Most common applications of probability and statistics in operation research are in decision theory, queuing models, inventory models, simulation and replacement models.

2.11 SUMMARY

In this unit, the concept of probability and statistics have been discussed. Types of probability and its theorems have been explained with the help of various examples. Concept of Random Variable is also discussed. The use of statistics and probability in OR is explained in brief.

2.12 SELF ASSESSMENT QUESTIONS

1. What do you understand by the term probability? Discuss its importance in business decision making?

2. Critically examine the priori definition of probability showing clearly the importance which the empirical version of probability makes over it.

3. Define independent and mutually exclusive events. Can two events be mutually exclusive and independent simultaneously? Support your answer with suitable example.

4. State the multiplication theorem of probability

5. Three unbiased coins are tossed. What is the probability of obtaining: All heads, Two heads, One head, At least one head, At least two heads, All tails.

6. Discuss about the role of statistics and probability in OR.

7. A vendor estimating from his past experiences the probability of his selling breads in a day. These are as follows:

No. of breads sold in a day:	0	1	2	3	4	5
Probability:	0.02	0.20	0.30	0.35	0.10	0.03

Find the mean number of breads sold in a day.

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BLOCK

2

PROGRAMMING TECHNIQUES – LINEAR PROGRAMMING AND APPLICATIONS

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BLOCK INTRODUCTION

In Block II you will learn about liner programming problem under which you will learn two important methods one is graphical method another is simplex method of solving Liner programming problem. Also in this block you will learn two special case of liner programming problem that is transportation problem and assignment model.

Unit-3 will discuss about graphical method of solving linear programming problem. Also in this unit you will learn formulation of linear programming problem

Unit-4 will discuss about simplex method of linear programming problem. Also you will learn about various solution conditions like feasible and infeasible solution, bounded and unbounded solution and sudoku solution etc.

Unit-5 will discuss about transportation problem under which various method of solving transportation like NWCR, LCM and VAM will be discuss. Also, method of getting feasible solution that is MODI method will be discuss along with some special cases like maximization problem degeneracy, prohibited and preferred routes etc.

Unit-6 will discuss about assignment problem. Under this you will learn about Hungerian method of solving assignment problem, along with some special cases like multiple solution, prohibited routes etc.

UNIT-3 LINEAR PROGRAMMING-GRAPHICAL METHOD

Structure

3.1 OBJECTIVES

After completing this unit, you will be able to:

1. Understand nature of linear programming problem.
2. Understand how it helps in decision making.
3. Understand how to formulate linear programming problem.
4. Graphical method of solving linear programming problem.

3.2 INTRODUCTION

Every business while working faces certain problems regarding allocation of resources, included men, machine, material, information etc. Most of these decisions are subject to various constraints, like shortage of men, improper availability of raw material, scheduling of machine work etc. Thus every business is supposed to work under such conditions. Thus, to maximize the profit there should a proper scientific technique that can solve all these problems and can optimize use of resources. Linear programming problem is such a tool that can help all such problems very easily and efficiently.

3.3 DEFINITION OF LINEAR PROGRAMMING

According to **Samuleson**, "The analysis of problems in which, linear functions of a number of variables are to be maximized (or minimized) when those variables are subject to a number of restrains in the form of a linear inequalities."

In the words of **Kohlars**, "A method of planning and operations involved in the construction of a model of a real situation containing the following elements: (i) variables representing the available choices, (ii) mathematical expressions (a) relating the variable to the controlling conditions, (b) reflecting the criteria to be used in the measuring the benefits derivable from each of the several possible plans, and (c) establishing the objective. The method may be so devised as to ensure the selection of the best of a large number of alternatives."

In the words of **Loomba**, "Linear programming is only one aspect of what has been called a system approach to management wherein all programmes are designed and evaluated in terms of their ultimate effects in the realization of business objective."

3.3.1 BASIC ASSUMPTIONS OF LINEAR PROGRAMMING

Linear programming problems (LPP) have four basic necessary assumptions. These are-

a. **Linearity-** A LPP model is based on the assumption that both objective and constraints are linear equations or inequalities, that means we can graphically represent them in the form of a straight line.

b. **Certainty-** In a LPP, it is always assumed that all parameters in the model for ex- resource availability, profit contribution, per unit cost etc. are deterministic or known with certainty and do not change during the period of study.

c. **Additivity-** It is assumed in a LPP, that the value of objective function and the total sum of resources must be equal to the sum of the per unit contribution or consumption from each decision variable. For example- the total cost of producing two products A & B must be equal to the sum of cost of producing A & B separately.

d. **Divisibility-** According to this assumption, the solution of any LPP in terms of values of decision variables and resources can take only non-negative values, i.e., decision variables can have a fractional value but cannot have a value less than zero.

3.4 APPLICATIONS OF LINEAR PROGRAMMING

Practically, linear programming is the most widely used quantitative technique in many fields such as business decision making, industrial applications, communication etc. A few broad application areas are as follows-

a. product mix problem

b. production scheduling problem

c. mixing or blending problem

d. rail road transportation problems

e. least cost shipping problems

f. financial mix problems

g. manpower planning

h. assignment of jobs

i. travelling salesmen problem

j. media selection problem

k. Diet selection problems

3.5 GENERAL MODEL OF LINEAR PROGRAMMING PROBLEM

Linear programming problem in general form with 'n' decision variable and 'm' constraints can be stated in the following form:

Optimize (Maximization or Minimization) $Z=c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots + c_n x_n$

Subject to linear constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n (\leq, =, \geq) b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n (\leq, =, \geq) b_2$$

$$\vdots$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n (\leq, =, \geq) b_m$$

and non-negative constraints $x_1, x_2, \ldots\ldots\ldots x_n \geq 0$

where, $x_1, x_2, \ldots, x_n$ are the decision variables whose values are to be determined.

Z , the linear function which is to be optimized (maximized or minimized) is called the Objective function and the constraints of the problem written in the form of simultaneous equation or inequalities are referred as constraints.

The constant coefficients c_j (j =1,2,…,n) represents the per unit contribution of decision variable x_j to the objective function.

The substitution coefficients a_{ij} (i= 1,2,…,m) (j = 1,2,….,n) represent the amount of resources consumed per unit of variable.

Constant b_i (i=1,2,…..,m) represents the requirement or availability of the ith constant.

3.6 MATHEMATICAL FORMULATION OF LPP

It is important to recognize a problem which can be handled by linear programming and then to formulate it in the form of a mathematical model. A two phase procedure is adopted for formulation of a Linear Programming Problem.

Phase 1- In this phase, first a detailed description of the problem is provided to have a clear and adequate understanding of the problem. Then the relevant objective is determined which is usually to be maximized or minimized in relation to profit or cost and so on. Finally the factors that restrict in the way of achieving objective are identified. In this way, the detailed explanation of the problem under consideration is given.

Phase 2- After giving a verbal description of the problem, the problem is transformed into mathematical model using the following steps-

a. Define the decision variables and identify the contribution coefficients (c_j) associated with each variable.

b. Formulate the objective function as a linear function of decision variables.

c. Identify the substitution coefficients (a_{ij}) for each decision variable.

d. Identify the availability of resources or requirements(b_i) for each type of resource.

e. Formulate the mathematical constraints for each resource as linear inequalities/ equalities in terms of decision variables.

f. Mention the non-negativity condition associated with decision variables.

The formulation procedure can be explained with the help of following example-

Example 3.1- A manufacturer of medicines is working on the production plan of two medicines A and B. There is sufficient ingredient available to manufacture 20000 bottles of medicine A and 40000 bottles of medicine B but the manufacturer has only 45000 empty bottles in which either A or B can be filled. Further it takes 3 hours to prepare material to fill 1000 bottles of A medicine, and it takes one hour to prepare material to fill 1000 bottles of B. There are 66 hours available for this operation. The profit per bottle of medicine A is Rs. 8 and Rs. 7 for each bottle of medicine B. Formulate this problem as LPP to determine how many bottles of A and B should be produced by manufacturer so as to maximize the profit.

Solution- Let us consider that the manufacturer should produce x_1 bottles of medicine A and x_2 bottles of medicine B.

Total Profit (in Rs.) can be calculated as $Z = 8x_1 + 7x_2$

There are only 45000 empty bottles to fill the medicine, therefore

$x_1 + x_2 \leq 45000$

Sufficient ingredient is available to produce 20000 bottles of medicine A and 40000 bottles of medicine B. therefore,

$x_1 \leq 20000$; $x_2 \leq 40000$

The time required to prepare x_1 bottles of medicine A = $\frac{3x_1}{1000}$ hours

The time required to prepare x_2 bottles of medicine B = $\frac{x_2}{1000}$hours

Total time available for this operation is 66 hours, therefore

$\frac{3x_1}{1000} + \frac{x_2}{1000} \leq 66$ or $3x_1 + x_2 \leq 66000$

The given problem can therefore be written as a mathematical LPP as follows-

Maximize $Z = 8x_1 + 7x_2$

Subject to the constraints

$x_1 + x_2 \leq 45000$

$3x_1 + x_2 \leq 66000$

$x_1 \leq 20000$

$x_2 \leq 40000$

and $x_1 \geq 0$; $x_2 \geq 0$

Example 3.2- A company has two types of quality inspectors grade- A and grade-B whom are assigned to inspect the quality of items produced. The firm has to inspect atleast 2000 items in a day of 8-hour working. Inspectors of grade A can inspect the items at a rate of 50 items per hour with the accuracy of 97%, whereas inspectors of grade B can inspect the items at a rate of 40 per hour with an accuracy of 95%. The wage rate of grade A inspectors is Rs.4.50 per hour and that of grade-B inspectors is Rs.2.50 per hour. Each time any inspector makes an error, it costs Rs. 1.00 to the company. The company has 10 inspectors of grade-A and 5 inspectors of grade-B available for inspection. Formulate this problem as a LPP to determine the number of inspectors of each grade to be selected in order to minimize the total cost of inspection.

Solution- Let us assume that the company should appoint x_1 inspectors of Grade-A and x_2 inspectors of Grade- B.

Total number of items inspected will be $(8 * 50)\, x_1 + (8 * 40)\, x_2 = 400\, x_1 + 320\, x_2$

However, the company requires at least 2000 items to be inspected daily, therefore $400\, x_1 + 320\, x_2 \geq 2000$ or $5\, x_1 + 4x_2 \geq 25$

The available number of inspectors are 10 of grade A and 5 of grade B, therefore

$x_1 \leq 10$; $x_2 \leq 5$

The cost of each inspector per hour can be calculated as follows

Grade A: Rs. 4.50 +1*50*3/100 = Rs.6.00 per hour

Grade B: Rs. 2.50 + 1*40*5/100 = Rs. 4.50 per hour

Total cost of inspection Z = 8 *6 x_1 + 8 *4.5 x_2 = $48x_1 + 36x_2$

Now the problem can be formulated as a mathematical LPP as follows-

Minimize Z= $48x_1 + 36x_2$

Subject to the constraints

$5x_1 + 4x_2 \geq 25$

$x_1 \leq 10$

$x_2 \leq 5$

and $x_1 \geq 0$; $x_2 \geq 0$

3.7 GRAPHICAL METHOD OF SOLVING LINEAR PROGRAMMING PROBLEM

Graphical method of solving linear programming problems is the simplest method of solving a LPP. But this method can only be used for two variable linear programming problems. Following are the steps of solving a LPP using graphical method.

Step-1: Formulate the problem in terms of mathematical equalities/inequalities for each constraint and an objective function.

Step-2: Plot each constraint on the graph using the following procedure- first consider each constraint as equality and then draw line for each constraint by obtaining two points for each constraint. In each equation of constraint put x1 =0 and find x2 then put x2 =0 and find x1, this way two points for each constraints can be obtained. A line is drawn by joining these points. So for each constraint, a straight line is plotted.

Step-3: Find the common feasible region for the problem- common feasible region is the area which satisfies all the constraints simultaneously. For 'greater than' constraints, the feasible region is the area that lies above the constraint lines. For 'less than' constraints, the feasible region is the area below the constraint lines. The feasible region also includes the points on the constraint line. The feasible regions are shown in figure 3.1

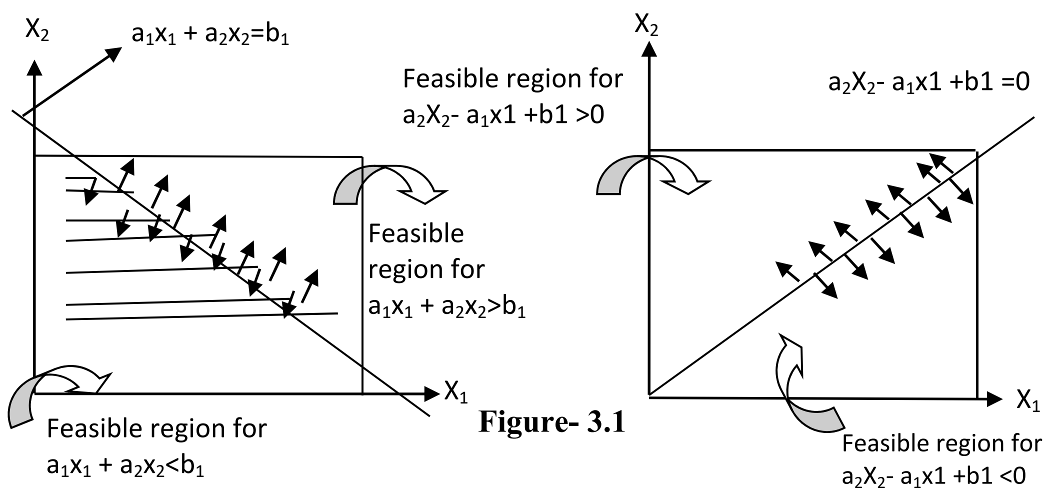


Figure- 3.1

Step-4: The point in common feasible region is identified that gives the optimum value of objective function. This point will be called as optimum point and the solution values will represent optimal solution to the problem. There are two ways to find the optimum point-

a. Corner point method- In this method, corner points (extreme points) of the common feasible region identified in the step -3 are determined. The value of objective function at each of these corner points is computed by submitting the coordinates of each corner point in the objective function. The corner point at which the value of objective function is maximum (for a maximization problem) or minimum (for a minimization problem) is called the optimum point and the solution values (coordinates) at this point represent the optimum solution.

   If two corner points have the same optimum value of objective function then it shows that there are two possible optimal solutions of the problem.

b. Iso-profit or Iso-cost method- In this method, a straight line (called as iso-profit/ iso-cost line) is drawn by giving a convenient value for objective function so that the line falls in the common feasible region. Move this isoprofit(isocost) line parallel to itself and farther (closer) from the origin until further movement takes this line completely out of the feasible region. The point in the feasible region which touches the highest possible isoprofit line (lowest possible isocost line) represents the optimum point and the coordinates of that point represent the optimal solution. The optimal value of objective function is ultimately obtained by putting the optimal solution values in the objective function.

Example 3.3- Two types of Television sets are produced with a profit of 6 units from each television of Type I and 4 units from each television of Type II. In addition, 2 and 3 units of raw materials are needed to produce one television of Type I and Type II respectively & 4 and 2 units of time are required to produce one television of Type I and Type II, respectively. If 100 units of raw materials and 120 units of time are available, how many units of each type of television should be produced to maximize profit and still meet all constraints of the problem? Formulate the problem as a Linear Programming Problem and solve using graphical method.

Solution- Let X_1, X_2 be the number of units of Type I and Type II television respectively.

Since each unit of Type I and Type II yields a profit of 6 and 4 units, this means that objective function can be written as

$$\text{Max } Z = 6X_1 + 4X_2$$

The raw material and time is available in limited quantity so there are constraints related to these which may be written as follows

$$2X_1 + 3X_2 \leq 100 \text{ ………… Raw material}$$

$4X_1 + 2X_2 \leq 120$ ………… Time

The mathematical formulation of the problem can be represented as-

Max $Z = 6X_1 + 4X_2$

Subject to constraints

$2X_1 + 3X_2 \leq 100$ ……….. Raw material

$4X_1 + 2X_2 \leq 120$ ………… Time

$X_1, X_2 \geq 0$ ..……….. Non-negativity

To solve it graphically, let us draw the straight line for all constraints for where the constraints are satisfied exactly. Graph the constraints as if it were equality.

Therefore, the 1st constraint i.e. $2X_1 + 3X_2 \leq 100$ becomes

$$2X_1 + 3X_2 = 100$$

To graph the 1st constraint, determine the set of points that satisfy this constraint.

Now, to determine (X_1, X_2) co-ordinates, set X_1=0 and find out X_2 and vice- versa.

Therefore, putting $X_1 = 0$ in $2 X_1 + 3X_2= 100$

We get, $3X_2 = 100$; $X_2 = 33.33$

And putting $X_2 = 0$ in $2X_1 + 3X_2 = 100$

We get, $2X_1 = 100$; $X_1 = 50$

Therefore, set of points is (50, 0) and (0, 33.33).

In the similar way determine the set of points for the second constraint $4X_1 + 2X_2 = 120$

Putting X_1 =0 in this constraint, we get

$2X_2 =120$; $X_2=60$

Putting $X_2 = 0$, we get

$4X_1 = 120$; $X_1 = 30$

Therefore, set of points is (30, 0) and (0,60).

Now draw the straight lines for both the constraints on the graph. The graph is shown in figure 3.2 below.

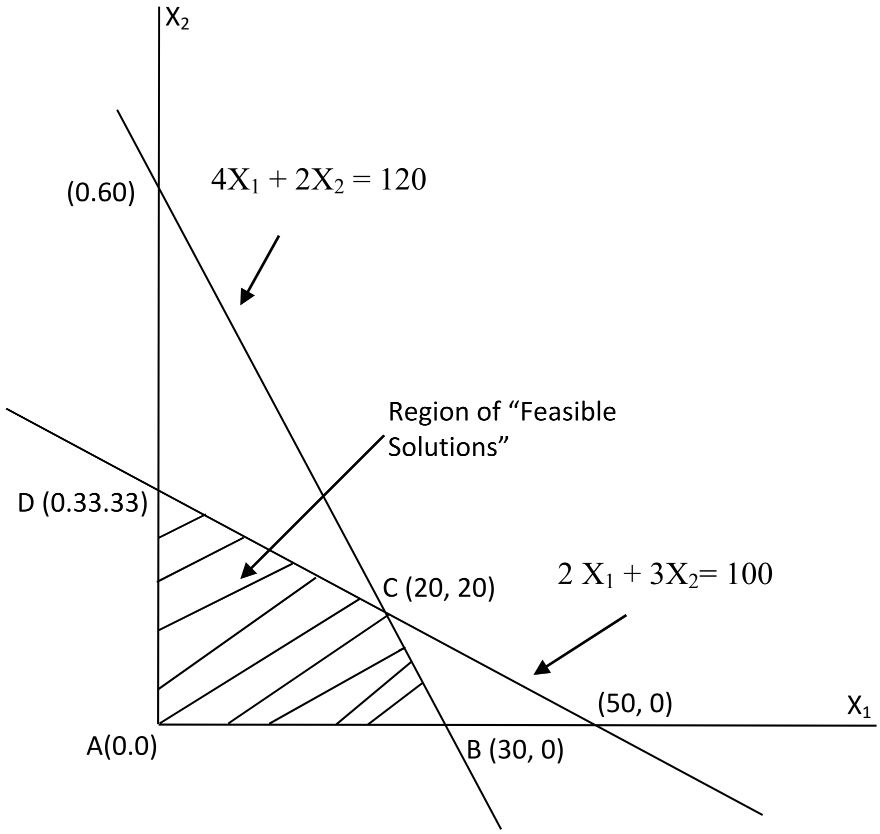


Figure 3.2

Consider the region which is common to all the lines i.e. $2X_1 + 3X_2 \leq 100$; $4X_1 + 2X_2 \leq 120$ and $X_1, X_2 \geq 0$. That region is called as "Feasible Region". It is shown as area ABCD in the figure.

To find out the optimal solution, substitute the set of extreme points in the objective function i.e.

$Z = 6X_1 + 4X_2$

The corner points are: A (0, 0); B (30, 0); C (20, 20); D (0, 33.33)

Substitute these values in $Z = 6X_1 + 4X_2$

A (0, 0) $Z = 6*0 + 4*0 = 0$

$Z = 0$

B (30, 0) $Z = 6*30 + 4*0 = 180$

$Z = 180$

C (20, 20) $\mathbf{Z = 6* 20 + 4*20 = 200}$

$\mathbf{Z = 200}$

D (0, 33.33) $Z = 6*0 + 4*33.33 = 133.32$

$Z = 133.32$

Z = 200 is the maximum value & is the optimal solution. Therefore, 20 units of Type I and 20 units of Type II should be produced to yield a maximum profit of 200 units.

Example 3.4- A catering manager is in the process of replacing the furniture in a canteen. He wishes to determine how many tables of type 'S' (seating 6) and how many of type 'T' (seating 10) to buy.

He has to work under the following constraints:

(1) The canteen must be able to accommodate at least 60000 people.

(2) The available floor space of the canteen is at most 63000 sq. meters.

He estimates that each type 'S' table needs 7 meters sq. of floor space while each type 'T' needs 9.

Advise the manager on how many tables of each type to buy if each type 'S' costs Rs. 100 and each type 'T' costs Rs. 190.

Solution: Let X1 be the number of units of table type S and X2 be the number of units of table type T. The LP model can be written as

Minimise $Z = 100x1 + 190x2$

subject to the constraints

$6X1 + 10X2 \geq 60000$ (People)

$7X1 + 9X2 \leq 63000$ (Floor space)

$X1, X2 \geq 0$

The Graph for the problem is indicated in Figure 3.3.

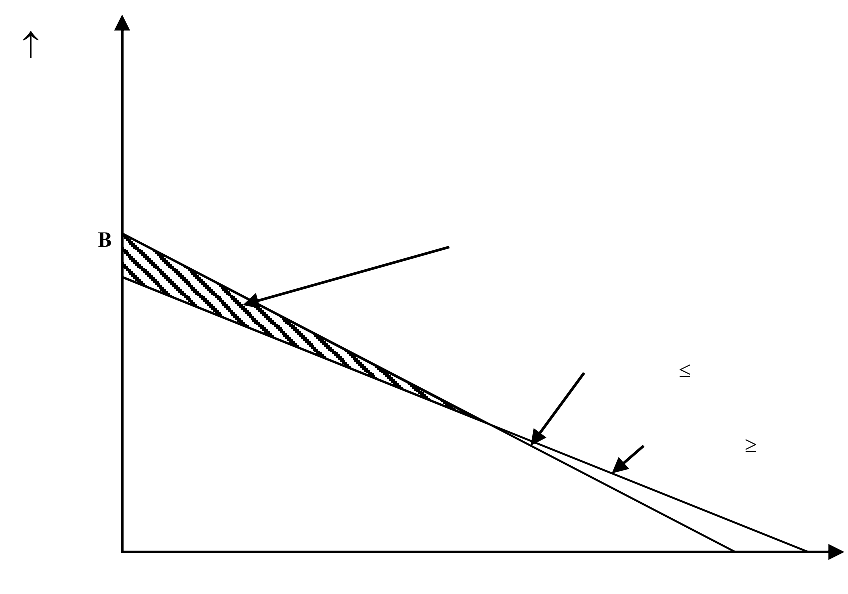


Figure 3.3

The optimum solution will lie on one of the corner points of the shaded area ABC

Calculation of Z (Z = 100X1 + 190X2) at corner points

A (5625, 2625); Z_A = 100 x 5625 + 190 x 2625= Rs. 1061,250

B (0, 7000); Z_B = 190 x 7000 = Rs. 1330,000

C (0, 6000); Z_C = 190 x 6000 = Rs. 1140,000

Corner point A signifies the optimum solution because the corresponding value of Z is minimum.

The optimum solution is 5625 units of Table type S & 2625 units of Table type T

The corresponding Total Cost = Rs. 1061250.

3.8 SOME BASIC TERMS

There are some basic terms that are used in solving linear programming problems.

a. **Basic Solution-** If a LPP has m constraints with n variables (n>m), a solution obtained by setting any (n-m) variables to zero and solving the remaining m equations in m unknowns is called a basic solution. Zero variables are called non-basic variables and remaining m variables are called basic variables and constitute basic solution.

b. **Feasible Solution-** A solution that satisfies all restrictions including non-negative restrictions of a LPP is called a feasible solution.

c. **Basic Feasible Solution-** A feasible solution to a LPP which is also basic solution is called a basic feasible solution.

d. **Optimal Solution-** A basic feasible solution that optimize (maximize or minimize) the objective function of LPP, is called optimal solution to the problem.

e. **Degenerate Solution-** A basic solution of a LPP is called degenerate if one or more basic variables become equal to zero.

3.9 SOME SPECIAL CASES

There are some special cases of Linear Programming Problem in which no solution is obtained or the solution becomes unbounded or sometimes even there are many optimal solutions possible. Here some special cases are given-

3.9.1 UNBOUNDED SOLUTIONS

When the value of a decision variable is permitted to increase infinitely without violating any of the feasibility conditions, then the solution is said to be unbounded. Thus an unbounded LPP occurs if it is possible to find arbitrarily large values of Z within the feasible region. It is not possible to find a single optimal solution to an unbounded problem though there are infinite feasible solutions for the same.

Example 3.5- Use graphical method to solve the following LP problem

Maximise Z = 3x1 + 3x2

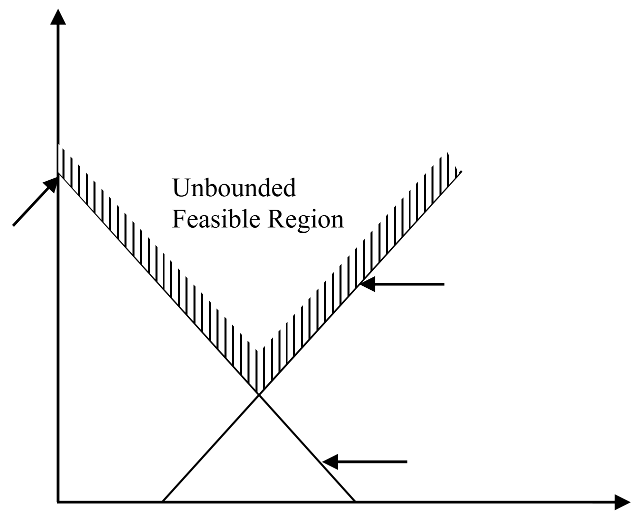
subject to the constraints

x1 - x2 ≤ 1

x1 + x2 ≥ 3

and x1, x2 ≥ 0

Solution: The problem is depicted graphically in Fig. 3.4. The solution space is shaded and is bound by A and B from below.



It is noted here that the shaded convex region (solution space) is unbounded. The two corners of the region are A = (0, 3) and B = (2, 1). The values of the objective function at these corners are:

Z(A) = 6 and Z(B) = 8.

But there exist number of points in the shaded region for which the value of the objective function is more than 8. For example the point (10,12) lies in the region and the function value at this point is 70 which is more than 8. Thus both the variables x1 and x2 can be made arbitrarily large and the value of Z also increased. Hence, the problem has an unbounded solution.

NOTE: An unbounded solution does not mean that there is no solution to the given LP problem, but implies that there exist an infinite number of solutions.

3.9.2 INFEASIBLE SOLUTION

Sometimes it is not possible to find a single solution that satisfies all the feasibility constraints. Such a problem does not have a feasible solution.

It is notable that in the case of infeasiblity it is not possible to have a single feasible solution whereas in the case of unboundedness, it is possible to have infinite feasible solutions, but not a single optimal solution.

Example 3.6- Use graphical method to solve the following LP problem:

Maximise Z = 6x1 - 4x2

subject to the constraints

$2x1 + 4x2 \leq 4$

$4x1 + 8x2 \geq 16$

and $x1, x2 \geq 0$

Solution: The problem is shown graphically in Fig. 3.5.

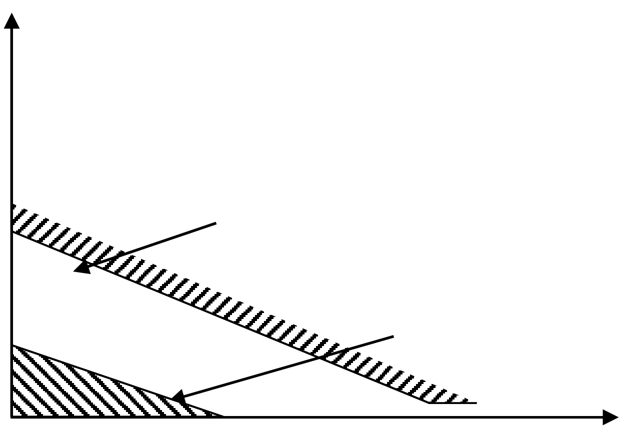


Figure 3.5

The two inequalities that form the constraint set are inconsistent. Thus, there is no set of points that satisfies all the constraints. Hence, there is no feasible solution to this problem.

3.9.3 MULTIPLE OPTIMAL SOLUTIONS

There can exist multiple optimal solutions for a linear programming problem if

i. The given objective function is parallel to a constraint that forms the boundary of the feasible solutions region

ii. The constraint should form a boundary on the feasible region in the direction of the optimal movement of the objective function. In other words, the constraint must be a binding constraint.

In case the slope of the objective function (represented by the iso-profit lines) is the same as that of a constraint, then multiple optimal solutions might exist.

Example 3.7- Solve graphically the following LPP:

Maximise Z = 8x1 + 16x2

Subject to the constraints

$x1 + x2 \leq 200$

$x2 \leq 125$

$3x1 + 6x2 \leq 900$

$x1, x2 \geq 0$

Solution:

The constraints are shown plotted on the graph in Figure 3.6. Also, iso-profit lines have been graphed. We observe that iso-profit lines are parallel to the equation for third constraint 3x1 + 6x2 = 900. As we move the iso-profit line farther from the origin, it coincides with the portion BC of the constraint line that forms the boundary of the feasible region. It implies that there are an infinite number of optimal solutions represented by all points lying on the line segment BC, including the extreme points represented by B (50, 125) and C (100, 100). Since the extreme points are also included in the solutions, we may disregard all other solutions and consider only these ones to establish that the solution to a linear programming problem shall always lie at an extreme point of the feasible region.

The extreme points of the feasible region are given and evaluated here.

Point	x1	x2	Z = 8x1 + 16x2	
0	0	0	0	
A	0	125	2000	
B	50	125	2400	} Maximum
C	100	100	2400	} Maximum
D	200	0	1600	

The point B and C clearly represent the optima.

In this example, the constraint to which the objective function was parallel, was the one which formed a side of the boundary of the space of the feasible region. As mentioned in condition (a), if such a constraint (to which the objective function is parallel) does not form an edge or boundary of the feasible region, then multiple solutions would not exist.

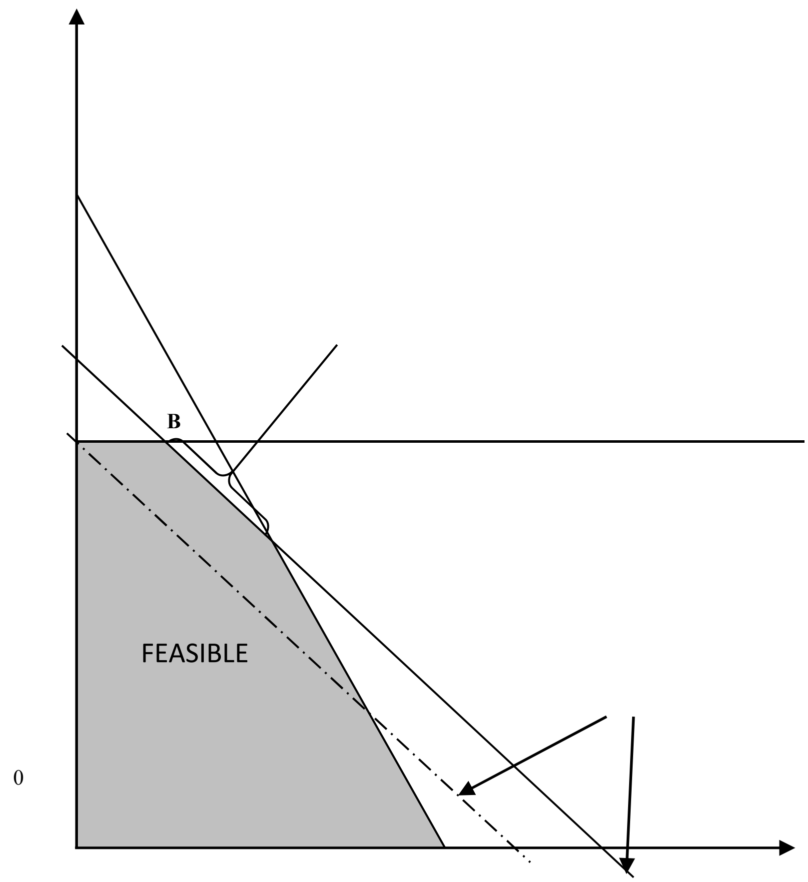


Figure 3.6

Example 3.8 Solve graphically:

Minimise Z = 6x1 + 14x2

Subject to

$5x1 + 4x2 \geq 60$

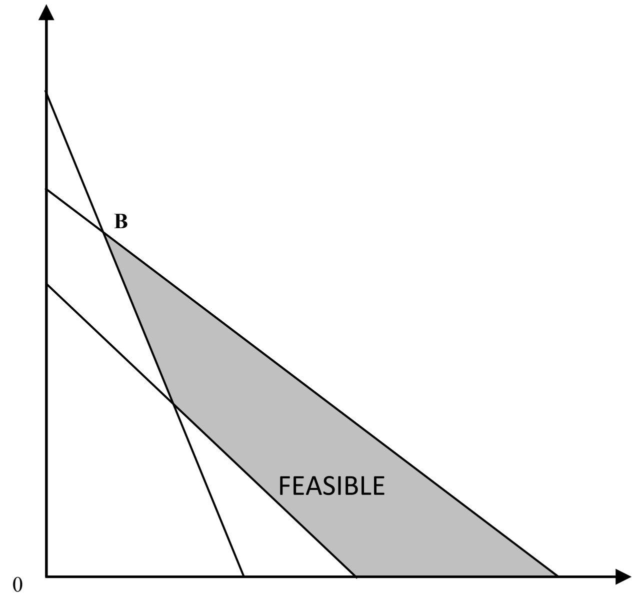
$3x1 + 7x2 \leq 84$

$x1 + 2x2 \geq 18$

$x1, x2 \geq 0$

Solution:

The restrictions in respect of the given problem are depicted graphically in Figure 3.7. The feasible area has been shown shaded. It may be observed here that although the iso-cost line is parallel to the second constraint line represented by 3x1 + 7x2 = 84, and this constraint does provide a side of the area of feasible solutions, yet the problem has a unique optimal solution, given by the point D. Here condition (b) mentioned earlier, is not satisfied. This is because, being a minimisation problem, the optimal movement of the objective function would be towards the origin and the constraint forms a boundary on the opposite side. Since the constraint is not a binding one, the problem does not have multiple optima.



We can show the uniqueness of the solution by evaluating various extreme points as done here.

Point	x1	x2	Z = 6x1 + 14x2	
A	8	5	118	
B	84/23	240/23	168	
C	28	0	148	
D	18	0	108	Minimum

The optimum solution is 108 units for $x_1 = 18$ and $x_2 = 0$.

3.10 SUMMARY

In this unit, the meaning, concept and graphical method of solving linear programming problems is discussed. Linear programming problems are the problems of minimizing or maximizing an objective function comprising of a set of independent variables (decision variables) which may be subjected to a set of constraints in the form of inequalities/ equalities on the decision variables. The term 'linear' indicates that the function is linear in nature and constraints are also represented by a set of linear equations/inequalities.

Linear programming has four basic assumptions of linearity, certainty, additivity and divisibility. It has many applications in field of manufacturing, business decisions of financial mix, product mix , assignment problems, transportation problems, diet problems etc.

To solve a linear programming problem, it has to be first formulated in the form of a mathematical model which includes one objective function represented in terms of a linear function of decision variables and a set of constraints represented as linear equation/ inequalities.

A LPP with two decision variables can be solved graphically. A two dimensional graph is plotted in which each constraint is represented by a line. A feasible area is identified that satisfies all the constraints. The value of objective function is calculated at extreme points of the feasible region. The point at which the value of objective function is optimized is considered as the final solution of the problem.

There can be some special cases of LPP where the solution can be infeasible, unbounded or multiple optimal solutions.

3.11 SELF-ASSESSMENT QUESTIONS

1. What do you understand by linear programming problems? What are the assumptions of linear programming?
2. Discuss various applications of linear programming.
3. Define following terms-
    a. Objective function
    b. Constraints
    c. Decision variables
    d. Non-negativity condition
4. XYZ factory manufactures two articles A and B. to manufacture the article A, a certain machine has to be worked for 1.5 hrs and in addition a craftsman has to work for 2 hrs. To manufacture the article B, the machine has to be worked for 2.5 hrs and in addition the craftsman has to work for 1.5 hrs. in a week a factory can avail of 80 hrs of machine time and 70 hrs of craftsman's time. The profit on each article A is Rs.50 and that on each article B is Rs.40 If all the article produced can be sold away, find how many of each kind should be produced to earn the maximum profit per week. Formulate the problem as LP model.
5. An electronic company is engaged in the production of two component C1 and C2 used in T.V. sets. Each unit of C1 cost the company Rs.25 in wages and Rs.25 in material, while each unit of C2 cost the company Rs.125 in wages and 75 in material. The company sells both product on one period credit term but the company's labor and material expenses must be paid in cash. The selling price of c1 is rs 150 per unit and of c2 is rs 350 per unit because of strong monopoly of the company of these components it is assumed that the company can sell at the prevailing price as many units as it produces the company production capacity is however limited by two consideration: first at the beginning of period one the company has the initial balance of 20000 second the company has available in each period 4000 hours of machine time and 2800 hours of assembly line. The production of each c1 requires 6 hours of machine time and 4 hours of assembly time whereas the production of each c2 requires 4

hrs. of machine time and 6 hrs. of assembly time. Determine how many units of C1 and C2 should be produced in order to maximize the profit ?

6. Vitamins V and W are found in two different foods f1 and f2. one unit of food f1 contains 2 unit of v and 5 unit of w. one unit of f2 contains 4 unit of v and 2 units of w. 1 unit of f1 and f2 cost rs. 30 and 25. the minimum daily requirement of vitamin v and w is 40 and 50 units assuming that anything in excess of daily minimum requirement of v and w is not harmful. Find out optimal mixture of f1 and f2 at the minimum cost.

7. A firm makes two product X & Y, and has a production capacity of 9 tons per day, X & Y requiring the same production capacity. The firm has a permanent contract to supply at least 2 tons of X and at least 3 tons of Y per day to another company. Each tons of X requires 20 machine hrs. production time and each tons of Y requires 50 machine hrs. production time; the daily maximum possible number of machine hrs is 360hrs. All the firm's output can be sold, and profit made is Rs.80 per tons of X and Rs.120 per tons of Y. it is require to determine the production schedule for maximum profit and to calculate this profit.

8. Define the following cases in LPP-

   a. Infeasible solution

   b. Multiple optimal solution

   c. Unbounded Solution

3.12 TEXT AND REFERENCES

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UNIT-4 LINEAR PROGRAMMING-SIMPLE METHOD

Unit Framework

4.1 OBJECTIVE

After completing this unit you will be able to:

- Describe the concept of simplex method.
- Build a simplex table and describe the components of the table.
- Solve a LPP with ‘less than type’ constraints using simplex method.
- Solve a LPP with ‘greater than type’ constraints using simplex method.
- Solve a LPP with ‘mixed constraints using simplex method.
- Describe special cases in LPP solved by simplex method.

4.2 INTRODUCTION

In the previous unit, graphical method of solving was explained. Graphical method has limited application in industrial problems as it can be applied for a two variable linear programming problem, whereas the number of variables in real industrial problems is substantially large. Therefore, another method known as ‘Simplex Method’ is used for solving linear programming problems with more than two variables. Simplex method involves an iterative process in which step by step procedure is adapted to reach from initial basic feasible solution to other basic feasible solutions and ultimately optimal solution.

4.3 PRINCIPLE OF SIMPLEX METHOD

Simplex Method was developed by George B. Dantzig in 1957 to solve such linear programming problems which involved more than two decision variables. Although simplex method is applicable to any LPP with any number of variables. It can deal with linear problems involving several thousand variables and several thousand linear constraints with the help of computer programs.

The simplex method is a systematic and step by step procedure for finding optimal solution of a linear programming problem. It is based on the fundamental principle that, “The optimal solution to a linear programming problem, if it exists, always occurs at one of the corner points of the feasible solution space.”

In simplex method, the search for the corner point always start at the point of origin which is also called as initial basic feasible solution (IBFS) and is one of the corners of the feasible solution space. This IBFS is then tested for optimality, i.e. whether the value of objective function is optimum or can be improved by moving to another corner point of the feasible solution space. If there is a possibility of improvement then reach to the next corner point and again test the solution for optimality. This iterative procedure is repeated until the optimal solution is determined.

4.4 BASIC TERMS USED IN SIMPLEX PROCEDURE

Slack and Surplus variables are additional variables that are introduced into the linear constraints of a linear program to convert them from inequality constraints to equality constraints. Slack variable is added to the left hand side of a ‘less than type’ constraint and Surplus variable is subtracted from the left hand side of a ‘greater than type’ constraint to convert the constraint into equality.

Standard form is the baseline format for all linear programs before solving for the optimal solution. In this form all constraints are written as equalities.

Basic Solution for a linear programming problem with m constaints and n , variables (n>m), the solution obtained by setting (n-m) variables equal to zero and solving for the remaining m variables is called basic solution. The (n-m) variables set equal to zero are called non-basic variables and other m variables are called basic variables.

Optimal solution is that basic feasible solution that optimizes (maximizes or minimizes) the value of objective function. These are the values assigned to the variables in the objective function to give the optimum value. The optimal solution would exist on the corner points of the graph of the entire model.

Simplex tableau is the table used to perform row operations on the linear programming model. This table comprises of a matrix with m unit columns for basic variables and values of these variables are given in quantity column. The format of the table is shown in table 4.1.

Zj Row- the values in this row under each variable represents the total contribution of outgoing profit when one unit of non basic variable is introduced into the basis in place of basic variable.

Net evaluation (cj – Zj)Row- contains the net profit (loss) that will result from introducing one unit of variable indicated in corresponding column.

Pivot (or Key)Column- the column with the largest positive value in the net evaluation row for a maximization problem or the largest negative value in the net evaluation row for a minimization problem is called key column or Pivot column. This indicates the variable that will enter into the basis in the next step of solution.

Pivot (or Key) Row-the row corresponding to the variable that will leave the basis in order to improve the solution and make room for the entering variable. The row with the minimum positive ratio(XB/ Xj) found by dividing the quantity column values by the pivot column values for each row, is the Pivot row.

Pivot variable (key element) is used in row operations to identify which variable will become the unit value and is a key factor in the conversion of the unit value.

4.5 COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

To solve a linear programming problem using the Simplex method the following computational steps are used:

- Formulate the problem into linear programming model
- Convert the problem into Standard form by introducing slack/surplus variables
- Set up the simplex tableau
- Identify the Pivot column and row
- Creating a new tableau
- Checking for optimality
- Identify optimal values

Step 1- Formulate the problem into linear programming model-

The given real life problem must be first formulated into a mathematical linear programming model including objective function and constraints as stated below-

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots + c_n x_n$

Subject to linear constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n \leq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n \leq b_2$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n \leq b_m$$

and non negative constraints $x_1, x_2, \ldots\ldots\ldots x_n \geq 0$

Step 2- Convert the problem into Standard form by introducing slack/surplus variables

Standard form is the basic format for all linear programs before solving for the optimal solution. In order to convert the original problem into standard form, slack variables are added to left hand side of the less than type constraints so that the constraints are converted into equalities. The slack variables are also added to objective function with a zero coefficient. It can be represented as follows-

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots + c_n x_n + 0.s_1 + 0.s_2 + \ldots.. + 0.s_m$

Subject to linear constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n + s_1 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n + s_2 = b_2$$

$$\begin{array}{cccc} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{array}$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n + s_m \leq b_m$$

where $x_1, x_2, \ldots\ldots\ldots x_n \geq 0$ and $s_1, s_2, \ldots\ldots, s_m \geq 0$

Step 3- Set up the simplex tableau

In this step, the problem is represented in a table form so that computations can be made in a simpler way. Simplex tableau represents the initial basic feasible solution, the constraints of the LPP and objective function in the same table format as shown below in table 4.1-

Table 4.1- Simplex Tableau

C_j			C_1	C_2	…	C_n	0	0	…	0	Min. Ratio X_B/X_j
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	…	X_n	S_1	S_2	…	S_m	
0	S_1	b_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	
0	S_2	b_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	
.	.	.	.	.	…		.	.	…	.	
.	.	.	.	.			.	.	…	.	
0	S_m	b_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	
Contribution loss per unit $Z = \sum C_{Bj}a_{ij}$			0	0	…	0	0	0	…	0	
Net evaluation per unit $C_j - Z_j$			c_1	C_2	…	C_n	0	0	…	0	

Basis   Main Body of Tableau

The first row in the table is called objective row, the values in this row (c_j) indicate the coefficient value of (m+n) variables in the objective function. The second row bears the column headings for the initial as well as subsequent simplex tables.

The first column C_B represents the coefficients of current basic variables in the objective function. Second column indicates the basic variables in the solution. In the initial simplex tableau these basic variables are the slack variables. The third column represents solution values for basic variables.

The main body of the simplex tableau represents the coefficients of all variables (decision variables and slack variables) in the constraints set of equations.

Each row in this part represents each constraint in the problem. The second last row in the table is indicating Z_j values which are obtained by adding the products of entries in the C_B column and the entries of corresponding column $Z_j = \sum C_{Bj}a_{ij}$. In initial simplex tableau Z values remain zero as C_B values are zero. Z value in the solution column gives the current value of objective function. The final row in the tableau represents net evaluation C_j - Z_j values. These values are used to determine whether the solution is optimal or not. In the initial simplex tableau C_j - Z_j values are same as C_j values.

Step 4- Identify the Pivot column and row-

The simplex method moves from the initial basic solution to a better solution in each iteration till the optimal solution is obtained. In order to reach on a new and better solution, one basic variable (called outgoing variable) is replaced by a new non-basic variable (called as incoming variable). The column corresponding to incoming variable is called as pivot column and the row corresponding to outgoing variable is called pivot row. To identify pivot column, find out the largest positive number in the net evaluation row (C_j - Z_j) of the first table. The column corresponding to this value is pivot column and the corresponding variable is incoming variable. Next the outgoing variable is determined by computing ratio shown in the last column of simplex tableau. It is computed by dividing the values of solution column by the pivot column values. The row having the smallest non-negative number in the ratio column is called pivot row and the corresponding row variable is called outgoing variable. The number that lies at the intersection of the pivot column and pivot row is called as pivot variable or key element.

Step 5- Creating a new tableau

To create a new simplex tableau, new values are computed in the following way-

i) Compute the new values for the pivot row in main body of the tableau by dividing every element in that row by the pivot variable (key element). Solution value for this row in the basis also gets changed by dividing the existing value by key element. This revised row now can be called as replacement row.

ii) Compute new values for remaining rows by using the formula:

New value = Old value – [corresponding row element in pivot column of old tableau × corresponding value in the replacement row of new tableau]

iii) New values in the C_B, Basic Variable and X_B columns of Basis are entered in the new table. Outgoing variable is replaced by incoming variable, change C_B values according to basic variables also write new solution X_B values. Solution value in the basis for rows other than replacement row also gets changed by using the same formula.

Step 6- Check for optimality

For the revised simplex tableau, compute the Zj values and Cj-Zj values. If all the values in Cj – Zj row are either zero or negative, an optimal solution has been attained. If any of the value in this row is positive, repeat the step 4, 5 and 6 again until an optimum solution is obtained.

To illustrate the computational procedure of simplex method following example is given.

Example 4.1 Solve the following LPP by simplex procedure.

Maximize Z = $2x_1 + 5x_2 + 7x_3$

Subject to constraints

$3x_1 + 2x_2 + 4x_3 \leq 100$

$x_1 + 4x_2 + 2x_3 \leq 100$

$x_1 + x_2 + 3x_3 \leq 100$

$x_1, x_2, x_3 \geq 0$

Solution- The given problem is already formulated as a LPP. First we convert the given LPP in standard form by introducing slack variables S_1, S_2, S_3 as follows-

Maximize Z = $2x_1 + 5x_2 + 7x_3 + 0.S_1 + 0.S_2 + 0.S_3$

Subject to constraints

$3x_1 + 2x_2 + 4x_3 + S_1 = 100$

$x_1 + 4x_2 + 2x_3 + S_2 = 100$

$x_1 + x_2 + 3x_3 + S_3 = 100$

$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$

Now, putting $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$, we obtained initial basic feasible solution, $S_1 = 100$, $S_2 = 100$, and $S_3 = 100$. Here S_1, S_2, and S_3 are basic variables and other variables x_1, x_2, x_3 are non-basic variables.

The initial simplex tableau is written as follows-

Table 4.2- Initial Simplex Tableau

C_j			2	5	7	0	0	0	Min. Ratio X_B/X_3
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	X_3	S_1	S_2	S_3	
0	S_1	100	3	2	**4***	1	0	0	**25** →
0	S_2	100	1	4	2	0	1	0	50

0	S_3	100	1	1	3	0	0	1	100/3
Contribution loss per unit $Z = \sum C_{Bj}a_{ij}$			0	0	0	0	0	0	
Net evaluation per unit $C_j - Z_j$			2	5	**7**↑	0	0	0	

Since all values in the Net evaluation (C_j - Z_j) row are not 0 or negative so the solution is not optimal. Therefore the solution is revised as shown in revised simplex tableau 4.3. Here incoming variable is X_3 as the net evaluation for this variable is largest positive and outgoing variable is S_1 as ratio (X_B/X_3) is minimum for row corresponding to S_1. The values are revised using the procedure mentioned in Step 4 and 5 of simplex method.

Table 4.3 Revised Simplex Tableau 1

C_j			2	5	7	0	0	0	Min.
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	X_3	S_1	S_2	S_3	Ratio X_B/X_2
7	X_3	25	3/4	1/2	1	1/4	0	0	50
0	S_2	50	-1/2	**3***	0	-1/2	1	0	**50/3** →
0	S_3	25	-5/4	-1/2	0	-3/4	0	1	-50
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		175	21/4	7/2	0	7/4	0	0	
Net evaluation per unit $C_j - Z_j$			-13/4	**3/2**↑	0	-7/4	0	0	

The solution in table 4.3 is also not optimal as Net evaluation C_j - Z_j is positive for X_2. Therefore again the tableau is revised. Incoming variable now will be X_2 and outgoing variable will be S_2 as the ratio (X_B/X_2) is minimum. The revised simplex tableau 2 is shown in table 4.4

Table 4.4 Revised Simplex tableau 2

C_j			2	5	7	0	0	0	Min. Ratio
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	X_3	S_1	S_2	S_3	
7	X_3	50/3	5/6	0	1	1/3	-1/6	0	
5	X_2	50/3	-1/6	1	0	-1/6	1/3	0	
0	S_3	100/3	-4/3	0	0	-5/6	1/6	1	
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		200	5	5	7	3/2	1/2	0	
Net evaluation per unit $C_j - Z_j$			-3	0	0	-3/2	-1/2	0	

Since all values in Net evaluation ($C_j - Z_j$) row are either zero or negative, therefore, the present solution is optimal solution.

The optimal solution is- $x_1 = 0$, $x_2 = 50/3$, $x_3 = 50/3$ and Maximum Z =200.

4.6 SIMPLEX PROCEDURE FOR A MINIMIZATION PROBLEM WITH 'GREATER THAN TYPE' CONSTRAINTS

The computational procedure of simplex method explained in section 4.5 is applicable for a LPP with maximization objective and 'less than type' (≤) constraints. There may be problems with constraints of 'greater than type' or 'equal to' (≥ or =) type and objective function may be of minimization. In such cases, some modification in the simplex method is needed. This is explained below-

Step 1- Formulate the given problem into mathematical form of LPP as shown below-

Minimize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots + c_n x_n$

Subject to linear constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n \geq b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n \geq b_2$$

$$\begin{matrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{matrix}$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n \geq b_m$$

where $x_1, x_2, \ldots\ldots\ldots x_n \geq 0$

Step 2- Convert the given formulated LPP with minimization objective and 'greater than type' constraints into standard form. To convert inequalities into equations subtract surplus variables to the left hand side of 'greater than type' constraints. The problem then takes the following form-

Minimize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots + c_n x_n + 0.s_1 + 0.s_2 + \ldots.. + 0.s_m$

Subject to linear constraints

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n - s_1 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n - s_2 = b_2$$

$$\begin{matrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{matrix}$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n - s_m = b_m$$

where $x_1, x_2, \ldots\ldots\ldots x_n \geq 0$ and $s_1, s_2, \ldots\ldots, s_m \geq 0$

An initial basic solution is obtained by putting $x_1 = 0, x_1 = 0, \ldots\ldots x_n = 0$. Thus the solution will be- $S_1 = -b_1, S_2 = -b_2, \ldots\ldots, S_m = -b_m$.

This solution is not feasible because it violates the non-negativity constraints, therefore, a modification in the simplex algorithm is needed.

Step 3- Introduce Artificial variables

A set of m new variables $A_1, A_2, \ldots\ldots, A_m$ called as Artificial Variables is introduced into the system. These artificial variables are included one by one to each constraint equation (m constraints). So the revised constraints equations can be written as follows:

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n - s_1 + A_1 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n - s_2 + A_2 = b_2$$

.   .   .   .

.   .   .   .

.   .   .   .

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{mn} x_n - s_m + A_m = b_m$

where $x_1, x_2, \ldots\ldots\ldots\ldots x_n \geq 0$; $s_1, s_2, \ldots\ldots, s_m \geq 0$ and $A_1, A_2, \ldots\ldots, A_m \geq 0$.

Now the standard LPP includes system of m equations with n +m + m variables (n decision variables, m surplus variables and m artificial variables). An initial basic solution can now be obtained by putting all $x_j = 0$ and $S_j = 0$, which is given as $A_1 = b_1$, $A_2 = b_2, \ldots\ldots, A_m = b_m$.

However, this solution does not provide solution to original problem, therefore, to obtain solution to the original problem; artificial variables are to be removed from the system.

There are two techniques to remove artificial variables from the solution.

a. Two Phase Method

b. Big M- Method

4.6.1 TWO PHASE METHOD

In this method, there are two phases-

Phase 1- In phase 1, we remove the artificial variables from the basis and introduce other variables.

Phase 2- In phase 2, the solution of phase 1 is used as the initial basic feasible solution and simplex procedure is applied to determine the optimal solution.

The procedure of two phase method is explained below-

Phase 1

Step 1- convert each constraint into equation by adding slack, surplus and artificial variables.

Step 2- Obtain new objective function (Z') by assigning costs -1 to each artificial variable and 0 cost to all other decision variables, slack and surplus variables. Thus

Z' = - (sum of the artificial variables)

Step 3- Using simplex method maximize the new objective function Z' subject to the constraints of original problem. The optimal value of the new objective function Z' will be equal to 0 as each of the artificial variables will be equal to 0. There can be three possibilities-

- If max, Z'<0 and at least one artificial variable is in the basis at a positive level, in such case, the given problem does not have any feasible solution.

- If max, Z'= 0 and at least one artificial variable is in the basis at a zero level, in such case, proceed to phase 2 for finding optimal solution.
- If max, Z'= 0 and no artificial variable is in the basis at a positive level, then also proceed to phase 2.

It is to be noted that new objective function Z' is always of maximization irrespective of the objective function in original problem.

Phase 2

In this phase, the optimal solution obtained in phase 1 is considered as initial basic feasible solution. Here the objective function of original problem (if minimization) is converted into maximization type by multiplying both sides of objective function with -1. Now there are only decision variables with actual costs and slack/surplus variables with zero costs remain in the objective function The problem then is solved using normal simplex method to find the optimal solution.

Example 4.2- Solve the following LPP

Minimize $Z = x_1 + x_2$

Subject to constraints

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0$

Solution- The given problem is of minimization with the greater than type constraints. First we convert the objective function into maximization type and convert the given LPP in standard form by introducing surplus variables S_1, S_2 and artificial variables A_1, A_2 as follows-

Maximize $Z' = -Z = -x_1 - x_2 + 0.S_1 + 0.S_2 - A_1 - A_2$

Subject to constraints

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Phase 1-

Writing the new objective function

$Z'' = 0.x_1 + 0.x_2 + 0.S_1 + 0.S_2 - A_1 - A_2$

Subject to the constraints as given in the standard form.

Now, putting $x_1 = 0$, $x_2 = 0$, $S_1 = 0$, and $S_2 = 0$ we obtained initial basic feasible solution, $A_1 = 4$, and $A_2 = 7$. The simplex tableau can be written as follows-

Table 4.5 Initial Simplex Tableau

C_j			0	0	0	0	-1	-1	Min. Ratio X_B/X_2
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	
-1	A_1	4	2	1	-1	0	1	0	4/1=4
-1	A_2	7	1	**7***	0	-1	0	1	**7/7=1** →
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		-11	-3	-8	1	1	-1	-1	
Net evaluation per unit $C_j - Z_j$			3	**8** ↑	-1	-1	0	0	

As shown in the table since all values in C_j - Z_j row are not negative or zero therefore the solution is not optimal. To revise the solution the new incoming variable will be X_2 and outgoing variable will be A_2. So A_2 will now be removed from the computations. The revised simplex tableau 1 is given below-

Table4.6 Revised Simplex Tableau 1

C_j			0	0	0	0	-1	Min. Ratio X_B/X_1
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	
-1	A_1	3	**13/7***	0	-1	1/7	1	21/13 →
0	X_2	1	1/7	1	0	-1/7	0	7
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		-3	-13/7	0	1	-1/7	-1	

Net evaluation per unit $C_j - Z_j$	**13/7** ↑	0	-1	1/7	0	

Here also net evaluation for X_1 is largest positive so x1 will be the incoming variable and A_1 will be the outgoing variable. Now A_1 will be dropped from the simplex table. The revised simplex tableau 2 is given below.

Table 4.7 Revised Simplex Tableau 2

C_j			0	0	0	0	Min. Ratio
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	
0	X_1	21/13	1	0	-7/13	1/13	
0	X_2	10/13	0	1	1/13	-2/13	
Contribution loss per unit $Z = \sum C_{Bj} a_{ij1}$		0	0	0	0	0	
Net evaluation per unit $C_j - Z_j$			**0**	0	0	0	

Since in the above table 4.7 all values in $C_j - Z_j$ row are zero and no artificial variable is left in the basis, therefore, now we can proceed for phase 2.

Phase 2-

In this phase the objective function will be rewritten by considering actual costs in objective function and deleting artificial variables.

Maximize Z' = -Z = -- $x_1 - x_2 + 0.S_1 + 0.S_2$

Now the new simplex tableau is written with the final solution of phase 1 as initial basic feasible solution.

Table 4.8 Final Solution

C_j			-1	-1	0	0	Min. Ratio
C_B	Basic	Value of	X_1	X_2	S_1	S_2	

	Variable B.V.	B.V. X_B				
-1	X_1	21/13	1	0	-7/13	1/13
-1	X_2	10/13	0	1	1/13	-2/13
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		-31/13	-1	-1	6/13	1/13
Net evaluation per unit $C_j - Z_j$			**0**	0	-6/13	-1/13

Since all net evaluation (Cj- Zj) values are either zero or negative, therefore, this solution is optimal solution.

The optimal solution is- $x_1 = 21/13$, $x_2 = 10/13$ and min Z = -Z' = 31/13.

4.6.2 BIG M-METHOD

In Big M-Method, a very large value -M for maximization problem (in case problem is of minimization convert into maximization by multiplying objective function with -1 on both sides) is assigned to each of the artificial variables and a zero value is assigned to each of the surplus variable in the objective function. Due to the presence of large value M, the objective function cannot be optimized, so we first remove the artificial variables from the initial basic solution, as soon as one artificial variable leaves the basis, it cannot enter into the basis again and it is not included in the subsequent simplex tableau. This procedure can be understood with the help of following example-

Example 4.3- Solve the following LPP using Big M- Method.

Minimize $Z = 4x_1 + 6x_2$

Subject to constraints

$x_1 + 2x_2 \geq 80$

$3x_1 + x_2 \geq 75$

$x_1, x_2 \geq 0$

Solution- The given problem is of minimization with the greater than type constraints. First we convert the objective function into maximization type and convert the given LPP in standard form by introducing surplus variables S_1, S_2 and artificial variables A_1, A_2 as follows-

Maximize $Z' = -Z = -4x_1 - 6x_2 + 0.S_1 + 0.S_2 - M.A_1 - M.A_2$

Subject to constraints

$x_1 + 2x_2 - S_1 \quad + A_1 \quad = 80$

$3x_1 + x_2 \quad - S_2 \quad + A_2 \quad = 75$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$ and M is a very large positive value.

Now, putting $x_1 = 0$, $x_2 = 0$, $S_1 = 0$, and $S_2 = 0$ we obtained initial basic feasible solution, $A_1 = 80$, and $A_2 = 75$. The simplex tableau can be written as follows-

Table 4.9 Initial Simplex Tableau

C_j			-4	-6	0	0	-M	-M	Min. Ratio X_B/X_1
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	
-M	A_1	80	1	2	-1	0	1	0	80
-M	A_2	75	**3***	1	0	-1	0	1	**25** →
Contribution loss per unit $Z = \sum C_{Bj} a_{ij1}$		-140M	-4M	-3M	M	M	-M	-M	
Net evaluation per unit $C_j - Z_j$			4M-4 ↑	3M-6	-M	-M	0	0	

Table 4.10 Revised Simple Tableau 1

C_j			-4	-6	0	0	-M	-M	Min. Ratio X_B/X_2
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	
-M	A_1	55	0	**5/3***	-1	1/3	1	-1/3	**33** →
-4	X_1	25	1	1/3	0	-1/3	0	1/3	75

Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$	-55M +100	-4	-(5M +4)/3	M	(-M+4)/3	-M	(M-4)/3
Net evaluation per unit $C_j - Z_j$		0	(5M-14)/3 ↑	-M	(M-4)/3	0	(-4M+4)/3

Table 4. 11 Final Simplex Tableau

C_j			-4	-6	0	0	-M	-M	Min. Ratio
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	
-6	X_2	33	0	1	-3/5	1/5	3/5	-1/5	
-4	X_1	24	1	2	1/5	-2/5	-1/5	2/5	
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		--294	-4	-6	14/5	2/5	-14/5	-2/5	
Net evaluation per unit $C_j - Z_j$			0	0	-14/5	-2/5	-M+14/5	-M+2/5	

Since all net evaluation (Cj- Zj) values are either zero or negative, therefore, this solution is optimal solution.

The optimal solution is- $x_1 = 24$, $x_2 = 33$ and min Z = - maximum Z' = 294.

4.7 SIMPLEX PROCEDURE FOR A MIXED CONSTRAINT PROBLEM

There can be a linear programming problem with mixed constraints i.e. less than type, greater than type and equal to constraints. In such problems, slack, surplus and artificial variables are added simultaneously. The problem can then be solved by either two phase method or Big M- method. The following example can explain the procedure of solving such problems.

Example 4.4 Solve the following LPP.

Maximize $Z = 2x_1 + 4x_2$

Subject to constraints

$2x_1 + x_2 \leq 18$

$3x_1 + 2x_2 \geq 30$

$x_1 + 2x_2 = 26$

$x_1, x_2 \geq 0$

Solution- The given problem is of maximization with mixed constraints. First we convert the given LPP in standard form by introducing slack, surplus variables S_1, S_2 and artificial variables A_1, A_2 as follows-

Maximize $Z = 2x_1 + 4x_2 + 0.S_1 + 0.S_2 - M.A_1 - M.A_2$

Subject to constraints

$2x_1 + x_2 + S_1 = 18$

$3x_1 + 2x_2 - S_2 + A_1 = 30$

$x_1 + 2x_2 + A_2 = 26$

$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$ and M is a very large positive value.

Now, putting $x_1 = 0$, $x_2 = 0$, and $S_2 = 0$ we obtained initial basic feasible solution, $S_1 = 18$ $A_1 = 30$, and $A_2 = 26$. The simplex tableau can be written as follows-

Table 4.12 Initial Simplex Tableau

C_j			2	4	0	0	-M	-M	Min. Ratio
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	X_B/X_2
0	S_1	18	2	1	1	0	0	0	18
-M	A_1	30	3	2	0	–1	1	0	15
-M	A_2	26	1	2*	0	0	0	1	13 →

Contribution loss per unit $Z = \sum C_{Bj} a_{ij1}$	56M	-4M	-4M	0	M	-M	–M
Net evaluation per unit $C_j - Z_j$		4M+2	4M+4 ↑	0	–M	0	0

Table 4.13 Revised Simple Tableau 1

C_j			2	4	0	0	-M	-M	Min. Ratio X_B/X_1
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	
0	S_1	5	3/2	0	1	0	0	–1/2	10/3
-M	A_1	4	2*	0	0	–1	1	–1	2 →
4	X_2	13	1/2	1	0	0	0	1/2	26
Contribution loss per unit $Z = \sum C_{Bj} a_{ij1}$		52-4M	2-2M	4	0	M	-M	2+M	
Net evaluation per unit $C_j - Z_j$			2M ↑	0	0	–M	0	-2-2M	

Table 4.14 Final Simple Tableau

C_j			2	4	0	0	-M	-M	Min. Ratio
C_B	Basic Variable B.V.	Value of B.V. X_B	X_1	X_2	S_1	S_2	A_1	A_2	

0	S_1	2	0	0	1	3/4	-3/4	1/4	
2	X_1	2	1	0	0	-1/2	1/2	-1/2	
4	X_2	12	0	1	0	1/4	-1/4	3/4	
Contribution loss per unit $Z = \sum C_{Bj}a_{ij1}$		52	2	4	0	0	0	2	
Net evaluation per unit $C_j - Z_j$			0	0	0	0	-M	-M-2	

Since all values in the net evaluation (Cj–Zj) row are either zero or negative, therefore the present solution is the optimal solution. The optimal solution is X1=2, X2=12, Z=52.

4.8SOME SPECIAL CASES

4.8.1 MULTIPLE OPTIMAL SOLUTIONS

It is possible that the solution of a Linear Programming Problem may not be unique. A given LPP may have more than one optimal solution. Such a case is called as the case of multiple optimal solutions. In case of the final optimal solution obtained by simplex tableau, if the (cj – zj) value for a non-basic variable is zero in the final table, then the solution is not unique and multiple solutions exist.

4.8.2 INFEASIBILITY

Infeasibility is the condition when the given problem does not have any feasible solution. When in the final solution, an artificial variable is in the basis with a positive solution value, then there is no feasible solution to the problem.

4.8.3 UNBOUNDEDNESS

A LPP is said to have an unbounded solution if the value of objective function can be increased (for maximization problem) or decreased (for minimization problem) without limit. In simplex method, when a solution is not the optimal solution and is to be further improved but there are no non-negative value in the minimum replacement ratio column in order to select the outgoing variable, then the solution is terminated and it indicates that the given problem has unbounded solution.

4.8.4 DEGENERACY

Degeneracy in a linear programming problem is the condition when one or more basic variable has a zero solution value. Basically for a n- variable, m-constraints problem, there are m basic and n-m non-basic variables, and the basic variables would assume positive values. But in case, one or more basic variables have a value equal to zero. Then the solution is called to be degenerate. Therefore, in the case of degeneracy, the number of non-zero variables would be smaller than the number of constraints, m.

While solving a LPP, if there is a tie between two or more basic variables for leaving the basis, i.e. minimum ratio is same for two rows or values of one or more basic variables in the solution X_B column become zero, it causes the problem of degeneracy. Degeneracy may be avoided by proper selection of the variable leaving the basis, but if the minimum ratio is zero, then the simplex iterations repeat indefinitely without arriving at optimal point.

4.9 SUMMARY

In this unit, simplex method of solving linear programming problems is presented. Simple method is a step by step procedure for finding the optimal solution of a linear programming problem. The process starts with a basic feasible solution which is tested for optimality, if the solution is not optimal, a better revised feasible solution is obtained, and the process of improving the solution continues and stops only when an optimal solution is obtained. To find a solution by simple method, a linear problem is converted into standard form where constraints are converted into equality constraints by including, slack, surplus or artificial variables. The initial simplex tableau is displayed indicating the initial basic feasible solution and this solution is tested for optimality by noting the signs of values in cj-zj row. If the solution is not optimal, it is revised as per the rules of revision of simplex table.

There can be various types of special cases in solving a linear problem by simplex method. A problem may have infeasible solution, when an artificial variable is left present in the final solution. A linear programming problem may have unbounded solution if the solution is not optimal and all replacement ratios are negative, or a mix of negative and infinity type. A solution may be degenerate, if some basic variables has a solution value equal to zero.

4.10 SELF-ASSESSMENT QUESTIONS

a. Explain the concept of simple method for solving linear programming problems.

b. Define the computational procedure of simplex method.

c. What are slack, surplus and artificial variables?

d. What is a pivot column, pivot row and pivot element?

e. Describe two phase method of solving linear programming problems.

f. Explain Big M-Method of solving linear problems.

g. Solve the following linear programming problems using simplex method.

Maximize $Z = 7x_1 + 14x_2$	Maximize $Z = 21x_1 + 15x_2$
Subject to the constraints	Subject to the constraints
$3x_1 + 2x_2 \leq 36$	$x_1 + 2x_2 \leq 6$
$x_1 + 4x_2 \leq 10$	$4x_1 + 3x_2 \leq 12$
$x_1, x2 \geq 0$	$x_1, x2 \geq 0$

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UNIT-5 TRANSPORTATION PROBLEMS

Unit Framework

5.1 OBJECTIVE

After completing this unit you will be able to:

1. Understand and formulate a transportation problem

2. Find out basic feasible solution of transportation problem by various methods

3. Find most optimum solution of a transportation method

4. Describe special methods like maximization, multiple optimum solution, degenerate transportation problem and prohibited & preferred routes and to solve these.

5.2 INTRODUCTION

The transportation problem is a special case of linear programming problem in which the objective is to transport a commodity or to provide services from several supply origin to different demand destination at a minimum total cost. The transportation problem deals with the transportation of a product manufactured at different plants or factories (called as sources or supply origin) to a number of different warehouses (called as demand destination). The objective is to satisfy the destination requirement within the plant capacity constraints at a minimum transportation cost. The transportation problem can easily be expressed by linear relationship as a linear programming problem and can be solved by simplex method but a special method is used to solve these problems which are known as transportation method.

5.3 GENERAL STRUCTURE

The transportation problem can be described as follows:

Suppose that a manufacturer of LCD has three plants A, B, and C, further, that the buyers are located in three locations X, Y, and Z where he has to supply them the goods. Also, assume that the demand in the three regions and the production in different plants per unit of time are known and equal in aggregate, and further that the cost of transportation for one unit of LCD from each plant to each location is given and constant. The problem is to determine the quantity each plant should transport t each warehouse to minimize total costs.

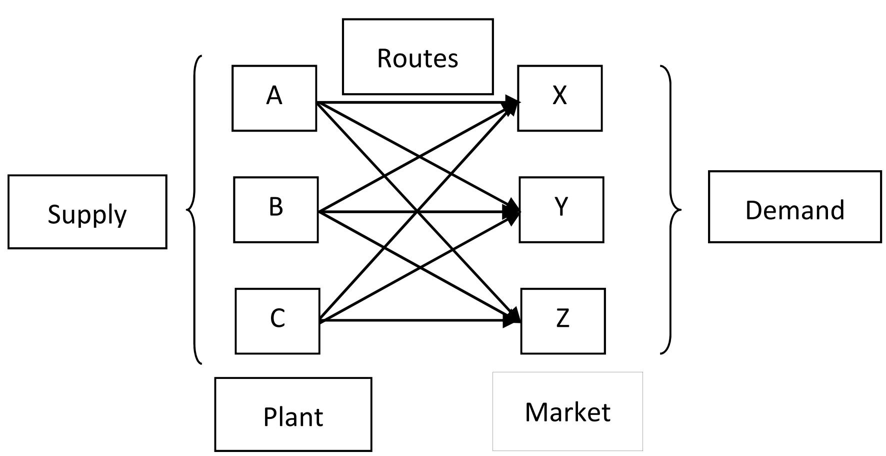

Figure 5.1: Transportation Problem

5.3.1 GENERAL STRUCTURE OF A TRANSPORTATION PROBLEM

The standard transportation problem can be expressed in the form of a table which displays the values of all the data coefficients (supply s_i, demand d_j and cost c_{ij}) associated with the problem.

The general transportation problem with m-sources and n-destinations is represented in the following table-

Sources	Destinations						Supply
	D_1	D_2	. . .	D_j	…	D_n	
S_1	X_{11} C_{11}	X_{12} C_{12}		X_{1j} C_{1j}		X_{1n} C_{1n}	s_1
S_2	X_{21} C_{21}	X_{22} C_{22}		X_{2j} C_{2j}		X_{2n} C_{2n}	s_2
. . .							. . .
S_i	X_{i1} C_{i1}	X_{i2} C_{i2}		X_{ij} C_{ij}		X_{in} C_{in}	s_i
. . .							. . .
S_m	X_{m1} C_{m1}	X_{m2} C_{m2}		X_{mj} C_{mj}		X_{mn} C_{mn}	s_m
Demand	d_1	d_2	…	d_j	…	d_n	$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

The problem is to distribute the available product to different destinations in such a way so as to minimize the total transportation cost for all the possible source-destination shipping patterns.

Here i = index for source; where i=1,2,3…..m j = index for destination; where j=1,2,3…..n

X_{ij} = numbers of units shipped per route from the source i to destination j for each route.

C_{ij} = Cost per unit of shipping from source i to destination j.

s_i = supply in units at source i.

d_j= demand in units at destination j.

5.4 MATHEMATICAL FORMULATION OF A TRANSPORTATION PROBLEM

Mathematically the transportation problem can be stated as follows-

Find x_{ij}(I = 1,2,…., m; j = 1. 2…… n)for which the total transportation cost

$z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij}.X_{ij}$ is minimized, subject to the restrictions

$$\left.\begin{array}{ll} \sum_{j=1}^{n} x_{ij} = s_i & \text{Where } i=1,2,3\ldots..m \\ \sum_{i=1}^{m} x_{ij} = d_j & \text{Where } j=1,2,3\ldots..n \end{array}\right\}$$

$\sum_{i=1}^{m} s_i = \sum_{j=1}^{n} d_j$

And, $x_{ij} \geq 0 \quad for\ all\ i = 1,2, \ldots..m; j = 1,2, \ldots \ldots.n$

thus, the transportation problem is a linear programming problem of special type, where we are required to find the values on m, n variables that minimize the objective function Z , satisfying (m+n) restrictions and non negative restrictions of variables.

5.5 FEW IMPORTANT DEFINITIONS

Let us describe few terms that are used in the transportation problems.

1. **A Feasible Solution-** A feasible solution of a transportation problem is a set of non-negative individual allocation ($x_{ij} \geq 0$) which satisfies the row and column sum restrictions.

2. **Basic Feasible Solution-** A basic feasible solution is that feasible solution where total number of possible allocations X_{ij} is exactly equal to m + n -1, i.e. summation of no. of rows and no. of columns -1.

3. **Optimum Solution-** An optimum solution is that feasible solution which minimizes the total transportation cost.

4. **Non-Degenerate basic feasible solution-** A feasible solution of m by n transportation problem is called non-degenerate basic feasible solution if

   (i) No. of positive allocations = m + n – 1

   (ii) all allocations are at independent positions.

5.6 PROCEDURE OF SOLVING TRANSPORTATION PROBLEMS

The following steps are used to solve the transportation problem.

1. Define the objective function in minimization terms along with the constraints of the problem.
2. Set up the transportation table with m rows (sources) and n columns (destinations). The total supply of sources should be equal to the total demand of destinations.
3. Develop an initial basic feasible solution to the problem.
4. Examine whether the IBFS is non- degenerate or not. The solution is called non-degenerate if the solution has allocations in (m +n -1) cells with independent positions.
5. Test the solution obtained in step 4 for optimality. It is done by calculating opportunity costs for all the empty cells. If the opportunity costs for all empty cells are found to be positive, then the solution is called a optimum solution.
6. If the solution is non-optimal, modify the solution by including allocations to an empty cell whose inclusion results in higher savings.
7. Repeat steps 5 & 6 until an optimum solution is obtained.

Example 5.1: A production house has to transport goods from four factories namely A, B, C and D to three warehouses namely X, Y and Z. the cost of transportation from each factory to each warehouse is given in the following table. The available supply at each factory and the requirement at each warehouse is also mentioned in the table. Determine the optimum transportation schedule so as to minimize the total cost of transportation.

Transportation cost /unit

Factory	Warehouse			Supply
	X	Y	Z	
A	2	7	4	5
B	3	3	1	8
C	5	4	7	7
D	1	6	2	14
Demand	7	9	18	34

5.6.1 METHODS OF FINDING INITIAL BASIC FEASIBLE SOLUTION

The given problem has an objective of minimization with the given supply and demand constraints. The transportation cost table is given where the total supply is equal to total demand that means the problem is a balanced problem.

Now we have to find an initial Basic Feasible solution. There are various methods of finding IBFS.

5.6.1.1 METHOD-1 NORTH WEST CORNER RULE

This is the simplest method to find the IBFS. It is called as NWCR method because it uses the North-West corner cell i.e. upper left corner cell to start allocating units in the solution.

Step I- Assign as many units as possible to the north-west corner cell.

Step II- the satisfied row or column is crossed out and the amount of supply and demand are adjusted accordingly.

Step III- look for the north-west corner cell in the remaining table and repeat the step I and II until all demand and supply conditions are met.

The above problem can be solved by NWCR method as follows.

Step I- first the allocations are made to the top most left corner cell. The maximum possible amount in this cell is 5 (since Minimum of s1=5 and d1 =7 is 5).

Step II- the supply for first row is met therefore this row is crossed out and the demand for first column is reduced to d1 =2 (7-5 =2).

Step III- now the north-west corner cell is cell (2,1) in the original table. The maximum number of allocations possible in this cell is 2 (since minimum of s2 =8 and d1 =2 is 2). Now allocation for first column is complete, therefore first column is crossed out and the supply of second row is reduced to s2 = 6 (8-2=6). The NW corner cell now is cell (2,2). The maximum allocations possible to this cell is 6 (since minimum of s2 =6 and d2 =9 is 6). This way the supply of row 2 is complete therefore this row is crossed out and the demand of column 2 is reduce to d2 =3 (9-6=3). The NW corner cell is now cell(3,2). The maximum allocations possible for this cell is 3(since minimum of s3 =7 and d2 =3 is 3). Now the demand of column2 is complete therefore column 2 is crossed out and supply of row 3 is reduced to s3= 4(7-3 =4). The north-west corner cell in the remaining table is cell (3,3). Maximum no. of possible allocations to this cell is 4 (since minimum of s3 =4 and d3 = 18 is 4). This way the supply of row 3 is met and the demand of column 3 is reduced to d3 =14 (18-4 =14). Row 3 is then crossed out. The only cell left is cell (4,3) where demand is d3= 14 and supply is also s4 =14. This will complete the allocation and the resulting feasible solution is shown in the table below.

The values in () are the number of allocated units on each route.

Factory	Warehouse			Supply
	X	Y	Z	
A	2 (5)	7	4	5
B	3(2)	3(6)	1	8
C	5	4(3)	7(4)	7
D	1	6	2(14)	14
Demand	7	9	18	34

The total cost of transportation for this feasible solution can be calculated by multiplying each individual allocation by its corresponding cost and adding this product for all allocated cells.

Total cost = 5*2 + 2*3 + 6*3 +3*4 + 4*7 +14*2 =Rs.102.

5.6.1.2 METHOD- 2 LEAST COST METHOD

In this method, the units are first allocated to the cell with lowest cost in the matrix. Following steps are used in this method.

Step I- write the transportation problem in a tabular form.

Step II- Assigns as much as possible to the cell with the smallest unit cost.

Step III- the satisfied row or column is crossed out and the amount of supply and demand are adjusted accordingly.

Step IV- Look for the uncrossed out cell with the smallest unit cost and repeat the process until all demand and supply condition is met.

The above example can be solved by least square method in the following way-

The least cost cell in the given transportation cost matrix is cell (2,3) with cost Rs.1 and cell (4,1) with cost Rs.1. we choose the cell (2,3) as we can allocate maximum amount 8 to this cell. Leaving this cell we find that there is lowest cost Rs.1 in cell (4,1) where we allocate the maximum amount 7. Continuing in this way, we get the required feasible solution as shown in the table.

Factory	Warehouse			Supply
	X	Y	Z	
A	2	7(2)	4(3)	5
B	3	3	1 (8)	8

C	5	4(7)	7	7
D	1(7)	6	2(7)	14
Demand	7	9	18	34

The total transportation cost of this feasible solution is

Z = 2*7 + 3*4 + 8 *1 + 7 *1 + 7*1 + 7*2= Rs. 83.

The cost for this solution is less than the cost obtained with the solution by NWCR method.

5.6.1.3 METHOD-3 VOGEL'S APPROXIMATION METHOD (VAM)

VAM is an improved version of the least cost method that generally, but not always, produces more efficient starting solutions.

Step I- Write the TP in the tabular form

Step II- For each row (column), determine a penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (column)

Step III- Identify the row or column with the largest penalty. Break ties arbitrarily. Allocate as much as possible to the variable with the least unit cost in the selected row or column. Adjust the supply and demand, and cross out the satisfied row or column.

Step IV- if one row (column) with positive supply (demand) remains uncrossed out; determine the basic variable in the row (column) by the least cost method. Stop otherwise, go to step II.

The above example can be solved by using Vogel's Approximation Method in the following way-

Since the cost matrix is already given, therefore we can start from step II.

The penalty for each row and column is calculated and represented in the following table.

Factory	Warehouse			Supply	**Penalty**
	X	Y	Z		
A	2	7	4	5	**2**
B	3	3	1(8)	8	**(2)**
C	5	4	7	7	**1**

NODE-129/NODE-106

D	1	6	2	14	**1**
Demand	7	9	18		
Penalty	**1**	**1**	**1**		

Since the maximum penalty (2) is corresponding to row 1 & 2 both. Maximum amount that can be allocated in lowest cost cell of row 1 i.e. cell(1,1) is 5 and maximum 8 can be allocated to the lowest cost cell (2,3) in Row 2, therefore we choose maximum penalty for row 2 first and allocate maximum amount i.e. 8 to cell (2,3) in row 2. The supply for this row is then complete, so this row will be eliminated from the matrix and demand for column 3 is reduced to 10 (18-8=10). The new matrix thus obtained is shown below.

Factory	Warehouse			Supply	**Penalty**
	X	Y	Z		
A	2	7	4	5	**2**
C	5	4	7	7	**1**
D	1	6	2(10)	14	**1**
Demand	7	9	10		
Penalty	**1**	**2**	**(2)**		

Now the maximum penalty is corresponding to row 1 and column 2 & 3, out of which the maximum possible allocation is 10 in least cost cell (3,3) in column 3. Therefore the allocation of 10 units is made in this cell (3,3). The demand for the column 3 is now complete, so this column will now be eliminated from the matrix and demand for row 3 is reduced to 4 (14-10=4). The new matrix with remaining demand and supply is as follows-

Factory	Warehouse		Supply	**Penalty**
	X	Y		
A	2	7	5	**5**
C	5	4	7	**1**
D	1(4)	6	4	**(5)**

Demand	7	9		
Penalty	**1**	**2**		

In this matrix, the maximum penalty is corresponding to row 1 & 3, out of which maximum possible allocations is 4 to the least cost cell (3,1) in row 3. Therefore, the allocation in cell (3,1) is made for 4 units. The supply of row 3 is now complete, so this row can be crossed out from the matrix and supply of column 1 is reduced to 3 (7-4 =3). The new matrix with remaining demand and supply is given below-

Factory	Warehouse		Supply	**Penalty**
	X	Y		
A	2 (3)	7	5	**5**
C	5	4	7	**1**
Demand	3	9		
Penalty	**3**	**3**		

Here the maximum penalty is for row 1 and the maximum possible allocation is 3 in the least cost cell of row 1 i.e. cell(1,1). The demand for column 1 is now complete therefore this column will be removed from the matrix and supply of row 1 will be reduced to 2 (5 -3 =2).

Now only one column 2 is remaining in the matrix, therefore the penalty is no more calculated and the allocation is done according to least cost method. Allocation is made first to least cost cell (2,1) for 7 units and then 2 units are allocated to the remaining cell (1,1). This is shown in the table shown below-

Factory	Warehouse	Supply
	Y	
A	7 (2)	2
C	4 (7)	7
Demand	9	

Thus, the required feasible solution obtained through VAM is shown in the following table.

Factory	Warehouse			Supply
	X	Y	Z	
A	2(3)	7(2)	4	5
B	3	3	1 (8)	8
C	5	4(7)	7	7
D	1(4)	6	2(10)	14
Demand	7	9	18	34

The total cost of transportation with VAM is

Z = 3*2 + 2*7 + 8*1 +7*4 + 4*1 + 10*2 = Rs. 80.

The cost of transportation for the solution obtained through VAM is lesser than the cost with the solutions obtained with the previous two methods, therefore, VAM is considered to be the best amongst all methods of finding initial feasible solution.

5.6.2 TEST FOR OPTIMALITY

Once the initial basic feasible solution is obtained, the solution is then tested for optimality that means we check whether the obtained solution minimizes the transportation cost or not. There are two methods for checking optimality, these are (i) stepping stone method and (ii) Modified Distribution (MODI) method. Out of these two the one which is widely used is Modified Distribution Method.

5.6.2.1 MODIFIED DISTRIBUTION (MODI) METHOD-

To obtain the optimal solution of a minimization transportation problem a systematic procedure is followed. This iterative procedure of determining the optimal solution is known as MODI Method.

The following steps are involved in this method-

1. Determine an initial basic feasible solution using any one of the three methods given below: a) North West Corner Rule b) Matrix Minimum Method c) Vogel Approximation Method
2. Determine the values of dual variables, u_i and v_j, by assuming one value of either u or v as zero and using $u_i + v_j = c_{ij}$ relationship for all occupied (allocated) cells.
3. Compute the opportunity cost using $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for all empty cells.

4. Check the sign of each opportunity cost. a) If the opportunity costs of all the unoccupied cells are either positive or zero, the given solution is the optimal solution. On the other hand, b) if one or more unoccupied cell has negative opportunity cost, the given solution is not an optimal solution and further savings in transportation cost are possible.

5. Select the unoccupied cell with the smallest negative opportunity cost as the cell to be included in the next solution.

6. Draw a closed path or loop for the unoccupied cell selected in the previous step. Please note that the right angle turn in this path is permitted only at occupied cells and at the original unoccupied cell.

7. Assign alternate plus and minus signs at the unoccupied cells on the corner points of the closed path with a plus sign at the cell being evaluated.

8. Determine the maximum number of units that should be shipped to this unoccupied cell. The smallest value with a negative position on the closed path indicates the number of units that can be shipped to the entering cell. Now, add this quantity to all the cells on the corner points of the closed path marked E plus signs, and subtract it from those cells marked with minus signs. In this way, an unoccupied cell becomes an occupied cell.

9. Repeat the whole procedure until an optimal solution is obtained.

Example 5.1 continued:

Step 1. The initial basic feasible solution for example 5.1 obtained through Least cost Method is given below-

Factory	Warehouse			Supply
	X	Y	Z	
A	2	7(2)	4(3)	5
B	3	3	1 (8)	8
C	5	4(7)	7	7
D	1(7)	6	2(7)	14
Demand	7	9	18	34

Total transportation cost = 2*7 + 3*4 + 8 *1 + 7 *1 + 7*1 + 7*2= Rs. 83.

Step 2. Now we determine a set of dual variables u_i and v_j, using $u_i + v_j = c_{ij}$ for each occupied cell. This is shown in the table below-

We first choose $v_3 = 0$ (since column 3 has maximum no. of occupied cells).

Now, $c_{13} = u_1 + v_3$, since $c_{13} = 4$ and $v_3 = 0$ therefore, $u_1 = 4$

Similarly $c_{23} = u_2 + v_3$, since $c_{23} = 1$ and $v_3 = 0$ therefore, $u_2 = 1$

$c_{43} = u_4 + v_3$, since $c_{43} = 2$ and $v_3 = 0$ therefore, $u_4 = 2$.

$c_{12} = u_1 + v_2$, since $c_{12} = 7$ and $u_1 = 4$ therefore, $v_2 = 3$.

$c_{32} = u_3 + v_2$, since $c_{32} = 4$ and $v_2 = 3$ therefore, $u_3 = 1$.

$c_{41} = u_4 + v_1$, since $c_{41} = 1$ and $u_4 = 2$ therefore, $v_1 = -1$

Factory	Warehouse			u_i
	X	Y	Z	
A	2	7(2)	4(3)	4
B	3	3	1 (8)	1
C	5	4(7)	7	1
D	1(7)	6	2(7)	2
v_j	-1	3	0	

Step 3. Now we compute the opportunity cost using $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for all empty cells.

The calculated opportunity cost is shown in the lower left corner of each empty cell.

Factory	Warehouse			u_i
	X	Y	Z	
A	+ 2 [-1]	7(2)	– 4(3)	4
B	3 [3]	3 [-1]	1 (8)	1
C	5 [5]	4(7)	7 [6]	1
D	- 1(7)	6 [1]	2(7) +	2
v_j	-1	3	0	

Step 4. Since opportunity cost for two cells (1,1) and (2,2) is negative, therefore the solution is not optimal. It is therefore required to revise the solution.

Step 5, 6, 7. The smallest negative opportunity cost is -1 in two cells (1,1) and (2,2). We choose cell (1,1) as c_{11} is smaller than c_{22} and make a closed loop starting from cell (1,1) as shown in the above table. This loop takes right turn only at occupied cells. Alternate + and – sign is allocated to the corner cells starting with a + sign in the original empty cell.

Step 8. The minimum no. of allocations among –ve signed cells i.e. cell (1,3) and cell (4,1) is 3. This no. of allocation is added to + signed cells and reduced from –ve signed cells. Hence the revised solution obtained is as given below.

Factory	Warehouse		
	X	Y	Z
A	2(3)	7(2)	4
B	3	3	1 (8)
C	5	4(7)	7
D	1(4)	6	2(10)

Step 9. We again check the optimality for this revised solution by repeating steps 2 to 5.

We again calculated variables u_i and v_j for each row and column by assuming $v_3 = 0$. We used cells (1,1), (1,2), (2,3), (3,2), (4,1), and (4,3) for relation $u_i + v_j = c_{ij}$ to calculate all u and v as shown below

Factory	Warehouse			u_i
	X	Y	Z	
A	2(3)	7(2)	4	3
B	3	3	1 (8)	1
C	5	4(7)	7	0
D	1(4)	6	2(10)	2
v_j	-1	4	0	

Again we calculated opportunity costs for all empty cells using $\Delta_{ij} = c_{ij} - (u_i + v_j)$ as shown below.

Factory	Warehouse			u_i
	X	Y	Z	
A	2(3) +	7(2) -	4 [1]	3
B	3 [3]	+ 3 [-2]	- 1 (8)	1
C	5 [2]	4(7)	7 [7]	0
D	1(4) -	6 [0]	2(10) +	2
v_j	-1	4	0	

Since there is one cell (2,2) with negative opportunity cost so this solution is also not optimal. In order to revise the solution, a loop is formed starting from cell (2,2) and ending at same, which takes right turn only at occupied cells as shown in the table above. Alternate + and – signs are allotted to the corner cells of the loop starting with a + sign in the original empty cell (with negative opportunity cost). The lowest amount of allocations among all –ve signed cells is 2 which is added to the allocations of +ve signed cells and reduced from the allocations of –ve signed cells. Thus the revised solution is obtained which is shown in the table below.

Factory	Warehouse		
	X	Y	Z
A	2(5)	7	4
B	3	3(2)	1 (6)
C	5	4(7)	7
D	1(2)	6	2(12)

Again this revised solution is tested for optimality by calculating u_i and v_j for each row and column respectively and obtaining opportunity costs for all empty cells in the same manner as done earlier. The values are shown in the table below.

Factory	Warehouse			u_i
	X	Y	Z	
A	2(5)	7 [2]	4 [1]	3
B	[3] 3	3 (2)	1 (6)	1
C	[4] 5	4(7)	[7] 7	0
D	1(2)	6 [2]	2(12)	2
v_j	-1	2	0	

Since all the opportunity cost values for empty cells are positive therefore this solution is the optimal solution. The total opportunity cost with this solution is

Z =2*5 +3*2 +1*6 +4*7 + 1*2 +2*12 = Rs. 76.

Thus the final solution of the given transportation solution is

From factory A to warehouse X transport 5 units.

From factory B to warehouse Y transport 2 units.

From factory B to warehouse Z transport 6 units.

From factory C to warehouse Y transport 7 units.

From factory D to warehouse X transport 2 units.

From factory D to warehouse Z transport 12 units.

And the total transportation cost is Rs. 76.

5.7 DEGENERACY IN TRANSPORTATION PROBLEMS

Degeneracy is the condition in the solution of a m by n transportation problem where the number of independent individual allocation cells is less than m +n -1. Degeneracy may occur in two stages- at the initial stage and at an intermediate stage at some subsequent iteration. Whenever a solution is degenerate, it cannot be further tested for optimality as it will not be then possible to compute all values of dual variables u_i and v_j. Therefore, it is necessary to resolve the degeneracy problem.

5.7.1 DEGENERACY AT INITIAL STAGE

If the number of independent individual allocation cells in the initial basic feasible solution is less than m+n-1, it is called as degeneracy at initial stage. To resolve degeneracy in such cases, we make use of a artificial quantity (α) in one or more independent empty cells so that the total number of allocated cells becomes equal to m +n -1. This artificial quantity $\alpha \to$ is so small (α 0) that it does not make any effect on supply and demand constraints and the total transportation cost. The independent cell in which α allocation is made is considered to be an occupied cell however that does not affect the transportation schedule. It only helps in resolving the degeneracy. An independent cell is the cell from which no loop can be traced. A α can be assigned to any of the independent cells preferably to the cell with lowest cost.

Example 5.2- Solve the following transportation problem-

From	To						Supply
	1	2	3	4	5	6	
A	5	3	7	3	8	5	3
B	5	6	12	5	7	11	4
C	2	1	3	4	8	2	2
D	9	6	10	5	10	9	8
Demand	3	3	6	2	1	2	17

Solution- The given transportation problem can be solved for initial basic feasible solution by VAM .

The IBFS obtained through VAM is shown below-

From	To						Supply	Penalties
	1	2	3	4	5	6		
A	**5**	**3(1)**	**7**	**3**	**8**	**5 (2)**	3/3/1/0	0/0/0/-
B	**5 (3)**	**6**	**12**	**5**	**7(1)**	**11**	4/4/4/4/1/0	0/0/0/0/1
C	**2**	**1**	**3 (2)**	**4**	**8**	**2**	2/0	1/-
D	**9**	**6 (2)**	**10(4)**	**5 (2)**	**10**	**9**	8/8/8/8	1/1/1/1/1
Demand	3/3/3/3/0	3/3/3/2/2	6/4/4/4/4	2/2/2/2	1/1/1/1/0	2/2/0	17	
Penalties	3/0/0/④/-	2/3/③/0/0	④/3/3/2/2	1/2/2/0/0	1/1/1/3/③	3/④/-		

Since the number of occupied cells in the solution is 8 which is one less tham m+ n-1 = 9. Therefore the solution is a degenerate solution.

We cannot test optimality unless we resolve the degeneracy, therefore a very small amount α is allocated to the cell (2,4) to get 9 occupied cells in the solution.

The new solution after removing degeneracy is shown in the following table. The optimality for this solution is tested by computing u_i and v_j and calculating opportunity cost for all empty cells as shown in the same table.

From	To						u
	1	2	3	4	5	6	
A	[3] **5**	**3(1)**	[0] **7**	[1] **3**	[4] **8**	**5 (2)**	-3
B	**5 (3)**	[0] **6**	[2] **12**	**5 (α)**	**7(1)**	[3] **11**	0
C	[4] **2**	[2] **1**	**3 (2)**	[6] **4**	[8] **8**	[1] **2**	-7
D	[4] **9**	**6 (2)**	**10(4)**	**5 (2)**	[3] **10**	[1] **9**	0
v	5	6	10	5	7	8	

Since opportunity cost for all the empty cells is either 0 or +ve. Therefore the solution is optimal solution. A α in the solution will not have any affect on total transportation cost as α =0.

Total transportation cost = 1*3 + 2*5 +3*5 + 0*5 + 1*7 +2*3 + 2*6 + 4 *10 + 2* 5 = Rs.103.

5.7.2 DEGENERACY AT AN INTERMEDIATE STAGE

Degeneracy at an intermediate stage of testing optimality occurs when the inclusion of an empty cell in the solution vacates two or more occupied cells simultaneously. The case is explained with the help of following example.

Example 5.3- Solve the following transportation problem.

From	To				Supply
	A	B	C	D	
X	7	3	8	6	60
Y	4	2	5	10	100
Z	2	6	5	1	40
Demand	20	50	50	80	200

Solution- The Initial Basic Feasible Solution of the problem is obtained through NW Corner rule as shown in the following table. It is not optimal as opportunity cost for two empty cells i.e. cell (1,3) and cell (2,1) is negative. Therefore, the solution is revised by including cell (1,3) with most negative opportunity cost in the solution as shown in the table given below.

From	To				Supply	u
	A	B	C	D		
X	7 (20)	3 (40) -	[2] 8	[-5] 6 +	60	0
Y	[-2] 4	2(10) +	5 (50)	10 (40) -	100	-1
Z	[5] 2	[13] 6	[9] 5	1 (40)	40	-10
Demand	20	50	50	80	200	
v	7	3	6	11		

Total cost = 7*20 + 3*40 + 2*10 +5*50 +10*40 + 1*40 = Rs.970

The obtained revised solution is shown in the table below. In this solution two occupied cells are vacated due to inclusion of cell (1,3) therefore the number of allocated cells is 5, which is less than m +n-1 =6. Therefore, the solution is degenerate.

From	To				Supply
	A	B	C	D	
X	7 (20)	3	8	6 (40)	60
Y	4	2 (50)	5 (50)	10	100
Z	2	6	5	1 (40)	40
Demand	20	50	50	80	200

To resolve the degeneracy we allocate a small amount α in cell (1,2). The solution now is non-degenerate. To test the optimality u and v values are computed and opportunity costs for empty cells are calculated. Since there is one cell (2,1) with negative opportunity cost, so the solution is further revised.

From	To				Supply	u
	A	B	C	D		
X	7 (20) -	3 (α) +	[2] 8	6 (40)	60	0
Y	[-2] 4 +	2 (50) -	5 (50)	[5] 10	100	-1
Z	[0] 2	[8] 6	[4] 5	1 (40)	40	-5
Demand	20	50	50	80	200	
v	7	3	6	6		

Total Transportation cost = 7*20 + 6*40 + 2*50 +5*50 + 1*40 = Rs.770

The revised solution is given in the following table. It is also tested for optimality and it is found that opportunity cost for all empty cells is positive, therefore the solution is optimal.

From	To				Supply	U
	A	B	C	D		
X	[2] 7	3 (20)	[2] 8	6 (40)	60	0
Y	4(20)	2 (30)	5 (50)	[5] 10	100	-1
Z	[2] 2	[8] 6	[4] 5	1 (40)	40	-5
Demand	20	50	50	80	200	
v	5	3	6	6		

Total transportation cost =3*20 + 6*40 + 4*20 + 2*30 +5*50 + 1*40 = Rs.730

The final solution is

Transport 20 units from source X to destination B.

Transport 40 units from source X to destination D.

Transport 20 units from source Y to destination A.

Transport 30 units from source Y to destination B.

Transport 50 units from source Y to destination C.

Transport 40 units from source Z to destination D.

5.8 SPECIAL CASES IN TRANSPORTATION PROBLEMS

5.8.1 UNBALANCED TRANSPORTATION PROBLEMS

These are the cases when the aggregate supply available with sources is not equal to the aggregate demand required by all destinations. Such problems are called as unbalanced transportation problems. In such cases, we cannot solve the problem using the methods discussed in the chapter above. To solve the problem, first the aggregate supply and aggregate demand is balanced by adding dummy source/destination (add a dummy destination if aggregate demand is less than aggregate supply or add a dummy source if aggregate supply is less than aggregate demand). The costs associated with dummy source/destination are equal to zero and the available demand/ supply is equal to the amount of shortage.

Once the problem is converted into balanced transportation problem, it can be solved in the same manner using various methods.

5.8.2 TRANSPORTATION PROBLEMS WITH PROHIBITED ROUTES

Sometimes one or more routes in a given transportation problem may not be available or prohibited. There can be various reasons behind prohibiting a route such as blocked road or unfavorable weather condition. In such situation, no allocation is to be done on the prohibited routes. To ensure that, we assign a very large cost of transportation represented by M to each prohibited route and then solve the problem using same method. By adding a large cost, such prohibited routes are automatically eliminated from the solution because we always solve transportation problem for minimizing the cost.

5.8.3 TRANSPORTATION PROBLEMS WITH MULTIPLE OPTIMAL SOLUTIONS

A transportation problem may have more than one optimal solution. In such case, all solutions have same total transportation cost that is least. In a transportation problem, an optimal solution is found when opportunity costs of all empty cells are either zero or positive. If all values of opportunity costs for empty cells are positive than the solution is the only unique optimal solution, but if one or more of the empty cells have zero opportunity cost, then multiple optimal solutions exist. To obtain an alternate optimal solution, a closed loop is formed starting with the empty cell with zero opportunity cost in the same way as a solution is revised. The revised solution thus obtained is the alternate optimal solution for the given transportation problem.

To explain above three types of cases, the following example is being taken.

Example 5.4 In the following transportation problem it is given that route from source 1 to destination 1 and from source 3 to destination 3 is blocked due to some reason. Solve the transportation problem to find optimal solution. If there are multiple solutions, then find them also. The transportation costs from various sources to various destinations along with the available supply and demand are given in the following table.

Source	Destination			Supply
	1	2	3	
1	10	12	7	180
2	14	11	6	100
3	9	5	13	160
4	11	7	9	120
Demand	240	200	220	

Solution- The above transportation problem is unbalanced as the total demand is 660 but the total supply is 560. Thus to make it balanced we add a dummy supply i.e. source of 100 units as shown in the table below. The transportation cost with this dummy source will be zero.

Also the route from source 1 to destination 1 and from source 3 to destination 3 is blocked, therefore the cost in these two routes is considered to be very large and denoted by M.

Source	Destination			Supply
	1	2	3	
1	M	12	7	180
2	14	11	6	100
3	9	5	M	160
4	11	7	9	120
Dummy	0	0	0	100
Demand	240	200	220	

Now the problem is a balanced problem. Hence the IBFS is obtained from VAM which is shown in the following table. It is tested for optimality, where opportunity cost for prohibited routes are not calculated as M is very large so for any value of u+v the opportunity cost Δ = M – (u +v) will be positive only.

Source	Destination			Supply	u
	1	2	3		
1	**M**	[1] **12**	**7 (180)**	180	15
2	**14 (60)**	[1] **11**	**6 (40)**	100	14
3	[0] **9** +	**-5 (160)**	**M**	160	9
4	**11 (80)** -	+ **7 (40)**	[6] **9**	120	11
Dummy	**0 (100)**	[4] **0**	[8] **0**	100	0
Demand	240	200	220		
V	0	-4	-8		

Total cost = 7*180 + 14*60 + 6*40 + 5*160 + 11*80 + 7*40 +0*100 = Rs. 4300

Since opportunity cost Δ for all empty cells is either 0 or positive, so the solution is the optimal solution. Total cost of transportation for this solution is Rs. 4300. But this is not unique optimal solution because for one cell (3,1) opportunity cost Δ is zero therefore it indicates that there is one more optimal solution possible for this problem.

To obtain alternate optimal solution, a closed loop from cell (3,1) is formed and the solution is revised which is given in the next table. The total cost for this new revised solution is also Rs. 4300 which is same as the previous solution.

Source	Destination			Supply	u
	1	2	3		
1	**M**	[1] **12**	**7 (180)**	180	15
2	**14 (60)**	[1] **11**	**6 (40)**	100	14
3	**9 (80)**	**5 (80)**	**M**	160	9
4	[0] **11**	**7 (120)**	[6] **9**	120	11
Dummy	**0 (100)**	[4] **0**	[8] **0**	100	0
Demand	240	200	220		
V	0	-4	-8		

Total cost = 7*180 + 14*60 + 6*40 + 5*80 + 9*80 + 7*120 +0*100 = Rs. 4300

5.9 MAXIMIZATION TRANSPORTATION PROBLEM

The typical transportation problem is always with minimization type objective. However, there may be cases where the problem may contain profits instead of costs. In such cases, the objective of the problem will be of maximization. Such cases where the objective is to maximize the values, the transportation problem is called maximization transportation problem. In that case, the transportation method of solving problem is applied in the same way except the difference that first the problem is converted into minimization type. In order to convert the maximization problem into minimization type, the values of original profit matrix are subtracted from the highest profit value in the matrix. Once the problem is converted in minimization matrix, it is solved in the same way to obtain the optimal solution. At the end the total profit is calculated by using the original profit values.

In case a maximization problem is unbalanced, the problem is first balanced and then it is converted into the minimization problem.

Example 5.5 Solve the given problem for maximizing the profit.

Per unit profit in Rs.

Factory	Market				Supply
	W	X	Y	Z	
A	12	18	6	25	200
B	8	7	10	18	500
C	14	3	11	20	300
Demand	180	320	100	400	

Solution- The given problem is a balanced transportation problem but the values given in the matrix are profits which are to be maximized. So in order to apply transportation algorithm we should first convert the profit matrix into minimization matrix. To convert into minimization matrix, we will subtract all profit values from the highest profit in the matrix i.e. 25. Thus the revised minimization matrix is given below.

Minimization matrix

Factory	Market				Supply
	W	X	Y	Z	
A	13	7	19	0	200
B	17	18	15	7	500
C	11	22	14	5	300
Demand	180	320	100	400	

Now the IBFS is obtained through VAM which is shown in the following table.

Factory	Market				Supply
	W	X	Y	Z	
A	13	7 (200)	19	0	200
B	17	18 (100)	15	7 (400)	500
C	11(180)	22(20)	14(100)	5	300
Demand	180	320	100	400	

The initial solution obtained is non-degenerate. Therefore, it is tested for optimality.

One cell (3,3) is having negative opportunity cost. So a closed loop is formed from this cell and the solution is revised.

Factory	Market				Supply	u
	W	X	Y	Z		
A	17 13	7 (200)	20 19	4 0	200	0
B	17 17	18 (100) +	17 15	7 (400) -	500	11
C	11(180)	22(20) -	14(100)	-6 5 +	300	15
Demand	180	320	100	400		
V	-4	7	-1	-4		

Revised Solution

Factory	Market				Supply	u
	W	X	Y	Z		
A	11 13	7 (200)	14 19	4 0	200	0
B	4 17	18 (120)	-1 15+	7 (380) -	500	11
C	11(180)	6 22	14(100) -	5(20) +	300	9
Demand	180	320	100	400		
V	2	7	5	-4		

Further in this solution one more empty cell (2,3) is with negative opportunity cost. Therefore, the solution is again revised. The revised solution is shown below.

Factory	Market				Supply	u
	W	X	Y	Z		
A	13 [11]	7 (200)	19 [15]	0 [4]	200	0
B	17 [4]	18 (120)	15 (100)	7 (280)	500	11
C	11(180)	22 [6]	14 [1]	5(120)	300	9
Demand	180	320	100	400		
V	2	7	5	-4		

Since opportunity cost for all empty cells is positive therefore this solution is optimal solution.

Total profit = 18*200 + 7*120+ 10*100 +18*280 + 14* 180 + 20*120 = Rs. 15400.

The optimal solution is

From factory A to Market X transport 200 units.

From factory B to Market X transport 120 units.

From factory B to Market Y transport 100 units.

From factory B to Market Z transport 280 units.

From factory C to Market W transport 180 units.

From factory C to Market Z transport 120 units.

5.10 SUMMARY

This unit explains the concept of transporation problems which are special kind of linear programming problem. A general transportation problem has a number of sources and destination. A certain amount of supply is available at each source. Similarly, each destination has a certain demand/requirements. The transportation problem represents amount of goods to be transported from different sources to destinations so that the transportation cost is minimized without violating the supply and demand constraints. The transportation problem is solved in two phases. In the first phase, basic feasible solution is obtained and optimum solution is determined in the second phase. The three methods to determine basic feasible solution are North West Corner Method, Least Cost Method or Matrix Minimum Method and Vogel's Approximation Method (VAM) In order to determine optimum solution Modified Distribution (MODI) Method is used. Transportation problems may involve various types of special cases which are also discussed in this unit.

5.11 SELF ASSESSMENT QUESTIONS

1. What is a transportation problem? How can the initial basic feasible solution of a transportation problem be obtained?
2. Explain various methods of obtaining initial basic feasible solution of a transportation problem.
3. Show how to balance the transportation problem if it is unbalanced. Also explain the method to check the optimality of an initial basic feasible solution.
4. Solve the following transportation problem for finding initial basic feasible solution-

	Warehouses			
Company	A	B	C	**Supply**
W	10	8	9	15
X	5	2	3	20
Y	6	7	4	30
Z	7	6	9	35
Requirement	25	26	49	100

5. The Sigma transport company ships truckloads of food grains from three sources viz. X, Y, Z to four mills viz. A, B, C, D respectively. The supply and the demand together with the unit transportation cost per truckload on the different routes are described in the following transportation table. Assume that the unit transportation costs are in hundreds of dollars. Determine the optimum shipment cost of transportation using MODI method.

	Mill				
Source	A	B	C	D	**Supply**
X	10	2	20	11	15
Y	12	7	9	20	25
Z	4	14	16	18	10
Demand	5	15	15	15	

5.12 TEXT AND REFERENCES

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UNIT-6 ASSIGNMENT PROBLEM

Unit Framework

6.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the concept of Assignment problem
- Formulate an assignment problem
- Solve the assignment problem by Hungarian Method
- Solve the unbalanced assignment problems

- Modify the assignment problems with maximization objective into the assignment problem with minimization objective and solve them.

6.2 INTRODUCTION

Assignment problems are the special type of linear programming problems where the objective is to determine optimum allocation of jobs/tasks to an equal number of persons/facilities in such a way that the cost/time of completing all the jobs is minimum. Sometimes the objective may also be to maximize the profit by way of allocating jobs to different persons. The problem of assignment occurs because available resources such as men, machines etc. have different degrees of efficiency for performing various activities, therefore, cost, time, profit or loss of performing the different activities is also different. In simple words, an assignment problem can be expressed as, “How should the assignments of jobs to workers be made so as to optimize the given objective”. Some of the problem where the assignment technique may be useful are assignment of workers to machines, salesman to different sales areas.

6.3 STRUCTURE OF AN ASSIGNMENT PROBLEM

The assignment problem in the general form can be stated in the form of n × n matrix where there are *n* facilities and *n* jobs. The effectiveness of each facility for each job is different, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimised (Maximised or Minimised).

The assignment matrix is shown in the following table, where c_{ij} is the cost of assigning i^{th} facility(person) to j^{th} job-

Persons	Jobs					
	1	2	. . .	j	….	n
1	C_{11}	C_{12}		C_{1j}		C_{1n}
2	C_{21}	C_{22}		C_{2j}		C_{2n}
. . .						

I	C_{i1}	C_{i2}		C_{ij}		C_{in}
. . .						
Dn	C_{n1}	C_{n2}		C_{mj}		C_{nn}

6.4 DIFFERENCE BETWEEN A TRANSPORTATION AND AN ASSIGNMENT PROBLEM

In an assignment problem, the supply of each resource (same as source) and the demand of each job (same as destination) is taken to be unity. It is due to the fact that assignments are to be made on a one to one basis. Therefore, it is similar to a transportation problem where, m = n and all s_i and d_j (I = 1,2,3,…m; j = 1,2,3,…n) are unity and each x_{ij} i.e. number of allocations in one cell is limited to two values 0 and 1.

6.5 SOLUTION OF AN ASSIGNMENT PROBLEM

The assignment problem can be solved by the following four methods:

a) Complete enumeration method

b) Simplex Method

c) Transportation method

d) Hungarian method

6.5.1 COMPLETE ENUMERATION METHOD

In this method, a list of all possible assignments among the given resources and activities is prepared. Then an assignment involving the minimum cost, time or distance or maximum profits is selected. If two or more assignments have the same minimum cost, time or distance, the problem has multiple optimal solutions. This method can be used only if the number of assignments is less. It becomes unsuitable for manual calculations if number of assignments is large

6.5.2 SIMPLEX METHOD

This can be solved as a linear programming problem as discussed in chapter 4 and as such can be solved by the simplex algorithm.

The mathematical formulation of an assignment problem can be represented as follows-

Minimize z $= \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}.X_{ij}$, subject to the restrictions

$$\left.\begin{array}{ll} \sum_{j=1}^{n} x_{ij=} \ s_i = 1 & \text{Where i=1,2,3…..m} \\ \sum_{i=1}^{n} x_{ij=} \ d_j = 1 & \text{Where j=1,2,3…..n} \end{array}\right\}$$

This means that each person should be assigned to only one job and each job must be assigned to only one person.

In such problem, there are n× n decision variables and n+n = 2n equalities.

6.5.3 TRANSPORTATION METHOD

As assignment is a special case of transportation problem, it can also be solved using transportation model discussed in unit 5. The solution obtained by applying this method would be degenerate. This is because the optimality test in the transportation method requires that there must be m+n-1= (2n-1) basic variables. For an assignment problem of order n x n there would be only n basic variables in the solution because here n assignments are required to be made. This degeneracy problem of solution makes the transportation method computationally inefficient for solving the assignment problem.

6.5.4 HUNGARIAN ASSIGNMENT METHOD

The Hungarian method of assignment provides us with an efficient means of finding the optimal solution. The Hungarian method is based upon the following principles:

(i) If a constant is added to every element of a row and/or column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem or vice versa.

(ii) The solution having zero total cost is considered as optimum solution.

Hungarian method of assignment problem (minimization case) can be summarized in the following steps:

Step I: Subtract the minimum cost of each row of the cost (effectiveness) matrix from all the elements of the respective row so as to get first reduced matrix.

Step II: Similarly subtract the minimum cost of each column of the cost matrix from all the elements of the respective column of the first reduced matrix. This is first modified matrix.

Step III: Starting with row 1 of the first modified matrix, examine the rows one by one until a row containing exactly single zero elements is found. Make any assignment by making that zero in or enclose the zero inside a box □. Then cross

(X) all other zeros in the column in which the assignment was made. This eliminates the possibility of making further assignments in that column.

Step IV: When the set of rows have been completely examined, an identical procedure is applied successively to columns that is examine columns one by one until a column containing exactly single zero element is found. Then make an experimental assignment in that position and cross other zeros in the row in which the assignment has been made.

Step V: Continue these successive operations on rows and columns until all zeros have been either assigned or crossed out and there is exactly one assignment in each row and in each column. In such case optimal assignment for the given problem is obtained.

Step VI: There may be some rows (or columns) without assignment i.e. the total number of marked zeros is less than the order of the matrix. In such case proceed to step VII.

Step VII: Draw the least possible number of horizontal and vertical lines to cover all zeros of the starting table. This can be done as follows:

1. Mark (√) in the rows in which assignments has not been made.
2. Mark column with (√) which have zeros in the marked rows.
3. Mark rows with (√) which contains assignment in the marked column.
4. Repeat 2 and 3 until the chain of marking is completed.
5. Draw straight lines through marked columns.
6. Draw straight lines through unmarked rows.

By this way we draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once. It should, however, be observed that in all n x n matrices less than n lines will cover the zeros only when there is no solution among them. Conversely, if the minimum number of lines is n, there is a solution.

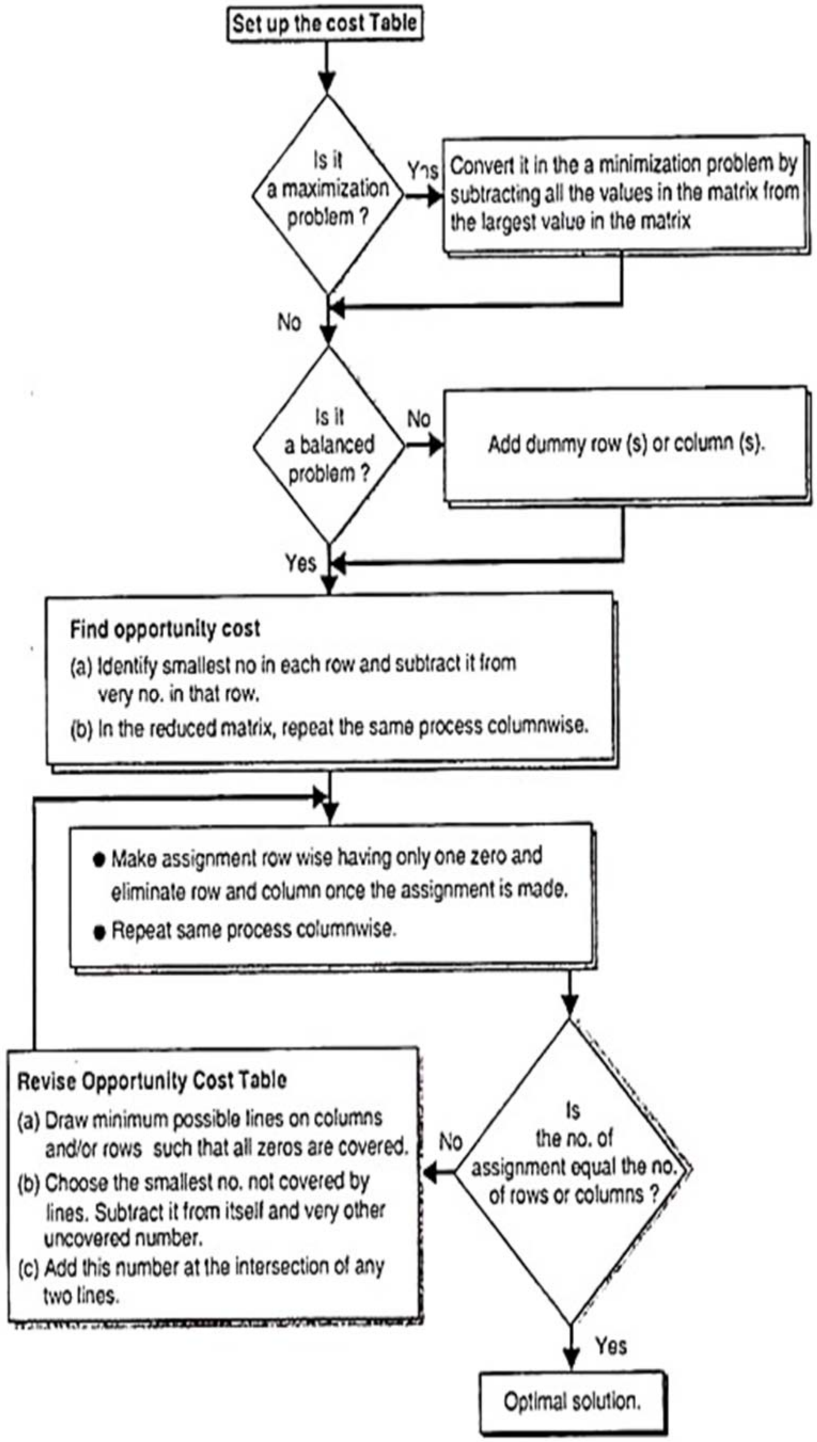
Step VIII: In this step, select the smallest element, say X, among all the elements not covered by any of the lines of the table; and subtract this value X from all of the non-covered elements in the matrix and add X to all those elements that lie at the intersection of the horizontal and vertical lines, thus obtaining the second modified cost matrix.

Step IX: Repeat Steps IV, V and VI until we get the number of lines equal to the order of matrix I, till an optimum solution is attained.

Step X: We now have exactly one boxed zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeros is the optimum assignment.

This procedure is shown in the form of a flow chart as below-

Flow chart of steps in the Hungarian Mehod



The above technique is explained by taking the following example-

Example 6.1 A manager has four sales executives, and four sales areas to perform sales. The sales executives differ in efficiency and the sales areas also differ in their sales target levels. The number of days each sales executive would take to finish sales target of each sales area is given in the effectiveness matrix below. How should the sales areas be allocated, one to a sales executive, so as to minimize the total no. of days to finish sales target?

	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

Solution

Step I : Subtracting the smallest element in each row from every element in that row, we get the first reduced matrix.

0	18	9	3
9	24	0	22
23	4	3	0
9	16	14	0

Step II: Next, we subtract the smallest element in each column from every element in that column; we get the second reduced matrix.

Step III: Now we test whether it is possible to make an assignment using only zero distances.

0	14	9	3
9	20	0	22
23	0	3	0
9	12	14	0

(a) Starting with row 1 of the matrix, we examine rows one by one until a row containing exactly single zero elements are found. We make an assignment (indicated by □) to that cell. Then we cross all other zeros in the column in which the assignment was made.

(b) When the set of rows has been completely examined an identical procedure is applied successively to columns. Starting with Column 1, we examine columns until a column containing exactly one remaining zero is found. We make an assignment in that position and cross other zeros in the row in which the assignment was made. It is found that no additional assignments are possible. Thus, we have the optimal assignment as shown in the following table.

[0]	14	9	3
9	20	[0]	22
23	[0]	3	~~0~~
9	12	14	[0]

The minimum total no. of days to finish sales target are computed as

Optimal assignment	No. of days
A – I	8
B – III	4
C – II	19
D – IV	10
Total	**41 days**

Example 6.2 A dairy plant has five milk tankers I, II, III, IV & V. These milk tankers are to be used on five delivery routes A, B, C, D, and E. The distances (in kms) between dairy plant and the delivery routes are given in the following distance matrix

	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

How the milk tankers should be assigned to the chilling centers so as to minimize the distance travelled?

Solution

Step I: Subtracting minimum element in each row we get the first reduced matrix as

30	0	45	60	70
15	0	10	40	55
30	0	45	60	75
0	0	30	30	60
20	0	35	45	70

Step II: Subtracting minimum element in each column we get the second reduced matrix as

30	0	35	30	15
15	0	0	10	0
30	0	35	30	20
0	0	20	0	5
20	0	25	15	15

Step III: Row 1 has a single zero in column 2. We make an assignment by putting □around it and delete other zeros in column 2 by marking X. Now column1 has a single zero in row 4 we make an assignment by putting and cross the other zero which is not yet crossed in row 4. Column 3 has a single zero in row 2; we make an assignment and delete the other zero which is uncrossed. Now we see that there are no remaining zeros; and row 3, row 5, column 5 and column 4 has no assignment. Therefore, we cannot get our desired solution at this stage.

30	[0]	35	30	15
15	⊗	0	10	0
30	⊗	35	30	20
[0]	⊗	20	⊗	5
20	⊗	25	15	15

Step IV: Draw the minimum number of horizontal and vertical lines necessary to cover all zeros at least once by using the following procedure

1. Mark (√) row 3 and row 5 as having no assignments and column 2 as having zeros in rows 3 and 5.
2. Next we mark (√) row 2 because this row contains assignment in marked column 2. No further rows or columns will be required to mark during this procedure.
3. Draw line L_1 through marked col.2.
4. Draw lines L_2 & L_3 through unmarked rows.

Step V: Select the smallest element say X among all uncovered elements which is X = 15. Subtract this value X=15 from all of the values in the matrix not covered by lines and add X to all those values that lie at the intersections of the lines L_1, L_2 & L_3.

Applying these two rules, we get a new matrix

15	0	20	15	0
15	15	0	10	0
15	0	20	15	5
0	15	20	0	5
5	0	10	0	0

Step VI: Now reapply the test of Step III to obtain the desired solution.

15	⊗	20	15	[0]
15	15	[0]	10	⊗
15	[0]	20	15	5
[0]	15	20	⊗	5
5	⊗	10	[0]	⊗

Since no zeros are left and there is one assignment in each row and column, so we have reached to the optimal assignment.

The assignments are

A → V B → III C → II D → I E → I

Total Distance= 200 + 130 + 110 + 50 + 80 = 570.

6.6 CASE OF MULTIPLE OPTIMAL SOLUTIONS

Sometimes, it is possible to have two or more ways to assign/cross out zeros in the final reduced matrix for a given problem. This implies that there are more than one optimal solutions. In such cases, the total cost of assignment remains the same for all possible optimal solutions. Decision maker can choose any of the optimal solution as per their preference in such case. Example 6.3 is used to explain the case of multiple optimal solutions

6.7 MAXIMIZATION CASE IN ASSIGNMENT PROBLEM

Sometimes, assignment problems are derived with an objective of maximization. For example- maximization of profit, output or revenue instead of minimizing cost or time. In such case, Hungarian method is not applicable directly, then it is required to first convert the maximization problem into equivalent minimization problem. A maximization problem can be converted into minimization by subtracting every number in the original payoff table from the highest number in that table. After conversion into minimization problem, the Hungarian method can be applied in the same way.

This type of assignment problem is illustrated in the following example-

Example 6.3 A marketing manager has five salesmen and sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows. Find the assignment of salesmen to districts that will result in maximum sales.

Salesman	District				
	1	2	3	4	5
A	32	38	40	28	40
B	40	24	28	21	36
C	41	27	33	30	37
D	22	38	41	36	36
E	29	33	40	35	39

Solution- The given maximization problem is converted into minimization problem by subtracting from the highest sales value (i.e., 41) with all elements of the given table.

Conversion to Minimization Problem

Salesman	District				
	1	2	3	4	5
A	9	3	1	13	1
B	1	17	13	20	5
C	0	14	8	11	4
D	19	3	0	5	5
E	12	8	1	6	2

Now reduce the matrix row-wise

Salesman	District				
	1	2	3	4	5
A	8	2	0	12	0
B	0	16	12	19	4
C	0	14	8	11	4
D	19	3	0	5	5
E	11	7	0	5	1

Now reduce matrix column wise

Salesman	District				
	1	2	3	4	5
A	8	0	0	7	0
B	0	14	12	14	4
C	0	12	8	16	4
D	19	1	0	0	5
E	11	5	0	0	1

Now assigning zeros row wise and columnwise we get the following table

Salesman	District				
	1	2	3	4	5
A	8	[0]	⊗	7	⊗
B	[0]	14	12	14	4
C	⊗	12	8	16	4
D	19	1	[0]	⊗	5
E	11	5	⊗	[0]	1

Since row 3 has no assignment so checking and revising the solution for optimality by covering all zero with least number of horizontal and vertical lines as shown below.

	District				
	1	2	3	4	5
A	8	[0]	⊗	7	⊗
B	[0]	14	12	14	4
C	⊗	12	8	16	4
D	19	1	[0]	⊗	5
E	11	5	⊗	[0]	1

Since the total number of lines is 4 which is less than the order of the matrix i.e.5 therefore, the solution is revised by subtracting the smallest uncovered element i.e.4 from all uncovered elements and adding it to the cells lying at intersection of horizontal and vertical lines. The revised matrix is written below.

Salesman	District				
	1	2	3	4	5
A	12	0	0	7	0
B	0	10	8	10	0
C	0	8	4	12	0
D	23	1	0	0	5
E	15	5	0	0	1

Now, as shown in the above table, number of lines drawn = Order of matrix, hence optimality is reached. Next the zeros are assigned, there are four alternative assignments due to presence of zero elements in cells (B, 1), (B, 5), (C, 1), (C, 5) and (D, 3). (D, 4), (E, 3) (E, 4). these four optimal solutions are shown below-

Alternate Solution 1

Salesman	District				
	1	2	3	4	5
A	12	[0]	⊗	7	⊗
B	[0]	10	8	10	⊗
C	⊗	8	4	12	[0]
D	23	1	[0]	⊗	5
E	15	5	⊗	[0]	1

Alternate Solution 2

Salesman	District				
	1	2	3	4	5
A	12	[0]	⊗	7	⊗
B	⊗	10	8	10	[0]
C	[0]	8	4	12	⊗
D	23	1	[0]	⊗	5
E	15	5	⊗	[0]	1

Alternate Solution 3

Salesman	District				
	1	2	3	4	5
A	12	[0]	~~0~~	7	~~0~~
B	[0]	10	8	10	~~0~~
C	~~0~~	8	4	12	[0]
D	23	1	~~0~~	[0]	5
E	15	5	[0]	~~0~~	1

Alternate Solution 4

Salesman	District				
	1	2	3	4	5
A	12	[0]	~~0~~	7	~~0~~
B	~~0~~	10	8	10	[0]
C	[0]	8	4	12	~~0~~
D	23	1	~~0~~	[0]	5
E	15	5	[0]	~~0~~	1

Total sales made by salesmen in all districts for all the four optimal solutions is same and shown below-

32	38	40	28	40
40	24	28	21	36
41	27	33	30	37
22	38	41	36	36
29	33	40	35	39

Total sales for solution 1 = 38+40+37+41+35 =191

Total sales for solution 2 = 38+41+36+41+35 =191

Total sales for solution 3 = 38 +40+37+40 +36 =191

Total sales for solution 4 = 38 +41+36+40 +36 =191

6.8 UNBALANCED ASSIGNMENT PROBLEM

As discussed earlier the solution procedure of an assignment problem requires that the matrix should be a square matrix or in other words, number of rows in the effectiveness matrix should be equal to the number of columns in that matrix. However, sometimes the number of rows (persons or facilities) is not equal to the number of columns (jobs or machines). Such an assignment problem, is called as unbalanced assignment problem.

To solve unbalanced assignment problems, we first balance the problem by adding dummy rows/columns in the matrix with zero value in each cell. If number of rows in original matrix is more, then dummy columns are added and if number of columns is more, then dummy rows are added. After having equal number of rows and columns in the matrix, Hungarian method is applied to further solve the problem. To illustrate such case , the following example is taken-

Example- 6.4- A company has five machines that are used for four jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following Table. Determine the optimal assignment.

Machines	Jobs			
	I	II	III	IV
A	0	9	5	16
B	12	11	10	10
C	4	12	6	9
D	5	17	12	10
E	7	12	7	12

Solution- Since the given problem is a unbalanced problem, so we add a dummy column (job) in the given matrix in order to solve the problem.

Machines	Jobs				
	I	II	III	IV	Dummy
A	0	9	5	16	0
B	12	11	10	10	0
C	4	12	6	9	0
D	5	17	12	10	0
E	7	12	7	12	0

Now applying row reduction and column reduction the following matrix is obtained.

Machines	Jobs				
	I	II	III	IV	Dummy
A	0	0	0	7	0
B	12	2	5	1	0
C	4	3	1	0	0
D	5	8	7	1	0
E	7	3	2	3	0

Drawing minimum number of horizontal and vertical lines to cover all the zeros, we see that no. of lines i.e. 3 is less than the order of the matrix i.e.5. So revising the matrix by subtracting lowest element i.e. 1 amongst uncovered elements from all uncovered elements and adding it to the intersection cells. Thus a new matrix obtained is shown below-

Machines	Jobs				
	I	II	III	IV	Dummy
A	0	0	0	8	1
B	11	1	4	1	0
C	3	2	0	0	0
D	4	7	6	1	0
E	6	2	1	3	0

In this matrix also the number of lines covering all zeros is less than 5. So one more time the matrix is revised in the same way. Now the smallest uncovered element is 1 which is subtracted from all uncovered elements and added to the intersection cells. The revised table is shown below-

Machines	Jobs				
	I	II	III	IV	Dummy
A	0	0	1	9	2
B	10	0	4	1	0
C	2	1	0	0	1
D	3	6	6	1	0
E	5	1	1	3	0

In this matrix also the number of lines covering all zeros is less than 5. So one more time the matrix is revised in the same way. Now the smallest uncovered element is 1 which is subtracted from all uncovered elements and added to the intersection cells. The revised table is shown below-

Machines	Jobs				
	I	II	III	IV	Dummy
A	0	1	1	9	3
B	9	0	3	1	0
C	2	2	0	0	2
D	2	6	5	1	0
E	4	1	0	3	0

Since the number of lines covering zeros is equal to the size of matrix, therefore, optimality is reached. So now assignment can be made in the following way.

Machines	Jobs				
	I	II	III	IV	Dummy
A	[0]	1	1	9	3
B	9	[0]	3	1	⊗
C	2	2	⊗	[0]	2
D	2	6	5	1	[0]
E	4	1	[0]	3	⊗

The optimal solution of the problem is

A to I, B to II, C to IV, E to III, and Machine D will remain empty or unoccupied as jobs are 4 only but machines are 5.

6.9 RESTRICTIONS IN ASSIGNMENT PROBLEM

Sometimes, it is possible that one person/machine is incapable of performing certain task/job. Such a case is called the restriction case in assignment problems. To solve such problems, a very high cost (represented by M) is considered in the cell where assignment is restricted. This prohibits the entry of the restricted pair of resource-activity in the final solution of the problem.

The following example can be used to understand such a case-

Example 6.5- In a plant layout, four different machines M1, M2, M3 and M4 are to be erected in a machine shop. There are five vacant areas A, B, C, D and E. Because of limited space, Machine M2 cannot be erected at area C and Machine M3 cannot be erected at area A. The cost of erection of machines is given in the Table. Find optimal assignment schedule.'

Machines	Area				
	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	--	10	9
M3	--	11	14	11	7
M4	14	8	12	7	8

Solution- the given problem is an unbalanced problem, therefore one dummy machine Md is added to the matrix with zero cost of erection. Also the costs in cell(2,C) and (3,A) is taken as M, where m is a very large number. The cost matrix can be now written as shown below-

Machines	Area				
	A	B	C	D	E
M1	9	11	15	10	11
M2	12	9	M	10	9
M3	M	11	14	11	7
M4	14	8	12	7	8
Md	0	0	0	0	0

Now subtracting the minimum element of each row and each column from the elements of that row/column. We obtain the folllowing matrix after row /column minimization-

Machines	Area				
	A	B	C	D	E
M1	[0]	2	6	1	2
M2	3	[0]	M	1	⊗
M3	M	4	7	4	[0]
M4	7	1	5	[0]	1
Md	⊗	⊗	[0]	⊗	⊗

Now assigning the zeros, as shown in the above table, we have found the optimal solution. The solution can be written as-

M1 to A, M2 to B, M3 to E, M4 to D, C area will remain vacant and the minimum cost of erection will be 9+9+7+7 = 32.

6.10 SUMMARY

In this unit, assignment problem has been discussed. Assignment problems are special type of transportation problems in which allocations are made on one to one basis. In assignment problems, the demand/supply of each destination/source is one. Assignment problems are solved by a special method known as Hungarian Method. The Hungarian method is discussed in detail in this unit. Also various special cases in assignment problems such as multiple optimal solutions, unbalanced problem and restrictions in assignment have been discussed.

6.11 SELF ASSESSMENT QUESTIONS

1. What is an Assignement problem? What are its applications?

2. Give an example to show that assignment problems are special cases of transportation problems.

3. Describe the Hungarian method for solving assignment problems.

4. What is an unbalanced assignment problem? How can Hungarian method be applied in such a problem.

5. A marketing manager wants to assign salesman to four cities. He has four salesmen of varying experience. The possible profit for each salesman in each city is given in the following table. Find out an assignment which maximizes the profit.

Salesman	Cities			
	A	B	C	D
1	25	27	28	38
2	28	34	29	40
3	35	24	32	33
4	24	32	25	28

6. Solve the following problem to determine which machine should be assigned to each worker. The given values in the table are costs (in Rs. Hundred) associated with each machine-worker combination.

Machines	Workers				
	A	B	C	D	E
M1	3	8	2	10	3
M2	8	7	2	9	7
M3	6	4	2	7	5
M4	8	4	2	3	5
M5	9	10	6	9	10

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Master of Business Administration

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Operations Research

BLOCK

3

PROGRAMMING TECHNIQUES – FURTHER APPLICATIONS

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BLOCK INTRODUCTION

In Block III you will learn about advanced programming techniques in Operations Research. This block includes four units that cover various applications of programming techniques.

Unit-7 will discuss about goal programming which deals with the problems having multiple objectives. This unit will help in understanding difference between linear programming and goal programming and how the goal programming problems can be solved.

Unit-8 will discuss about integer programming which is used to solve the problems where the values of decision variables are restricted to integers such as problems in financial management and plant location.

Unit-9 will discuss about dynamic programming which is used to solve problems that involve many stages and highly complex interrelations.

Unit-10 will discuss about Non-linear programming which is used when some or all variables in a given problem are curvilinear. In most of the practical cases, either objective or constraints or both are non-linear but due to the level of complexity involved sometimes, the problems are approximated to linear programming problems.

UNIT-7 GOAL PROGRAMMING

Unit Framework

7.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the meaning and concept of goal programming
- Differentiate between goal programming and linear programming
- Formulate goal programming problems
- Solve goal programming problems with the help of simplex method

7.2 INTRODUCTION

Linear programming problem is based on the assumption that the decision making has a single, quantiifiable objective such as maximization of profit/output/sale or minimization of cost/time. It is impossible in LPP to have multiple goals unless they can be measured in the same units. Often there are situations where instead of having single objective, multiple objectives or goals may be set. Such as maximization of market share, high product quality, maintaining full employment, providing quality ecological management etc. may be multiple goals for a firm.

For example, for increasing sale or profit, we require more risky investment. The objective will be to maximize return one hand and minimize risk on the other hand. In such situations we need different techniques that seek a compromise solution based on the relative importance of each objective. Various programming techniques such as linear and integer programming have the shortcoming that they may have only one objective at a time. An important technique has been developed to supplement linear programming. This technique is known as Goal programming which uses simplex method for finding optimum solution to a multidimensional linear objective function with linear constraints.

7.3 MEANING AND CONCEPT OF GOAL PROGRAMMING

Goal programming is a mathematical programming technique which considers the constraints of a linear programming problem as various goals in the objective function. In this case, optimization means reaching as close as possible to achieve the goals in order of priority, as determined by decision maker.

Goal programming can be applied to single or multiple goals, however it is greatly useful in the case when multiple goals are conflicting and cannot be satisfied simultaneously.

In goal programming, management is made to set some estimated targets for each of their goals and to rank them in order of their importance. The goal programming is then used to minimize the deviations from the set targets. It starts with the goal of prime importance and then continues to achieve a less important goal until the achievement of any less important goal cause management to fail to realise a more important goal.

7.4 DIFFERENCE BETWEEN LINEAR PROGRAMMING AND GOAL PROGRAMMING

In linear programming, one goal is selected as an objective function and the other goals are specified as constraints. Any solution to a problem must satisfy all constraints prior to the optimization of objective function. Whereas in goal programming, each goal enters the problem formulation as an equality constraint which contains slack variables, indicating either under achievement or overachievement of goals. The objective function then contains these deviational variables, and a solution will attempt to minimize them in order of priority. In this way, goal programming allows for full or partial achievement of goals, while linear programming requires complete satisfaction of all goals represented as constraints.

7.5 APPLICATIONS OF GOAL PROGRAMMING

Goal programming has limited applications. It was first applied by Charnes and other for advertising and media planning. In this application, a goal programming model was developed for utilizing in solving advertising and media planning problems.

Goal programming concept was also used for manpower planning problems. Since it is a relatively new technique, its true potential is yet to be determined. Goal programming is applicable to the following areas.

(i) Marketing- In marketing problems, where conflicting goals are present such as maximization of market share and minimization of advertising cost, maximize profit margin per item sold.

(ii) Inventory control- In problems of inventory control, where it is necessary to minimize the number of stock-outs and to minimize the storage cost.

(iii) Production- In problems of production, where it is necessary to minimize time of manufacture, minimize cost, maximize quality control and maximize resource utilization.

7.6 FORMULATION OF A GOAL PROGRAMMING PROBLEM

The following are main steps involved in formulation of a Goal Programming problem.

Step-1:- Find out the decision variables of the major decision.

Step-2:- Formulate all the goals of the problem. These are generally fixed by the desire of the decision maker or limited resources or constraints.

Step-3:- Express each goal in terms of constraint equations by introducing negative and positive deviations i.e, D_U or d^-&d^+ or D_O

For each decision goal, we introduce a positive deviation variable d^+ and a negative deviation variable d^- while d^+ denotes how much the decision has exceeded the goal, and d^- denotes how far the decision is from the goal

Here, we have $d^+ \geq 0$, $d^- \geq 0$, and $d^+ \cdot d^- = 0$.

$d_i^- = \max(0, g_i - Z_i)$, $d_i^+ = \min(0, Z_i - g_i)$, where g_i is the aspiration level of the ith objective goal Z_i

Step-4:- Fix the priority level of each goal/s (only in case of multiple goals) and put these in the objective/ achievement function.

Example 7.1: A company manufactures two type of products X and Y. The per unit profit on X is Rs. 250 and per unit profit on Y is Rs. 150. There is sufficient demand for both of these products in the market so the company can sell all the units produced at the prevailing prices. However, the production capacity of the company is limited due to the common production facility of scarce resources. These scarce resources pertain to assembly department and finishing department. The per unit processing time and capacity of each department is shown in the following table-

Type of product	Hours required to process each product		
	In assembly department	In finishing department	Unit profit contribution
X	1	1	150
Y	3	1	250
Hours available in each department per day	60	40	

Management of the company feels that a daily profit of Rs. 6000 would be satisfactory and wishes to determine the product mix that would yield this rate of profit contribution. Formulate this as a Goal programming model.

Solution- Let x_1 and x_2 be the number of units of X and Y to be produced.

d_i^- = underachievement of the target profit.

d_i^+ = overachievement of the target profit.

The profit goal in the model can be written as a goal constraint-

$150\ x_1 + 250\ x_2 + d_i^- + d_i^+ = 6000$

The goal programming model can be now written as

Minimize $Z = d_i^- + d_i^+$

Subject to the constraints

$x_1 + 3x_2 \leq 60$ (assembly hours constraint) } resource

$x_1 + x_2 \leq 40$ (finishing hours constraint) } constraints

$150\ x_1 + 250\ x_2 + d_i^- + d_i^+ = 6000$ (target profit) goal constraints

$x_1, x_2, d_i^-, d_i^+ \geq 0$

7.7 GRAPHICAL METHOD OF GOAL PROGRAMMING

Goal programming problems can be solved by graphical method in a similar manner as the linear programming problems are solved. The only difference is that in linear programming, the graphical method is used to maximize or minimize an objective function with one goal, whereas in goal programming, it is used to minimize total deviation from a set of multiple goals.

The following are the steps of solving a goal programming problem by graphical method-

Step 1: Formulate the linear goal programming problem.

Step 2: construct the graph of all the goals in terms of decision variables. Plot lines for each goal by identifying two points for each goal considering each goal as an equation with deviation. Indicate deviations range by arrows for each goal line.

Step 3: identify the goal line corresponding to the goal with highest priority. Locate the feasible region with respect to this goal at the first priority level.

Step 4: Move to the goal having the next priority and determine the best solution space with respect to the goal at this priority level provided that best solution do not degrade the solution already achieved for higher priority goals.

Step 5: Repeat step 4 untill all priority levels have been investigated.

Step 6: Identify the optimal solution which corresponds to the most acceptable 'best values' located in step 5.

Example 7.2:- A small furnishing company manufactures tables & chairs. Each chair requires 8 man hours labor while each table requires 10 man hours of labor. If only 80 man hour are available each week and the owner of the company would neither hire additional labor nor utilize overtime, formulate the linear goal programming problem & solve it. Both the table & chair fetch a profit of Rs.100 each. The owner has target to earn a profit of Rs.2000 per week. Also he would like to supply 10 chairs if possible per week to a sister concern.

Solution :-

Priority P_1→ to avoid hiring extra labor or utilize overtime.

Priority P_2→ to reach a profit goal of Rs.2000 a week.

Priority P_3→ to supply 10 chair a week to the sister concern.

Let x_1&$x_2$represents the number of chair & tables to be manufactured per week.d_i^- & d_i^+ denotes the amount by which the ith goal is under achieved & over achieved respectively. The formulation of the problem is,

Min z=$P_1 d_1^+$ +$P_2 d_2^-$ + $P_3 d_3^-$

Subject to the constraints,

$80x_1$+ $10x_2$ + d_1^- - d_1^+ = 80 (man labour goal)

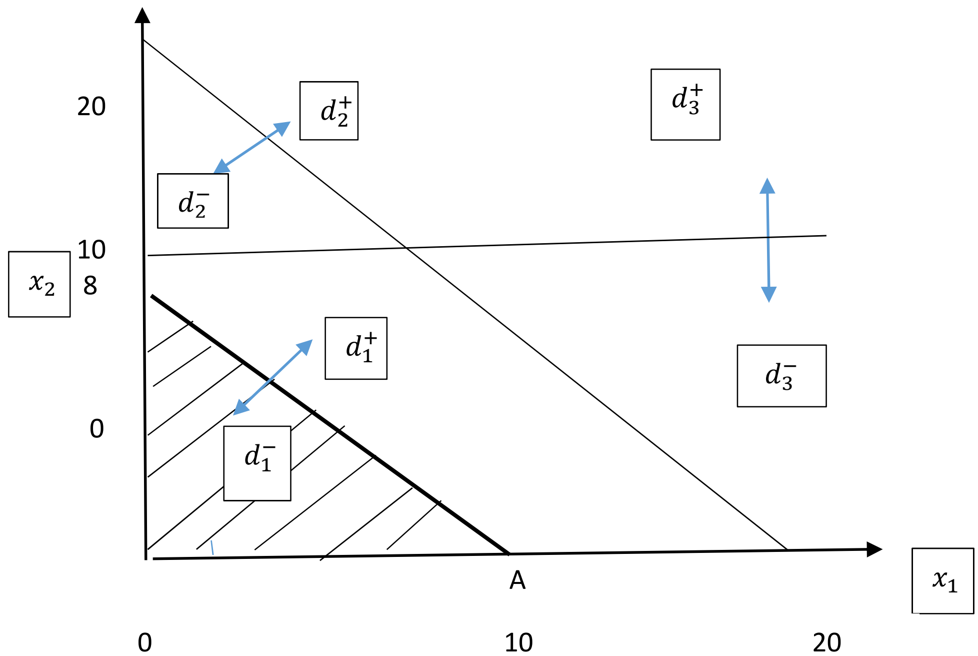
$100x_1$ + $100x_2$ + d_2^- - d_2^+ =2000 (profit goal)

x_2 + d_3^- - d_3^+ = 10 (chair goal), All variables ≥ 0 &d_i^- x d_i^+ =0

To achieve the first goal d_1^+ must be minimized. We may set d_1^+=0 as its minimum value. The region to satisfy, x_1,x_2≥0 &d_1^+=0 is shown shaded in the figure.

The next goal is profit goal for which we have to minimize d_2^-.It cannot be minimized to zero because it would degrade our previous solution. We can take it to the minimum possible value. Also to achieve the last goal, we must minimize d_3^-.But doing so we would not like to degrade goals already achieved. As a result the final solution is given by point A (x_1=10,x_2=0)

The optimal solution is x_1=10,x_2=0. Goal I is fully met, goal II is missed by 1000,goal III is missed by 10.



7.8 SIMPLEX METHOD OF SOLVING GOAL PROGRAMMING PROBLEM

Goal programming problems can be solved by simplex method in the same manner as simple linear programming problems are solved. This can be understood with the help of following example 7.3 using simplex method.

Example 7.3- A firm produces two products X and Y. Product X yields profit of Rs. 40 per unit, whereas product Y yields profit of Rs. 35 per unit. The raw material requirement for per unit of X and Y is 2 kg and 4 kg respectively and labour requirement is 3 hours for each. The firm has only 60 kg of raw material and 96 hours of labour available. Solve the problem as a goal programming problem with the goal to earn profit of Rs.1400.

Solution- In this problem, let us first define variables-

x_1 = no. of units of X to be produced

x_2 = no. of units of Y to be produced

d_i^-= underachievement of the target profit.

d_i^+= overachievement of the target profit.

The profit goal in the model can be written as a goal constraint-

$40\,x_1 + 35x_2 + d_i^- + d_i^+ = 1400$

The goal programming model can be now written as

Minimize $Z = d_i^- + d_i^+$

Subject to the constraints

$2x_1 + 3x_2 \leq 60$ (Raw material constraint) } resource

$4x_1 + 3x_2 \leq 96$ (labour hours constraint) } constraints

$40\,x_1 + 35\,x_2 + d_i^- + d_i^+ = 1400$ (target profit) goal constraints

$x_1, x_2, d_i^-, d_i^+ \geq 0$

Let us assume S_1 and S_2 as slack variables for resource constraints.

Initial simplex table can be written in the following manner-

Table 9

Basis		x_1	x_2	S_1	S_2	$\mathbf{d_i^-}$	$\mathbf{d_i^+}$	Bi	b_i/a_{ij}
S_1	0	2	3	1	0	0	0	60	30
S_2	0	4*	3	0	1	0	0	96	24 ←
$\mathbf{d_i^-}$	1	40	35	0	0	1	-1	1400	35
c_j		0	0	0	0	0	0		
Solution		0	0	60	96	1400	0		
Δ_j		↑ - 40	-35	0	0	0	1		

Table 10

Basis		x_1	x_2	S_1	S_2	$\mathbf{d_i^-}$	$\mathbf{d_i^+}$	Bi	b_i/a_{ij}
S_1	0	0	3/2 *	1	-1/2	0	0	12	8 ←
x_1	0	1	3/4	0	1/4	0	0	24	32
$\mathbf{d_i^-}$	1	0	5	0	-10	1	-1	440	88
c_j		0	0	0	0	1	0		
Solution		24	0	12	0	440	0		
Δ_j		0	↑ -5	0	10	0	1		

Table 11

Basis		x_1	x_2	S_1	S_2	d_i^-	d_i^+	Bi
S_1	0	0	1	2/3	-1/3	0	0	8
x_1	0	1	0	-1/2	1/2	0	0	18
d_i^-	1	0	0	-10/3	-25/3	1	-1	400
c_j		0	0	0	0	1	0	
Solution		18	8	0	0	400	0	
Δ_j		0	0	10/3	25/3	0	1	

In table 3, all Δ_j values are 0 or +ve, so the table represents the optimal solution. The solution is $x_1 = 18$, $x_2 = 8$, $d_i^- = 400$ and $d_i^+ = 0$, and profit = Rs. 1000, which shows that profit goal is underachieved by Rs. 400.

7.9 SUMMARY

This unit explains the concept of goal programming. Goal programming problems are the problems where more than one objective are involved. In such problems, goals are prioritised with respect to their relative importance and goals with higher priority are achieved first. The unit explains the method to formulate the goal programming problems. Simplex method for solving goal programming problems is also explained in the unit.

7.10 SELF ASSESSMENT QUESTIONS

1. What do you understand by goal programming?
2. How goal programming is different to linear programming?
3. Explain the method to formulate goal programming problems.
4. A hospital has various departments that are requesting for a new emergency room. The number of beds required by each department is different. The hospital is planning to use a maximum of 15000 square feet area for the new emergency room. The hospital has established the following goals in the order of importance-

Department	No. of Beds requested	Cost per Bed	Area per Bed (sq. ft.)	Peak Requirement (max. number of patients)
A	5	12600	474	3
B	20	5400	542	18
C	20	8600	438	15

i) Avoid overspending of the budget of Rs. 300000.

ii) Avoid room requiring more than 15000 sq. ft. area.

iii) Meet the peak requirements and

iv) Meet the departmental requirements.

Formulate as a goal programming problem.

5. A company produces three items A, B and C. All items are processed in one manufacturing plant. Production of A, B, and C requires 2, 3 , and 1 hours respectively. The plant capacity is 40 hours per week. The maximum sales of A, B, and C is 10, 10 and 12 units per week respectively. The manager has established following goals according to their preference-

i) Avoid underutilization of production capacity.

ii) Meet the order of XYZ stores for 7 units of B and 5 units of C per week.

iii) Avoid overtime operation of the plant beyond 10 hours.

iv) Achieve the sales goal of 10 units of A, 10 units of B and 12 units of C.

v) Minimise the overtime operation as much as possible.

Formulatie this as a goal programming problem.

7.11 TEXT AND REFERENCES

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UNIT-8 INTEGER PROGRAMMING

Unit Framework

8.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the meaning and concept of integer programming
- Learn about various types of integer programming
- Formulate integer programming problems
- Solve integer programming problems by cutting plane method.
- Solve integer programming problems by branch and bound method.

8.2 INTRODUCTION

Linear programming problems involve minimization or maximization of a given objective function subject to a certain set of constraints. In such problems, the variables involved are non-negative and may assume values which are fractional as well as integer. However in some cases, fractional solutions are unrealistic, in such situations the variables may be allowed to take only integer values. For example- many business problems require the assignment of machines, men and material to production activities in integer quantities. The restriction that the decision variables must have integer values has led to the development of special programming techniques. Such problems are called integer programming problems and the technique is called integer programming.

In this chapter, first, we will discuss integer-programming formulations. It will provide insight into the scope of integer-programming applications and give some indication of why many practitioners feel that the integer-programming model is one of the most important models in management science. Second, we consider basic approaches that have been developed for solving integer and mixed-integer programming problems.

8.3 MEANING AND CONCEPT OF INTEGER PROGRAMMING

Integer Programming problems are special type of linear programming problems, in which some or all the variables are restricted to non-negative integer values. In a typical linear programming problem, decision variables are assumed to be continuous variables which can also take fractional values. For example- 25/6 units of a product or ¾ units of labour, but such condition is not applicable in every problem. There are many situations where decision variables can take only integer values such as in production problems, manufacturing is mostly scheduled in lots or batches, transportation of goods involves a discrete number of trucks etc. where fractional values are meaningless. Therefore, it is necessary to use integers as decision variables in such situations. This necessitates the use of new programming models called integer programming models.

8.4 TYPES OF INTEGER PROGRAMMING PROBLEMS

There are three types of Integer Programming problems (IPP)-

a. Pure integer programming problems- a pure integer programming problem is the case in which all decision variables are required to have integer values. Such problems are also called as all-integer programming problems.

b. Mixed integer programming problems- an integer programming problem in which some but not all of the decision variables are required to have integer values is called mixed integer programming problem.

c. Zero-one integer programming problems- integer programming problems in which all decision variables are required to have integer solution values of zero or one only are called ‘0-1’ integer programming problems. These are special type of IPP where variables are limited to two logical values such as yes or no, pass or fail etc. which are represented as 0 or 1. A good example of such 0-1 IPPs are Assignment problems.

Example 8.1- A factory produces two types of purses X and Y. Per unit profit on X and Y is Rs. 200 and Rs. 300 respectively. The company has two machines A and B which are used to produce X and Y. Machine A can work for a total of 17 hours per day where as machine B can work at most for 15 hours a day. Both purses can be produced by both machines. The time required for producing one unit of X on machine A is 2 hours and 3 hours for producing by machine B. Similarly producing one unit of Y requires 4 hours on machine A and 3 hours on machine B. How many units of X and Y should be produced in a day to maximize the profit. Formulate it as an integer programming problem.

Solution- Let the daily production of purse X and purse Y be x_1 and x_2 units respectively.

The profit will then be $Z = 200x_1+300x_2$

The IPP can be formulated as:

Maximize $Z = 200x_1+300x_2$

Subject to the constraints

$2x_1 + 4x_2 \leq 17$

$3x_1 +3x_2 \leq 15$

$x_1, x_2 \geq 0$ and integer

Example 8.2- In a factory, there are three workers and three jobs, the cost matrix for each worker-job combination is given below:

Worker	Job		
	J1	J2	J3
W1	4	7	8
W2	6	5	10
W3	7	7	9

Formulate the problem as an Integer Programming problem to determine which job should be assigned to whom so as to minimise the cost of completing all the jobs.

Solution- the given problem can be expressed as a 0-1 integer programming problem in which x_{ij}'s represent assignment of i^{th} worker to the j^{th} job.

The objective function can be written as-

Minimize $Z = 4x_{11} + 7x_{12} +8x_{13} + 6x_{21} + 5x_{22} + 10x_{23} + 7x_{31} + 7x_{32} + 9x_{33}$

Subject to the constraints

$x_{11} + x_{12} + x_{13} =1$ $\quad$ $x_{11} + x_{21} + x_{31} =1$

$x_{21} + x_{22} + x_{23} =1$ $\quad$ $x_{12} + x_{22} +x_{32} =1$

$x_{31} + x_{32} + x_{33} =1$ $\quad$ $x_{13} + x_{23} + x_{33} =1$

$x_{ij} \geq 0$ for i= 1, 2, 3 and j= 1, 2, 3

$x_{ij} = 0$ (if assignment is not made) or

$x_{ij} =1$ (if assignment is made)

8.5 APPLICATIONS OF INTEGER PROGRAMMING MODELS

There are various types of problems in which integer programming models are applied. Some of the applications of integer programming models are as follows-

a. Assignment problems- Assignment problems are most of the time 0-1 type integer programming problems where 0 may be used as no assignment and 1 may be used as assignment made.

b. Capital Budgeting problems- Many times, companies face the problem of selecting one or more developmental projects or investment opportunities among several projects. In such cases also the problem may be treated as 0-1 integer programming problems where 0 may be assigned to a project not selected and 1 may be assigned to a project that is selected.

c. The fixed charge problem- Many problems in real life situations involve a combination of fixed and variable costs. Fixed cost changes only when the certain capacity level is exceeded such as in case of machine scheduling. A new machine may be required only in a case when production exceeds, such cases can also be treated as 0-1 integer programming problems where 0 may be assigned if no new machine is used and 1 may be assigned if new machine is required.

d. Facility location problem- one of the applications of integer programming problem is facility location problem. In such problems, it is decided where to locate a facility. A company may have various potential options to locate a facility. A location if chosen to set up the facility then a 1 may be assigned, on the other hand if the location is not selected, a zero may be assigned.

8.6 METHODS TO SOLVE INTEGER PROGRAMMING PROBLEMS

However, the integer programming problems can be solved by using regular graphical or simplex methods of solving linear programming problems and then rounding off the fractional values obtained in optimal solution, but this method can give infeasible solutions many time due to the deviations caused by rounding off the actual fractional values in the optimal solution. Therefore a systematic procedure is developed to solve integer programming problems and obtaining optimal solutions to such problems. There are two such methods to solve IPPs- cutting plane algorithm and Branch and Bound method. These are mentioned below-

8.6.1 CUTTING PLANE ALGORITHM

In this method, first we obtain the solution assuming the given problem as a regular linear programming problem using simplex method. If the solution has all integer values for the decision variables then the optimal solution itself is the solution for integer programming problem. Otherwise, we introduce a cut using a constraint in the optimal solution which has a fractional right hand side solution (b_j) value. The cut is introduced to optimal solution of LPP and the revised solution is obtained using simplex method, with some modification. Cuts are introduced until all the variables assume integer values.

The algorithm can be illustrated by using the formulated problem in example 8.1 above.

Example 8.3- Solve the problem given in example 8.1 by using cutting plane algorithm.

Solution- The problem is given as-

Maximize $Z = 200x_1+300x_2$

Subject to the constraints

$2x_1 + 4x_2 \leq 17$

$3x_1 +3x_2 \leq 15$

$x_1, x_2 \geq 0$ and integer

The solution to above problem assuming no restriction of integer values on decision variables can be obtained by using simplex method of solving LPP as shown below-

After introducing slack variables

Max Z= 200x1+300x2+0SI+0S2

subject to

2x1+4x2+SI =17

3x1+3x2 +S2=15

and x1,x2,SI,S2≥0

Simplex Table 1		*Cj*	200	300	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***S*1**	***S*2**	**MinRatio** ***XBx*2**
*S*1	0	17	2	4	1	0	17/4=4.25→
*S*2	0	15	**3**	3	0	1	15/3=5
***Z*=0**		***Zj***	**0**	**0**	**0**	**0**	
		Cj-Zj	200	300↑	0	0	

Simplex Table 2		*Cj*	200	300	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***S*1**	***S*2**	**MinRatio *XBx*1**
*x*2	300	4.25	0.5	1	0.25	0	4.25/0.5=8.5
***S*2**	0	2.25	**1.5**	0	-0.75	1	2.25/1.5=1.5 →
***Z*=1275**		***Zj***	**150**	**300**	**75**	**0**	
		Cj-Zj	50↑	0	-75	0	

Simplex Table 3		*Cj*	200	300	0	0
B	***CB***	***XB***	x_1	x_2	S_1	S_2
x_2	300	3.5	0	1	0.5	0.3333
x_1	200	1.5	1	0	–0.5	0.6667
***Z*=1350**		***Zj***	**200**	**300**	**50**	**33.3333**
		Cj-Zj	0	0	–50	–33.3333

Since all Cj-Zj≤0

Hence, non-integer optimal solution is arrived with value of variables as : $x_1=1.5, x_2=3.5$

Max Z=1350
To obtain the integer valued solution, we proceed to construct Gomory's fractional cut, with the help of x_2-row as follows:

$3.5=1x_2+0.5S_1-0.3333S_2$

$(3+0.5)=(1+0)x_2+(0+0.5)S_1+(-1+0.6667)S_2$

The fractional cut will become

$-0.5=Sg_1-0.5S_1-0.6667S_2 \rightarrow$ (Cut-1)

Adding this additional constraint at the bottom of optimal simplex table. The new table so obtained is

Revised Simplex Table-1		C_j	200	300	0	0	0
B	C_B	X_B	x_1	x_2	S_1	S_2	Sg_1
x_2	300	3.5	0	1	0.5	-0.3333	0
x_1	200	1.5	1	0	-0.5	0.6667	0
Sg_1	0	-0.5	0	0	-0.5	(-0.6667)	1
Z=1350		Zj	200	300	50	33.3333	0
		Cj- Zj	0	0	-50	-33.3333	0
		Ratio= $(C_j- Z_j)/Sg_{1,j}$ and $Sg_{1,j}<0$	---	---	100	50↑	---

Revised Simplex Table-2		Cj	200	300	0	0	0

B	C_B	X_B	x_1	x_2	S_1	S_2	Sg_1
x_2	300	3.75	0	1	0.75	0	-0.5
x_1	200	1	1	0	-1	0	1
S_2	0	0.75	0	0	0.75	1	-1.5
Z=1325		Zj	200	300	25	0	50
		Cj- Zj	0	0	-25	0	-50
		Ratio	---	---	---	---	---

Since all Cj- Zj ≤ 0

Hence, non-integer optimal solution is arrived with value of variables as : $x_1=1, x_2=3.75$
Max Z=1325
To obtain the integer valued solution, we proceed to construct Gomory's fractional cut, with the help of x_2-row as follows:

$3.75=1x_2+0.75S_1-0.5Sg_1$
$(3+0.75)=(1+0)x_2+(0+0.75)S_1+(-1+0.5)Sg_1$
The fractional cut will become

$-0.75=Sg_2-0.75S_1-0.5Sg_1 \rightarrow$ (Cut-2)
Adding this additional constraint at the bottom of optimal simplex table. The new table so obtained is

Revised Simplex Table-3		Cj	200	300	0	0	0	0
B	CB	X_B	x_1	x_2	S_1	S_2	Sg_1	Sg_2
x_2	300	3.75	0	1	0.75	0	-0.5	0
x_1	200	1	1	0	-1	0	1	0

S_2	0	0.75	0	0	0.75	1	-1.5	0
Sg_2	0	-0.75	0	0	(-0.75)	0	-0.5	1
Z=1325		Zj	20 0	300	25	0	50	0
		Cj- Zj	0	0	-25	0	-50	0
		Ratio=(Cj- Zj)/Sg_2,j and Sg_2,j<0	---	---	33.3333↑	---	100	---

Revised Simplex Table- 4		Cj	200	300	0	0	0	0
B	C_B	X_B	x_1	x_2	S_1	S_2	Sg_1	Sg_2
x_2	300	3	0	1	0	0	-1	1
x_1	200	2	1	0	0	0	1.6667	-1.3333
S_2	0	0	0	0	0	1	-2	1
S_1	0	1	0	0	1	0	0.6667	-1.3333
Z=1300		Zj	200	300	0	0	33.3333	33.3333
		Cj- Zj	0	0	0	0	-33.3333	-33.3333
		Ratio	---	---	---	---	---	---

Since all Cj- Zj ≤ 0

Hence, integer optimal solution is arrived with value of variables as:

x1=2,x2=3
Max Z=1300
The integer optimal solution found after 2-cuts.

8.6.2 BRANCH AND BOUND METHOD

The common approach to solve integer programming problems, is to round off the optimal non-integer solution to integer solution, however, this approach has two major limitations. First is that the rounded off solution may not be feasible as it may not be easy to ascertain the direction in which rounding is done. Second, there is no gurantee that the rounded off solution will be optimal integer solution. Therefore several algorithms have been developed to solve integer programming problems, out of which one important method is branch and bound method. In this method, a certain procedure is applied differently to different kinds of problems to find the optimal solution..

Following are the steps of applying branch and bound method-

Step 1- Solve the original problem using Linear programming. If the solution satisfies the integer constraints, the solution obtained is the optimal integer solution, if not, this value provides an initial upper bound.

Step 2- Find an feasible integer solution that to be used as a lower bound. For that usually rounding down each variable is required.

Step 3- Branch on one variable from step 1 that does not have an integer value. Split the problem into two sub-problems based on integer values that are immediately above and below the non-integer value. For example, if $x_1 = 5.85$ was in the final optimal linear solution, introduce the constrains $x_1 \geq 6$ in the first sub problem and $x_1 \leq 5$ in the second sub problem.

Step 4- Create nodes at the top of these new branches by solving the new problems.

Step 5- (a) if the branch gives a solution to Linear Programming Problem that is not feasible, terminate the branch.

(b) if the branch gives a solution to Linear Programming Problem that is feasible, but not integer solution go to step 6.

(c) if the branch gives a feasible integer solution, examine the value of the objective solution has been reached, if it is not equal to the upper bound, but exceeds the lower bound, set it as the new lower bound and go to step 6. Finally, if it is less than the lower bound, terminate the branch.

Step 6- Examine both branches again and set the upper bound equal to the maximum value of the objective function at all find nodes. If the upper bound equals the lower bound, stop. If not, go back to Step 3.

Example 8.4- Solve the following integer programming problem by branch and bound method.

Minimize Z= $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1 + 5x_2 \leq 30$

$x_1, x_2 \geq 0$ and integer

Solution- The given problem can be solved by graphical method of solving Linear Programming Problem. The solution obtained is

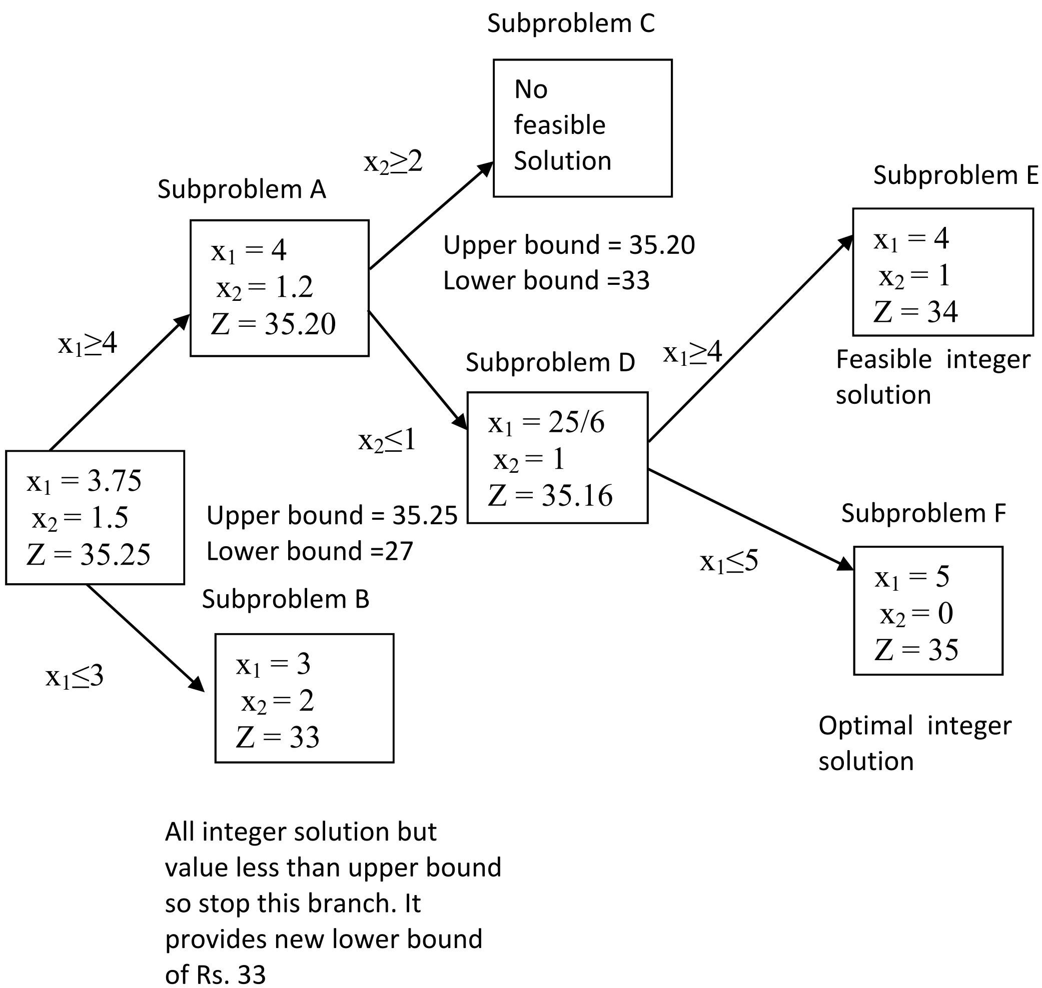
$x_1 = 3.75$, $x_2 = 1.5$ and profit = 35.25

Since the solution obtained is a non-integer solution therefore it is not valid solution. Using branch and bound method, we will consider this profit value of 35.25 as an initial upper bound.

Rounding down the solution gives $x_1 = 3$ and $x_2 = 1$ with a profit value of 27, which is feasible and can be used as a lower bound.

Now we get sub problems A and B as shown in the diagram below-

Let us first choose x_1 for branching.



Sub-problem A

Minimize Profit= $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

Sub-problem B

Minimize profit = $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1 + 5x_2 \leq 30$

$x_1 \leq 3$

Solving these two sub problems we get the following solution

Sub problem A's Solution- $x_1 = 4$ and $x_2 = 1.2$ with a profit value of Rs. 35.20

Sub problem B's solution- $x_1 = 3$ and $x_2 = 2$ with a profit value of Rs. 33

This information is presented in the diagram shown above.

Now the branch of sub problem B can be terminated as it has all integer solution (same as step 5-c). The profit value of Rs. 33 is now the new lower bound. Sub-problem A's branch can be searched further as the solution is non-integer. So the second upper bound is profit value of Rs. 35.20. Now the sub-problem A is further divided into two new sub-problems C & D where branching of x_2 is done. It is shown in the diagram above.

Sub-problem C

Minimize Profit= $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_2 \geq 2$

Sub-problem D

Minimize profit = $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_2 \leq 1$

Sub problem C has no feasible solution because the first two constraints are violated if the last two constraints are observed. Therefore this branch is terminated.

The solution for sub problem D is $x_1 = 25/6$ and $x_2 = 1$with a profit value of Rs. 35.16. This is a non integer solution which gives another upper bound of profit value Rs. 35.16.

Subproblem D is further divided into two branches E and F as x_1 had a non integer value in this solution so again branching for x_1 is done.

Sub-problem E

Minimize Profit= $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

Sub-problem F

Minimize profit = $7x_1 + 6x_2$

Subject to constraints

$2x_1 + 3x_2 \leq 12$

$6x_1+5x_2 \leq 30$	$6x_1 + 5x_2 \leq 30$
$x_1 \geq 4$	$x_1 \geq 4$
$x_1 \leq 4$	$x_1 \leq 5$
$x_2 \leq 1$	$x_2 \leq 1$

Solving these two sub problems we get the following solution

Sub problem E's Solution- $x_1 = 4$ and $x_2 = 1$with a profit value of Rs. 34

Sub problem F's solution- $x_1 = 5$ and $x_2 = 0$ with a profit value of Rs. 35

The stopping rule for branching process is that we continue until the new upper bound is less than or equal to lower bound or no further branching is possible. Here no further branching is possible so the optimal solution is obtained at sub-problem F's solution i.e. $x_1 = 5$ and $x_2 = 0$ with a profit value of Rs. 35.

8.7 SUMMARY

This unit explains the meaning and concept of integer programming. Integer programming problems are such special type of linear programming problems where all or some of the variables are restricted to be integer variables. There are pure or mixed integer problems depending upon whether all or some variables are integer variables. A integer programming problem can be 0-1 type of problem where the variables can take only either of these two values. The unit discusses cutting plane and branch and bound methods of solving integer programming problems.

8.8 SELF ASSESSMENT QUESTIONS

1. Explain the concept of integer programming. Give example for pure and mixed integer programming problems.
2. "It may not be possible to obtain an optimal integer solution by rounding off the continuous optimum of a linear programming problem."Comment.
3. Discuss the cutting-plane method for solving integer programming problems.
4. Discuss the branch and bound method of solving an integer programming problem.
5. An exporter of ready-made garments makes two types of shirts- X and Y. he makes a profit of Rs. 10 and Rs. 40 per shirt on X and Y respectively. He has two tailors- A and B who stitch these shirts. Tailors A and B can devote at most 7 hours and 15 hours per day respectively. Both these shirts are to be stitched by both the tailors. Tailor A and Tailor B spend 2 hours and 5 hours respectively in stitching a X shirt and 4 hours and 3 hours, respectively in stitching a Y shirt. How many shirts (integer values only) of both types should be stitched in order to maximize daily profits?

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UNIT-9 DYNAMIC PROGRAMMING

Unit Framework

9.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the concept of dynamic programming.
- Understand the difference between linear programming and dynamic programming.
- Learn various terms used in dynamic programming.
- Explain various applications of dynamic programming.
- Formulate and solve a dynamic programming problem.

9.2 INTRODUCTION

Dynamic programming is a quantitative approach used for solving certain type of decision problems. It deals with large problems in which there are large number of decision variables and are thus difficult to solve in one go. Dynamic programming technique adapts a systematic procedure where a complex problem is broken down into smaller sub-problems and determines the optimal combination of decisions.

9.3 MEANING AND CONCEPT OF DYNAMIC PROGRAMMING

Dynamic programming is a mathematical technique that deals with the optimization of multistage decision processes. The term ‘programming’ here refers to selecting an optimal allocation of resources, and it is called as ‘dynamic’ because it is used for the problems where decisions are taken at several stages. This technique was developed by Richard Bellman in early 1950. Another name that can be used in place of dynamic programming is ‘recursive optimization’.

According to Bellman, “An optimal policy has the property that, whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision processes.”

Dynamic programming obtains solutions by working backward from the end of the problem towards the beginning. It divides the problem into number of decision stages, the outcome of a decision at a given stage affects the decision at each of the following stages. Dynamic programming involves the following four steps-

1. Divide the original problem in sub-problems or stages.
2. Solve the last stage of the problem for all possible conditions or states.
3. Move backward from the last stage and solve each stage successively. This is done by determining optimal policies from that stage to the end of the problem.
4. Obtain optimal solution to the original problem by solving all stages sequentially.

9.4 DIFFERENCE BETWEEN DYNAMIC PROGRAMMING AND LINEAR PROGRAMMING

There are two major differences between dynamic programming and linear programming. First, unlike LPP , there is no standard mathematical formulation of dynamic programming problem. Therefore, there is no algorithm such as Simplex Method, that can be implemented to solve all problems. Rather, dynamic programming is a technique where a complex problem is divided into sequence of sub-problems, which are then evaluated by stages. Second, in Linear Programming, problems are solved in a single stage or in one-time period. Whereas, dynamic programming determines the optimal solution over a one year time hoizon by breaking the problem into twelve smaller one-month time horizon problems and solves each problem optimally. Therefore, it uses multistage approach to problem solving.

Various types of problems can be solved using dynamic programming concept, such as- shortest route problems, resource distribution problems, assignment problems, evaluating investment opportunities etc.

9.5 TERMS USED IN DYNAMIC PROGRAMMING

Some important terms used in dynamic programming are explained below-

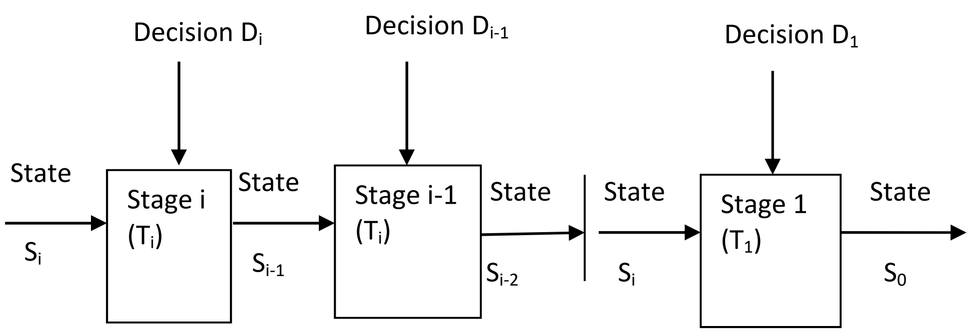
a. **Stage-** stage refers to each point where a decision is to be made. For example- in a shortest route problem, a stage may be group of cities with a common property or in a salesman allocation problem, each territory represented a stage.

b. **State-** State refers to the conditions describing the problem at each stage, in the form of specific values of state variables. For example- in a shortest route problem, the state at any stage is a specific city.

c. **Policy-** Policy refers to a decision making rule which permits a feasible sequence of decisions. A policy transforms the state at a given stage into a state linked with the next stage. The policy in a shortest route problem may be the selection of the routes to the next group of cities.

d. **Optimal Policy-** Optimal policy is the policy that optimizes the value of a criterion, objective or a return function. Starting in any given state of any stage, the optimal policy depends only upon that state and not upon how it was reached. The optimal decision at any stage is in no way dependent on the previous history of the system.

The general relationship between state, stage and policy can be represented by the following diagram-



At any stage i, the next state S_{i-1} is a function only of the state S_i at stage i and the decision D_i made at that stage. The state transformation function T_i transforms state Si to state S_{i-1}, given decision Di. It can be written as $S_{i-1} = T_i(S_i, D_i)$.

In the dynamic programming problems, solution starts by determining optimal policy for the last stage, labelled as stage 1 as it is analyzed first. Then we move backward stage by stage to find the optimal policy for each state of every stage i given the optimal policy or each state of stage i-1. The optimal policy for the entire problem is determined when the process ends at the initial stage. This procedure is called backward recursion.

9.6 APPLICATIONS OF DYNAMIC PROGRAMMING

There are various applications of dynamic programming in the field of management.

a. Production- In production field, dynamic programming is used for production scheduling and employment smoothening in the period of fluctuating demand requirements.

b. Allocation of resources to various alternative uses, such as assigning salesman to various sales territories, capital budgeting problems etc.

c. Selecting advertising media.

d. Spare part level determination to guarantee high efficiency utilization of expensive equipment.

e. Scheduling methods for routine and major overhauls on complex machinery.

f. Investment of cash surplus for a temporary time period in a way so that the investments will mature by the time cash is needed.

g. Equipment replacement policies to determine at which stage equipment is to be replaced.

h. Deciding locations.

i. Control of temperatures and catalysts in a chemical process.

9.7 FORMULATION AND SOLUTION OF A DYNAMIC PROGRAMMING PROBLEM

In order to formulate a decision making problem as a dynamic programming problem and solve that the following steps are used-

Step 1- Define the problem variables, determine the objective function to be minimized or maximized, also specify the problem constraints.

Step 2- Define the stages of the particular multistage decision problem. Determine the state variables, whose values constitute the state at each stage, and what constitutes the decisions required (policy) at each stage. Specify the state transformation function, the relationship by which the state at one stage is determined as a function of the state and decision at the next stage.

Step 3- Develop the recursion relationship, the optimal return function, which permits computation of the optimal policy given the state at any stage. Specify the optimal return function at stage 1, since this function is usually slightly different in form from the general optimal return function for the other stages.

The recursive relationship (for minimization) for state S_n at stage n may be formulated as follows- $f_n * (s_n) = \overset{min}{x_n}\{C_n(S_n, x_n) + f_{n+1}^{n}(s_{n+1})\}$

Where n = index for current stage (n =1, 2,..... N)

n+1= previous stage (remember that successive stages are numbered in descending order)

S_n = state of the system in the current stage

S_{n+1} = state in the previous stage

f_n (S_n) = total payoff for each alternative, starting from state S_n in stage n to the end of process

$f_n * (s_n)$ = optimal total payoff from state S_n in stage n.

$f_{n+1}^{n}(s_{n+1})$ = optimal total payoff obtained in the previous stage.

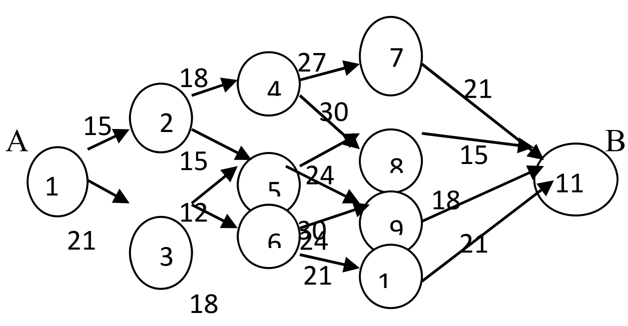
$C_n(S_n, x_n)$= immediate payoff in stage n when decision is made for a specific value S_n of the state variable.

x_n= decision among alternatives made at state n in the stage under consideration.

Step 4- Construct tables to clearly show the required values and calculations for each stage. Solve manually or by developing the required computer program.

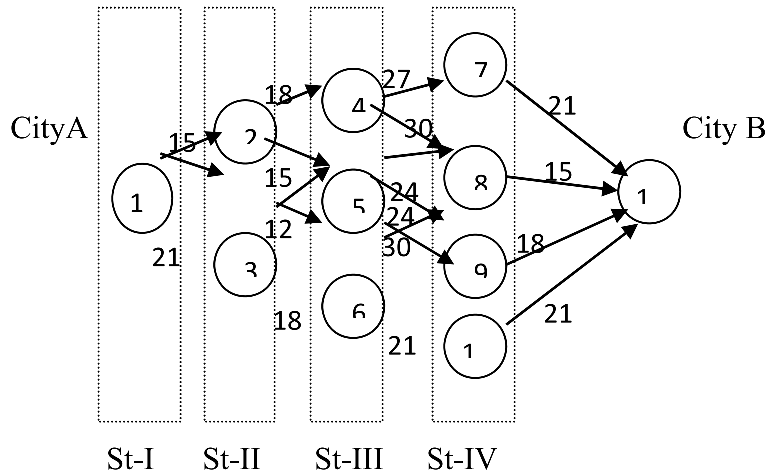
Step 5- The optimal solution is obtained for the original problem when all the stages have been sequentially solved.

Example 9.1 A traveller want to travel from city A to city B. There is no direct link between the two cities and the distance between the cities is very long. The traveller has decided to break the journey thrice during the travel. There are various options available at each break of journey. these are shown in the figure given below. Each circle in the figure represents a node for exchange where traveller has to leave the old means of transport and shift to a new one. The travel time in hours is indicated along with every route in the given figure. The traveller has to decide the route to travel to city B that involves minimum travelling time.



Solution- The first step here is to divide the problem into stages. In the given problem, a stage is a point where the traveller has to leave old mean of transport and decide next means of transport.

The given problem can be divided into four stages as shown in the figure given below. The traveller will have to choose a new means of transportation at four stations or stages. Each stage is considered as one sub-problem. In each stage there can be only one exchange point as shown in the figure. Various exchange points in each stage can be termed as states.



To solve the problem, we move backward from city B to city A, we determine the shortest path to city B in stage IV from stage III. This process is then repeated until city A is reached.

Analysis for stage IV- If the traveller has to reach city B, then he must be in one of the exchange points (states) 7, 8, 9 or 10 of stage IV. It is shown in the figure that there is only one way to reach city B from these states. The results for stage IV can be represented in the following table.

State	Altenative routes	Travel time (hrs.)	Best route time (hrs.) $f_4 * (s_n)$
7	7-11	21	21
8	8-11	15	15
9	9-11	18	18
10	10-11	27	27

Analysis for stage III- Moving backward to stage III, here three states 4, 5, and 6 are there. From each of these state there is a possibility to reach city B, but we will examine the route till states of stage IV. The following table shows the result for this stage.

State	Altenative routes	Travel time to stage IV (hrs.)	Travel time to city B	Total Travel time $f_3 * (s_n)$	Best route
4	4-7	27	21	48	

	4-8	30	15	45	√
5	5-8	24	15	39	√
	5-9	30	18	48	
6	6-9	24	18	42	√
	6-10	21	27	48	

For state 4,

$f_3 * (4)$=min{ $(C_3(4,7) + f_4(7)), (C_3(4,8) + f_4(8))$}= min {(27 + 21), (30+15)}

= min(48, 45) = 45

For state 5,

$f_3 * (5)$=min{ $(C_3(5,8) + f_4(8)), (C_3(5,9) + f_4(9))$}=min {(24 + 15), (30+18)}

= min(39, 48) = 39

For state 6,

$f_3 * (6)$=min{ $(C_3(6,9) + f_4(9)), (C_3(6,10) + f_4(10))$}=min{(24+ 18), 21+27)}

= min(42, 48) = 42

Analysis for Stage II- - Moving backward to stage II, here two states 2 and 3 are there. From each of these state there is a possibility to reach city B, but we will examine the route till states of stage III. The following table shows the result for this stage.

State	Altenative routes	Travel time to stage III (hrs.)	Travel time from stage III to city B	Total Travel time $f_2 * (s_n)$	Best route
2	2-4	18	45	63	
	2-5	15	39	54	√
3	3-5	12	39	51	√
	3-6	18	42	60	

For state 2,

$f_2 * (2)$=min{ $(C_2(2,4) + f_3(4)), (C_2(2,5) + f_3(5))$}=min {(18 + 45), (15+39)}

= min(63, 54) = 54

For state 3,

$f_2 * (3)$=min{ $(C_2(3,5) + f_3(5)), (C_2(3,6) + f_3(6))$}=min {(12 +39), (18 +42)}

= min(51, 60) = 51

Analysis for Stage I- Moving backward to stage I, there is only one state 1. Here we will examine the route till states of stage II. The following table shows the result for this stage.

State	Altenative routes	Travel time to stage II (hrs.)	Travel time from stage II to city B	Total Travel time $f_1 * (s_n)$	Best route
1	1-2	15	65	69	√
	1-3	21	51	72	

For state 1,

$f_1 * (1)$=min{ $(C_1(1,2) + f_2(2)), (C_1(1,3) + f_2(3))$}=min {(15 + 65), (21+51)}

= min(69, 72) = 69.

Now we have obtained the optimal solution. Moving forward from stage I to stage IV, we decided that traveller should move from state 1 to state 2, from state 2 to state 5, from state 5 to state 8 and from state 8 to state 11. It gives the minimum total travel time i.e. 69 hours.

9.8 SUMMARY

This unit explains the meaning and concept of dynamic programming. Dynamic programming solves complex problems by breaking them in smaller sub-problems which are called stages. The outcome of a decision at a given stage affects the decision at each of the following stages. Dynamic programming is different to linear programming in that there is no standard mathematical formulation of a dynamic programming problem unlike as of a LPP. The unit also covers the method of formulation and solution of dynamic programming problems. Applications of dynamic programming problems have also been discussed in this unit.

9.9 SELF ASSESSMENT QUESTIONS

1. Explain dynamic programming. How is it different from linear programming?
2. What is a stage in dynamic programming?
3. Explain various steps involved in solving dynamic programming problems.

4. Discuss various applications of dynamic programming.
5. Explain the following terms used in dynamic programming.

   a) stages, b) states, c)Payoff function, d) Recursive relationship

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UNIT-10 NON-LINEAR PROGRAMMING

Unit Framework

10.1 OBJECTIVES

After completing this unit, you will be able to:

- Understand the concept of Non-linear programming.
- Differentiate between linear and non-linear programming.
- Learn various applications of non-linear programming.
- Understand problem classification in non-linear programming.

10.2 INTRODUCTION

Non-linear Programming is that form of programming in which some or all of the variables are curvilinear. In other words, this means that either the objective function or constraints or both are not in linear form. In most of the practical situations, we encounter with non-linear programming problems but for computational purposes we approximate them as linear programming problems. Even then there may be some non-linear programming problems which may not be fully solved by presently known methods.

10.3 MEANING AND CONCEPT OF NON-LINEAR PROGRAMMING

A Non-linear program (NLP) is similar to a linear program in that it is composed of an objective function, general constraints and variable bounds. The difference is that a nonlinear program a nonlinear program includes at least one nonlinear function, which could be the objective function, or some or all of the constraints.

Thus, the general nonlinear program is stated as:

Maximize $Z = f(x1, x2, \ldots, xn)$,

subject to:

$g_1(x1, x2, \ldots, xn) \leq b1$,

. .

. .

. .

$g_m(x1, x2, \ldots, xn) \leq bm$,

Where each of the constraint functions g_1 through g_m is given.

Here either the objective function is nonlinear and/or the feasible region is determined by nonlinear constraints.

Many real systems are basically nonlinear, e.g. modelling the drop in signal power with distance from a transmitting antenna, so it is important to develop such optimization algorithms that may be able to solve them. Nonlinear models are inherently *much* more difficult to optimize as compared to Linear models.

10.4 SOME REAL APPLICATIONS OF NONLINEAR PROGRAMMING PROBLEMS

The following simplified examples illustrate how nonlinear programs can arise in practice.

a. **Portfolio Selection-** An investor has Rs.50000 and two potential investments. Let x_j for j = 1 and j = 2 denote his allocation to investment j in Thousands of Rs.. From historical data, investments 1 and 2 have an expected annual return of 20 and 16 percent, respectively. Also, the total risk involved with investments 1 and 2, as measured by the variance of total return, is given by $2x_1^2 + x_2^2 + (x_1+x_2)^2$, so that risk increases with total investment and with the amount of each individual investment. The investor would like to maximize his expected return and at the same time minimize his risk. Clearly, both of these objectives cannot, in general, be satisfied simultaneously. There are several possible approaches. For example, he can minimize risk subject to a constraint imposing a lower bound on expected return. Alternatively, expected return and risk can be combined in an objective function, to give the model:

Maximize $f(x) = 20x_1 + 16x_2 - \theta[2x_1^2 + x_2^2 + (x_1 + x_2)^2]$,

subject to:

$g_1(x) = x_1 + x_2 \leq 5$,

$x_1 \geq 0, x_2 \geq 0$.

The nonnegative constant θ reflects his tradeoff between risk and return. If $\theta = 0$, the model is a linear program, and he will invest completely in the investment with greatest expected return. For very large θ, the objective contribution due to expected return becomes negligible and he is essentially minimizing his risk.

b. Water Resources Planning- In regional water planning, sources emitting pollutants might be required to remove waste from the water system. Let x_j be the pounds of Biological Oxygen Demand (an often-used measure of pollution) to be removed at source j. One model might be to minimize total costs to the region to meet specified pollution standards:

Minimize $\Sigma_{j=1}^{n} f_j(x_j)$,

subject to: $\Sigma_{j=1}^{n} a_{ij} x_j \geq b_i$ (i = 1, 2, . . . , m)

$0 \leq x_j \leq u_j$ (j = 1, 2, . . . , n), where $f_j(x_j)$ = Cost of removing x_j pounds of Biological Oxygen Demand at source j, b_i = Minimum desired improvement in water quality at point i in the system, a_{ij} = Quality response, at point i in the water system, caused by removing one pound of Biological Oxygen Demand at source j, u_j = Maximum pounds of Biological Oxygen Demand that can be removed at source j.

c. Constrained Regression- A university wishes to assess the job placements of its graduates. For simplicity, it assumes that each graduate accepts either a government, industrial, or academic position. Let N_j = Number of graduates in year j (j = 1, 2, . . . , n), and let G_j, I_j, and A_j denote the number entering government, industry, and academia, respectively, in year j ($G_j + I_j + A_j = N_j$). One model being considered assumes that a given fraction of the student population joins each job category each year. If these fractions are denoted as λ1, λ2, and λ3, then the predicted number entering the job categories in year j is given by the expressions $\hat{G}_j = \lambda_1 N_j$, $\hat{I}_j = \lambda_2 N_j$, $\hat{A}_j = \lambda_3 Nj$.

A reasonable performance measure of the model's validity might be the difference between the actual number of graduates G_j, I_j and A_j entering the three job categories and the predicted numbers $\hat{G}_j$, $\hat{I}_j$, and $\hat{A}_j$, as in the least-squares estimate:

Minimize Xn j=1 [(G j − Gˆ j) 2 + (Ij − ˆIj) 2 + (Aj − Aˆ j) 2], subject to the constraint that all graduates are employed in one of the professions. In terms of the fractions entering each profession, the model can be written as:

Minimize $\Sigma_{j=1}^{n} [(G_j - \lambda_1 N_j)^2 + (I_j - \lambda_2 N_j)^2 + (A_j - \lambda_3 N_j)^2]$,

subject to: $\lambda_1 + \lambda_2 + \lambda_3 = 1$,

$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0$.

This is a nonlinear program in three variables λ1, λ2, and λ3.

10.5 DIFFERENCE BETWEEN LINEAR AND NON-LINEAR PROGRAMMING

It is comparatively difficult to optimize nonlinear programming models than linear programming models. There are various reasons behind that. These are explained in the sections below-

10.5.1 LOCAL OPTIMUMVS GLOBAL OPTIMUM IN CASE OF NONLINEAR PROGRAMMING

Numerical methods for solving nonlinear programs have limited information about the problem, typically only information about the current point (and stored information about past points that have been visited). The usual available information is (i) the point ***x*** itself, (ii) the value of the objective function at ***x***, (iii) the values of the constraint functions at ***x***, (iv) the gradient at ***x*** (the derivative in a 1-variable problem), and (v) the Hessian matrix (i.e. the second derivative in a 1-variable problem). This is enough information to recognize when you are at a local maximum or minimum, but there is no way of knowing whether there exists a different

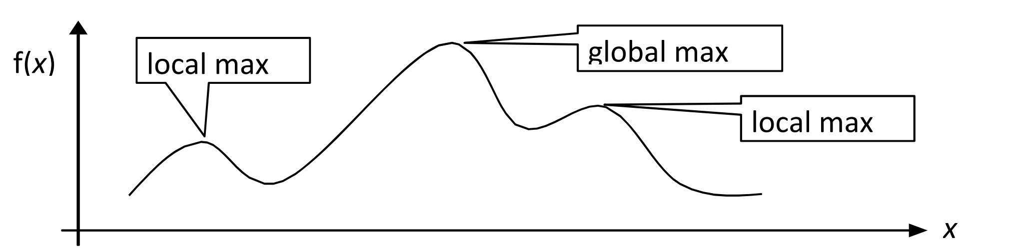


Figure 10.1: Local vs. global maximum for a 1-variable problem.

and better local maximum, or even how to proceed towards it. This also means that there is no way to easily determine where the global optimum is. These ideas are illustrated for a 1-variable unconstrained problem in Fig. 10.1. Formally, a *local optimum* is a feasible point that has a better value than any other feasible point in a small neighbourhood around it. The global optimum, on the other hand, is the point with the best value of the objective function anywhere in the feasible region. Note that the global optimum will also be a local optimum.

10.5.2 OPTIMIZATION MAY NOT BE ON EXTREME POINTS IN NONLINEAR PROGRAMMING PROBLEMS-

Geometrically, nonlinear programs can behave much differently from linear programs, even for problems with linear constraints. In the figure shown below 10.2, the portfolio-selection example given above has been plotted for several values of the tradeoff parameter θ. For each fixed value of θ, contours of constant objective values are concentric ellipses. As Fig. 10.2 shows, the optimal solution can occur:

a) at an interior point of the feasible region;

b) on the boundary of the feasible region, which is not an extreme point; or

c) at an extreme point of the feasible region.

As a consequence, procedures, such as the simplex method, that search only extreme points may not determine an optimal solution.

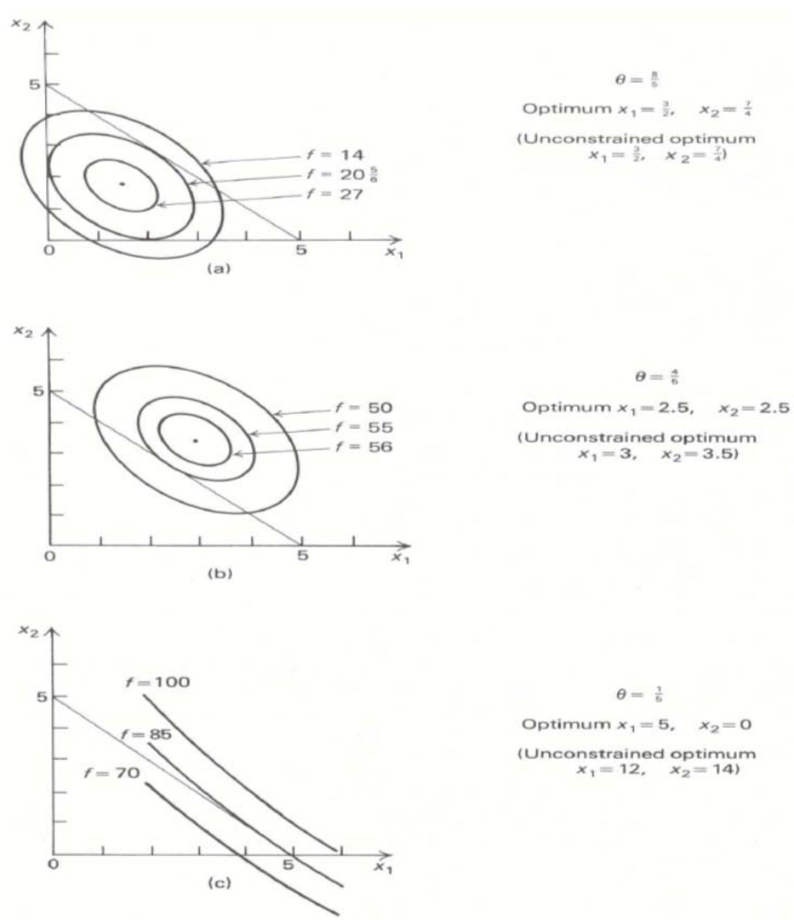
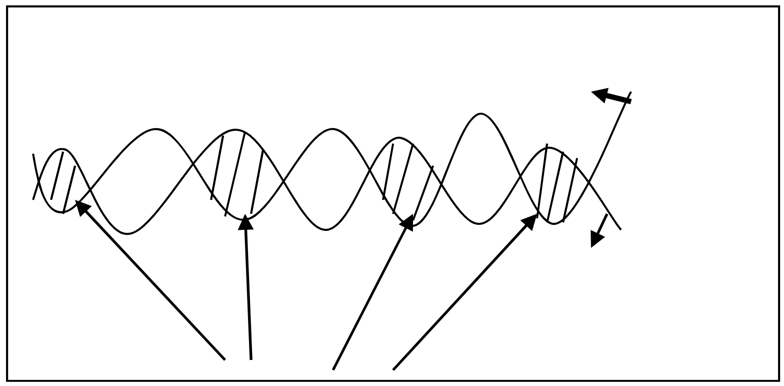


Figure 10.2 Portfolio Selection Example for various values of θ

NODE-129/NODE-106

10.5.3 POSSIBILITY OF MULTIPLE DISCONNECTED FEASIBLE REGIONS IN NONLINEAR PROGRAMS

Many nonlinear constraints can take twist and curve in such a manner that there may be a possibility of multiple different feasible regions. So even if the optimum solution within a particular feasible region is found, there is a possibility that there is some other disconnected (also called *discontiguous*) feasible region that haven't been found and explored. This is illustrated in Fig 10.3 in which the small arrows indicate the feasible sides of the nonlinear inequalities.



Feasible Regions

Figure 10.3

10.5.4 NONLINEAR MODELS MAY HAVE DIFFICULTY IN FINDING FEASIBLE STARTING POINT

In linear programming, it is possible to either find a point that satisfies all of the constraints (and hence is feasible), or it will accurately determine that no feasible points exist anywhere. However, in NLP, there is no guarantee of such condition. Most NLP procedures try to minimize some measure of the total infeasibility,

e.g. the sum of squared constraint violations. For example, if the constraint is $c(\boldsymbol{x}) \leq 12$ and at the current point $c(\boldsymbol{x}) = 15$, then the constraint violation is 3 and the squared violation is 9. If a point at which the sum of the squared constraint violations is zero is found, then the global optimum is reached and a feasible point is found. However this is in itself a nonlinear programming problem to solve, and also faces the difficulty.

10.5.5 NONLINEAR MODELS INVOLVE A HUGE BODY OF VERY COMPLEX MATHEMATICAL THEORY AND NUMEROUS SOLUTION ALGORITHMS

Nonlinear functions have a much wider range of behaviours and characteristics than linear functions. But it does make it hard to know how to solve the problem at hand. It is very difficult to decide which algorithm to apply?

Also whether all steps are being followed correctly or not?

The reasons why NLP is so much harder than LP given above are mainly based on the theory of nonlinear functions.

10.6 PROBLEM CLASSIFICATION

Many of the nonlinear-programming solution procedures that have been developed do not solve the general problem

Maximize f (x),

subject to:

$g_i(x) \leq b_i$ (i = 1,2,...,m),

but rather some special case. For reference, let us list some of these special cases:

1. Unconstrained optimization: f general , m = 0 (noconstraints).
2. Linear programming:

   $f(x) = \sum_{j=1}^{n} c_j x_j$, $g_i(x) = \sum_{j=1}^{n} a_{ij} x_j$ (i = 1,2,...,m),

   $g_{m+i}(x) = -x_i$ (i = 1,2,...,n).
3. Quadratic programming:

   $f(x) = \sum_{j=1}^{n} cjxj + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} x_i x_j$ (Constraints of case 2),

   (q_{ij} are given constants).
4. Linear constrained problem:

   f (x) general, $gi(x) = \sum_{j=1}^{n} a_{ij} x_j$ (i = 1,2,...,m), (Possibly $x_j \geq 0$ will be included as well).
5. Separable programming:

   $f(x) = \sum_{j=1}^{n} f_j(x_j)$, $g_i(x) = \sum_{j=1}^{n} g_{ij}(x_j)$ (i = 1,2,...,m);

   i.e., the problem ''separates'' into functions of single variables. The functions f_j and g_{ij} are given.
6. Convex programming: f is a concave function. The functions g_i(i = 1,2,...,m) (In a minimization problem, are all convex. f would be a convex function.)

Note that cases 2, 3, and 4 are successive generalizations. In fact linear programming is a special case of every other problem type except for case 1.

10.7 INDUSTRIAL APPLICATIONS OF NON-LINEAR PROGRAMMING

There are various applications of non-linear programming in different industries. Some of the applications are mentioned below-

1. Petrochemical Industry: Product blending, Refinery unit optimization, The unit design to multi-plant production, Distribution planning.

2. Nonlinear network: Electric power dispatch, hydroelectric reservoir management, Problems involving traffic flow in urban transportation networks, Routing.
3. Economic Planning: Dynamic econometric models, Static equilibrium models, Submodels of larger planning system.
4. Miscellaneous: Resource allocation, Computer aided design, Solution of equilibrium models, Data analysis & least square formulations, modeling human or organizational behavior.

10.8 SUMMARY

This unit explains the concept of non-linear programming. Non-linear programming is used when the objective function and the constraints are not linear in nature. Linear relationships may be applied to approximate non-linear constraints but limited to some range, because approximation becomes poorer as the range is extended. Thus, the non-linear programming is used to determine the approximation in which a solution lies and then the solution is obtained using linear methods. The unit helps in understanding various applications of non-linear programming and difference between linear and non-linear programming.

10.9 SELF ASSESSMENT QUESTIONS

1. What is non-linear programming?
2. Explain the difference between linear and non-linear programming.
3. Discuss various real life applications of non-linear programming.
4. Explain the problem classification in non-linear programming.
5. Discuss the concept of local vs global optimum in case of non-linear programming.

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Uttar Pradesh Rajarshi Tandon
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Master of Business Administration

BBA-117

Operations Research

BLOCK

4

INVENTORY AND WAITING LINE MODELS

UNIT-11

Inventory Control – Deterministic Models

UNIT-12

Inventory Control – Probabilistic Models

UNIT-13

Queuing Models

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BLOCK INTRODUCTION

In Block IV you will learn about concept of inventory management and control. In this block, you will understand about various deterministic and probabilistic models of inventory control. Also concept of queuing theory will be discussed in the later part of this block and various queuing models are explained in this block.

Unit-11 will discuss about basic concept of inventory management, objectives and functions of inventory management. In this unit you will learn about various types of inventories, inventory system and deterministic models of inventory control.

Unit- 12 helps in understanding the probabilistic inventory models. In this unit, the concept of re-order level and buffer stock will be introduced to understand the practical inventory control problems.

Unit- 13 will discuss about various types of queuing problems. In this unit, standard queuing techniques and models will be explained to help in solving the various queuing problems.

UNIT-11 INVENTORY CONTROL-DETERMINISTIC MODELS

Unit Framework

11.1 OBJECTIVES

After completing this unit you will be able to:

- Understand the concept of inventory and inventory costs
- Examine the functions of inventory in business.
- Understanding basic EOQ model and its assumptions
- Introducing the deterministic inventory models

11.2 INTRODUCTION

Inventory may be defined as any resource that has certain value and which can be used at a later time, when the demand for the item will arise. "Inventory can be defined as the stock of goods, commodities or other resources that are stored at any given period for future production."

These are idle goods or materials that are held by an organization for use sometime in the future. Items carried in inventory include raw materials, purchased parts, components, subassemblies, work-in-process, finished goods, and supplies. One reason organizations maintain inventory is that it is rarely possible to predict sales levels, production times, demand, and usage needs exactly. Thus, inventory serves as a buffer against uncertain and fluctuating usage and keeps a supply of items available in case the items are needed by the organization or its customers.

Inventory management is the process of efficiently overseeing the constant flow of units into and out of an existing inventory. This process usually involves controlling the transfer in of units in order to prevent the inventory from becoming too high, or dwindling to levels that could put the operation of the company into jeopardy. Competent inventory management also seeks to control the costs associated with the inventory, both from the perspective of the total value of the goods included and the tax burden generated by the cumulative value of the inventory.

11.2.1 NEED FOR KEEPING INVENTORY

The following are basic reasons for keeping an inventory:

1. ***Meeting production requirements-*** Raw materials, components and parts are required for producing finished goods. A manufacturing organization keeps stocks to meet the continuous requirements of production. Work in progress (WIP) inventory constitutes a major portion of production related inventory.

2. ***Support operational Requirements-*** To support production operations, inventory is required for repairs, maintenance and operations support e.g. spare parts, consumables such as lubrication oils, welding rods, chemicals etc.

3. ***Customer Service Consideration-*** Products like equipment, machinery and appliances require replacement of spare parts and other consumables for trouble free and smooth operation. Suppliers maintain an inventory of these parts to extend after sales services to its customers. Availability of spare parts when required at customer end is crucial for customer satisfaction and may be used as a tool for competitive advantage.

4. ***Hedge against future expectations-*** Inventory takes care of shortages in material or product availability or due to an unanticipated increase in the price of products.

5. ***Balancing supply and demand-*** The production and the consumption cycle never match. The sudden requirement of products in large quantities may not be satisfied since production cannot be taken so soon. In such a case, the products are manufactured in advance in anticipation of a sudden demand and kept in stock for supply during the peak period.

6. ***Periodic variation***- For seasonal products, the demand is at peak in certain periods while it is lean/little for the rest of the year. Production runs in the factory are taken based on the average demand for the year. Excess production in the lean periods is kept in inventory to take care of high demand. In cases where raw material is available seasonally, the products are manufactured and stocked as inventory to meet the demand of the finished product throughout the year. e.g agricultural produce.

7. ***Economies of scale***- Products are manufactured at focused factories to achieve economies of scale. This is done because of the availability of the latest technology, raw materials and skilled labour. Hence the product is kept in store for distribution to consumption centers as and when it is required.

8. ***Other reasons for keeping inventory include:***

   - To take advantage of quantity discounts.

   - As a necessary part of production process e.g. the maturing of whisky/ wine products.

   - Case of critical and strategic products such as petroleum or cereals (strategic food reserve).

   - As a contractual obligation e.g. the oil sector.

11.3 OBJECTIVES OF INVENTORY MANAGEMENT

Inventory Management has the following objectives-

a. To minimize the capital blocked in inventories.

b. To keep the minimum consumption of value of materials.

c. To maintain timely record of inventories of all items.

d. To provide basis for short-term and long-term planning of inventory.

e. To ensure timely action for replenishment.

f. To maintain a safeguard against variations in raw material lead time.

g. To help in facing demand fluctuations.

h. To reduce surplus stock.

i. To protect inventory from theft, waste, loss or damage.

j. To allow flexibility in production scheduling.

11.4 ADVANTAGES OF INVENTORY MANAGEMENT

The potential benefits of the application of inventory management concepts are many and include the following:

1. Provides both internal and external customers the required service levels

in terms of quantities and the order rate fill (timing).

2. Ascertains present and future requirements for all types of inventory to avoid overstocking or under- stocking.
3. Keeps costs at the minimum by variety reduction, economic lot sizes and analysis of costs incurred in obtaining and keeping inventories.
4. Provides upstream and downstream inventory visibility or service in the supply chain.
5. Reduces the risk of loss due to change in prices of items stocked at the time of making the stock.
6. Eliminates the possibility of duplicate ordering.
7. Helps the firm in ensuring smooth functioning of various departments by maintaining reasonable stocks.

11.5 CLASSIFICATION OF INVENTORY

Inventory can be broadly defined into two categories: Direct inventory and Indirect inventory.

1. **Direct Inventory-** It includes those items which play a direct role in the manufacturing and are the integral part of finished product. These can be of three types-

   a. **Raw materials:** Everything the crafter buys to make the product is classified as raw materials. That includes leather, dyes, snaps and grommets. The raw material inventory only includes items that have not yet been put into the production process.

   b. **Work in process:** This includes all the leather raw materials that are in various stages of development. For the leather crafting business, it would include leather pieces cut and in the process of being sewn together and the leather belts and purse etc. that are partially constructed.

      In addition to the raw materials, the work in process inventory includes the cost of the labor directly doing the work and manufacturing overhead. Manufacturing overhead is a catchall phrase for any other expenses the leather crafting business has that indirectly relate to making the products. A good example is depreciation of leather making fixed assets.

   c. **Finished goods:** When the leather items are completely ready to sell at craft shows or other venues, they are finished goods. The finished goods inventory also consists of the cost of raw materials, labor and manufacturing overhead, now for the entire product.

2. **Indirect Inventory-** It include those items which are required for manufacturing but do not become the integral part of finished goods. They can be divided into following categories-

a. **Cycle stocks.** The cycle stock induced by batching alternates between an upper level when a batch has just arrived and a lower level just before the arrival of the next batch. Cycle stocks mostly attribute to economies of scale of purchasing and transportation, and technological restrictions in production.

b. **Pipeline stocks.** Order processing times, production, and transportation rates contribute to pipeline stocks, also called process inventories. Materials that are in process, in transport, and in transit to another processing unit belong to pipeline stocks.

c. **Safety stocks.** The safety stock is interpreted as the expected inventory just before the next replenishment arrives. It is caused by the uncertainty of demand, processing time, yield and other factors. And its major function is to protect business performance from forecasting errors.

d. **Speculative stocks.** Expected price increase may result in earlier supply than would have been experienced under constant price, meaning there are more inventories on hand than actual demand at certain period of time, the redundant inventory is speculative inventory. And additionally, stimulated by the possible higher selling price, speculative stock may also appears.

e. **Anticipation stocks.** Some products are characterized with seasonal demand, this fact, rather than expectations, generate anticipation stocks. A time varying demand pattern asks for balancing of overtime and inventory carrying cost in order to deal with the demand peak. Companies with significant seasonality find it more efficient to use smaller plants and produce prior to demand, which obviously means accumulation of inventory.

11.6 INVENTORY COSTS

These can broadly be categorized into 4 groups.

a) Holding costs/carrying costs- These are costs incurred because a firm owns or maintains inventories e.g.:

i) Opportunity cost of capital i.e. money tied up in stock is not available to be used or invested elsewhere e.g. interest foregone.

ii) Storage/warehouse costs such as personnel, equipment, running the warehouse….etc.

iii) Insurance against fire, theft …etc.

iv) Security costs such as security personnel, alarm systems, electric fencing…etc.

v) Perishability costs. These are for perishable items such as edibles (bread, milk, fruits, vegetables, newspapers and other periodicals; tax reports, fresh flowers… etc).

vi) Obsolescence costs: These are usually due to being overtaken by technology. This is quite prevalent in the electronics industry e.g. computers, mobile phones.

vii) Pilferages.

viii) Spillages for liquids and gases etc.

ix) Damages e.g. breakages (fragile items like glass, tiles etc.)

b) **Ordering / Procurement costs-** These are the costs of getting the items into the firm's inventory. They are typically incurred each time an order is made.

For merchandising firms they are called ordering costs while for manufacturing firms they are called set-up costs (mobilizing factors of production). In order to understand ordering costs let us look at the ordering cycle model below;

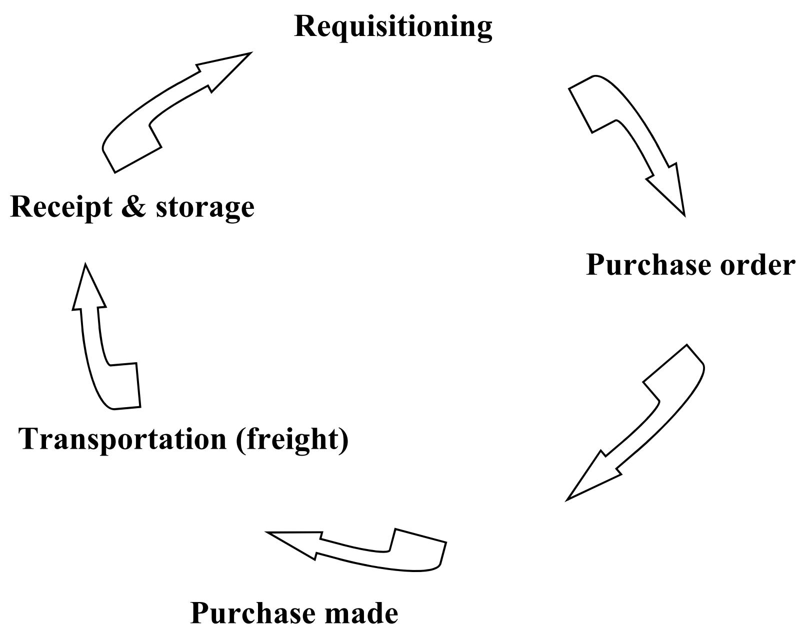

Fig: Ordering cycle model

Examples:

i) Purchasing dept costs: include personnel, equipments, communication costs (phone, internet, fax), consumables (stationery, ribbons, ink, etc.)

ii) Transportation costs.

iii) Insurance on transit.

iv) Taxes e.g. customs duty for an imported item.

v) Clearing and forwarding charges.

vi) Handling costs e.g. loading, pilferages, damages (breakages & spillages)

vii) Exchange rate differentials. In the case of imported products, valuation is done on the basis of the current exchange rates in the market. Any fluctuations may increase or decrease the value of the product. Due to this, there is the risk of selling stocks at prices lower than the landed costs.

c) Shortage costs- These are incurred as a result of the item not being in stock.

Examples:

i) Loss of goodwill; could lead to loss of customers.

ii) Contribution lost, due to not making a sale.

iii) Back order costs - these are costs of dealing with disappointed customers.

iv) Costs of idle resources e.g.:- production personnel being paid when there's a raw material missing.

v) Cost of having to speed up orders e.g. personnel working overtime, using a faster transportation mode (and hence more costly)

d) Purchase Costs- This is what is paid to the supplier /seller by the buyer in exchange of the product.

Inventory is usually a large investment for many firms. It is normally the second largest item in the balance sheet among the assets after fixed assets. Thus, inventory should only be held if the benefits (service to customers) exceed the inventory costs. Also in inventory modeling, purchase cost is a relevant factor to inventory policy due to availability of quantity discounts.

Thus, for inventories, (Total Cost) TC = Purchase costs + Holding costs + Ordering costs + Shortage costs

The objective of any inventory management systems or models is to minimize these total costs.

11.7 ECONOMIC ORDER QUANTITY

Concept of 'Economic Order Quantity (EOQ)' was first developed by Ford W. Harris in order to balance costs of holding too much stock against that of ordering in small quantities too frequently. According to him, "Economic order quantity is the size of the order representing standard quality of material and it is the one for which the aggregate of the costs of procuring the inventory and costs of holding the inventory is minimum."

11.8 DETERMINISTIC INVENTORY MODELS

Inventory models in which it is assumed that the rate of demand for the item is constant or nearly constant are called deterministic inventory models.

Basic Assumptions of Deterministic Models are given below-

1. Total no. of items required for one year is known and certain.
2. Demand is constant.
3. Orders are received instantaneously.
4. Ordering cost remains same. It does not depend on order size.
5. The purchase price does not vary during the period of consideration. However price can vary as a function of order quantity.
6. There is sufficient space and handling capacity to allow the procurement of any quantity.

11.8.1 BASIC ECONOMIC ORDER QUANTITY MODEL

The economic order quantity (EOQ) model is applicable when the demand for an item has a constant, or nearly constant, rate and when the entire quantity ordered arrives in inventory at one point in time. The constant demand rate assumption means that the same number of units is taken from inventory each period of time, such as 5 units every day, 25 units every week, 100 units every 4-week period, and so on. Following are the assumptions of Basic EOQ Model-

1. Demand is constant and known with certainty.
2. Lead-time is constant and known with certainty
3. There are no shortages - hence no shortage cost (no stock-outs).
4. All items for a given order arrive in one batch or at the same time. i.e. simultaneous or instantaneous arrival. In particular they do not arrive gradually.
5. Purchase cost is constant i.e no discounts, hence in these models, purchase cost is irrelevant since total purchase cost is the same regardless of the quantity ordered.
6. Holding cost per unit p.a. is constant. This implies that total holding cost is an increasing linear function of quantity of stock in the year.
7. Ordering cost per order is constant irrespective of size of order.

Definition of the variables

Let Q - Order quantity per order (the unknown / a decision variable)

D – Annual Demand

C_o - ordering cost per order placed

C_h - Holding cost per unit per year

C_p - Unit purchase cost

i - Holding cost expressed as a percentage of unit cost of item.

Note: $C_h = C_p$ x i

The cost formulae are as follows;

i) Ordering Cost = Annual no. of orders x C_o

$$= \frac{D}{Q} \, x \, Co$$

ii) Holding Cost = Average stock in the year x C_h or Average stock in the year x C_p x i

Thus to obtain average stock in the year we need to examine the receipt and usage profiles of stock through time.

Receipt & usage profile through time

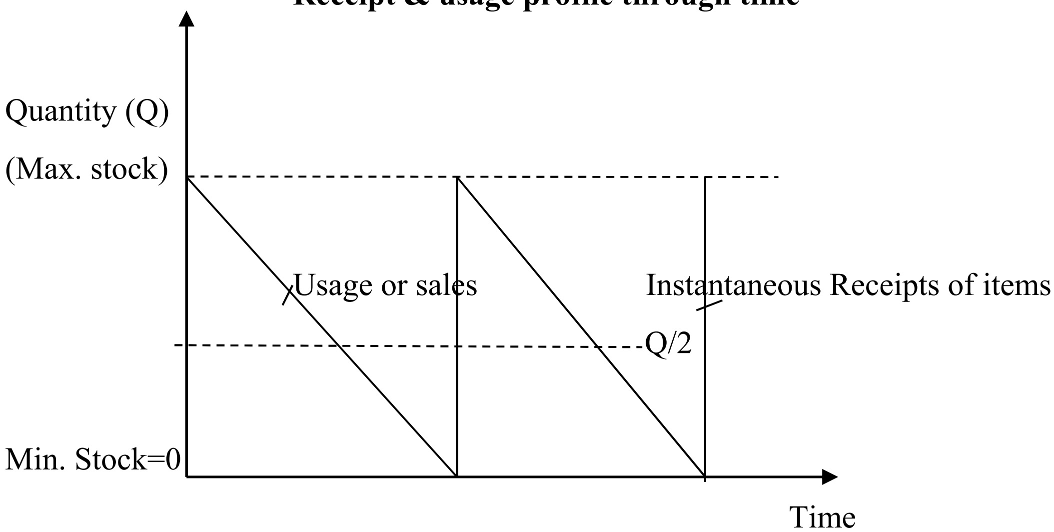

Figure- 11.1

Average stock = (Maximum stock + Minimum stock) / 2

= (Q +0)/2 = Q/2

Thus Holding cost = $Q/2 \, C_h = Q/2 \, Cp * i$

At min.TC

$$Q/2 \, C_h = \frac{D}{Q} C_o$$

To obtain the EOQ we make Q the subject as follows;

EOQ = $\sqrt{\frac{2DCo}{Cpi}}$

Or

EOQ = = $\sqrt{\frac{2DCo}{Ch}}$

Calculus approach

TC = Purchase Cost + Holding Cost + Ordering Cost (recall there are no shortage costs)

In symbolic form:

$$TC = DC_p + \frac{QCh}{2} + \frac{D}{Q}C_o$$

The objective now is to find Q which minimizes TC. We apply fist order and second order conditions as follows:

FOC: $\frac{dTC}{dQ} = \frac{C_h}{2} - DC_oQ^{-2} = 0$

$$\frac{Ch}{2} - DCo/Q^2 = 0$$

$C_h/2 = DC_o/Q^2$

Re-arranging:

$Q^2 = 2DC_o/C_h$

Hence $Q = \sqrt{\frac{2DCo}{C_h}}$

To confirm the turning point is a minimum, we apply SOC as follows;

SOC:

$d^2TC/dQ^2 = 2DC_oQ^{-3} = 2DC_0/Q^3 > 0$ i.e +ve, since D, Q, C_o are all positive values. Hence turning point is minimum.

Example 11.1- Acme Logistics Ltd has found that annual quantity for a given item is 4000 units. The cost of placing an order is Rs. 5000 and the price per unit is Rs. 2000. Inventory holding cost percentage is 20% of purchase cost. Find the optimal Quantity to order (EOQ).

Solution- Given that annual demand, D=4000 units; cost of ordering, Co = Rs.5000; carrying cost percentage, i =20% of unit cost; unit purchase cost, Cp=Rs. 200. Then;

$$EOQ = \sqrt{\frac{2DCo}{C_p * i}}$$

$$= \sqrt{\frac{2 \times 4000 \times 5000}{200 \times 0.2}} = 1000 \text{ units}$$

Example 11.2- A firm has determined that for a particular item X, the purchase cost is Rs. 36 per order and Rs. 2 per unit of X. Its inventory carrying cost is 9 percent of the average inventory. The demand of X is 10000 units per annum. The firm wants to determine-

i) What should be the economic order quantity?

ii) What is the optimal number of orders?

Solution- Given that- D = 10000 units per Annum

C_0 = Rs. 36 per order

Cost of one part C = Rs.2

Inventory carrying cost (as % of value of average inventory = 0.09

Total inventory carrying cost over a period = (Q/2)*C_p * i = (Q/2) *2* 0.09 = 0.09Q

Total Ordering Cost = No. of orders per period * ordering cost per order = 10000/Q * 36

Total Cost = 10000* 2 + 360000/Q +0.09Q

Differentiating w.r.t. Q $\frac{\partial^2(TC)}{\partial Q} = \frac{-360000}{Q^2} + 0.09$

Keeping it equal to zero,

$$\frac{360000}{Q^2} = 0.09$$

Q = 2000 units.

At Q = 2000,

$$\frac{\partial^2(TC)}{\partial Q^2} = -720000/Q^2 > 0$$

i.e. TC is minimum when Q =2000, therefore, EOQ =2000 units.

ii) Optimal number of orders = Demand/ EOQ = 10000/2000 = 5 orders.

11.8.2 EOQ WITH GRADUAL DELIVERIES

In this model, orders are assumed to be supplied at a uniform rate in place of instantaneous supply. Following are the assumptions of this model-

1. Demand, inventory carrying cost and ordering cost for a material can be estimated with certainty.
2. No safety stock, and items supplied at a uniform rate (p) and used at a fixed rate (d) and materials are entirely used up when the next order begins to arrive.
3. No discounts on bulk volume purchase.
4. There is no stock out costs.
5. Supply rate (p) is greater than usage (d) rate.

6. Production begins immediately after production set up.

Definition of the variables

Variables used in this model in addition to the model I are –

p = rate at which items are supplied.

d = rate at which items are used.

Cost Formula-

Maximum inventory level = inventory accumulation rate × period of delivery

$$= (p - d)\frac{Q}{p}$$

Minimum inventory = 0

Average inventory = ½ (Maximum inventory level + Minimum inventory level)

$$= \frac{1}{2}\left[(p-d)\frac{Q}{p} + 0\right] = \frac{Q}{2}\left[\frac{p-d}{p}\right]$$

Annual inventory carrying cost = $\frac{Q}{2}\left[\frac{p-d}{p}\right] C_h$

Annual ordering cost = orders per year × ordering cost

$$= \left(\frac{D}{Q}\right)C_o$$

Total annual stocking cost = $\frac{Q}{2}\left(\frac{p-d}{p}\right) C_h + \left(\frac{D}{Q}\right)C_o$

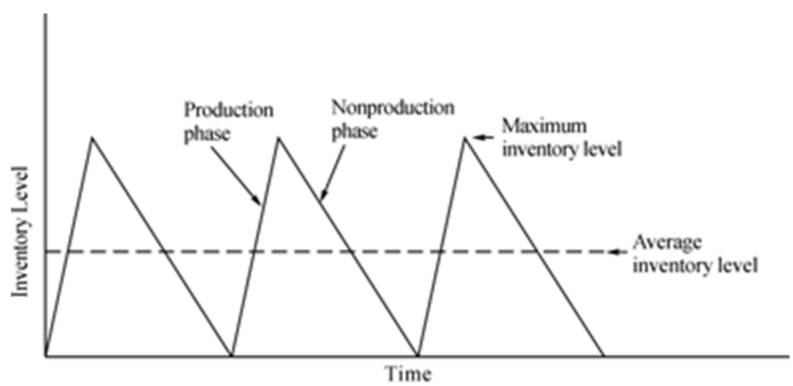

Figure 21.2

Economic order quantity in this model occurs when the annual ordering costs are exactly equal to annual carrying costs.

Therefore,

$$\frac{Q}{2}\left[\frac{p-d}{p}\right] C_h = \left(\frac{D}{Q}\right)C_o$$

$$\frac{Q^2}{2}\left(\frac{p-d}{p}\right) C_h = DC_o$$

$$Q^2 = \frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)$$

Economic Order Quantity $Q = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}$

Example 11.3- A firm consumes 10000 units of a particular item in a year. Each unit costs Rs. 32. The production set-up cost is Rs. 55 and holding cost is 12.5% of the value of inventory. Rate of supply (Replenishment rate) is uniform 120 units per day. Assuming 250 working days, calculate the economic order quantity and total inventory cost.

Solution- Given that D =10000 units per year, C = Rs. 32 per unit.

C_o= Rs. 55 per set up, C_h = 12.5% of Rs.32 = Rs. 4 per year.

p= 120 units per day, d = 10000/250 = 40 units per day.

EOQ = $\sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)} = \sqrt{\frac{2*10000*55}{4}\left(\frac{120}{120-40}\right)}$ = 642.26 units

Total cost $= \frac{Q}{2}\left(\frac{p-d}{p}\right) C_h + \left(\frac{D}{Q}\right) C_o$

$= \frac{642.26}{2}\left(\frac{120-40}{120}\right) 4 + \left(\frac{10000}{642.26}\right) 55$ = Rs. 1713.15 per year.

11.8.3 EOQ MODELS WITH QUANTITY DISCOUNTS

In this deterministic model, it is assumed that there are quantity discounts available while purchasing in large quantities. These may also be referred as 'quantity discounts'. To take the advantage of reduced product cost, the organization must decide between EOQ and the Quantity Discount (Q_D). The following are the assumptions of this model-

1. Demand, inventory carrying cost and ordering cost for a material can be estimated with certainty.
2. Quantity discounts do exist. The prices are precisely determined.
3. There are no stockout costs.
4. Average inventory levels- Q/2 (if model I prevails)

   - $\frac{Q}{2}\left(\frac{p-d}{p}\right)$(if model II prevails)

Definition of variables-

In addition to all the variables defined in previous two models, there are additional variables-

TIC = Total annual inventory cost

C = Acquisition cost of either purchasing or producing one unit of material.

Cost Formulas-

Annual acquisition cost = Annual demand × Acquisition cost per unit

$$= D \times C$$

Total inventory costs = Total stocking cost + Annual acquisition cost

$$TIC = TSC + DC$$

In case model I prevails,

$$EOQ = \sqrt{\frac{2DCo}{C_h}}, \quad TIC = \left(\frac{Q}{2}\right) C_h + \left(\frac{D}{Q}\right) C_o + DC$$

In case model II prevails,

$$EOQ = \sqrt{\frac{2DC_0}{C_h}\left(\frac{p}{p-d}\right)}, \quad TIC = \left(\frac{Q}{2}\right)\left(\frac{p-d}{p}\right) C + \left(\frac{D}{Q}\right) C_o + DC$$

Example 11.4- A firm knows that in order to get a discount of 10%, it has to place an order of minimum quantity of 500 units. From the past records, it is known that 8 orders each of size 200 were placed last year. The ordering cost is Rs. 500 per order, inventory carrying cost is 40 % of the inventory value and cost per unit is Rs. 400. Should the firm place an order of minimum 500 units? What will be the effect of this decision?

Solution- Given that,

D = 8*200 = 1600 units per year

C_o= Rs. 500 per order, C = Rs. 400 per unit, C_h = 40% of 400 = Rs.160.

$$EOQ = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2*1600*500}{160}} = 100\, units$$

$$TIC = \left(\frac{Q}{2}\right) C_h + \left(\frac{D}{Q}\right) C_o + DC = \frac{100}{2} * 160 + \frac{1600}{100} * 500 + 1600 * 400 = Rs. 656000$$

Proposed inventory cost with discounts,

C_o= Rs. 500 per order, C = Rs. 360 per unit, C_h = 40% of 360 = Rs. 144

$$TIC = \left(\frac{Q}{2}\right) C_h + \left(\frac{D}{Q}\right) C_o + DC = \frac{500}{2} * 144 + \frac{1600}{500} * 500 + 1600 * 360 = Rs. 613600$$

Since total inventory cost with discount is less than the total inventory cost with EOQ model I, therefore, the firm should place an order of minimum 500 units to avail discount of 10% it will save Rs. 42,400 (656000- 613600) for the firm.

11.8.4 EOQ MODEL WITH SHORTAGE

This model takes into account that shortage may exist. A shortage or stockout is a demand that cannot be supplied. In many situations, shortages are undesirable and should be avoided. However, there are other cases in which it may be desirable—from an economic point of view—to plan for and allow shortages. In this model a type of shortage known as a backorder is considered. In a backorder situation, it is assumed that when a customer places an order and discovers that the supplier is out of stock, the customer waits until the new shipment arrives, and then the order is filled. Frequently, the waiting period in backordering situations is relatively short; thus, by promising the customer top priority and immediate deliver when the goods become available, companies may be able to convince the customer to wait until the order arrives. In these cases, the backorder assumption is valid.

Definition of variables-

C_b = Backorder cost per unit

S= the number of backorders that are accumulated

T = time between receipt of orders (cycle time)

Q-S = number of shortages per order

When a new shipment of size Q is received, then the inventory system for the backorder case has the following characteristics:

(1) If S backorders exist when a new shipment of size Q arrives, the S backorders are shipped to the appropriate customers, and the remaining Q-S units are placed in inventory. Therefore, Q-S is the maximum inventory level.

(2) The inventory cycle of T days is divided into two distinct phases; t_1 days when inventory is on hand and orders are filled as they occur and t_2 days when there are stockouts and all new orders are placed on backorder. (Figure-11.3)

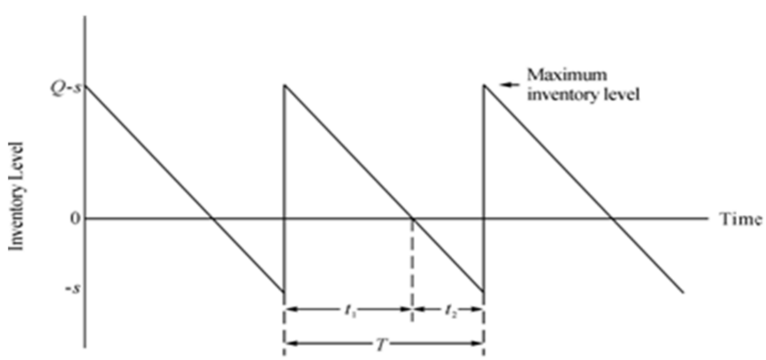

Figure 11.3

With a maximum inventory of Q-S units, t_1 days will have an average inventory of (Q-S)/2. No inventory is carried for the t_2 days in which backorders are experienced. Thus, over the total cycle time of $T=t_1+t_2$ days, we can compute the average inventory level as follows:

$$\text{Average inventory level} = \frac{\frac{1}{2}(Q-S)t_1 + 0t_2}{t_1 + t_2} = \frac{\frac{1}{2}(Q-S)t_1}{T}$$

Since the maximum inventory is Q-S and d represents the constant daily demand, we have $t_1 = \frac{Q-S}{d}$ days

That is, the maximum inventory level of Q-S units will be used up in (Q-S)/d days. Since Q units are ordered each cycle, we know the length of a cycle must be

$$T = \frac{Q}{d} \text{ days}$$

the average inventory level can therefore be expressed as follows:

$$\text{Average inventory level} = \frac{\frac{1}{2}(Q-S)[(Q-S)/d]}{Q/d} = \frac{(Q-S)^2}{2Q}$$

The formula for the annual number of orders placed using this model is identical to that for the EOQ model. With D representing the annual demand, we have

$$\text{Annual number of orders} = \frac{D}{Q}$$

To find the average backorder level, We have an average number of backorders during the period t_2 of 1/2 the maximum number of backorders or 1/2S. Since we do not have any backorders during the t_1 days we have inventory, we can calculate the average backorder level in a manner similar to the computation of average inventory level. Using this approach, we have

$$\text{Average backorder level} = \frac{0t_1 + (S/2)t_2}{T} = \frac{(S/2)t_2}{T}$$

Since we let the maximum number of backorders reach an amount S at a daily rate of d, the length of the backorder portion of the inventory cycle is $t_2 = \frac{S}{d}$

Therefore,

$$\text{Average backorder level} = \frac{(S/2)(S/d)}{Q/d} = \frac{S^2}{2Q}$$

The total annual cost (TC) for the inventory model with backorders becomes

$$TC = \frac{(Q-S)^2}{2Q}C_h + \frac{D}{Q}C_0 + \frac{S^2}{2Q}C_b$$

Given the cost estimates C_h, C_0, and C_b and the annual demand D, the minimum-cost values for the order quantity Q* and the planned backorders S* are as follows:

$$Q^* = \sqrt{\frac{2DC_0}{C_h}\left(\frac{C_h + C_b}{C_b}\right)} \qquad S^* = Q^*\left(\frac{C_h}{C_h + C_b}\right)$$

Example 11.5- A Radio Components Company has a product for which the assumptions of the inventory model with backorders are valid. Information obtained by the company is as follows:

D = 2000 units per year

I = 0.20

C = Rs.50 per unit

C_h = IC = (0.20)(Rs.50) = Rs.10 per unit per year

C_0 = Rs.25 per order

The company is considering the possibility of allowing some backorders to occur for the product. The annual backorder cost has been estimated to be Rs.30 per unit per year.

$$Q^* = \sqrt{\frac{2(2000)(25)}{10}\left(\frac{10+30}{30}\right)} = 115$$

and

$$S^* = 115\left(\frac{10}{10+30}\right) = 29$$

If this solution is implemented, the system will operate with the following properties:

Maximum inventory = Q – S = 115 – 29 = 86

Cycle time = $T = \frac{Q}{D}(250) = \frac{115}{2000}(250) = 14.4$ working days

The total annual cost is

$$\text{Holding cost} = \frac{(86)^2}{2(115)}(10) = Rs.322$$

$$\text{Ordering cost} = \frac{2000}{115}(25) = Rs.435$$

$$\text{Backorder cost} = \frac{(29)^2}{2(115)}(30) = \underline{Rs.110}$$

Total cost = Rs.867

If the company had chosen to prohibit backorders and had adopted the regular EOQ model, the recommended inventory decision would have been

$$Q^* = \sqrt{\frac{2(2000)(25)}{10}} = \sqrt{10{,}000} = 100$$

This order quantity would have resulted in a holding cost and an ordering cost of Rs.500 each or a total annual cost of Rs.1000 – Rs.867 = Rs.133 or 13.3% savings in cost from the no-stockout EOQ model. The preceding comparison and conclusion are based on the assumption that the backorder model with an annual cost per backordered unit of Rs.30 is a valid model for the actual inventory situation. If the company is concerned that stockouts might lead to lost sales, then the savings might not be enough to warrant switching to an inventory policy that allowed for planned shortages.

11.9 SUMMARY

This unit explains the concept of inventory, various types of inventory, costs associated with inventory, advantages of keeping inventory etc. The unit also discusses about deterministic models of inventory control such as Economic Order Quantity Models with shortage and discounts.

11.10 SELF ASSESMENT QUESTIONS

1. What are the functions of inventory? How it is beneficial for an organization?
2. What are different types of inventories?
3. Explain various types of costs associated with inventory.
4. What are deterministic models of inventory control?
5. What do you understand by Economic Order Quantity? What are various assumptions behind EOQ model?
6. A Production unit uses Rs.10,000 worth of an item during the year. The production units estimated the ordering cost as Rs.25 per order and holding cost as 12.5 percent of the average inventory value. Determine the optimal order size, number of orders per year, time period per order and total cost.

11.11 TEXT AND REFERENCES

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UNIT-12 INVANTORY CONTROL-PROBABILISTIC MODELS

Unit Framework

12.1 OBJECTIVES

After completing this unit you will be able to:

- Understand the concept of probabilistic inventory control
- Understand the single period and multi period inventory models
- Understand the concept of safety stock
- Understand the fixed order quantity system
- Learn about the periodic review system

12.2 INTRODUCTION

In unit XI, Simple deterministic inventory models were explained. In those models each and every influencing factor is known with certainty. But, in real business situations complete certainty does not occur always. Therefore, in this unit, some practical situations of inventory problems where the condition of certainty is not applicable will be discussed. The major influencing variables involved in the inventory problems are price, demand and lead time. Other factors such as carrying cost, ordering and stock-out costs are also involved in the inventory problems, but these can be estimated by using averaging. Even price can also be averaged out to reflect the condition of certainty. But sometimes, price fluctuations are too much in the market and hence they influence the inventory decisions. Similarly, variability in demand or consumption of an item as well as the variability in lead time influences the overall inventory policy.

12.3 CONCEPT OF RE-ORDER LEVEL AND SAFETY STOCK

Re-order level in the inventory management represents the minimum stock level when the store must place purchase request to the purchase department for the supply of stock by way of issuing fresh orders. This level should be determined in a manner so that fresh supply is obtained before the stock runs out. The re-order level can be defined as that level of inventory when fresh order should be placed with the suppliers to get additional inventory equal to the Economic Order Quantity.

Re-order Level = (Average daily usage of inventory) × (Lead time in days)

Here, lead time is the time between ordering a stock and actually receiving that stock.

Safety Stock or Buffer Stock is the minimum additional stock with the store that serve as a cushion to meet an unanticipated increase in usage resulting from various uncontrollable factors such as high demand or delay in receipt of incoming inventory. This stock may also be used in case of any excessive in-process rejection or rejection at the time of receipt or damage of stock etc. Safety Stock = (maximum lead time- average lead time) × consumption rate

If the safety stock level is kept inadequately low, the inventory carrying charges would be low but stock out may be experienced frequently and the stock out costs would be very high. However, if the large safety stock is maintained, it will reduce the risk of stock out but the inventory carrying costs would be high. Therefore, a balance between stock out costs and inventory carrying costs is necessary to arrive at the optimal safety stock.

12.3 INVENTORY MODELS WITH PROBABILISTIC DEMAND

In most situations, demand is probabilistic because only probability distribution of future demand is known not the exact value of demand. This probability distribution of future demand is determined by analyzing past data. In these type of situations, such inventory policies are chosen that minimize the expected costs. Expected costs are obtained by summating the product of actual cost of a situation and the probability of occurrence of a particular situation for all types of situations that may take place. The probability distribution can be discrete or continuous. Perishable items, seasonal items and fashion goods are examples of probabilistic demand.

Inventory models with probabilistic demand can be of two types- inventory models for single period and inventory models for multi periods.

12.4 SINGLE PERIOD PROBABILISTIC MODEL

The single-period probabilistic inventory model refers to inventory situations in which one order is placed for the product; at the end of the period, the product has either sold out, or there is a surplus of unsold items that will be sold for a salvage value. The single-period inventory model is applicable in situations involving seasonal or perishable items that cannot be carried in inventory and sold in future periods. Seasonal clothing (such as bathing suits and winter coats) is typically handled in a single-period manner. In these situations, a buyer places one preseason order for each item and then experiences a stockout or holds a clearance sale on the surplus stock at the end of the season. No items are carried in inventory and sold the following year. Newspapers are another example of a product that is ordered one time and is either sold or not sold during the single period. While newspapers are ordered daily, they cannot be carried in inventory and sold in later periods. Thus, newspaper orders may be treated as a sequence of single-period models; that is, each day or period is separate, and a single-period inventory decision must be made each period (day). Since we order only once for the period, the only inventory decision we must make is how much of the product to order at the start of the period. Because newspaper sales is an excellent example of a single-period situation, the single-period inventory problem is sometimes referred to as the **newsboy problem.**

Obviously, if the demand were known for a single-period inventory situation, the solution would be easy; we would simply order the amount we knew would be demanded. However, in most single-period models, the exact demand is not known. In fact, forecasts may show that demand can have a wide variety of values. If we are going to analyze this type of inventory problem in a quantitative manner, we will need information about the probabilities associated with the various demand values.

Definition of variables

D= demand of an item in units (a random variable)

Q= the number of units stocked (or to be purchased)

C_1= Over-stocking cost (also known as over-ordering cost). This is an opportunity loss associated with each unit left unsold.

$= C + C_h - V$

C_2= Under-stocking cost (also known as under-ordering cost). This is an opportunity loss due to not meeting the demand.

$= S - C - C_h/2 + C_s$

where C is the unit cost price; C_h, the unit carrying cost for the entire period; C_s, the shortage cost; S, the unit selling price and, V, the salvage value.

For any quantity in stock Q, only D units are consumed. Then for specified period of time, the cost associated with Q units in stock is either:

a) $(Q-D)C_1$, where D, the number of units used or demanded is less than or equal to the number of units Q, in stock, i.e., $D \leq Q$

b) $(D-Q)C_2$, where the number of units required is greater than the number of units in stock, i.e. D>Q.

Since, the demand D is random variable, its probability distribution of demand is known. p(D) denotes the probability that the demand is D units, such that total probability is one, i.e.,

$$p(0) + p(1) + \dots\dots + p(D) + \cdots = \sum_{D=0} p(D) = 1$$

The total expected cost is the sum of expected cost of under-stocking and overstocking. It is given as

$$f(Q) = C_1 \sum_{D=0}^{Q} (Q-D)p(D) + C_2 \sum_{D=Q+1}^{\infty} (D-Q)p(D)$$

If Q* is the optimal quantity stocked, then the total expected cost f(Q*) will be minimum. Thus, if we stock one unit more or less than the optimal quantity, the total expected cost will be higher than the optimal. Thus,

$$f(Q^*+1) = C_1 \sum_{D=0}^{Q^*+1} (Q^*+1-D)\, p(D) + C_2 \sum_{D=Q^*+2}^{\infty} (D-Q^*-1)p(D)$$

$$= C_1 \sum_{D=0}^{Q^*} (Q^*-D)\, p(D) + C_2 \sum_{D=Q^*+1}^{\infty} (D-Q^*)p(D) + C_1 \sum_{D=0}^{Q^*} p(D) - C_2 \sum_{D=Q^*+1}^{\infty} p(D)$$

$$= f(Q^*) + (C_1 + C_2) \sum_{D=0}^{Q^*} p(D) - C_2$$

$$f(Q^*+1) - f(Q^*) = (C_1 + C_2)\, p(D \leq Q^*) - C_2 \geq 0$$

where $p(D \leq Q^*) = \sum_{D=0}^{Q^*} p(D)$, a cumulative probability.

Similarly, $f(Q^*-1) - f(Q^*) = C_2 - (C_1 + C_2)\, p(D \leq Q^*-1) \geq 0$

Therefore, the optimal stock level Q* follows the following relationship. It can be written as-

$$p(D \leq Q^*-1) \leq \frac{C_2}{C_1 + C_2} \leq p(D \leq Q^*)$$

Example 12.1- A firm stocks a particular seasonal product at the beginning of the season and cannot re-order. The item costs Rs. 25 each and the firm sells it at Rs. 50 each. For any item that cannot be met on demand, the firm has estimated a goodwill cost of Rs.15. Any item unsold will have a salvage value of Rs. 10. Holding cost during the period is estimated to be 10 per cent of the price. The probability distribution of demand is as follows:

Units Stocked:	2	3	4	5	6
Probability of demand:	0.35	0.25	0.20	0.15	0.05

Determine the optimal number of items to be stocked.

Solution- the probability distribution of demand function can be given as shown in the following table-

Units Stocked	Probability of Demand P (D =Q)	Cumulative probability P (D ≤Q)
2	0.35	0.35
3	0.25	0.60
4	0.20	0.80
5	0.15	0.95
6	0.05	1.00

Table 12.1 : Probability Distribution of Demand

Given that S = Rs.50, C =Rs.25, Ch = 0.10*25 = 2.5, V=10, Cs = Rs.15

Therefore, C1 = C + Ch- V = 25 +2.5-10 = 17.5

C2 = S –C –Ch/2 + Cs = 50-25-2.5/2 +15 = 38.75

$$\frac{C_2}{C_1 + C_2} = \frac{38.75}{17.5+38.75} = 0.69$$

Looking at Table 12.1, this ratio lies between cumulative probabilities of 0.60 and 0.80 which in turn reflect `the values of Q as 3 and 4. That is,

$P(D \leq 3)=0.60<0.69<0.80 = P(D \leq 4)$.

Therefore, the optimal number of units to stock is 4 units.

12.5 MULTI-PERIOD PROBABILISTIC MODELS

In a multi-period model, the inventory system operates continuously with many repeating periods or cycles; inventory can be carried from one period to the next. Whenever the inventory position reaches the reorder point, an order for Q units is placed. Since demand is probabilistic, the time the reorder point will be reached, the time between orders, and the time the order of Q units will arrive in inventory cannot be determined in advance. Therefore, lead time also become variable. The variation in demand and/or in lead time imposes risks. The effects of demand and lead time variation can be set off by absorbing risks in carrying larger inventories, called buffer stocks or safety stocks. The larger are the safety stocks, the greater is the risk, in terms of the funds tied up in inventories, the possibility of obsolescence and so on. However, the risk of running out of stock is minimized. While minimizing the risk of out of stock , the risk of inventories can be minimized by reducing the buffer inventories which in turn lead to increase in the risk of poor inventory service. Therefore, the objective is to find a rational decision model for balancing these risks.

The inventory pattern for the order-quantity, reorder-point model with probabilistic demand will have the general appearance shown in Figure 12.1. The increases or jumps in the inventory level occur whenever an order of Q units arrives. The inventory level decreases at a non-constant rate based on the probabilistic demand. A new order is placed whenever the reorder point is reached. At times, the order quantity of Q units will arrive before inventory reaches zero. However, at other times, higher demand will cause a stockout before a new order is received. As with other order-quantity, reorder-point models, the manager must determine the order quantity Q and the reorder point *r* for the inventory system.

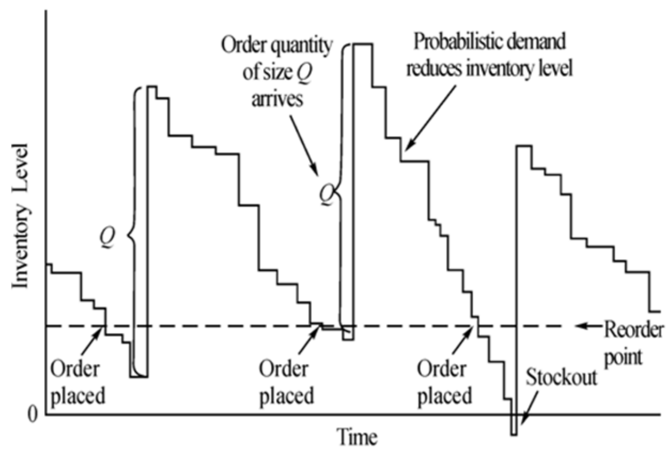
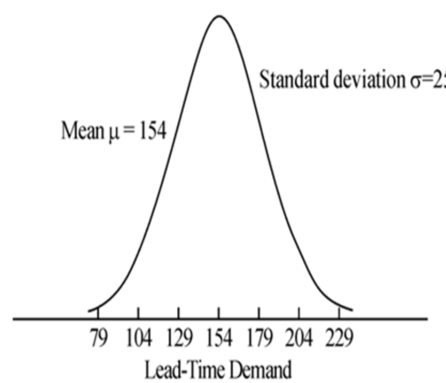


Figure 13.1

Example 12.2- Dabur purchases an item for which the ordering cost is Rs.12 per order, one item costs Rs.6, and Dabur uses a 20% annual holding cost rate for its inventory (C_h $=IC$=0.20×Rs.6 = Rs.1.20). Dabur, which has over 1000 different customers, experiences a probabilistic demand.The lead time for a new order is 1 week. Historical sales data indicate that demand during a 1-week lead-time can be described by a normal probability distribution with a mean of 154 items and a standard deviation of 25 items. The normal distribution of demand during the lead time is shown in Figure 12.2. Since the mean demand during 1-week is 154 units, Dabur can anticipate a mean or expected annual demand of 154 units per week ×52 weeks per year = 8008 units per year. Determine how much to order and when to order so that a low-cost inventory policy can be maintained.



Solution- Since the annual demand is known as 8008 units per year, applying EOQ model, we can get the Economic order Quantity,

$$Q^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2(8008)()12}{(1.20)}} = 400 \text{ units}$$

We have established the 400-unit order quantity by ignoring the fact that demand is probabilistic. Using Q^* = 400, Dabur can anticipate placing approximately

D/Q^* = 8008/400 = 20 orders per year with an average of approximately 250/20 = 12.5 working days between orders.

In case when demand for a product is probabilistic, a manager who will never tolerate a stockout is being somewhat unrealistic because attempting to avoid stockouts completely will require high reorder points, high inventory levels, and an associated high holding cost.

Suppose in this case that Dabur management is willing to tolerate an average of one stockout per year. Since Dabur places 20 orders per year, this implies management is willing to allow demand during lead time to exceed the reorder point one time in 20, or 5% of the time. This suggests that the reorder point *r* can be found by using the lead-time demand distribution to find the value of *r* for which there is only a 5% chance of having a lead-time demand that will exceed it. This situation is shown graphically in Figure 12.3.

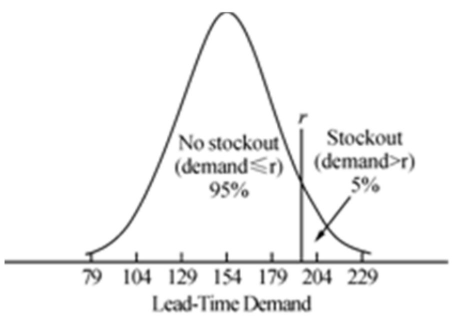


Figure 12.4

From the standard normal probability distribution table , we know that the *r* value is 1.645 standard deviations above the mean. Therefore, for the assumed normal distribution for lead-time demand with μ=154 and σ=25, the reorder point *r* is r = 154+1.645(25) = 195

If a normal distribution is used for lead-time demand, the general equation for *r* is

$r = \mu + z\sigma$

where z is the number of standard deviations necessary to obtain the acceptable stockout probability.

Thus, the recommended inventory decision is to order 400 units whenever the inventory level reaches the reorder point of 195. Since the mean or expected demand during the lead time is 154 units, the 195-154=41 units serve as a safety stock, which absorbs higher-than-usual demand during the lead time. Roughly 95% of the time, the 195 units will be able to satisfy demand during the lead time. The anticipated annual cost for this system is as follows:

Ordering cost $(D/Q)C_0$ = (8000/400)12 = Rs.240.00

Holding cost, normal inventory $(Q/2)C_h$ = (400/2)(1.20) = Rs.240.00

Holding cost, safety stock $(41)C_h$ = 41(1.20) = Rs.49.20

Total cost = Rs.529.20

If Dabur. could have assumed that a known, constant demand rate of 8008 units per year existed for the light bulbs, then Q^* =400, r =154, and a total annual cost of Rs.240+Rs.240= Rs.480 would have been optimal. When demand is uncertain and can only be expressed in probabilistic terms, a larger total cost can be expected. The larger cost occurs in the form of larger holding costs due to the fact that more inventory must be maintained to limit the number of stockouts. For Dabur, this additional inventory or safety stock was 41 units, with an additional annual holding cost of Rs.49.20.

12.6 FIXED ORDER QUANTITY SYSTEM

This system is also known as perpetual inventory system or re-order point inventory system or Q-system or two- bin system. In this type of system, the order quantity is fixed and the frequency of order vary with respect to the demand fluctuations. There are four quantities that are used as critical decision rules-

a. Maximum Inventory- This is the level of stock above which the stock should not be allowed to rise. The purpose of this level is to cut excess investment.

b. Minimum Inventory- This is the inventory level below which stock is not allowed to fall. When this level is reached, it triggers of urgent action to bring forward delivery of the next order. This level is also known as safety stock.

c. The re-order point- This is the level at which ordering action is taken for the material to be delivered before stock falls below the minimum. Two main factors are involved in deciding the re-order level, (i) the average usage rate, and (ii) the lead time.

   Re-order point = Average usage rate × lead time

d. The order size- The replenishment order quantity is in variably the EOQ.

Under fixed order quantity system, a fixed quantity is ordered each time but the time intervals may vary. It is explained through the following figure 12.4.

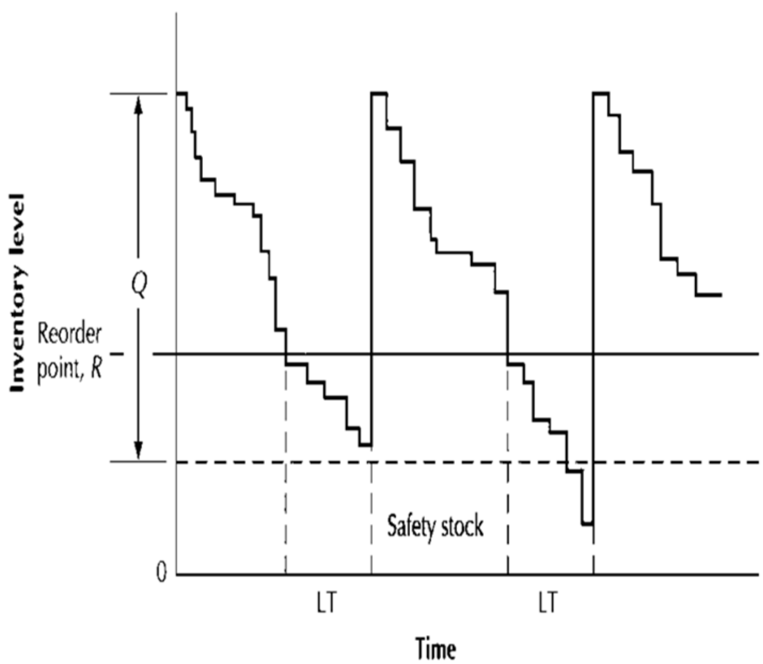


Figure 12.5

Advantages of fixed order quantity system-

- Lower stocks on average.
- Items ordered in economic quantities via the EOQ calculation.
- Somewhat more responsive to fluctuations in demand.
- Automatic generation of a replenishment order at the appropriate time by comparison of stock level against re-order level.
- Appropriate for widely differing types of inventory within the same firm.

Disadvantages of fixed order quantity system-

- Many items may reach re-order level at the same time, thus overloading the re-ordering system.
- Items come up for re-ordering in a random fashion so that there is no set sequence.
- In certain circumstances (e.g. variable demand, ordering costs etc), the EOQ calculation may not be accurate.

Example 12.3- A firm has an annual demand for 1000000 units of a certain item. Each unit of that item costs Rs. 1 to the firm. The firm follows fixed order quantity system. The ordering cost is Rs. 12.5 per order and carrying cost of inventory is Rs. 0.25 per unit. The usage rate is almost constant throughout the year. Past lead times- 15 days, 25 days, 13 days, 14 days, 30 days, 17 days. Find out the EOQ, Safety stock, Normal lead time consumption, Reorder level and average inventory.

Solution- Given that, D = 10,00,000 units/year

C_o = Rs. 12.5 per order, C_h = Rs. 0.25 per unit/year

$$\text{EOQ } Q^* = \sqrt{\frac{2DC_0}{C_h}} = \sqrt{\frac{2*1000000*12.5}{0.25}} = 10000 \text{ units}$$

Excluding the high lead times of 25 days and 30 days, the lead time varies from 14 days to 19 days. Thus optimal lead time is around 15 days.

Safety Stock = (Maximum lead time – Normal or average lead time) × monthly consumption

$$= \left(\frac{30-15}{30}\right)\times \frac{1000000}{12} = 41.667 \text{ units approx.}$$

Therefore, the optimal safety stock can be around 42000 units.

Normal lead time usage = Normal lead time × monthly consumption

= 15/30 × 1000000/12 = 41.667 units approx.

Re-order level = Safety stock + Normal lead time usage

= 42000 + 41.667 = 83.667 units

Maximum inventory level = Safety stock + EOQ = 42000 +10000 = 52000 units

Average inventory level = ½ (Safety Stock + Maximum Inventory)

= ½ (42000 + 52000) = 47000 units.

12.7 PERIODIC REVIEW SYSTEM

This system is sometimes called the constant cycle system or replenishment inventory system or p- system or fixed order interval system. The system has the following characteristics:

- Stock levels for all parts are reviewed at fixed intervals e.g. every fortnight.
- Where necessary a replenishment order is issued.
- The quantity of the replenishment order is not a previously calculated EOQ, but is based upon; the likely demand until the next review, the present stock level and the lead time.
- The replenishment order quantity seeks to bring stocks up to a predetermined level.
- The effect of the system is to order variable quantities at fixed intervals as compared with the re-order level system, where fixed quantities are ordered at variable intervals.

Replenishment level = Average usage + Lead time + safety stock

Order quantity = Replenishment level – Stock available

The figure 12.5 explains the working of periodic review system.

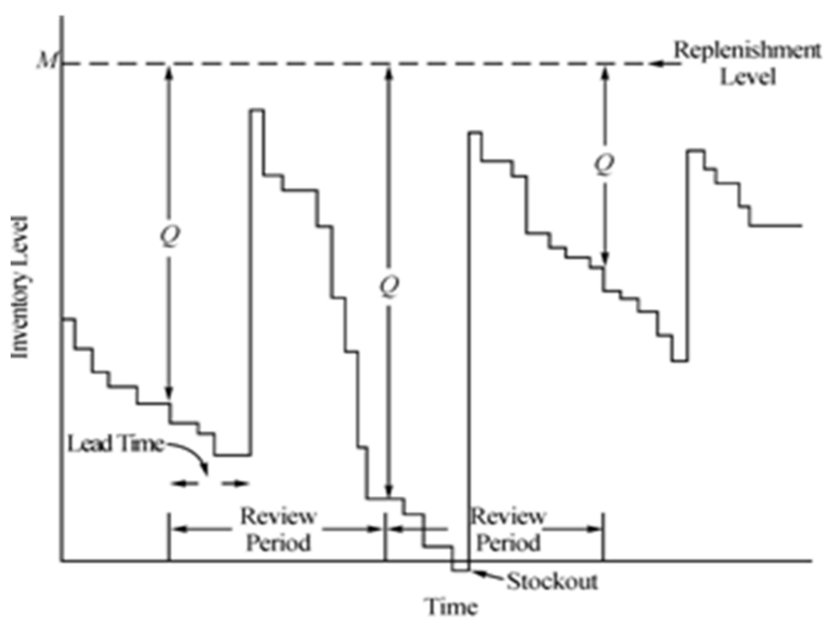


Figure 12.6

Advantages of periodic review system-

- All stock items are reviewed periodically so there is more chance of obsolete items being eliminated.
- Economies in placing orders may be gained by spreading the purchasing office load more evenly.
- Larger quantity discounts may be obtained when a range of stock items are ordered at the same time from a supplier.
- Because orders will always be in the same sequence, there may be production economies due to more efficient production planning being possible and lower set up costs. This often a major advantage and results in the frequent use of some form of periodic review system in production control systems in firms where there is a preferred sequence of manufacture, so that full advantage can be gained form the predetermined sequence implied by the periodic review system.

Disadvantages of periodic review system-

- In general larger stocks are required, as re-order quantities must take account of the period between reviews as well as lead times.
- Re-order quantities are not at the optimum level of a correctly calculated EOQ.
- Less responsive to changes in consumption. If the rate of usage change shortly after a review, a stock-out may well occur before the next review.
- Unless demands are reasonably consistent, it is difficult to set appropriate periods for review.

Example 12.4- For a periodic review system. The following data is given-

Annual usage = 14000 units, Cost per unit = Rs.10, Supplier's minimum quantity =1000 units, normal lead time=10 days, maximum lead time = 15 days, maximum consumption = 1.20(average consumption).

Determine the safety stock, replenishment level and average inventory level.

Solution- Since supplier's minimum quantity is 1000 units, maximum no. of orders to cover the annual demand will be 14000/1000 = 14 orders.

Hence, the review period should be $1/14^{th}$ of a year or 26 days. It may be more convenient to have a review after 1 month so considering Review period = 1 month.

Safety stock= Maximum consumption rate (review period + maximum lead time)- Normal Consumption rate (review period + normal lead time)

$$= 1.20 * 14000/12 (1+15/30) – 14000/12 (1 +10/30)$$

$$= 2100 -1555 = 545 \text{ units or can be rounded off to 550 units}$$

Replenishment level =Average rate of consumption (Review period + normal lead time)+ safety stock)

=14000/12 (1 + 10/30) + 550 = 2105 units approx.

The maximum inventory level = safety stock + order quantity = 550 + 14000/12 = 1710 units

Average inventory level = ½ (maximum inventory level + safety stock)

= ½ (1710 +550) = 1130 units.

12.8 SUMMARY

This unit has highlighted the role of inventory control under the actual business conditions of uncertainty desirable by any probability distribution. Various inventory models under the conditions of probabilistic demand and/or probabilistic lead time have been illustrated with the help of various examples. Two major Inventory Control systems have been described- Fixed order quantity system and Periodic Review system with the help of examples.

12.9 SELF ASSESSMENT QUESTIONS

1. Explain probabilistic models of inventory control.
2. What are the characteristics of the re-order level system of inventory control?
3. What are the three levels commonly calculated for use in re-order level systems?
4. Define the periodic review system and explain how it differs from the re-order level systems?
5. What are the major advantages and disadvantages of the re-order level system?

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UNIT-13 QUEUING MODELS

Unit Framework

13.1 OBJECTIVES

After completing this unit you will be able to:

- Understand the situations in which queuing problems are generated.
- Understand basic queuing system and its characteristics
- Examine various types of queuing models
- Solve queuing problems in single channel and multiple channel systems
- Understand the benefits and limitations of queuing theory

13.2 INTRODUCTION

Many times you might have to wait in long queues at supermarket's checkout counters or the ticket window in a movie theatre or cashier's window in a bank etc. In these waiting line conditions, the waiting time for customers is undesirable but unavoidable. Sometimes you might have left these waiting lines without availing the service or for seeking service elsewhere. Such situations when a customer leaves the waiting line may have direct consequences for the organization. Also long waiting lines and queues are signs of losing business. Therefore, long queues are important concern for a manager in today's times.

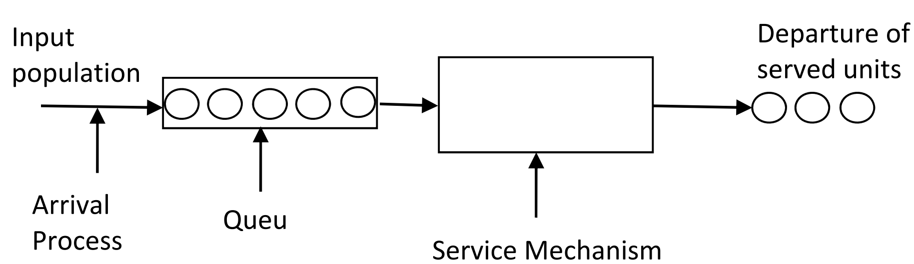
Queues are not only formed at a service facility, but also in manufacturing a queue may be formed such as parts are waiting to be served at a manufacturing operation or machines are waiting to be repaired at a maintenance workshop or trucks are waiting to unload their cargo at unloading dock. Most of the times, queue is formed whenever no. of customers exceeds the number of service facilities or when the service is not rendered immediately to a customer on his arrival at service facility. Another case may be when some services stand idle when the total no. of service facilities exceeds the number of customers requiring service. But in many situations, waiting lines cause significant congestion and increase in operating costs. Therefore, a manager has to decide on appropriate level of service which is neither too low nor too high. Queuing theory attempts to maintain an economic balance between cost of providing service and the cost associated with the wait required for that service.

Queuing theory or waiting line theory is a class of OR models that help in reducing the length of waiting lines. Queuing theory was developed by A.K. Erlang, in the early 20th century when he began a study of congestion and waiting times in completion of telephone calls.

In this unit, we will discuss various models of waiting lines that help managers in evaluating the cost and effectiveness of service systems.

13.3 BASIC QUEUING SYSTEM

The basic queuing process can be described as a process in which the customers arrive for service at a service counter, wait for their turn in the queue if the server is busy in the service of other customer and are served when the server gets free. Finally the customer leave the system as soon as he is served. The general structure of a queuing system is shown in the figure below-



Basic Structure of Queuing System

13.3.1 ELEMENTS OF QUEUING SYSTEM

Various elements of a queuing system are:

i) The input arrival process

ii) Queue discipline (structure)

iii) Service Mechanism

i) The input arrival process- The input describes the pattern in which customers arrive for service. The arrival may be classified as per various bases.

a) According to source- The source of customers in a queue can be infinite or finte. For example, the potential customers in a supermarket can be all people of a city or state. The number of people in a city or state being very large, it can be considered as an infinite source. While, the number of machine in the queue waiting for annual maintenance in a factory may be finite.

b) According to numbers- The customers in a queue may arrive for service individually or in groups. For example- customers reaching in a library may be individuals, whereas customers reaching in a restaurant with family are the example of batch arrivals.

c) According to time- customers in a queuing system, may arrive in a regular pattern (deterministic pattern) or in a random way. The queuing models wherein customers' arrive in a certain pattern are called as deterministic models. Such models are easier to handle. When the arrival pattern is random, the customers reaching the system per unit time might be described by a probability distribution. The frequently employed assumption, which adequately supports many real world situations, is that the arrivals are poisson distributed. For a poisson distribution, the mean arrival rate in queuing model is represented by symbol λ. In such a distribution, the mean is equal to the variance. The probability density function for poisson probability distribution is given as $P\ (n) = \frac{e^{-\lambda}(\lambda)^n}{n!}$ where n = 0, 1, 2, …

n is number of units which can arrive per unit of time.

ii) Queue discipline (Structure)- Next element in a queuing system, is the queue structure or queue discipline. A queue refers to the customers waiting for service, it does not include the customer being served. In some situations, a service system is unable to accommodate more than the required number of customers at a time. No further customers are allowed to enter until space becomes available to accommodate new customers. Such types of situations are referred to as finite (or limited) source queue. Examples of finite source queues are cinema halls, restaurants, etc. On the other hand, if a service system is able to accommodate any number of customers at a time, then it is referred to as infinite (or unlimited) Queue.

For example, in a sales department, here the customer orders are received; there is no restriction on the number of orders that can come in, so that a queue of any size can form. In many other situations, when arriving customers experience long queue(s) in front of a service facility, they often do not enter the service system even though additional waiting space is available. The queue length in such cases depends upon the attitude of the customers. For example, when a motorist finds that there are many vehicles waiting at the petrol station, in most of the cases he does not stop at this station and seeks service elsewhere.

The queue discipline is the order or manner in which customers from the queue are selected for service. There are a number of ways in which customers in the queue are served. Some of these are:

(a) **Static queue disciplines** are based on the individual customer's status in the queue. Few of such disciplines are:

First Come First Served (FCFS) service discipline- If the customers are served in the order of their arrival, then this is known as the First Come First Served (FCFS) service discipline. Prepaid taxi queue at airports where a taxi is engaged on a first-come, first-served basis is an example of this discipline.

Last-come-first-served (LCFS)- Sometimes, the customers are serviced in the reverse order of their entry so that the ones who join the last are served first. For example, assume that letters to be typed, or order forms to be processed accumulate in a pile, each new addition being put on the top of them. The typist or the clerk might process these letters or orders by taking each new task from the top of the pile. Thus, a just arriving task would be the next to be serviced provided that no fresh task arrives before it is picked up.

(b) **Dynamic queue disciplines** are based on the individual customer attributes in the queue. Few of such disciplines are:

Service in Random Order (SIRO) Under this rule customers are selected for service at random, irrespective of their arrivals in the service system. In this every customer in the queue is equally likely to be selected. The time of arrival of the customers is, therefore, of no relevance in such a case.

Priority Service Under this rule customers are grouped in priority classes on the basis of some attributes such as service time or urgency or according to some identifiable characteristic, and FCFS rule is used within each class to provide service. Treatment of VIPs in preference to other patients in a hospital is an example of priority service.

For the queuing models that we shall consider, the assumption would be that the customers are serviced on the first-come-first-served basis.

(iii) **Service Mechanism-** The service is provided by a service facility (or facilities). This may be a person (a bank teller, a barber, a machine (elevator, gasoline pump), or a space (airport runway, parking lot, hospital bed), to mention just a few. A service facility may include one person or several people operating as a team. There are two aspects of a service system

(a) The configuration of the service system and (b) The speed of the service.

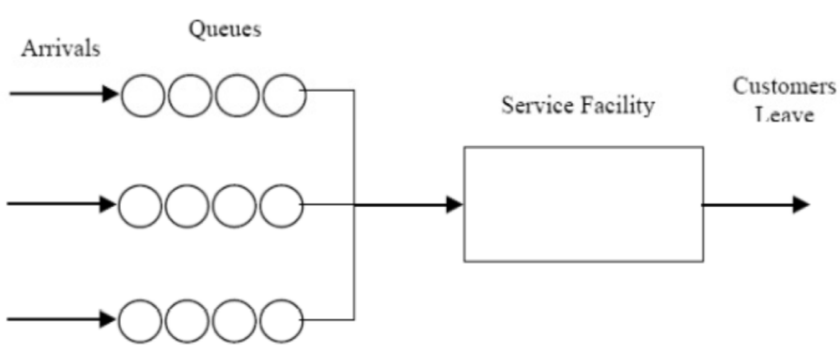
(a) Configuration of the service system- The customers' entry into the service system depends upon the queue conditions. If at the time of customers' arrival, the server is idle, then the customer is served immediately. Otherwise the customer is asked to join the queue, which can have several configurations. By configuration of the service system we mean how the service facilities exist. Service systems are usually classified in terms of their number of channels, or numbers of servers .

Single Server – Single Queue- The models that involve one queue – one service station facility are called single server models where customer waits till the service point is ready to take him for servicing. Students arriving at a library counter are an example of a single server facility.

Single Server – Single Queue Mode

Single Server – Several Queues

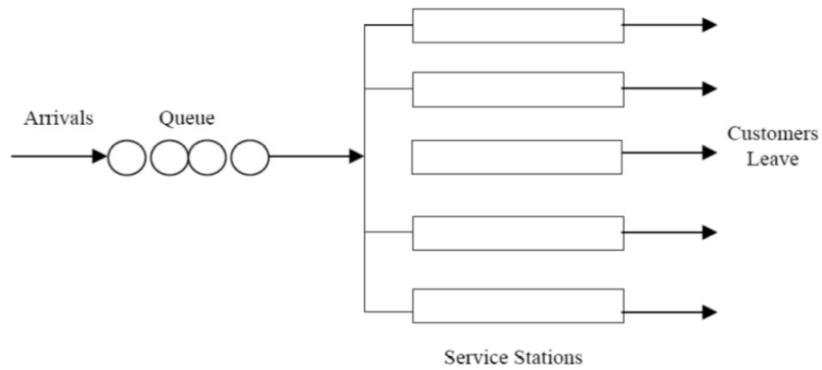
In this type of facility there are several queues and the customer may join any one of these but there is only one service channel.



Single Server – Several Queues

Several (Parallel) Servers – Single Queue

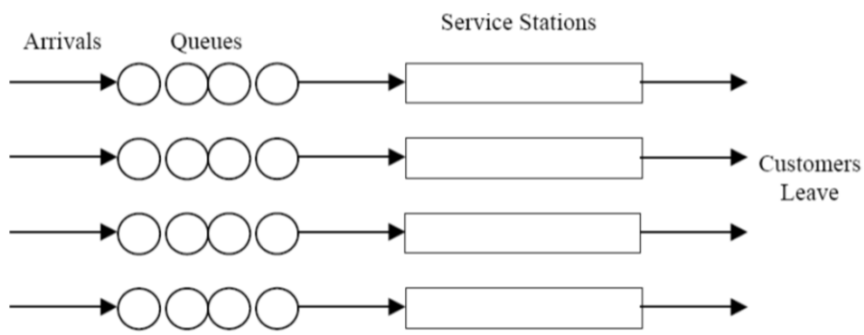
In this type of model there is more than one server and each server provides the same type of facility. The customers wait in a single queue until one of the service channels is ready to take them in for servicing.



Several, Parallel Servers – Single Queue Model

Several Servers – Several Queues

This type of model consists of several servers where each of the servers has a different queue. Different cash counters in an electricity office where the customers can make payment in respect of their electricity bills provide an example of this type of model.

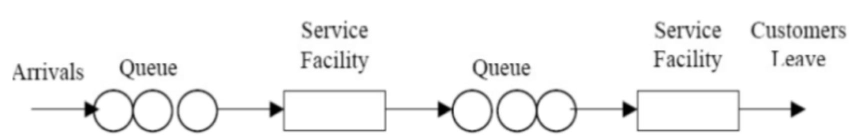


Several, Parallel Servers – Several Queues Model

Service facilities in a series

In this, a customer enters the first station and gets a portion of service and then moves on to the next station, gets some service and then again moves on to the next station. …. and so on, and finally leaves the system, having received the complete service.

For example, machining of a certain steel item may consist of cutting, turning, knurling, drilling, grinding, and packaging operations, each of which is performed by a single server in a series.



Multiple Servers in a Series

(b) **The speed of the service-** In a queuing system, the speed with which service is provided can be expressed in either of two ways—as service rate and as service time.

- The service rate describes the number of customers serviced during a particular time period.

- The service time indicates the amount of time needed to service a customer.

- Service rates and times are reciprocal of each other and either of them is sufficient to indicate the capacity of the facility.

Thus if a cashier can attend, on an average 5 customers in an hour, the service rate would be expressed as 5 customers/hour and service time would be equal to 12 minutes/customer. Generally, we consider the service time only. If these service times are known exactly, the problem can be handled easily. But, as generally happens. if these are different and not known with certainty, we have to consider the distribution of the service times in order to analyze the queuing system. Generally, the queuing models are based on the assumption that service times are exponentially distributed about some average service time.

13.3.2 CHARACTERISTICS OF QUEUING SYSTEM

The analysis of a queuing system helps in understanding various operating characteristics of a service system. Some common operating characteristics of a service system are-

a) **Queue Length (L_q)-** The average number of customers waiting in a queue represents the length of queue. Short queues represent either good customer service or high service capacity. Similarly long queues indicate either low server capacity or low efficiency.

b) **Waiting time in queue (W_q)-** It is the average time a customer has to wait in the queue to get served. Long waiting lines do not necessarily mean long waiting time if the service rate is fast. But if the waiting lines seem long, it may lead to customer dissatisfaction and potential loss of revenue. Long waiting lines indicate a need to check the service rate of the system.

c) **Number of customers in the system (L_s)-** It is the average number of customers in a queue and those being served. Large value of this statistic indicates congestion and potential customer dissatisfaction. It shows that there is a need to add more capacity.

d) **Total time in the system (W_s)-** It is the average time a customer spends in the service system from the entry into the system to exit from the system. Large time in the system indicates problem with customer, server efficiency or capacity.

e) **Server facility utilization (ρ)-** The collective utilization of the service facilities reflects the percentage of time the facilities are busy. If the facilities remain idle, it may have adverse effects on the other characteristics.

13.3.3 CUSTOMER BEHAVIOUR IN A QUEUE

Another thing to consider in the service system, is the behavior or attitude of the customers entering the queuing system. On this basis, the customers may be classified as being (a) patient, or (b) impatient. If a customer, on arriving at the service system stays in the system until served, no matter how much he has to wait for service is called a patient customer. Machines arrived at the maintenance shop in a plant are examples of patient customers. Whereas the customer, who waits for a certain time in the queue and leaves the service system without getting service due to certain reasons such as a long queue in front of him is called an impatient customer.

There are some interesting observations of human behavior in queues:

- **Balking–** Some customers even before joining the queue get discouraged by seeing the number of customers already in service system or estimating the excessive waiting time for desired service, decide to return for service at a later time. In queuing theory this is known as balking.
- **Reneging-** customers after joining the queue, wait for sometime and leave the service system due to intolerable delay, so they renege. For example, a customer who has just arrived at a grocery store and finds that the salesmen are busy in serving the customers already in the system, will either wait for service till his patience is exhausted or estimates that his waiting time may be excessive and so leaves immediately to seek service elsewhere.
- **Jockeying -** Customers who switch from one queue to another hoping to receive service more quickly are said to be jockeying.

13.3.4 SOME DEFINITIONS

Mean arrival rate (λ): The mean arrival rate in waiting line situation is defined as the expected number of arrivals occurring in a time interval of length unity and denoted by λ.

Mean time between arrivals is represented by $(1/\lambda)$.

Mean servicing rate (μ): The mean servicing rate for a particular servicing station is defined as the expected number of services completed in a time interval of length unity, given that the servicing is going on throughout the entire time unit and denoted by μ.

Mean time per customer served is represented by $(1/\mu)$.

Average service utilization rate (ρ): The average utilization of service facility, sometimes called as traffic density denoted by $\rho = \lambda/\mu$ or $\lambda/k\mu$. where k is the number of servers.

13.4 QUEUING MODELS

Queuing models can be divided into types- deterministic models and probabilistic models. If the customer arrives in a fixed pattern with known intervals and service times, the queuing model is a deterministic model. However, there may be cases in reality where the one or some elements of a queuing system may be probabilistic in nature, in such cases probabilistic queuing models are used.

13.4.1 DETERMINISTIC QUEUING MODEL

In a deterministic queuing model, customer arrive in the queuing system at regular interval and the service time for each customer is also known and fixed.

For example- Customer arrive in a bank at teller's counter every 5 minutes. Therefore, the interval between two consecutive customers is 5 minutes. Further it is known that the teller takes exactly 5 minutes to serve a customer. So the service time is also 5 minutes. In such a situation arrival rate and service rate are each equal to 12 customers per hour. Thus, there will never be a queue and the teller will always be busy with work.

In another case, if the teller takes 6 minutes to serve each customer, then the service rate will be 10 customers per hour, which will be less than the arrival rate. Therefore, the teller will always be busy and the queue length will increase continuously with the passage of time.

In another situation, if the teller takes 4 minutes to serve each customer, then the service rate will be 15 customers per hour, which is more than the customer arrival rate. Therefore, the teller will be busy (12/15 = 4/5)th time of total time and remain idle 1/5 th time of total time and there will be no queue.

Let arrival rate be λ customers per unit of time and service rate be μ customers per unit of time. Then,

if $\lambda>\mu$, the queue will be formed which will increase indefinitely, the server will always remain busy and the service system will eventually fail.

If $\lambda\leq\mu$, there will be no queue, server will remain idle for $1- \lambda/\mu$ portion of the total time.

The ratio λ/μ is called as service utilization rate ρ, therefore,

if $\rho >1$, the service system will eventually fail, and

if $\rho\leq1$, the system will work and will remain idle for $1- \rho$ portion of the time.

13.4.2 PROBABILISTIC QUEUING MODEL

The condition of arrival at a fixed rate and uniform service rate is not practically possible always. Such conditions are applicable only in case of highly automated systems. In a service system where humans are involved, arrival and service time becomes variable and uncertain. Therefore, probabilistic queuing models are used in such cases.

The probabilistic queuing models are based on assumption of variable arrival rates and servicing time. There can be various types of probabilistic models. Some popular probabilistic models are:

- Poisson- exponential, single server model- infinite population
- Poisson- exponential, single server model- finite population
- Poisson- exponential, multiple server model- infinite population
- Poisson- Erlangian, single server model- infinite population

Different models in queuing theory are classified by using special (or standard) notations described initially by D.G.Kendall in 1953 in the form (*a/b/c*). Later A.M.lee in 1966 added the symbols *d* and *e* to the kendall notation. Now in the literature of queuing theory the standard format used to describe the main characteristics of parallel queues is as follows: *(a/b/c): (d/e)* Where, a= Arrivals distribution, b= Service time (or departures) distribution, c= Number of service channels (servers), d= Queue (or service) discipline and e= Maximum number of customers allowed in the system (in queue plus in service).

13.5 MODEL 1 : (M/M/1) : (FIFO/ ∞) MODEL

This is the queuing model with poisson arrival, exponential service time, and single service channel with infinite capacity. The service discipline is first come first serve.

This model is based on certain assumptions about the queuing system.

(i) Arrivals are described by Poisson probability distribution and come from an infinite population.

(ii) Single waiting line and each arrival waits to be served regardless of the length of the queue (i.e. infinite capacity) and no balking or reneging.

(iii) Queue discipline is first come first serve.

(iv) Single server and service time follows exponential distribution.

(v) Customer arrival is independent but the arrival rate (average number of arrivals) does not change over time.

a) Expected number of customers in the system (customers in the line plus the customer being served) $L_s = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\,(1-\rho)\rho^n, \quad 0 < \rho < 1$ (since $P_n = (1-\rho)\,\rho^n$)

$$= (1-\rho)\sum_{n=0}^{\infty} nP_n = \rho\,(1-\rho)\sum_{n=0}^{\infty} n\,\rho^{n-1} = \rho\{1 + 2\,\rho + 3\,\rho^2 + \cdots\}$$

$$= (1-\rho)\left\{\frac{\rho}{(1-\rho)^2}\right\} = \frac{\rho}{(1-\rho)} = \frac{\lambda}{\mu-\lambda} \quad (\text{since } \rho = \frac{\lambda}{\mu})$$

$$L_s = \frac{\lambda}{\mu-\lambda}$$

b) Expected number of customers waiting in the queue (Queue Length)

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} nP_n - \sum_{n=1}^{\infty} P_n = \sum_{n=0}^{\infty} nP_n - \left[\sum_{n=0}^{\infty} P_n - P_0\right]$$

$$= L_s - (1 - P_0) = \frac{\rho}{(1-\rho)} - 1 + (1-\rho)$$

$$L_q = \frac{\lambda}{\mu - \lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

c) Expected waiting time for customer in the queue

$$W_q = \int_0^{\infty} t.\left\{\frac{d}{dt}\,\emptyset_w(t)\right\} dt$$

$$= \int_0^{\infty} t.\lambda\,(1-\rho)e^{-\mu(1-\rho)t}dt$$

Integrating by parts $W_q = \lambda\,(1-\rho)\left[\frac{te^{-\mu(1-\rho)t}}{-\mu\,(1-\rho)} - \frac{e^{-\mu(1-\rho)t}}{\mu^2\,(1-\rho)^2}\right] = \lambda\left(1-\frac{\lambda}{\mu}\right)\frac{1}{(\mu-\lambda)^2}$

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{L_q}{\lambda}$$

d) Expected waiting time for a customer in the system (waiting and service)

W_s=Expected waiting time in queue + Expected service time $=W_q + \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$

$W_s = \frac{1}{\mu-\lambda} = \frac{L_s}{\lambda}$

e) The variance (fluctuation) of queue length

$var\ (n) = \sum_{n=1}^{\infty} n^2 P_n - (\sum_{n=1}^{\infty} nP_n)^2 = \sum_{n=1}^{\infty} n^2 P_n - (L_s)^2 = \sum_{n=1}^{\infty} n^2\,(1-\rho)\rho^n - \left(\frac{\rho}{1-\rho}\right)^2$

$$var\ (n) = (1-\rho)[\,1.\rho^2 + 2^2.\rho^2 + 3^2.\rho^3 + \cdots.] - \left(\frac{\rho}{1-\rho}\right)^2$$

$$var\ (n) = \frac{\rho}{(1-\rho)^2} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

f) Probability that the queue is non empty

P (n>1) = 1 – P_0 – P_1 = 1- $\left(1-\frac{\lambda}{\mu}\right)$- $\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right) = \left(\frac{\lambda}{\mu}\right)^2$

g) Probability that the number of customers, *n* in the system exceeds a given number *k*.

P (n ≥ k) = $\sum_{n=k}^{\infty} P_k = \sum_{n=k}^{\infty}(1-\rho)\rho^k = (1-\rho)\rho^k \sum_{n=k}^{\infty} \rho^{n-k}$

$$= (1-\rho)\rho^k[\,1+\ \rho+\ \rho^2+\cdots] = \frac{(1-\rho)\rho^k}{(1-\rho)} = \ \rho^k\ = \ \left(\frac{\lambda}{\mu}\right)^k$$

$$\text{P (n > k)} = \ \left(\frac{\lambda}{\mu}\right)^{k+1}$$

h) Expected length of non empty queue

$$\text{L} = \frac{Expected\ length\ of\ waiting\ line}{prob\ (n>1)} = \frac{L_q}{P\ (n>1)} = \ \frac{\lambda^2/\mu}{(\lambda/\ \mu)^2}\,(\mu-\ \lambda)$$

$$= \ \frac{\lambda}{\mu-\lambda}$$

Example 13.1- At a currency exchange bureau, on average a customer arrives every 5 minutes and takes 4 minutes to be served. Considering the assumptions of a single channel queuing model, determine the following:

a) Average no. of arrivals per minute (λ)

b) Service rate(μ)

c) The traffic intensity

d) Fraction of the time the service point/cashier is busy

e) The probability the cashier is busy

f) Expected no. of customers in the system

g) The average length of the queue (no. of customers waiting in the queue)

h) The mean time a customer spends in the system

i) The mean time a customer spends in the queue

Solution-

a) Arrivals per minute

$$\lambda = \frac{no.\ of\ arrivals}{time\ taken} = {}^1\!/_5\ or\ 0.2\ arrivals\ \ per\ \text{min}$$

b) Service rate

$$\mu = \frac{units\ of\ \ services}{units\ of\ \ time} = {}^1\!/_4\ or\ 0.25\ customers\ \ per\ \ \text{min}$$

c) Traffic intensity

$$\rho = \frac{\lambda}{\mu} = \frac{0.2}{0.25} = 0.8$$

d) Fraction of time cashier is busy

80% of the time

e) Probability that cashier is busy = 0.8

f) Expected no. of customers in the system

$$L_s = \frac{\lambda}{\mu - \lambda}$$

$$L_s = \frac{0.2}{0.25 - 0.2} = 4 \textit{ customers}$$

g) Average length of the queue

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$L_q = \frac{(0.2)^2}{0.25(0.25 - 0.2)} = \frac{0.04}{0.0125} = 3.2 \textit{ customers}$$

h) The mean time a customer spends in the system

$$W_s = \frac{1}{\mu - \lambda}$$

$$W_s = \frac{1}{0.25 - 0.2} = 20 \text{ min}$$

i) The mean time a customer spends in the queue

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$$W_q = \frac{0.2}{0.25(0.25 - 0.2)} = 16 \text{ min}$$

13.6 MODEL 2 : (M/M/1) : (FIFO/ N) MODEL

This model is based on the similar assumptions as the previous model except that in this model, the input population is finite. Suppose that no more than *M* customers can be accommodated at any time in the system due to certain reasons. Thus any customer arriving when the system is already contains *M* customers does not enter the system and is lost. For example, a finite queue may arise due to physical constraint such as emergency room in a hospital; one man barber shop with certain number of chairs for waiting customers, etc.

In such a system, the inter-arrival rate of customers follows exponential distribution with mean=1/λ. If there are n customers in the system, then there are *M-n* customers in the arriving state and the total average rate of arrivals in the system is λ(M-n). Such a system is self regulating, that means when the system ges busy, with many customers in the queue, then the rate at which additional customers arrive is automatically reduced which results in lowering the further congestion in the system.

For a system, with a capacity of M customers, 1/ λ as the average inter-arrival time between successive arrivals, and μ as the service rate, the operating characteristics are given as follows:

a) Probability that the system will be idle-

$$P_0 = \left[\sum_{i=0}^{M}\left(\frac{M!}{(M-i)!}\right)\left(\frac{\lambda}{\mu}\right)^{i}\right]^{-1}$$

b) Probability that there shall be n customers in the system-

$$P_n = \begin{cases} P_0\left(\frac{\lambda}{\mu}\right)^{n}\frac{M!}{(M-n)!}, & 0 < n \le M \\ 0, \quad n > M \end{cases} \qquad 0 < n \le M$$

c) Expected length of the queue-

$$L_q = M - \frac{\lambda + \mu}{\lambda}\,(1 - P_0)$$

d) Expected number of customers in the system-

$$L_s = L_q + (1 - P_0) = M - \frac{\mu}{\lambda}\,(1 - P_0)$$

e) Expected waiting time of a customer in the queue-

$$W_q = \frac{L_q}{\mu(1-P_0)} = \frac{1}{\mu}\left(\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda}\right)$$

f) Expected time a customer spends in the system-

$$W_s = W_q + \frac{1}{\mu} = \frac{1}{\mu}\left[\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda} + 1\right]$$

Example 13.2- A factory owns five machines in which breakdowns occur at random and the average time between the breakdowns is 2 days. Assuming that the repairing capacity of the workman is one machine a day and the repairing time is distributed exponentially. Determine the following-

(a) The probability that the service facility will be idle.

(b) The probability of various number of machines (0 through 5) to be, and being repaired.

(c) The expected length of the queue.

(d) The expected number of machines waiting to be, and being repaired.

(e) The expected time that a machine shall wait in the queue to be repaired and

(f) The expected time a machine shall be idle for reason of waiting to be repaired and being repaired.

Solution- Given that, M = 5 machines

μ = 1 machine per day

λ = ½ machine per day

$\rho = \lambda/\mu = 0.5$

The probability that the service facility will be idle

$$P_0 = \left[\sum_{i=0}^{M}\left(\frac{M!}{(M-i)!}\right)\left(\frac{\lambda}{\mu}\right)^i\right]^{-1}$$

$$P_0 = [1 + 5 * 0.5 + 20 * 0.25 + 60 * 0.125 + 120 * .0625 + 120 * 0.03125]^{-1}$$

$$= [27.25]^{-1} = 0.0367$$

The probability of various number of machines (0 through 5) to be, and being repaired-

$$P_n = P_0 \left(\frac{\lambda}{\mu}\right)^n \frac{M!}{(M-n)!}$$

$$P_0 = 0.0367$$

$$P_1 = 0.0367 * 0.5 * 5 = 0.09175$$

$$P_2 = 0.0367 * 0.25 * 20 = 0.183500$$

$$P_3 = 0.0367 * 0.125 * 60 = 0.275250$$

$$P_4 = 0.0367 * 0.0625 * 120 = 0.275250$$

$$P_5 = 0.0367 * 0.03125 * 120 = 0.137620$$

The expected length of the queue-

$L_q = M - \frac{\lambda+\mu}{\lambda}(1 - P_0)$= 5 – $\frac{0.5+1}{0.5}(1 - 0.0367)$= 5 – 2.89 = 2.11 machine

The expected number of machines waiting to be, and being repaired -

$$L_s = M - \frac{\mu}{\lambda}(1 - P_0) = 5 - \frac{1}{0.5}(1 - 0.0367) = 5 - 1.93 = 3.07\ machines$$

Expected time that a machine shall wait in the queue to be repaired -

$$W_q = \frac{1}{\mu}\left(\frac{M}{1-P_0} - \frac{\lambda+\mu}{\lambda}\right) = \frac{1}{1}\left(\frac{5}{1-0.0367} - \frac{0.5+1}{0.5}\right) = 5.1905 - 3 = 2.19\ days$$

The expected time a machine shall be idle for reason of waiting to be repaired and being repaired $W_s = W_q + \frac{1}{\mu} = 2.19 + \frac{1}{1} = 3.19\ days$

13.7 MODEL 3 : (M/M/S) : (FIFO/ ∞) MODEL

In the previous two models, there was an assumption that system has only one server facility. In this model, this assumption is not considered and it is assumed that there are multiple servers in the system which are parallel in offering service. One of the example of such system may be a hospital with more than one doctor. In such a system, there may be a separate queue for each server or customers form a single queue from which they are picked up for the service.

The model is based on following assumptions-

(a) The arrival of customers follows poisson probability distribution with average arrival rate λ.

(b) The service time has an exponential distribution with a mean service rate μ.

(c) There are S service stations, each provides same service. Mean combines service rate of all the servers is Sμ.

(d) A single waiting line is formed.

(e) The input population is infinite.

(f) The service is offered on a first come first serve basis.

(g) The arrival rate is smaller than the combined service rate of all S facilities i.e. Sμ.

The utilization rate is $\rho = \lambda/S\mu$.

The probability that the system shall be idle-

$$P_0 = \left[\sum_{i=0}^{S-1}\left(\frac{(\lambda/\mu)^i}{i!}\right) + \frac{(\lambda/\mu)^S}{S!\,(1-\rho)}\right]^{-1}$$

The Probability that there shall be n customers in the system-

$$P_n = (P_0)\frac{(\lambda/\mu)^n}{n!} \quad , \qquad when\, n \le S$$

$$P_n = (P_0)\frac{(\lambda/\mu)^n}{S!\,S^{n-S}} \quad , \qquad when\, n > S$$

Expected length of the queue-

$$L_q = \frac{(\lambda/\mu)^S \rho}{S!\,(1-\rho)^2}\,(P_0)$$

Expected number of customers in the system-

$$L_s = L_q + \frac{\lambda}{\mu}$$

Expected waiting time of a customer in the queue-

$$W_q = \frac{L_q}{\lambda}$$

Expected time a customer spends in the system-

$$W_s = W_q + \frac{1}{\mu}$$

Example 13.3- A service station has five servers each of whom can service a machine in 2 hours on an average. The machines are registered at a single counter and then sent for servicing to different servers. The average arrival rate of machines at the service station is 2 machines per hour. Assuming that machine arrival is poisson distributed and the service time is exponentially distributed, determine-

(a) Utilization rate, (b) probability that the system shall be idle, (c) probability that there shall be 3 machines at the service centre, (d) probability that there shall be 8 machines in the service centre, (e) expected number of machines waiting in the queue, (f) expected number of machines in the service centre, (g) average waiting time in the queue and (h) average time spent by a machine in waiting and getting serviced.

Solution- Given that μ = 1/2 machine per hour

λ = 2 machine per hour and S= 5

(a) Utilization rate- ρ = λ/Sμ = 2/0.5*5= 0.8

(b) probability that the system shall be idle-

$$P_0 = \left[\sum_{i=0}^{S-1}\left(\frac{(\lambda/\mu)^i}{i!}\right) + \frac{(\lambda/\mu)^S}{S!\,(1-\rho)}\right]^{-1}$$

$$P_0 = \left[\sum_{i=0}^{5-1}\left(\frac{(2/0.5)^i}{i!}\right) + \frac{(2/0.5)^5}{5!\,(1-0.8)}\right]^{-1}$$

$$\sum_{i=0}^{5-1}\left(\frac{\left(\frac{2}{0.5}\right)^i}{i!}\right) = 4 + \frac{4}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} + \frac{4^4}{4!} = \frac{103}{3}$$

$$and\ \frac{\left(\frac{2}{0.5}\right)^5}{5!\,(1-0.8)} = \frac{4^5}{120 * 0.2} = \frac{128}{3}$$

$$P_0 = \left[\frac{103}{3} + \frac{128}{3}\right]^{-1} = 77^{-1} = 0.0130$$

(c) The Probability that there shall be 3 machines in the service centre-

$$P_n = (P_0)\frac{(\lambda/\mu)^n}{n!} \quad , \qquad when\ n \le S$$

$$P_3 = (P_0)\frac{(2/0.5)^3}{3!} = 0.0130 * \frac{64}{6} = 0.1387$$

(d) The Probability that there shall be 8 machines in the service centre-

$$P_n = (P_0)\frac{(\lambda/\mu)^n}{S!\,S^{n-S}} \quad , \qquad when\, n > S$$

$$P_n = (P_0)\frac{(2/0.5)^8}{5!\,5^{8-5}} = 0.0130 * \frac{4^8}{120 * 125} = 0.0568$$

(e) Expected number of machines waiting in the queue-

$$L_q = \frac{(\lambda/\mu)^S \rho}{S!\,(1-\rho)^2}\,(P_0)$$

$$L_q = \frac{(2/0.5)^5 0.8}{5!\,(1-0.8)^2} * 0.0130 = 2.22\, machines$$

(f) Expected number of machines in the system-

$$L_s = L_q + \frac{\lambda}{\mu} = 2.22 + 2/0.5 = 6.22 \text{ machines}$$

(g) Expected waiting time in the queue-

$$W_q = \frac{L_q}{\lambda} = \frac{2.22}{2} = 1.11\, hours$$

(h) Expected time a machine spends in the system-

$$W_s = W_q + \frac{1}{\mu} = 1.11 + \frac{1}{0.5} = 3.11\, hours$$

13.8 MODEL 4: (M/EK/1) : (FIFO/∞) MODEL

In this model, the arrival of customers in a service system is assumed to follow poisson distribution and service time is assumed to follow Erlang distribution. The model consists of single server with infinite capacity with first come first serve service discipline.

Erlang service distribution assumes the service as consisting of k phases, each phase taking an average time $1/\mu$. So the arrival and departure of one unit in the system increases or decreases k phases in the system. If at a time, m units are waiting in the queue and one unit is in service which has to complete s phases, then the total number of phases n in the system can be given as- n = mk + s.

Utilization rate of such a system ρ will be λ/kμ. Various operating measures in such model can be calculated as-

(a) Probability that the system shall be idle-

$$P_0 = 1 - \rho k$$

(b) Probability that there shall be n customers in the system-

$$P_n = P_0 \sum_{m,i,s} \rho^m * (-1)^i \binom{m}{i}\binom{m+s-1}{s}$$

Where m, i and s are such that, 0≤m≤∞, 0≤i≤m, 0≤s≤∞, for which m +ik+ s= n

(c) Expected number of customers in the queue-

$$L_q = \frac{k+1}{2k} * \frac{\lambda^2}{\mu(\mu-\lambda)}$$

(d) Expected number of customers in the system-

$$L_s = \frac{k+1}{2k} * \frac{\lambda^2}{\mu(\mu-\lambda)} + \frac{\lambda}{\mu}$$

(e) Average waiting time of a customer in the queue-

$$W_q = \frac{L_q}{\lambda} = \frac{k+1}{2k} * \frac{\lambda}{\mu(\mu-\lambda)}$$

(f) Expected waiting time of a customer in the system-

$$W_s = W_q + \frac{1}{\mu} = \frac{k+1}{2k} * \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu}$$

Example 13.4- A clinic has a doctor examining patients for general check-up. The doctor averages 4 minutes on each phase of the check-up although the distribution of time spent on each phase is Erlangian, if each patient goes through four phases in the check-up and if the arrival of patients in the system is poisson distributed at an average rate of 3 patients per hour, what is the average time spent by a patient waiting in the doctor's office? What is the average time spent in the examination? What is the most probable time spent in the examination?

Solution- Since it is a (M/Ek/1): (FIFO/∞) problem,

and λ= 3/60 = 1/20 patients per minute, no. of phases k = 4,

mean service time per phase = 4 minutes. o

So mean service time per patient = 4 * 4 = 16 minutes

Hence μ= 1/16

(a) average time spent by a patient waiting in the doctor's office-

$$W_q = \frac{k+1}{2k} * \frac{\lambda}{\mu(\mu-\lambda)} = \frac{4+1}{8} * \frac{(1/20)}{\frac{1}{16}(\frac{1}{16}-\frac{1}{20})} = 40 \, minutes$$

(b) average time spent in the examination-

1/μ = 1/16 = 16 minutes

(c) most probable time spent in the examination $= \frac{k-1}{\mu k} = \frac{4-1}{\frac{1}{16}*4} = 12 \, minutes$

13.9 ADVANTAGES OF QUEUING THEORY

Queuing theory has various advantages as listed below-

(a) Queuing theory have been applied for solution of a large number of problems such as scheduling of aircrafts, scheduling of issue and return of tools to workmen, scheduling of mechanical transport fleets etc.

(b) It attempts to formulate, interpret and predict for purposes of better understanding the queues and for the scope to introduce remedies such as adequate service with tolerate waiting.

(c) It provides models that are capable of influencing arrival pattern of customers or determines the most appropriate amount of service.

(d) It relates to study the behaviours of waiting lines via mathematical techniques utilizing concept of stochastic process.

13.10 LIMITATIONS OF QUEUING THEORY

Various limitations of queuing theory are as follows-

(a) Queuing problems are complex in nature and cannot be easily solved due to the presence of uncertainty element.

(b) The observed distributions of service times and the time between arrivals cannot be fitted by mathematical distributions every time which is generally assumed in various queuing models.

(c) In multi-server problems, analysis becomes more complex when the departure from one queue becomes arrival in the another queue.

(d) Queuing models with variable arrival or service times are so complicated in nature that they cannot be handled by formulae and mathematical models. In such cases simulation is used.

13.11 SUMMARY

This unit explains queuing theory which deals with situations where customers arrive, wait for service, get the service and leave the system. Customers who can be human or non-human may arrive individually or in group at known or unknown intervals form one or more queues and move in a particular order to the service station/s providing service whose speed may be fixed or variable. In this unit, we learned about various queuing models with single server or multiple servers, finite capacity or infinite capacity. Operating characteristics of these queuing systems include queue length, system length, waiting time in the queue, total time in the system and server idle time. All these characteristics for various queuing systems have been discussed in this unit. At the end, advantages and limitations of queuing theory have also been explained.

13.12 SELF ASSESSMENT QUESTIONS

1) What is queuing theory? What type of characteristics should be analyzed in a queuing system?

2) Explain general structure of the queuing system.

3) What are the basic characteristics of a queuing system?

4) What do you understand by queue discipline? Explain various types of queue discipline.

5) What is service system? Explain about service system structure in detail.

6) Distinguish between deterministic and probabilistic queuing models.

7) Products arrive at a computing centre in passion distribution with a mean arrival rate of 25 per hour. The average computing job requires 2 minutes for terminal time. Calculate the average number of problems for computer use and the percentage of times an arrival can walk right in without having to wait.

8) In a bank with a single server, there are two chairs for waiting customers. On an average one customer arrives every 10 minutes and each customer takes 5 minutes for getting served. Making suitable assumptions, calculate the probability that an arrival will get a chair to sit down, the probability that an arrival will have to stand and expected waiting time of a customer.

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BLOCK

5

GAME THEORY AND SIMULATION

UNIT-14

Corporative Situations : Game Theory

UNIT-15

Simulation

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BLOCK INTRODUCTION

In Block V you will learn about concept of game theory which is commonly used in taking decisions in real corporative situations. It is used in competing situations which arise out of conflicts of interest. You will also learn about simulation which is an important tool in OR that converts a real business problem in mathematical model and then results of the model are compared with the reality.

Unit-14 will discuss about the concept of game theory and its managerial applications. In this unit, various types of games, different assumptions of games, and methods of solving games are explained. Various managerial applications of game theory are also discussed in this unit.

Unit-15 helps in understanding the concept of simulation. In this unit various types of simulation, its advantages and applications are discussed.

UNIT-14 CORPORATIVE SITUATIONS : GAME THEORY

Unit Framework

14.1 OBJECTIVES

After completing this unit you will be able to:

- Understand decision making in corporative situations.
- Examine conflict situations when one decision maker competes with another decision maker.
- Understand concepts of game theory.
- Learn various procedures of solving two- player zero sum games.
- Compute the value of games with pure and mixed strategies.
- Understand the limitations of game theory.

14.2 INTRODUCTION

In business, managers take decisions under various types of constraints or uncertainty of environmental conditions (states of nature). But there are many competitive situations in business when the manager's decisions not only depend upon the selected values of decision variables, states of nature, resource constraints but also on the decisions taken by competitors. In such cases, the outcome of a decision taken by manager depends not only on his decision but also on the competitor's decision. Such conflict and corporative situations are referred as 'games'. Game theory is a widely used tool of operations research that is used to take decisions in such corporative and competitive situations.

14.3 BASIC CONCEPT OF GAME THEORY

Game Theory is the study of decision making under competition. In other words, Game Theory is the study of optimal decision making under competition when one individual's decisions affect the outcome of a situation for all other individuals involved. It attempts to mathematically capture behavior in competitive situations, or games, in which an individual's success in making choices depends on the choices of others.

Game Theory seeks to determine the rival's most profitable counter strategy to one's own best moves and to decide the appropriate defensive measures. For example- if two firms are locked up in a competition to maintain market share, then a price cut by the first firm will trigger the second firm to involve in price cut too. This will in turn effect the sales and profit of first firm which will again have to develop a counter strategy to face the challenge given by second firm. The game will thus continue.

Historically, Game Theory was developed in twentieth century but Its use and development accelerated during and after the Second World War. Game theory has wide applications in various fields. Some of the applications are given below-

a. It is used by Poker and Chess players to win their games.

b. Army uses this technique to plan war strategies.

c. It is also used to analyze various activities such as legal and political negotiations and economic behavior.

d. In situations where individuals have conflicting objectives, Game theory is used to take decisions.

e. For wage negotiations between unions and firms or in negotiations between two nations, game theory can be used.

14.3.1 TERMINOLOGY OF GAMES

Some basic terms which are used in game theory are defined below-

1. Game- Game represents a conflict between two or more parties/ competitors/ persons with certain predetermined rules.

   A conflict situation can be called as game if it has following properties-

   - The no. of competing players is finite.
   - Players act rationally and intelligently.

- Each player has a finite number of moves/choices/ courses of action available.
- All information about the game i.e. different strategies of other players and the amount of gain/loss with respect to a particular move are known to each player.
- Players select their choices/ strategies simultaneously.
- Players make decision without direct communication.
- The maximizing player attempts to maximize the gain and minimizing player wants to minimize his losses.
- The pay-off/outcome is fixed in advance.

2. Players- the participants taking decisions are called players.
3. Course of action- Each player has a finite number of possible options or strategies to choose from. These options are called courses of action.
4. Outcome/ Payoff- Every combination of course of action has an outcome. These are also called as payoffs. It is either gain or loss to each player. All outcomes are known in advance. However the resulting outcome of the game also known as value of game is known only after selection of course of action by players.

   Pay-off Matrix- Pay-off matrix is a tabular representation of available courses of actions to the players and the payoffs for each combination of courses of action. In a pay-off matrix, the numbers of rows in the matrix represent the number of courses of actions of player A which is always written on the left of the matrix, and the number of columns represent the number of courses of actions to player B that is written on the top of the matrix. The matrix always shows the payments or gains to player A. if there is a negative entry in any cell of the matrix that represents that the payments are to be made by A to B. thus for player B matrix always shows the losses. A sample pay of matrix for player A can be written in the following manner-

PlayerB

		1	2	3	....	n
	1	a_{11}	a_{12}	a_{13}	....	a_{1n}
Player A	2	a_{21}	a_{22}	a_{23}	....	a_{2n}
	3	a_{31}	a_{32}	a_{33}	....	a_{3n}
	.	....	......	......	....	....
	m	a_{m1}	a_{m2}	a_{m3}	....	a_{mn}

Player A is called the maximizing player since A would like to maximize the gains, while player B is called minimizing player since he would like to minimize the losses.

5. Strategy- strategy of a player is a decision rule by which a player chooses the course of action that gives him best profit/least loss.

6. Pure Strategy- A pure strategy is a decision in advance that a particular course of action is used for all plays in every time period.

7. Mixed Strategy- A mixed strategy is the decision to choose a course of action for each play in accordance with some particular probability distribution. In such a case, decision maker chooses two or more courses of action with some specific probability so that the expected gain is maximized.

8. Value of game- The value of the game is maximum guaranteed payoff to player A (maximizing player) if both the players play with best strategies. It is represented by symbol ν and is unique.

9. Maximin criterion for optimality- According to this criterion, the player adopts a pessimistic approach and plays safe that means his strategy is always that course of action which results in best out of worst outcomes. Player A i.e. maximizing player always decides to play with the strategy that gives him maximum out of the minimum gains for his different courses of action. This is called Maximin Criterion. Symbolically for the payoff matrix shown above, the maximin value is given by $Max_i[\text{Min}_j.\ a_{ij}]$, where i represents the available courses of action for A and j represents the courses of action for B.

10. Minimax Criterion for optimality- Player B (minimizing player) also likes to play safe and chooses a strategy that gives him minimum out of maximum losses for his different courses of action. This is known as minimax criterion. Symbolically for the payoff matrix shown above, the minimax value for B is given by $Min_j[\text{Max}_i\ a_{ij}]$, where i represents the available courses of action for A and j represents the courses of action for B.

14.3.2 TYPES OF GAMES

There are several types of game models which can be classified on the basis of factors like the number of players involved, the sum of gains or losses, the number of strategies employed and the value of the game.

a. Two person and n-person game- In a game situation, if two players are in conflict, such a game is called two-person game. If there are more than two participants involved in the game situation, it is called a n-person game.

b. Zero sum or Constant sum game and non-zero sum game - If the sum of gains and losses to participating players is equal to zero, then the game is called a zero sum or constant sum game. For example- if in the game of chess, two players agree that the loser will pay Rs. 100 to the winner at the end of the game, then it will represent a zero-sum game as gain of winner exactly matches to the loss of loser. However, if the sum of gains and losses is not equal to zero, it would be obviously called a non-zero sum game.

c. Finite game and Infinite game- A game is said to be finite, if each player has the option of choosing from only a finite number of strategies. If there are infinite number of strategies available to players, it is called an infinite game.

d. Fair and Unfair game- If the value of the game is zero, it is called a fair game. Otherwise the game is said to be unfair. For a positive value of game, it is considered as fair to player A but unfair to B. for a negative value of game, it is considered as favourable to B but unfair to A.

14.3.3 FORMS OF GAMES

Games can be represented in two forms- Normal form of games and Extensive form of games.

a. **Normal Form of Games-** In normal form of a game, information is given about the players who make relevant decisions, the strategies available to each player and the pay-offs which players receive at the end of the game based on the actions of all players in the game. For Example-

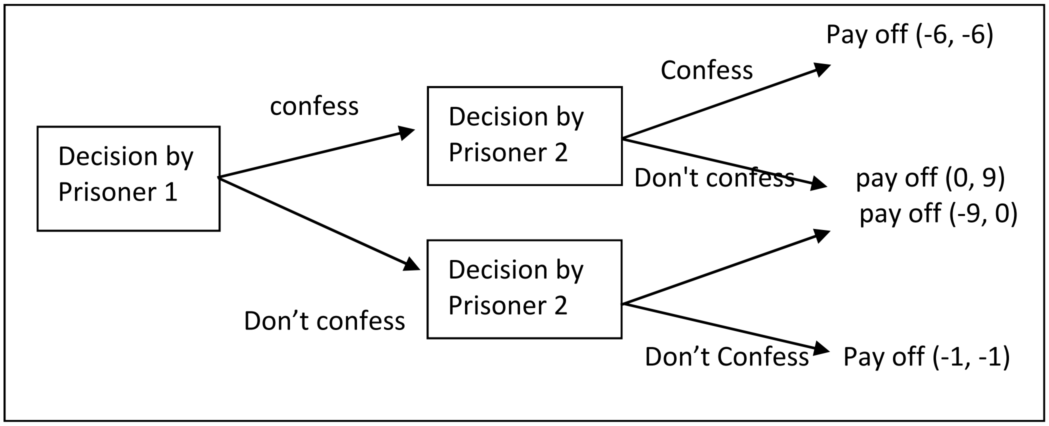
Prisoner's Dilemma- There are two suspects arrested for a crime. However, there is a lack of substantial evidence to prove the major charge unless one of them confesses. The police keeps the two suspects in separate cells and does the interrogation and explains the consequences. If neither confesses, then both will be convicted for a minor offence and sentenced to one month in prison. If both confesses, then both will be sent to prison for six months. If only one of them confesses and turns approver, then he will be released and the other will be given prison for nine months. Six months for crime and additional three months for obstructing in the course of justice.

This problem can be represented in the normal form of game as there are two players, each have three two options to choose from either confess or don't confess. Pay-off to each player for various combinations of strategies is also given. The game in the normal form can be shown as-

	Prisoner 2		
	Strategies	Confess	Don't Confess
Prisoner 1	Confess	-6, -6	0, -9
	Don't Confess	-9, 0	-1, -1

This is not a zero-sum game.

b. Extensive Form of Games- Games can be represented in extensive form as a decision tree. In this form the various strategies available to player 1 are shown as branches coming out of node 1, here player 1 takes the decision and then the alternatives for player 2 are shown at the subsequent nodes where player 2 takes the decision without knowing the strategy selected by player 1. The outcome at each leaf node is the payoff associated with that combination. The problem of prisoner's dilemma can be represented in extensive form in the following manner-



14.4 SOLUTION OF PURE STRATEGY TWO-PLAYER, ZERO-SUM GAMES

For solving a pure- strategy, two player game, the maximizing player selects his best strategy using maximin criterion, while minimizing player decides on optimal strategy using minimax criterion. The game is solved when the maximin value equals the minimax value.

14.4.1 SADDLE POINT

The position in the payoff matrix when the maximin value(maximum of row minimums) coincides with the minimax value (minimum of column maximum values) is called a saddle point. The cell value at this saddle point is called the value of the game. All pure strategy games have a saddle point where as a mixed strategy game does not have a saddle point. Therefore, in a pure strategy game,

Maximin for player A = Minimax for Player B.

14.4.2 DETECTING A SADDLE POINT

The saddle point in a game can be detected using following steps-

a. Develop the payoff matrix for the game.
b. Select the minimum elements in each row and encircle them.

c. Select the maximum value in each column and enclose them in small square boxes.

d. A point which is enclosed within a circle as well as the square is a saddle point. The value at saddle point is maximin value for maximizing player A and negative of the minimax value for minimizing player B.

If a payoff matrix has more than one saddle point then there are more than one solution of the game.

Example 14.1- There are two firms A and B. firm A has two strategies possible where as firm B has three strategies to choose from. The corresponding payoffs to A with respect to these strategies are given in the following pay-off matrix.

		Firm B		
	Strategy	B1	B2	B3
Firm A	A1	6	8	6
	A2	4	12	2

Determine the optimal strategies for both the firms.

Solution- In the given problem of game, firm A is the maximizing player and firm B is the minimizing player. If firm A chooses A1 strategy, firm B will choose either Strategy B1 or B3 as a counter strategy and the payoff to A will be 6. On the other hand if firm A chooses strategy A2, firm B will choose strategy B3 giving a payoff of 2. So, the better strategy for A will be A1 where it gets higher payoff.

The game can be solved by finding a saddle point as shown in the table below.

			Firm B		Row Minimum
	Strategy	B1	B2	B3	
Firm A	A1	6	8	6	6 6
	A2	4	12	2	2
Column Maximum		6	12	6	

Since there are two positions where maximin value is equal to minimax value that means there are two saddle points. The solution of the game will be-

- The best strategy for A is A1.
- Best strategy for firm B is either B1 or B3.
- The value of the game to firm A is 6 and -6 for firm B.

Example 14.2- The payoff matrix for a two player zero sum game is given below. Find the optimal strategy for each player and the value of game.

		Firm B				
	Strategy	I	II	III	IV	V
Firm A	I	9	6	1	8	0
	II	6	5	4	6	7
	III	2	4	3	3	8
	IV	5	6	2	2	1

Solution- To find the saddle point, circling the row minimums and putting squares on column maximums, we get the following table-

		Firm B				
	Strategy	I	II	III	IV	V
Firm A	I	[9]	[6]	1	[8]	(0)
	II	6	5	[(4)]	6	7
	III	(2)	4	3	3	[8]
	IV	5	[6]	2	2	(1)
Column Maximum		9	6	4	8	8

Since the matrix has a saddle point in cell (2,3), thus the solution to the game can be written as-

- Best strategy for A is II.
- Best strategy for B is III.
- Value of the game to A is 4 and -4 to B.

14.5 SOLUTION OF MIXED STRATEGY TWO-PLAYER, ZERO-SUM GAMES

If a game does not have a saddle point, the two players cannot use maximin and minimax strategies (pure) as their optimal strategy for all plays.

Then the best strategies are mixed strategies. A mixed strategy represents a combination of two or more strategies that are selected at a time. In this case, players play according to predetermined set which consists of probabilities corresponding to each of their pure strategies. There are various methods to solve a mixed strategy game. Let us first discuss the method of solving a 2×2 matrix game with no saddle point.

14.5.1 ALGEBRAIC METHOD OF SOLVING A 2×2 MIXED STRATEGY GAME

Consider a two-player zero-sum game with the following pay-off matrix:

		Player B	
	Strategy	I	II
Player A	I	a_{11}	a_{12}
	II	a_{21}	a_{22}

If this game does not have any saddle point, the game is a mixed strategy game. To find the optimal strategies, players will decide the probability with which they choose their course of action.

Let us consider in this game player A plays with strategy I with a probability p and play with strategy II with a probability 1-p. Also we assume player B plays with strategy I and II with respective probabilities q and 1-q.

The expected payoff to player A when player B plays with either strategy I or II throughout the game can be written as-

Payoff to A when B plays with I = $a_{11}p + a_{21}(1-p)$

Payoff to A when B plays with II = $a_{12}p + a_{22}(1-p)$

In order to have nullify the effect of choice of strategy of player B on the payoff to player A, the player A must decide the value of p in such a way that

Payoff to A when B plays with I = Payoff to A when B plays with II

i.e. $a_{11}p + a_{21}(1-p) = a_{12}p + a_{22}(1-p)$

$p = \frac{a_{22} - a_{21}}{(a_{11}-a_{12}) + (a_{22}-a_{21})}$ and $1-p = \frac{a_{11} - a_{12}}{(a_{11}-a_{12}) + (a_{22}-a_{21})}$

Similarly by equating the payoff to player B, for whatsoever choice of strategies of A, we can get,

$a_{11}q + a_{12}(1-q) = a_{21}q + a_{22}(1-q)$

$q = \frac{a_{22} - a_{12}}{(a_{11}-a_{12}) + (a_{22}-a_{21})}$ and $1-q = \frac{a_{11} - a_{21}}{(a_{11}-a_{12}) + (a_{22}-a_{21})}$

Value of the game can be found by substituting the value of p in any one of the expression for the payoff to A.

$\nu = a_{11}p + a_{21}(1-p)$, putiinng the value of p and rearranging we get,

$$\nu = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

14.5.1.1 DOMINANCE RULE

If a mixed strategy game involves more than two rows/ columns, then by eliminating a strategy which is inferior to another as never to be used, the size of the matrix can be reduced. Such a strategy which is inferior to some another strategy is dominated by the superior strategy. This concept of dominance is used to reduce the size of the matrix where no saddle point exists to a 2×2 size matrix.

Following are the rules of dominance-

1. If all elements in one row (i^{th} row) of a payoff matrix are less than or equal to the corresponding elements of the other row (j^{th} row), then the player A (maximizing player) will never choose the i-th strategy over j-th strategy or in other words i-th strategy will be dominated by j-th strategy. Hence the i^{th} row can be eliminated from the matrix.
2. If all elements in one column (r^{th} column) of a payoff matrix are more than or equal to the corresponding elements of the other column (s^{th} column), then the player B (minimizing player) will never choose the r-th strategy over s-th strategy or in other words r-th strategy will be dominated by s-th strategy. Hence the r^{th} column can be eliminated from the matrix.
3. A pure strategy may also be dominated if it is inferior to an average of two or more other pure strategies.

Example 14.3- Solve the following game-

		Player B		
	Strategy	I	II	III
Player A	I	-1	-2	8
	II	7	5	-1
	III	6	0	12

Solution- Let us first find if there exists any saddle point in the payoff matrix.

		Player B			
	Strategy	I	II	III	Row Minimum
Player	I	-1	(-2)	8	-2

A	II	7	5	-1	-1
	III	6	0	12	0
Column Maximum		7	5	12	

Since there is no saddle point, and the size of the game is more than 2 ×2, therefore, we try to reduce the size of the matrix by using dominance principle.

Since all the elements of row I are less than the corresponding elements of row III, therefore, row III dominates row I, so row I can be eliminated from the matrix. Now the reduce matrix is-

Player B

	Strategy	I	II	III
Player A	II	7	5	-1
	III	6	0	12

Again since all the elements of column I are greater than all corresponding elements of column II, therefore, column I is dominated by column II and hence can be removed from the matrix. Thus the following reduced 2×2 matrix is obtained.

Player B

	Strategy	II	III
Player A	II	5	-1
	III	0	12

Now we can apply algebraic method for solving this 2×2 game.

Let us assume player A plays with strategies II and III with respective probabilities p and 1-p. and player B plays with strategies II and III with respective probabilities q and 1-q. Using the formulas obtained for p and q in the above section, we can get

$$p = \frac{a_{22} - a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{12 - 0}{(5+1) + (12-0)} = \frac{12}{18} = \frac{2}{3}$$

$$1 - p = 1 - 2/3 = \frac{1}{3}$$

$$q = \frac{a_{22} - a_{12}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{12 - (-1)}{(5 - (-1)) + (12-0)} = \frac{13}{18}$$

$1\text{-}q = 1 - \frac{13}{18} = \frac{5}{18}$

The value of the game will be $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{5*12-(-1)*0}{(5-(-1))+(12-0)} = \frac{60}{18} = \frac{10}{3}$

The optimal strategies for A is $(0, \frac{2}{3}, \frac{1}{3})$ and for B is $(0, \frac{13}{18}, \frac{5}{18})$.

14.5.2 GRAPHICAL METHOD FOR SOLVING 2× N OR M× 2 GAMES

Graphical method is used to solve 2× n or m× 2 games. In those games with mixed strategy where only two pure strategies (undomianted) are available for one of the players, graphical method can be used. Since the optimal strategies for both players assign non-zero probabilities to the same number of pure strategies, therefore if one player has only two pure strategies, the other will also use two strategies only. Graphical method helps in finding which two strategies should be used by other player. This way, the game is reduced to size 2× 2 which can be solved by algebraic method explained in section 14.5.1.

Let us consider a 2 × n game which has no saddle point whose pay off matrix is as follows-

			B				
			1	2	3	…	N
A	P	1	a_{11}	a_{12}	a_{13}	…	a_{1n}
	1-p	2	a_{21}	a_{22}	a_{23}	…	a_{2n}

If p and 1-p are the probabilities with which the player A uses his pure strategies. Then

The expected pay off to player A for different pure strategies used by player B can be written as-

Expected Payoff to A when B uses strategy 1 = $a_{11}p + a_{21}(1-p) = (a_{11} - a_{21})p + a_{21}$

Expected Payoff to A when B uses strategy 2 = $a_{12}p + a_{22}(1-p) = (a_{12} - a_{22})p + a_{22}$

Expected Payoff to A when B uses strategy 3 = $a_{13}p + a_{23}(1-p) = (a_{13} - a_{23})p + a_{23}$

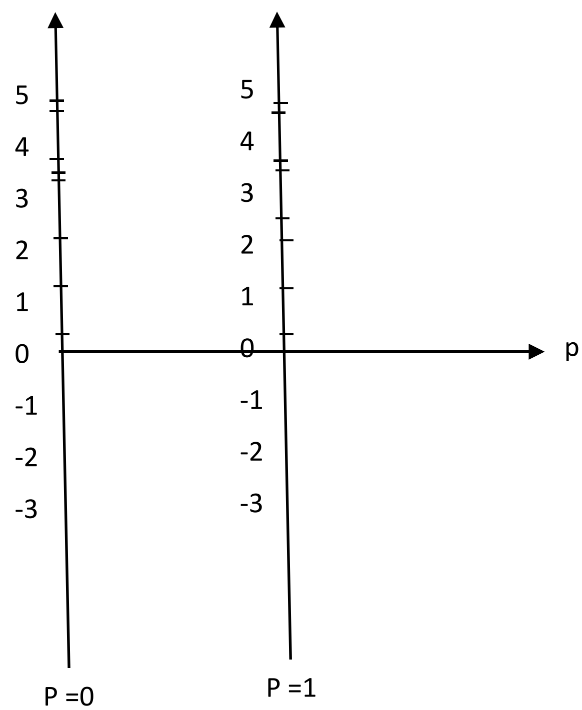
. . . .

Expected Payoff to A when B uses strategy n = $a_{1n}p + a_{2n}(1-p) = (a_{1n} - a_{2n})p + a_{2n}$

Using the Maximin criterion for mixed strategies game, player A will select that value of p which will maximize his minimum expected payoffs. In order to find this value, straight lines for expected payoffs to A corresponding to various strategies of B as a linear function of p.

Method to plot lines-

To plot the expected payoff lines, draw two parallel lines one unit apart and mark a scale on each as shown in the figure below.



These two lines represent the strategies available to A. Then lines representing each of B's strategies are drawn by obtaining the expected payoff values to A by putting p =0 and p =1 for each linear function representing expected payoffs. The value of expected payoff corresponding to p =0 is marked on scale 1 and value of expected payoff corresponding to p =1 is marked on scale 2. By joining these two points, a line is plotted. Similarly other lines are plotted for expected payoffs too.

The lower boundary to these lines will give the minimum expected payoff as function of p. the highest point on this lower boundary represents the maximin point and give maximum expected payoff to A. then we determine only two strategies for player B corresponding to those two lines which pass through this maximin point. In this way the game is reduced to 2×2 game which can be easily solved by algebraic method explained in 14.5.1.

In the same way, m×2 games can be solved, except that in that case in place of maximum point on lower boundary, the minimum point on the upper boundary is considered as the optimal point.

Example 14.4- Solve the following game-

B

1	2	3	4

A	1	1	3	-3	7
	2	2	5	4	-6

Solution- This is a 2×4 game. Let us first check the saddle point.

		1	2	3	4	Row minimum
	1	1	3	(–3)	[7]	–3
A	2	[2]	[5]	[4]	(–6)	–6
Column Max.		2	5	4	7	

Since there is no saddle point, it is a mixed strategy game. Using the dominance principle, Column 2 can be dominated by column 1 and hence column 2 can be removed. So the reduced matrix can be written as-

			B		
			1	2	4
	P	1	1	–3	7
A	1–p	2	2	4	–6

further reducing the size is not possible, so we shall use graphical method to solve the problem.

Let us assume that player A will play with strategies 1 and 2 with probabilities p and 1-p.

The expected pay off to A for different pure strategies used by player B can be written as-

$E(A)_{B1} = 1*p + 2(1-p) = -p + 2$ ------ (1)

$E(A)_{B3} = -3p + 4(1-p) = -7p+4$ -------(2)

$E(A)_{B4} = 7P + -6(1-p) = 13p-6$ -------(3)

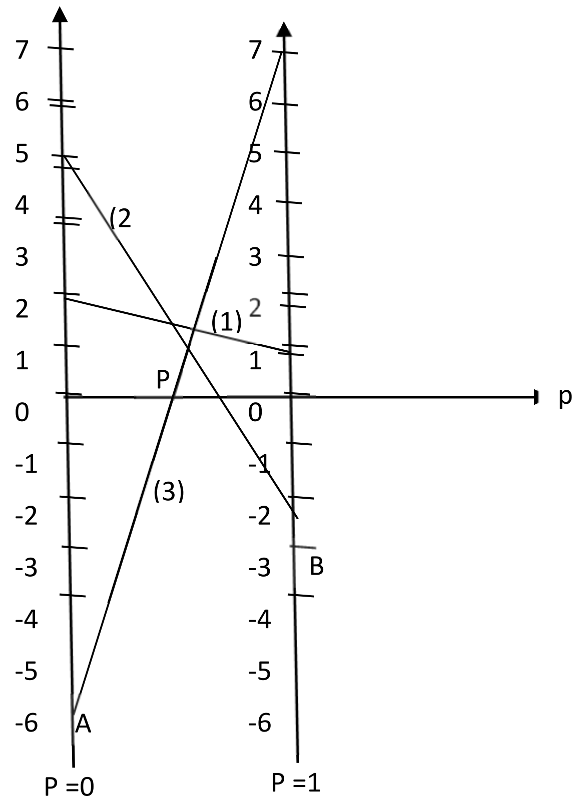
Now the graph is plotted for the three payoff linear equations as shown below-

First putting p =0 and then putting p =1 in each of the above equations we get following points-

$E(A)_{B1} = 2$ when p=0 and $E(A)_{B1} = 1$ when p =1

$E(A)_{B3} = 4$ when p=0 and $E(A)_{B3} = -3$ when p =1

$E(A)_{B4} = -6$ when p=0 and $E(A)_{B4} = 7$ when p =1



Now the lowest boundary to plotted lines (1), (2) and (3) is represented by APB. The highest point on this lowest boundary is P and represents the maximin point at which the expected payoff to A will be maximum at this point. Since it is at the intersection of lines (2) and (3) which are corresponding to Player B strategy 3 and 4. So the game can be reduced to following 2×2 game.

			B	
			q	1–q
			3	4
A	p	1	–3	7
	1–p	2	4	–6

If q and 1-q are the probabilities with which player B plays with strategies 3 and 4 respectively.

Now using the algebraic method formulas we can compute the values of p and q as follows-

$$p = \frac{a_{22} - a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{-6-4}{(-3-7)+(-6-4)} = \frac{-10}{-20} = \frac{1}{2}$$

$$1\text{-}p = 1\text{-} \tfrac{1}{2} = \frac{1}{2}$$

$$q = \frac{a_{22} - a_{12}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{-6-7}{(-3-7)+(-6-4)} = \frac{13}{20}$$

$1\text{-}q = 1 - \frac{13}{20} = \frac{7}{20}$

The value of the game will be $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})} = \frac{(-3)(-6) - 7*4}{(-3-7) + (-6-4)} = \frac{-10}{-20} = \frac{1}{2}$

The optimal strategies for A is $(\frac{1}{2}, \frac{1}{2})$ and for B is $(0, 0, \frac{13}{20}, \frac{7}{20})$.

			B
		1	2
A	1	2	7
	2	3	5
	3	11	2

Example 14.5- Solve the following game-

Solution- This is a 2×3 game with no saddle point.

Since there is no saddle point, it is a mixed strategy game of size 3×2.

so we shall use graphical method to solve the problem.

Let us assume that player B will play with strategies 1 and 2 with probabilities q and 1-q.

The expected pay off to B for different pure strategies used by player A can be written as-

$E(B)_{A1} = 2q + 7(1-q) = -5q + 7$ ------ (1)

$E(B)_{A2} = 3q + 5(1-q) = -2q + 5$ -------(2)

$E(B)_{A3} = 11q + 2(1-q) = 9q + 2$ -------(3)

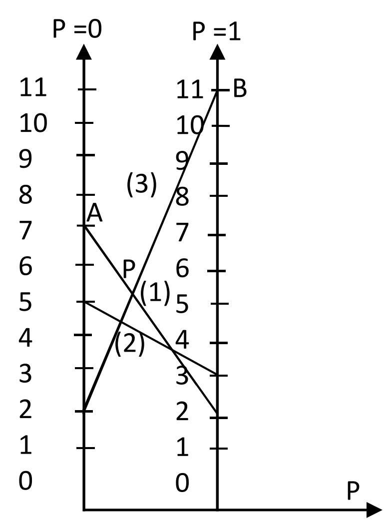
Now the graph is plotted for the three payoff linear equations as shown below-

First putting p =0 and then putting p =1 in each of the above equations we get following points-

$E(B)_{A1} = 7$ when p=0 and $E(B)_{A1} = 2$ when p =1

$E(B)_{A2} = 5$ when p=0 and $E(B)_{A2} = 3$ when p =1

$E(B)_{A3} = 2$ when p=0 and $E(B)_{A3} = 11$ when p =1



Now the upper most boundary to plotted lines (1), (2) and (3) is represented by APB. The lowest point on this upper most boundary is P and represents the minmax point at which the expected payoff to B will be maximum at this point. Since it is at the intersection of lines (1) and (3) which are corresponding to Player A strategy 1 and 3. So the game can be reduced to following 2×2 game.

			B	
			q	1–q
			1	2
A	P	1	2	7
	1–p	3	11	2

If p and 1-p are the probabilities with which player A plays with strategies 1 and 3 respectively.

Now using the algebraic method formulas we can compute the values of p and q as follows-

$$p = \frac{a_{22} - a_{21}}{(a_{11}-a_{12}) + (a_{22}-a_{21})} = \frac{2-11}{(2-7)+(2-11)} = \frac{-9}{-14} = \frac{9}{14}$$

$$1\text{-}p = 1\text{-}\ 9/14 = \frac{5}{14}$$

$$q = \frac{a_{22} - a_{12}}{(a_{11}-a_{12}) + (a_{22}-a_{21})} = \frac{2-7}{(2-7)+(2-11)} = \frac{5}{14}$$

$$1\text{-}q = 1\text{-}\frac{5}{14} = \frac{9}{14}$$

The value of the game will be $\nu = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}-a_{12}) + (a_{22}-a_{21})} = \frac{2*2-7*11}{(2-7)+(2-11)} = \frac{-73}{-14} = \frac{73}{14}$

The optimal strategies for A is ($\frac{9}{14}, 0, \frac{5}{14}$) and for B is (5/14, 9/14) .

14.5.3 LINEAR PROGRAMMING METHOD

For a m×n rectangular game when either m or n or both are greater than or equal to three, linear programming method is used.

Consider a two person zero sum game as explained below-

Player A has m courses of action (A_1, A_2, ….A_m) and player B has n courses of action (B_1, B_2, ….B_n). the payoff to player A corresponding to strategy A_i and B_j is a_{ij}. Mixed strategy for player A is defined by probabilities p_1, p_2, …., p_m, where $\sum_{i=1}^{m} p_i = 1$ and mixed strategy for player B is defined by probabilities q_1, q_2, …., q_n, where $\sum_{j=1}^{n} q_j = 1$.

This game can be defined as a Linear Programming Problem as below-

Determine unknown probabilities p_1, p_2, …., p_m with objective to maximize the value of game ν such that the following constraints are satisfied-

$a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + \ldots + a_{m1}p_m \geq \nu$

$a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + \ldots + a_{m2}p_m \geq \nu$

: :

$a_{1n}p_1 + a_{2n}p_2 + a_{3n}p_3 + \ldots + a_{mn}p_m \geq \nu$

$p_1 + p_2 + p_3 + \ldots + p_m \geq \nu$

$p_1, p_2, \ldots, p_m \geq 0$, ν is unrestricted in sign.

Dividing all the constraints by ν both the sides. We get

$a_{11}p_1/\nu + a_{21}p_2/\nu + a_{31}p_3/\nu + \ldots + a_{m1}p_m/\nu \geq 1$

$a_{12}p_1/\nu + a_{22}p_2/\nu + a_{32}p_3/\nu + \ldots + a_{m2}p_m/\nu \geq 1$

: :

$a_{1n}p_1/\nu + a_{2n}p_2/\nu + a_{3n}p_3/\nu + \ldots + a_{mn}p_m/\nu \geq 1$

$p_1/\nu + p_2/\nu + p_3/\nu + \ldots + p_m/\nu \geq 1$

Assuming $x_1 = p_1/\nu$, $x_2 = p_2/\nu, \ldots\ldots x_m = p_m/\nu$

Since the objective is to maximize the value ν , which is equivalent to minimize $1/v$.

New linear programming problem can be written as-

Minimize $Z = 1/\nu = x_1 + x_2 + x_3 + \ldots + x_m$

$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 + \ldots + a_{m1}x_m \geq 1$

$a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + \ldots + a_{m2}x_m \geq 1$

: :

$a_{1n}x_1 + a_{2n}x_2 + a_{3n}x_3 + \ldots + a_{mn}x_m \geq 1$

$x_1 + x_2 + x_3 + \ldots + x_m \geq 0$

Now this problem can be easily solved by simplex method explained in unit 4.

14.6 LIMITATIONS OF GAME THEORY

Game theory has some limitations also. These are mentioned below-

1. It is based on the onrealistic assumption that players have knowledge about their own payoffs and pauoffs of others. However, in reality players can only guess about the rival's strategies.
2. Game theory analysis becomes complicated and difficult in case of increase in number of players in a game. In a oligopoly situation, game theory cannot be very helpful.
3. The game theory is based on the assumption that players use maximin and minimax approach which shows that players are risk-averse. However, it may not be possible in many cases.

4. It is possible in real situation of oligopoly that players instead of choosing their strategies in uncertain condition, may share some information with each other to work out collusion. In such cases, mixed strategy may not be useful.

14.7 SUMMARY

A detailed introduction to the theory of games has been given in this unit. The basic conceptual framework and important terminology used in game theory is explained. The unit gives a deep understanding about the various types of games and solution method of pure and mixed strategy games. Various methods of solving rectangular games have been discussed with the help of various examples. In the end, limitations of game theory have been discussed.

14.8 SELF ASSESSMENT QUESTIONS

1. What is game theory? Discuss its importance in business decisions.
2. Discuss various types of games.
3. What are different forms of games?
4. Explain the following terms-
   a. Saddle Point
   b. Dominance principle
   c. Mixed strategy games
5. What are the limitations of game theory?
6. Solve the following games-

   a.

B

		1	2	3
A	1	5	7	11
	2	2	-1	8
	3	18	-6	10

b.

		B			
		1	2	3	4
A	1	1	7	3	4
	2	5	6	4	5
	3	7	2	0	3

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UNIT-15 SIMULATION

Unit Framework

15.1 OBJECTIVES

After completing this unit you will be able to:

- Understand the concept of Simulation
- Learn about various types of simulation.
- Understand approaches to run simulation
- Know about advantages and limitations of simulation
- Learn about various applications of simulation

15.2INTRODUCTION

Simulation is one amongst the widely used techniques of Operations Research. This technique is applied into all types of business problems related to all functional areas. One of its inherent characteristic is direct application on the gross data of past, due to which it is specially utilized for cross functional decision making. Simulation is a flexible technique along with the increasing applications of computers, it is becoming popular in every field. Simulation is frequently applied in the areas of risk analysis, capacity decisions, inventory, project management, market segmentation studies, resource planning, etc. where future is unpredictable or there are competitive situations. Simulation is widely used to analyse stochastic systems involving repeated decisions that interact with environment such as- weather forecast, aviation, design of vehicle safety system etc.

15.3 BUSINESS SIMULATION

In business simulation, a real business problem is replicated into a mathematical model. The model is then manipulated to get the results. These results are interpreted and compared with real conditions. Since the processes and relationships are complex, computer simulation is a costly exercise. In many business problems, manual calculations are not feasible. Even for a simple problem, the simulation calculations are too lengthy that manual handling is not feasible. Various computer simulation languages are available to solve business simulation problems.

15.4 TYPES OF SIMULATION

Simulations can be divided into four types-

1. Real Life Simulation- In this type of simulation, the real product is used to test how it will perform in a particular situation; for example- testing a car in adverse situation to check its safety features. Such simulations are expensive, time consuming, have little control on environment and difficult to repeat. However these simulations are used in the final testing of product like helmet for its crash worthiness.

2. Quasi-real Simulation- this type of simulation preserves some of the aspects of the real system whereas excludes the difficult or impossible to replicate aspects. In these type of systems the aspects related to human safety or involving very high cost are excluded such as human presence in crash simulation.

3. Laboratory Simulation- In such simulations, a model or replica of the real problem is simulated in a laboratory under controlled environment. In this type of simulation, it is possible to pause the simulation, repeat or reduce the speed and so on. For example- business gaming is a simulation in which we can stop the game at an interval.

4. Computer Simulation- This type of simulation uses well-designed computer programs. With the help of such programs, it is possible to run a simulation retaining all aspects of the real problem. This is not only cost effective but also a very efficient way of simulation provided that the problem can be programmed mathematically. This type of simulation has

several advantages such as time reduction. Model testing even when a system is not fully developed.

15.5 CONCEPT OF SIMULATION IN OR

Simulation is a very important tool in operations research. When the system is stochastic (time- probabilistic) for example Markov Chain, queues probabilistic inventory models etc. simulation is very useful. In such a case the events are randomly distributed. Since simulation requires generating and processing a large amount of data, computers are necessary for any business simulation. The first step to run a simulation is to develop a mathematical model. It is not necessary to develop a model with algebraic equations like LPP. Discrete functions can also be used in model similarly it is not required to know the exact relationship between variables. A simulation can be run as long as the output parameters corresponding to the combinations of input parameters are known.

15.6 STEPS IN SIMULATION

The following steps are involved in simulation-

1. Define the system in terms of input/output parameters and transformation process.
2. Identify possible states that a system can assume.
3. List all the events that change the state of the system (for example- arrival of customer, demand of a product etc.)
4. Simulation clock is necessary to identify the passage of time.
5. Random number generator that generates the time of occurrence of various events as per the probability distribution that the event is known to take place.
6. Describe a relationship or a formula for transition of state from one state to other.
7. Test the behavior of model using simple data.
8. Run the simulation.
9. Analyze the results of multiple runs.
10. Validate the simulation and use the results for decision making.

15.7 CATEGORIES OF SIMULATION

There are two categories of simulation. One is continuous simulation that runs over a time. These are typically used for aero plane design, study of vehicle trash, nuclear reaction etc. The other category is a discrete event simulation where changes in a state of the system occur instantaneously at random points in time as a result of discrete events. For example- a person entering queue, patient comes out of an operation theatre, replenishment of stock etc.

In business simulations most of the simulations are discrete event simulations. Therefore in this chapter only discrete event simulations are explained.

15.8 APPROACHES TO RUN SIMULATION

In simulation, the changes in states are studied over a time period when discrete random events occur and effect the state. Thus, the system is studied as the time progresses. There are two approaches for simulating the time advancement.

1. **Fixed time increment-** in this approach, time is advanced by a small fixed interval. At that moment the states of the system are updated. For this we need to identify what all events would happen during this incremented time interval. Then the resulting state of the system is studied and desired performance parameters like length of the queue, inventory in hand, bottlenecks in the system etc. are recorded. This approach has an advantage when the events are occurring too fast or a large number of discrete events are occurring in a short time period. In such a case, checking the state after every discrete event becomes too complex and may not be necessary. For example- monitoring the electronic exchange state over every single call may not be feasible while designing the capacity of the system. The primary interest is in proportion of time of congestion or average waiting time or average call duration etc.

2. **Next Event increment-** In this approach, simulation clock is incremented by a variable amount. When any discrete event takes place that changes the state of the system the time is incremented. So the clock is not recorded till the next event occurs. Thus, at every discrete event, clock record jumps, and the state of the system is updated and the system performance parameters measured. For example- in a queue simulation, the clock is incremented and the state of the system updated when either a customer arrives or service starts or service finishes or customer leaves the system etc.

15.9 ADVANTAGES AND LIMITATIONS OF SIMULATION

The business problems may be so complex that a manual mathematical model solution may not be feasible. In such cases, simulation is a good alternative. The main advantages of simulation are-

a. It is the only method available in many cases because of difficulty in measuring each parameter of an environment.

b. In such cases where developing a mathematical relationship is impossible and only discrete input-output relationship can be developed, simulation can be run effectively.

c. Actual running of a real system may be very expensive, in comparison to that, simulation is a cost effective option.

d. It is less time consuming.

e. It is a non-disruptive tool. It is not required to suspend the operation.

f. It is a very safe decision making tool specially in case of risky systems.

There are some limitations of simulation. These are-

a. It is not an optimization technique. Hence it does not provide an optimum solution.

b. Simulation answers are not precise as these are not based on mathematical models.

c. This technique may not be appropriate for all business situations.

d. Sometimes, simulation may be expensive as compared to a simplified OR model.

e. Unlike other OR models, simulation does not provide a solution. One has to extract the solution by observing multiple simulation runs.

15.10 APPLICATIONS OF SIMULATION

Some of the applications of simulations are mentioned below-

a. Inventory Control- In these cases, the lead time to get an order and the demand leading to consumption of the inventory, both may be random variables. This affects the inventory policies such as reorder level, safety stock, order quantity etc. simulation based on known/estimated probability distribution of lead time and consumption can be used in such cases to determine inventory policies.

b. Facility utilization- In such problems, the service time as well as arrival of customers may be probabilistic. Therefore, facility planning decisions such as service capacity, queue capacity, etc. may be affected. Hence simulation can be used in such cases. Some applications are – aero planes waiting for landing, calls waiting for connections, patients waiting for operation etc.

c. Production Process- In production process when number of operations are to be performed in a particular sequence by sharing common facilities and the time taken by operations is probabilistic, simulation can be used. Based on the simulation, the process sequence, capacity, layout etc. can be planned.

d. Resource planning- In a particular system, there can be many common resources, in such a case when the demand of a resource is probabilistic; simulation can be applied to decide about the amount of resource required.

e. Human Resource Planning- The requirement of manpower for various projects may change from time to time. No. of employees leaving jobs, absenting from work can also not be predicted exactly therefore, simulation can be used in such cases for taking decisions.

f. **Project Completion-** In any project, if the activity time is probabilistic, it becomes difficult to find the project completion time using other quantitative techniques. In such cases, simulation can be used for estimating the project completion duration with probability estimate.

g. **Financial Risk Analysis-** In this field, simulation is most commonly used. There can be capital investments with uncertain cash flows, fluctuating returns on investment, market uncertainties etc. in such probabilistic situations, simulation can be used to take decisions.

15.11 PROBABILISTIC SIMULATION AND RANDOM NUMBERS

When a system contains certain decision variables that can be represented by a probability distribution, the simulation model is used to study this type of system is called the probabilistic (Stochastic) simulation model. These models use random numbers to generate in certain events.

15.11.1 MONTE-CARLO SIMULATION

There are two types of simulation techniques- a) Monte-Carle technique , and b) System Simulation technique.

In Monte- Carlo technique, random numbers are used to solve problems requiring decision making under uncertainty and where mathematical formulation is not possible. System simulation is applied to those conditions where there is a reproduction of operating environment and the system allows the analysis of the response from the environment to alternative management actions. However, due to complexity involved in system simulation technique, it is not very popular. Monte- Carlo technique is generally used for simulation and is widely applied. The technique makes use of pure chances to construct a simulated version of the process in the same way as pure chance operates the original system. The simulation procedure involves following steps-

1. Define the problem by identifying objectives, and main factors (variables) to be considered.
2. Construct an appropriate model- identify parameters that influence the system, define a decision rule, identify type of distribution used, specify the manner in which time change.
3. Define starting conditions for simulation and specify the number of simulation runs to be made.
4. Experiment the model- Define a coding system that will correlate the factors identified in step 1 with the random numbers generated, select a random number generator, generate random numbers and correlate them with the factors, summarize and examine the results.
5. Evaluate the results and select best course of action.

15.11.2 RANDOM NUMBER GENERATION

For running a simulation, a sequence of random numbers is to be generated. This sequence help in choosing random observations from the probability distribution.

Random numbers can be generated with the help of random number tables. Computers can generate pseudo random numbers with the help of a computer programme. Random numbers in simulation are generated using the computer most of the times. For this the following options are available-

a. Store a table of random numbers in the memory of computer.

b. Construct an electronic device that can generate truly random numbers.

c. Utilize an arithmetic operation to compute a sequence of random.

Out of these methods, first two methods are practically less feasible, therefore, most o the time third method is used to generate random numbers. For example- Mid-square method may be used to compute a sequence of random numbers by choosing one arbitrary number of k digits (even), the next number in sequence is obtained by squaring the chosen number and then extracting the middle k digits of number. The process is then repeated. In the same way, there may be various operations that can be applied to generate random numbers.

15.12 GENERATION OF RANDOM OBSERVATIONS FROM A DISCRETE PROBABILITY DISTRIBUTION

In case of discrete distribution, either the probability distribution in terms of probability mass function (p.m.f.) is known or it is computed from the past data. Probability mass function is the discrete probabilities associated with all the possible values of random observations. It has a property that sum of all probabilities is 1 and each value of probability is between 0 and 1. The relative frequency could be used as the p .m.f. to simulate random observations. Once the p.m.f. is known, cumulative distribution function (c.d.f.) by computing cumulative values for observations is calculated. From the c.d.f. compute range of numbers of appropriate digits that are proportional to the p.m.f. for the observations. Then select random numbers one by one and search the interval in which the random numbers fall. Corresponding to the intervals, the observations are read from the table.

Example 15.1- A bakery keeps the stock of a popular brand of bread. The daily demand of bread has shown the following pattern.

Daily Demand	0	10	20	30	40	50	Total
Probability	0.01	0.2	0.15	0.56	0.12	0.02	1

Simulate the demand for next 10 days. Also find out the average demand per day. (use random numbers 41, 30, 20,59,79,35,22,83,46,56)

Solution- Since the probability of the demand is upto 1 and we need two decimals for the given probability values, we can use two digit random numbers from the random number table. Thus the discrete random numbers will range from 00 to 99. To depict the uniform probability distribution, we divide the numbers by 100, so that the uniform probability distribution range from 0 to 1. The computation is shown below-

Daily demand	Probability distribution	Cumulative probability distribution	Random number interval	Proportion of numbers in the interval out of 100
0	0.01	0.01	00	0.01
10	0.20	0.21	01-20	0.20
20	0.15	0.36	21-35	0.15
30	0.50	0.86	36-85	0.50
40	0.12	0.98	86-97	0. 12
50	0.02	1.00	98-99	0.02

There is one random number 00 out of 100 corresponding to demand of 0, giving probability of 0.01, similarly there are twenty random numbers from 01-20 out of 100 corresponding to the demand of 10, giving a probability of 0.20 and so on.

To simulate the demand, a two digit random number is selected, the number interval in which the selected number lies is checked and the corresponding demand is read from the above table. By using the given random numbers, we get the simulated demand as shown in the following table-

Day	1	2	3	4	5	6	7	8	9	10
Two digit random number	41	30	20	59	79	35	22	83	46	56
Interval	36-85	21-35	01-20	36-85	36-85	21-35	21-35	36-85	36-85	36-85
Simulated demand	30	20	10	30	30	20	20	30	30	30

Average Demand per day = $\frac{30+20+10+30+30+20+20+30+30+30}{10}$ = 25 units.

15.13 SUMMARY

This unit describes the concept of simulation and its applications in various fields. Various types of simulation, simulation approaches and business simulation in OR have been discussed. The unit also explains the steps in simulation, how random numbers are generated and simulation is applied for discrete probability distribution.

15.14 SELF ASSESSMENT QUESTIONS

1. Define the term simulation. What are different types of simulation?
2. How business simulation in OR takes place?
3. What are various approaches of simulation?
4. Discuss various advantages and limitations of simulation.
5. Explain various applications of Simulation.
6. What is Monte-Carlo Simulation? When is it applied?

15.15 TEXT AND REFERENCES

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