U.P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109
Operation Research

Block: 1 Formulation of Linear Programming **5 - 38**

Unit – 1 : Introduction to Operation Research **8 - 26**

Unit – 2 : Graphical Method to Solve LPP **27 - 38**

Block: 2 Simplex Method of Solving LPP **39 - 78**

Unit – 3 : Simplex Method **42 - 70**

Unit – 4 : Duality Problem in LPP **71 - 78**

Block: 3 Transportation Problem and Assignment Problem **79 - 118**

Unit – 5 : Representation of Transportation Problem & Assignment Problem as Linear Programming Problem **82 - 92**

Unit – 6 : Different Methods of Finding Initial Feasible Solution of a Transportation Problem **93 - 107**

Unit – 7 : Solution of Assignment Problem with using Hungarian Method **108 - 118**

Block: 4 Theory of Games **119 - 152**

Unit – 8 : Basic Concepts of Games Theory **122 - 133**

Unit – 9 : Dominance Rule, Equivalence of Rectangular Games with Linear Programming **134 - 152**

Course Preparation Committee

Dr. Shubra Katara — **Writer** ***(Block – 1 to 2)***
Department of Statistics, Bareily College, Bareily

Dr. D. C. Singh — **Writer** ***(Block – 3)***
Department of Statistics, Harishchandra P. G. College, Varanasi

Prof. V. K. Sehgal — **Writer** ***(Block – 4)***
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Prof. S. K. Pandey — **Writer** ***(Block – 4)***
Department of Statistics, Lucknow University, Lucknow

Prof. V. K. Sehgal — **Reviewer** ***(Block – 1, 3)***
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Dr. D. C. Singh — **Reviewer** ***(Block – 2, 4)***
Department of Statistics, Harishchandra P. G. College, Varanasi

Prof. R. C. Yadava — **Editor** ***(Block – 1)***
Department of Statistics, Banaras Hindu University, Varanasi

Dr. D. C. Singh — **Editor** ***(Block – 2)***
Department of Statistics, Harishchandra P. G. College, Varanasi

Prof. Umesh Singh — **Editor** ***(Block – 3)***
Department of Statistics, Banaras Hindu University, Varanasi

Prof. V. P. Ojha — **Editor** ***(Block – 4)***
Department of Statistics and Mathematics, D. D. U.,
Gorakhpur University, Gorakhpur

Dr. Shruti — **Course/SLM Coordinator**
School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109 **OPERATION RESEARCH**

First Edition: *March 2008* (Published with the support of the Distance Education Council, New Delhi)
Second Edition: ***July 2021***
ISBN :

Printed By: K.C.Printing & Allied Works, Panchwati, Mathura -281003.

Blocks & Units Introduction

The present SLM on ***Operation Research*** consists of nine units with four blocks.

The ***Block - 1 – Formulation of Linear Programming Problems,*** is the first block, which contains two units. It describes the concept of Linear Programming Problems and Graphical Method to Solve LPP.

Unit – 1 – Introduction to Operation Research, of the block describes definition and scope of operation research along with formulation of LPP with various examples.

Unit – 2- graphical Methods to Solve LPP, defines feasible and basic feasible solution and gives graphical method of solving LPP.

The ***Block - 2 – Simplex Method of Solving LPP,*** of operation research deals with Simplex Method of solving LPP and the Principles of Duality in the LPP. It consists of two units.

Unit – 3 – Simplex Method describes the simplex method of solving LPP with illustrations.

Unit – 4 – Duality Problem in LPP, gives the principle of duality in LPP along with simple problems based on duality theorem.

The ***Block - 3 – Transportation Problem and Assignment Problem, deals*** with transportation problem (TP) and assignment problem (AP). It contains three units.

Unit – 5 – Representation of Transportation Problem & Assignment Problem as linear Programming Problem, deals with representation of these problems as a special case of linear programming problem.

Unit – 6 – Different Methods of Finding Initial Feasible Solution of a Transportation Problem, describes the different methods of finding initial feasible solution to TP and MODI methods of finding optional solution of TP.

Unit – 7 – Solution of Assignment Problem with using Hungarian Method, gives the solution of AP using Hungarian method.

The ***Block - 4 – Theory of Games,*** is the last block of this SLM ,with its two units, which covers a very important concept known as theory of games. The game theory is a type of decision theory where all possible alternatives available to the opponent are considered before taking a decision.

J. Von Neumann, the renowned mathematician, gave the minimax (maximin) criterion to carry out the analysis of the problems in game theory. It is very important to understand the principles of game theory as these concepts give the algorithm to handler the actual problems of decision theory in competitive situations in industry or elsewhere.

Unit – 8 – Basic Concepts of Game Theory, deals with introduction to game theory, two-person zero-sum game, different types of strategies and games with or without a saddle point.

Unit – 6 – Dominance Rule, Equivalence of Rectangular Games with Linear Programming, deals with the theories of dominance rule, solution methods of games without saddle point and equivalence of rectangular games with linear programming.

At the end of every block/unit the summary, self assessment questions and further readings are given.

U.P. Rajarshi Tandon Open
University, Prayagraj

DECSTAT – 109
Operation Research

Block: 1 Formulation of Linear Programming

Unit – 1 : Introduction to Operation Research 8 - 26

Unit – 2 : Graphical Method to Solve LPP 27 - 38

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics, D. D. U. Gorakhpur University, Gorakhpur.

Dr. Shruti
Member-Secretary
School of Sciences, U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Shubra Katara **Writer**
Department of Statistics, Bareily College, Bareily

Prof. V. K. Sehgal **Reviewer**
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Prof. R. C. Yadava **Editor**
Department of Statistics, Banaras Hindu University, Varanasi

Dr. Shruti **Course/ SLM Coordinator**
School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109 **OPERATION RESEARCH**
©UPRTOU
First Edition: *March 2008* (Published with the support of the Distance Education Council, New Delhi)
Second Edition: ***July 2021***
ISBN :

Printed By: K.C.Printing & Allied Works, Panchwati, Mathura -281003.

Block & Units Introduction

The ***Block - 1 – Formulation of Linear Programming Problems,*** is the first block, which contains two units. It describes the concept of Linear Programming Problems and Graphical Method to Solve LPP.

Unit – 1 – Introduction to Operation Research, of the block describes definition and scope of operation research along with formulation of LPP with various examples.

Unit – 2- graphical Methods to Solve LPP, defines feasible and basic feasible solution and gives graphical method of solving LPP.

of games without saddle point and equivalence of rectangular games with linear programming.

At the end of every block/unit the summary, self assessment questions and further readings are given.

Unit-1 Introduction to Operation Research

Structure

1.1 Introduction
1.2 Objectives
1.3 Meaning of Operations Research
1.4 Phases of O.R. Problem
1.5 Operations Research Modeling Approach
1.6 Defining the problem and Gathering Data
1.7 Formulating a Mathematical Model
1.8 Deriving Solutions from the Model
1.9 Introduction to Linear Programming
1.10 Formulation of Linear Programming Problem with Examples
1.11 Exercises/ Solutions
1.12 Summary
1.13 Further Readings

1.1 Introduction

Since the advent of the industrial revolution, the world has seen a remarkable growth in the size and complexity of organizations. The artisans' small shops of an earlier era have evolved into the billion-dollar corporations of today. An integral part of this revolutionary change has been a tremendous increase in the division of labor and segmentation of management responsibilities in these organizations. The results have been spectacular. However, along with its blessings, this increasing specialization has created new problems, problems that are still occurring in many components of an organization to grow into relatively autonomous empires with their own goals and value systems, thereby loosing sight of how their activities and objectives mesh with those of the overall organization. What is best for one component frequently is detrimental to another, so the components may end up working at cross purposes. A related problem is that as the complexity and specialization in an organization increase, it becomes more and more difficult to allocate the available resources to the various activities in a way that is most effective for the organization as a whole. These kinds of problems and the need to find a better way to solve them provided the environment for the emergence of operations research (commonly referred to as OR).

The roots of OR can be traced back many decades, when early attempts were made to use a scientific approach in the management of organizations. However the beginning of the activity called operations research has generally been attributed to the military services early in World War II. Because of the war effort, there was an urgent need to allocate scarce resources to the various military operations and to the activities within each operations in an effective manner. Therefore the British and then the U.S. military management called upon a large number of scientists to apply a scientific approach to dealing with this and other strategic and tratical problems. In effect, they were asked to do research on (military) operations. These teams of scientists were the first OR teams. By developing effective methods of using the new tool of radar, these teams were instrumental in winning the Air Battle of Britain. Through their research on how to better manage convoy and antisubmarine operations, they also played a major role in winning the Battle of the North Atlantic. Similar efforts assisted the Island Campaign in the Pacific.

When the war ended the success of OR in the war effort spurred interest in applying OR outside the military as well. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations were again coming to the forefront. It was becoming apparent to a growing number of people including business consultants who had served on or with the OR terms during the war, that these were basically the same problems that had been faced by the military but in a different context. By the early 1950s these individuals had introduced the use of OR to a variety of organizations in business, industry and government. The rapid spread of OR soon followed.

At least two other factors that played a key role in the rapid growth of OR during this period cab be identified. One was the substantial progress that was made early in improving the techniques of OR. After the war many of the scientists who had participated on OR teams or Who had heard about this work were motivated to pursue research relevant to the field; important advancements in the state of the art resulted. A prime example is the simplex method for solving linear programming problems, developed by George Dantzig in 1947. Many of the standard tools of OR such as linear programming , dynamic programming queuing theory and inventory theory, were relatively well developed before the end of the 1950s.

A second factor that gave impetus to the growth of the field was the onslaught of the computer revolution. A large amount of computation is usually required to deal most effectively with the complex problems typically considered by OR. Doing this by hand would often be out of question. Therefore the development of electronic digital computers, with their ability to perform arithmetic calculations thousands or even millions of times faster than a human being can was a tremendous boom to OR. A further boost came in the 1980s with the development of increasingly powerful personal computes accompanied by good software packages for doing OR. This brought the use of OR within the easy reach of much larger numbers of people. Today literally millions of individuals have ready access to OR software. Consequently, a whole range of computers from mainframes to laptops now are being routinely used to solve OR problems.

1.2 Objectives

After reading this unit you must be able to understand:

- Definition and scope of operations research
- Definition of general linear programming problems
- Formulation of problems of LPP
- Various examples of LPP occurring in various fields.

1.3 Meaning of Operation Research

As is name implies, operations research involve “research on operations.” Thus, operations research is applied to problems that concern how to conduct and coordinate the operations (i.e., the activities) within an organization. The nature of the organization is essentially immaterial, and in fact, OR has been applied extensively in such diverse area as manufacturing, transportation, construction, telecommunication, financial planning, health care the military and public services to name just a few. Therefore the breadth of application is unusually wide.

The research part of the name means that operations research uses an approach that resembles the way research is conducted in established scientific fields. To a considerable extent, the scientific method is used to investigate the problem of concern. (In fact, the term management science sometimes is used as a synonym for operations research). In particular, the process begins by carefully observing and formulating the problem, including gathering all relevant data. The next step is to construct a scientific (typically mathematical) model that attempts to abstract the essence of the real problem. It is then hypothesized that this model is a sufficiently precise representation of the essential features of the situation that the conclusion (solutions) obtained from the model are also valid for the real problem. Next suitable experiments are conducted to test this hypothesis modify it as needed and eventually verify some form hypothesis. (This step is frequently referred to as model validation). Thus, in a certain sense operations research involves creative scientific research into the fundamental properties of operations. However there is more to it than this. Specifically OR is also concerned with the practical management of the organization. Therefore, to be successful, OR must also provide positive, understandable conclusions to the decision maker(s) when they are needed.

Still another characteristic of OR is its broad viewpoint. As implied in the preceding section, OR adopts an organizational point of view. Thus, it attempts to resolve the conflicts of interest among the components of the organization in a way that is best for the organization as a whole. This does not imply that the study of each problem must give explicit consideration to all aspect of the organization; rather the objectives being sought must be consistent with those of the overall organization. An additional characteristic is that OR frequently attempts to find a best solution (referred to as an optimal solution) for the problem under consideration. Rather than simply improving the status quo, the goal is to indentify a best possible course of action. Although it must be interpreted carefully in terms of the practical needs of management, this “search for optimality” is an important them in OR.

All these characteristics lead quite naturally to still another one. It is evident that no single individual should he expected to be an expert on all the many aspects of OR work or the problems typically considered: this would require a group of individuals having diverse backgrounds and skills. Therefore, when a full-fledged OR study of a new problem is undertaken, it is usually necessary to use a team approach. Such an OR team typically needs to include individuals who collectively are highly trained in mathematics, statistics and probability theory, economics, business administration, computer science, engineering and the physical science, the behavioral science, and the special techniques of OR. The team also needs to have the necessary experience and variety of skills to give appropriate consideration to the many ramifications of the problem throughout the organization.

1.4 Phases of O.R. Problem

One way of summarizing the usual (overlapping) phases of an OR study is the following:

1. Define the problem of interest and gather relevant data.
2. Formulate a mathematical model to represent the problem.
3. Develop a computer based procedure for deriving solutions to the problem from the model.
4. Test the model and refine it as needed.
5. Prepare for the ongoing application of the model as prescribed by management.

6. Implement.

Each of these phases will be discussed in turn in the following sections.

Most of the award-winning OR studies introduced in Table 1.1 provide excellent examples of how to execute these phases well. We will intersperse snippets from these examples throughout the chapter with reference to invite your further reading.

Defining the Problem and Gathering Data:

Most practical problems encountered by OR teams are initially described to them in a vague, imprecise way. Therefore the first order of business is to study the relevant system and develop a well defined statement of the problem to be considered. This includes determining such things as the appropriate objectives, constraints on what can be done interrelationships between the area to be studied and other areas of the organization, possible alternative courses of action, time limits for making a decision and so on. This process of problem definition is a crucial one because it greatly affects how relevant the conclusions of the study will be. It is difficult to extract a "right" answer from the "wrong" problem!

1.5 Operations Research Modeling Approach

The first thing to recognize is that an OR team is normally working in an advisory capacity. The team members are not just given a problem and told to solve it, instead, they are advising management (often one key decision maker). The team performs a detailed technical analysis of the problem and then presents recommendations to management. Frequently, the report to management will indentify assumptions or over a different range of values of some policy parameter that can be evaluated only by evaluates the study and its recommendations, takes into account a variety of intangible factors, and makes the final decision based on its best judgment. Consequently, it is vital for the OR team to get on the same wavelength as management, including identifying the "right" problem from management's viewpoint and to build the support of management for the course that the study is taking.

Ascertaining the approaching objectives is a very important aspect of problem definition. To do this, it is necessary first to identify the member (or members) of management who actually will be making the decision concerning the system under study and then to probe into this individual's thinking regarding the pertinent objectives. (involving the decision maker from the outset also is essential to build her or his support for the implementation of the study.)

By its nature, OR is concerned with the welfare of the entire organization rather than that of only certain of its components. An OR study seeks solutions that are optimal for the overall organization rather than sub-optimal solutions that are best for only one component. Therefore, the objectives that are formulated ideally should be those of the entire organization. However this is not always convenient. Many problems primarily concern only a portion of the organization, so the analysis would become unwieldy if the stated objectives were too general and if explicit consideration were given used in the study should be as specific as they can be while still encompassing the main goals of he decision maker and maintaining a reasonable degree of consistency with the higher level objective of the organization.

For profit – making organizations one possible approach to circumventing the problem of optimization is to use long run profit maximization (considering the time value of money) as the sole objective. The adjective long-run indicates that this objective provides the flexibility to consider activities that do not translate into profits immediately (e.g., research and development projects) but need to do so eventually in order to be worthwhile. This approach has considerable merit. This objective is specific enough to be used conveniently, and yet it seems to be broad enough to encompass the basis goal of profit making organizations. In fact some people believe that all other legitimate objectives can be translated into this one.

However, in actual practice, many profit making organizations do not use this approach. A number of studies of U.S. corporations have found that management tends to adopt the goal of satisfactory profits, combined with other objectives, instead of focusing on long run profit maximization. Typically some of these other objectives might be to maintain stable profits, increase (or maintain) one's share of the market, provide for product diversification, maintain stable prices, improve worker morale, maintain family control of he business, and increase company prestige. Fulfilling these objectives might achieve long-run profit , maximization, but the relationship may be sufficiently obscure that it may not be convenient to incorporate them all into this one objective.

1.6 Defining the Problem and Gathering Data

Furthermore, there are additional consideration involving social responsibilities that are distinct from the profit motive. The five parties generally affected by a business firm located in a single country are (1) the owners (stockholders, etc) who desire profits (dividends, stock appreciation and so on); (2) the employees who desire steady employment at reasonable wages; (3) the customers, who desire a reliable product at a reasonable price

(4) the suppliers who desire integrity and a reasonable selling price for their goods; and (5) the government and hence the nation, which desire payment of fair taxes and consideration of the national interest. All five parties make essential contributions to the firm, and the firm should not be viewed as the exclusive servant of any one party for the exploitation of others. By the same token, international corporations acquire additional obligations to follow social responsible practice. Therefore while granting that management's prime responsibility is to make profits (which ultimately benefits all five parties), we note that its broader social responsibilities also must be recognized.

OR teams typically spend a surprisingly large amount of time gathering relevant data about the problem. Much data usually are needed both to gain an accurate understanding of the problem and to provide the needed input for the mathematical model being formulated in the next phase of study. Frequently much of he needed data will not be available when the study beings, either because the information never has been kept or because what was kept is outdated or in the wrong form. Therefore, it often is necessary to install a new computer based management information formation system to collect the collect the necessary data on an on going basis and in the needed form. The OR team normally needs to enlist the assistance of various other key individuals in the organization to track down all the vital data. Even with this effort, much of the data may be quite "soft," i.e. rough estimates based only on educated guesses. Typically an or team will spend considerable time trying to improve the precision of the data and then will make do with the best that can be obtained.

Example 1.1: An OR study done for the San Francisco Police Department resulted in the development of a computerized system for optimally scheduling and deploying police patrol officers. The new system provided annual savings of $ 11 million, an annual $3 million increase in traffic citation revenues, and a 20 percent improvement in response times. In assessing the appropriate objective for this study, three fundamental objectives were identified:

1. Maintain a high level of citizen safety.
2. Maintain a high level of officer morale.
3. Minimize the cost of operations.

To satisfy the first objective, the police department and city government jointly established a desired level of protection. The mathematical model then unclosed the requirement that this level of protection be achieved. Similarly the model imposed the requirement of balancing the workload equitably among officers in order to work toward the second objective. Finally, the third objective was incorporated by adopting the long-term goal of minimizing the number of officers needed to meet the first two objectives.

1.7 Formulating A Mathematical Model

After the decision maker's problem is defined the next phase is to reformulate this problem in a form that is convenient for analysis. The conventional OR approach for doing this is to construct a mathematical model that represents the essence of the problem. Before discussing how to formulate such a model we first explore the nature of models in general and of mathematical models in particular.

Models or idealized representations, are an integral part of every life. Common examples include model airplanes, portraits, global, and so on. Similarly, models play an important role in science and business as illustrated by models of the atom, models of genetic structure, mathematical equations describing physical laws of motion or chemical reactions, graphs, organizational charts and industrial accounting systems. Such models are invaluable for abstracting the essence of the subject of enquiry, showing interrelationships and facilitating analysis. Mathematical models are also idealized presentations, but they are expressed in terms of mathematical symbols and expressions. Such laws of physics as $F = m\,a$ and $E = m\,c^2$ are familiar examples. Similarly, the mathematical model of a business problem is the system of equations and related mathematical expressions that describe the essence of the problem. Thus if there are n related quantifiable decisions to be made, they are represented as decision variables (say $x_1, x_2, \ldots\ldots\ldots x$) whose respective values are to be determined. The appropriate measure of performance (e.g. profit) is then expressed as a mathematical function of these decision variables (for example, $P = 3x_1, 2x_2 + \ldots + 5x_r$). This function is called the objective function. Any restrictions on the values that can be assigned to these decision variables are also expressed mathematically, typically by means of inequalities or equations. Such mathematical expression for the restrictions often are called constraints. The constant (namely the coefficients and right hand sides) in the constraints and the objective function are called the parameter of the model. The mathematical model might then say that the problem is to choose the values of the decision variables so as to maximize the objective function, subject to the specified constraints. Such a model, and minor variations of it, typifies the models used in OR.

Determining the appropriate values to assign to the parameters of the model (one value per parameter) is both a critical and a challenging part of the model-building process.

Mathematical models have many advantages over a verbal description of the problem. One advantage is that a mathematical model describes a problem much more concisely. This tends to make the overall structure of the problem more comprehensible, and it helps to reveal important cause and effect relationship. In this way it indicates more clearly what additional data are relevant to the analysis. However, we have to be cautious in using mathematical models. Such a model is necessarily an abstract idealization of the problem, so approximations and simplifying assumptions generally are required if the model is to be tractable (capable of being solved). Therefore, can must be taken to ensure that the model remains a validity of a model is whether the model predicts the relative effects of the alternative courses of action with sufficient accuracy to permit a sound decision. Consequently, it is not necessary to include unimportant details or factors that have approximately the same effect for all the alternative courses of action considered. It is not even necessary that the absolute magnitude of the measure of performance be approximately correct for the various alternatives, provided that their relative values are sufficiently precise. Thus, all that is required is that there be a high correlation between the prediction by the model and what would actually happen in the real world. To ascertain whether this requirement is satisfied, it is important to do considerable testing and consequent modifying of the model.

In developing the model, a good approach is to begin with a very simple version and then move in evolutionary fashion toward more elaborate models that more nearly reflect the complexity of the real problem. This process of model enrichment continues only as long as the model remains tractable. The basic trade-off under constant consideration is between the precision and the tractability of the model. (see selected reference 6 for a detailed description of this process.)

A crucial step in formulating an OR model is the construction of the objective function. This requires developing a quantitative measure of performance relative to each of the decision maker's ultimate objectives that were identified while the problem was being defined. If there are multiple objective, their respective measures commonly are then transformed and combined into a composite measure, called the overall measure of performance. The objective function is then obtained by expressing this measure as a mathematical function of the decision variables. Alternatively, there also are methods for explicitly considering multiple objectives simultaneously.

Example 1.2: An OR study done for Monsanto Corp. was concerned with optimizing production operations in Monsanto's chemical plants to minimize the cost of meeting the target for the amount of a certain chemical product (malefic anhydride) to be produced in a given month. The decisions to be made are the dial setting for each of the catalytic reactors used to produce this product where the setting determines both the amount produced and the cost of operating the reactor. The form of the resulting mathematical model is as follows:

Choose the values of the decision variables R_{ij}

(I = 1, 2, ……………,r , j = 1,2, ……..,s) so as to minimize

$$\sum_{i=j}^{r}\sum_{j=1}^{s} c_{ij}R_{ij,}$$

Subject to

$$\sum_{i=j}^{r}\sum_{j=1}^{s} p_{ij}R_{ij} \geq T$$

$$\sum_{j=1}^{s} R_{ij} = 1, for\ i = 1,2, \ldots., r$$

Where R_{ij} = 1 is reactor i is operated at setting j
= o otherwise

c_{ij} = cost for rector i at setting j

p_{ij} = Production of reactor i at setting j

T = production target

r = number of reactors

s = number of setting (including of position)

The objective function for this model is $\sum_{i=j}^{r}\sum_{j=1}^{s} C_{ij}R_{ij,}$. The constraints are given in the three lines below the objective function. The parameters are c_{ij}, p_{ij} and T. For Monsanto's application, this model has over 1,000 decision variables $R_{ij,}$ (that is rs > 1,000). Its use led to annual savings of approximately $2 million.

1.8 Deriving Solutions from the Model

After a mathematical model is formulated for the problem under consideration, the next phase in an OR study is to develop a procedure (usually a computer-based procedure) for deriving solutions to the problem from this model. It is relatively simple step, in which one of the standard algorithms (systematic solution procedures) of OR is applied on a computer by using one of a number of readily available software packages or by using mathematical methods.

A common theme in OR is the search for an optimal, or best, solution. Indeed, many procedures have been developed, and are presented in this book, for finding such solutions for certain kinds of problems. However it needs to be recognized that these solutions are optimal only with respect to the model being used. Since the model necessarily is an idealized rather than an exact representation of the real problem, there cannot be any utopian guarantee that the optimal solution for the model will prove to be the best possible solution that could have been implemented for the real problem. There just are too many imponderables and uncertainties associated with real problems. However, if the model is well formulated and tested the resulting solution should tend to be a good approximation to an ideal course of action for the real problem. Therefore rather than be deluded into demanding the impossible, you should make the test of the practical success of an OR study hinge on whether it provides a better guide for action than can be obtained by other means.

Documenting the process used for model validation is important. This helps to increase confidence in the model for subsequent users. Furthermore, if concern arise in the future about the model this documentations will he helpful in diagnosing where problems may lie.

Preparing to Apply the Model

What happens after the testing phase has been completed and an acceptable model has been developed? If the model is to be used repeatedly, the next step is to install a well documented system stem for applying the model as prescribed by management. This system will include the model solution procedure (including post optimality analysis), and operating procedures for implementation. Then even as personal changes the system can be called on at regular intervals to provide a specific numerical solution.

Implementation

After a system is developed for applying the model the last phase of an OR study is to implement this system as prescribed by management. This phase is a critical one because it is here, and only here, that the benefits of the study are reaped. Therefore, it is important for the OR team to participate in launching this phase both to make sure that model solutions are accurately translated to an operating procedure and to rectify any flaws in the solutions that are then uncovered.

The success of the implementation phase depends a great deal upon the support of both top management and operating management. The OR team is much more likely to gain this support if it has kept management well informed and encouraged management's active guidance throughout the course of the study. Good communications help to ensure that the study accomplishes what management a greater sense of ownership of the study, which encourages their support for implementation.

The implementation phase involves several steps . First the OR team gives operating management a careful explanation of the new system to be adopted and how it relates to operating realities. Next these two parties share the responsibility for developing the procedures required to put this system into operation. Operating management then sees that a detailed indoctrination is given to the personnel involved, and the new course of action is initiated. If successful the new system may be used for years to come. With this in mind the OR team monitors the initial experience with the course of action taken and seeks to indentify any modifications that should be made in the future.

Throughout the entire period during which the new system is being used, it is important to continue to obtain feedback on how well the system is working and whether the assumptions of the model continue to be satisfied. When significant deviations from the original assumptions occur, the model should be revisited to determine if any modifications should be made in the system. The post optimality analysis done earlier can be helpful in guiding this review process.

Upon culmination of a study, it is appropriate for the OR team to document its methodology clearly and accurately enough so that the work is reproducible. Explicability should be part of the professional ethical code of the operations researcher. This condition is especially crucial when controversial public policy issues are being studied.

Conclusions

Although the remainder of this course focuses primarily on constructing and solving mathematical models, in this discussion we have tried to emphasize that this constitutes only a portion of the overall process involved in conducting a typical OR study. The other phases described here also are very important to the success of the study. Try to keep in perspective the role of the model and the solution procedure in the overall process as you move through the subsequent chapters. OR is closely intertwined with the use of computers. In the early years, these generally were mainframe computers.

But now personal computers and workstations are being widely used to solve OR models.

In concluding this discussion of the major phases of an OR study, it should be emphasized that there are many exceptions to the “rules” prescribed in this chapter. By its very nature, OR requires considerable ingenuity and innovation, so it is impossible to write down any standard procedure that should always be followed by OR teams. Rather the preceding description may be viewed as a model that roughly represents how successful OR studies are conducted.

1.9 Introduction to Linear Programming

The development of linear programming has been ranked among the most important scientific advantages of the mid 20^{th} century, and we must agree with this assessment. Its impact since just 1950 has been extraordinary. Today it is a standard tool that has saved many thousands or millions of dollars for most companies or businesses of even moderate size in the various industrialized countries of the world; and its use in other sectors of society has been spreading rapidly. A major proportion of all scientific computation on computers is devoted to the use of linear programming. The most of allocating limited resources among competing activities in a best possible (i.e., optimal) way. More precisely, this problem involves selecting the level of certain activities that completer for scare resources that are necessary to perform those activities. The choice of activities levels then dictates how much of each resource will be consumed by each activity. The variety of situations to which this description applies is diverse, indeed, ranging from the allocation of production facilities to products to the allocation of national resources to domestic needs, from portfolio selection to the selection of shipping patterns, from agricultural planning to the design of radiation therapy, and so on. However the one common ingredient in each of these situations is the necessity for allocating resources to activities by choosing the levels of those activities.

Linear programming uses a mathematical model to describe the problem of concern. The adjective linear means that all the mathematical function in this model are required to be linear functions. The word programming does not refer here to computer programming; rather it is essentially a synonym for planning. Thus linear programming involves the planning of activities to obtain an optimal result, i.e. result that reaches the specified goal best (according to the mathematical model) among all feasible alternatives.

Although allocating resources to activities is the most common type of application linear programming has numerous other important applications as well. In fact any problem whose mathematical model tits the very general format for the linear programming model is a linear programming problem. Furthermore, a remarkably efficient solution procedure, called the simplex method, is available for solving linear programming problems of even enormous size. These are some of the reasons for the tremendous impact of linear programming in recent decades.

1.10 Formulation of a Linear Programming Problem

In 1947, George Dantzig and his Associates, while working in the U.S. department of Air Force, observed that a large number of military programming and planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variables were restricted to values satisfying a system of linear constraints (a set of linear equations/ for inequalities). A linear form is meant a mathematical expression of the type $a_1x_1 + a_2x_2+\ldots\ldots+a_nx_n$ where $a_1, a_2, \ldots.., a_n$ are constants and $x_1, x_2,\ldots.x_n$ are variables. The term 'Programming' refers to the process of determining a particular programme or plan of action. So Linear Programming (L.P.) is one of the most important optimization (maximization/ minimization) techniques developed in the field of Operations Research (O.R.).

The methods applied for solving a linear programming problem are basically simple problems, a solution can be obtained by a set of simultaneous equations. However a unique solution for a set of simultaneous equations in n variables $(x_1, x_2, \ldots\ldots x_n)$, at least one of them is non-zero, can be obtained if there are exactly n relations. When the number of relations is greater than or less than n, a unique solution does not exist but a number of trial solutions can be found. In various practical situations, the problems seen in which the number of relations is not equal to the number of variables and many of the relations are in the form of inequalities ($\leq or \ \geq$) to maximize (or minimize) a linear function of the variables subject to such conditions. Such problems are known as Linear Programming Problems (LPP).

Definition: The general LPP calls for optimizing (maximizing/ minimizing) a linear function of variables called the 'Objective Function' subject to a set of linear equations and / or inequalities called the 'Constraints' or 'Restrictions'.

Now it becomes necessary to present a few interesting examples to explain the real-life situations where LP problems may arise. The outlines of formulation of the LP problems are explained with the help of these examples.

Example 1.3: A firm can produce three types of cloth say: A, B and C. Three kinds of wool are required for it, say: red, green and blue wool. One unit length of type A cloth needs 2 meters of red wool and 3 meters of blue wool; one unit length of type B cloth needs 3 meters of red wool, 2 meters of green wool and 2 meters of blue wool; and one unit to type C cloth needs 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 8 meters of red wool, 10 meters of green wool and 15 meters of blue wool. It is assumed that the income obtained from one unit length of type A cloth is Rs. 3.00, of type B cloth is Rs. 5.oo and type C cloth is Rs. 4.oo.

Determine how the firm should use the available material so as to maximize the income from the finished cloth.

Formulation: It is often convenient to construct the Table (1) after understanding the problem carefully.

Quality of Wool	Types of Cloth			Total quantity of wool available (in meters)
	A (x_1)	B (x_2)	C (x_3)	
Red	2	3	0	8
Green	0	2	5	10
Blue	3	2	4	15
Income per unit length of cloth	Rs. 3.00	Rs. 5.00	Rs. 4.00	

Table 1

Let x_1, x_2 and x_3 be the quality (in meters) produced of cloth type A,B,C respectively. Since 2 meters of red wool are required for each meter of cloth A and x_1 meters of this type of cloth are produced, so $2x_1$ meters of red wool will be required for cloth A.

Similarly, cloth B requires $3x_2$ meters of red wool and cloth C does not require red wool. Thus total quality of red wool becomes:

$$2x_1 + 3x_2 + 0x_3 \text{ (red wool)}$$

Following similar arguments for green and blue wool,

$$0x_1 + 2x_2 + + 5x_3 \text{ (green wool)}$$

$$3x_1 + 2x_2 + 4x_3 \text{ (blue wool)}$$

Since not more than 8 meters of red, 10 meters of green and 15 meters of blue wool are available the variables x1, x2, x3 must satisfy the following restrictions:

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10 \quad \ldots\ldots\ldots\ldots\ldots (1)$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

Also negative quantities cannot be produced. Hence x_1, x_2, x_3 must satisfy the non negativity restrictions:

$$X_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0 \qquad \ldots\ldots\ldots\ldots\ldots (2)$$

The total income from the finished cloth is given by

$$P = 3x_1 + 5x_2 + 4x_3 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Thus the problem now become to find x_1, x_2, x_3 satisfying the restrictions (1) and (2) and maximizing the profit function P.

Example 1.4: A firm manufactures 3 products A, B and C. The profits are Rs. 3 Rs. 2 and Rs.4 respectively. The firm has 2 machines and below is the required processing time in minutes for each machine on each product.

Machine G and H have 2,000 and 2,500 machine- minutes, respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but no more than 150A's.

Setup an L.P. problem to maximize profit. Do not solve it.

Formulation: Let x_1, x_2, x_3 be the number of products A, B and C respectively.

Since the profits are Rs. 3, Rs. 2 and Rs.4 respectively the total profit gained by the firm after selling these three products is given by $P= 3x_1 + 5x_2+ 4x_3$.

Now the total number of minutes required in producing these three products at machine G and H are given by

$$4x_1 + 3x_2+ 5x_3 \text{ and } 2x_1 + 2x_2+ 4x_3 \qquad \text{respectively}$$

But there are only 2,000 minutes available at machine G and 2,500 minutes at machine H, therefore the restrictions will be

$$4x_1 + 3x_2 + 5x_3 \leq 2{,}000 \text{ and } 2x_1 + 2x_2 + 4x_3 \leq 2{,}500.$$

Also, since the firm manufactures 100A's 200B's and 50 C's but not more than 150A's, therefore further restrictions become:

$$100 \leq x_1 \leq 150,\ 200 \leq x_2 \geq 0 \text{ and } 50 \leq x_3 \geq 0.$$

Hence the allocation problem of the firm can be finally put in the form:

Find the value of x_1, x_2, x_3 so as to maximize

$P = 3x_1 + 5x_2 + 4x_3.$

Subject to the constraints:

$4x_1 + 3x_2 + 5x_3 \leq 2{,}000$

$2x_1 + 2x_2 + 4x_3 \leq 2{,}500$

$100 \leq x_1 \leq 150,\ 200 \leq x_2 \geq 0, 50 \leq x_3 \geq 0.$

1.11 Exercises/ Solutions

P-1.1) Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce and B for 16-ounce bottles. However, each can be used on both types with some loss of efficiency. The following data is available.

Machine	8-ounce bottles	16-ounce bottles
A	100/ minute	40/ minute
B	60/ minute	75/ minute

The machine can be run 8-hour per day, 5 day per week, profit on 8-ounce bottle is 15 paise and on 16- ounce is 25 paise. Weekly production of the drink cannot exceed 30,00,000 ounces and the market can absorb 25,000 eight-ounce bottles and 7,000 sixteen-ounce bottles per week. The planner wishes to maximize his profit subject of course, to all the production and marketing restrictions, formulate this as linear programming problem.

Hint: Let x_1 units of 8-ounce bottle and x_2 units of 16-ounce bottle be produced. Total profit of production planner is given by $P = 0.15x_1 + 0.25x_2$.

Since machine A and B both work 8 hours per day and 5 days per week the total, working time for machine A and B will become 2400 minutes per week. Therefore, time conditions will be

$$\frac{x_1}{100} + \frac{x_2}{40} \leq 2400\ (for\ machine\ A), \frac{x_1}{100} + \frac{x_2}{40} \leq 400\ (for\ machine\ B)$$

Restriction of total weekly production will be $8x_1 + 16x_2 \leq 3{,}00{,}000$.

P-1.2) Formulate the following linear programming problem.

A used-car dealer wishes to stock up his lot to maximize his profit. He can select cars A, B and C which are valued wholesale at Rs. 5000, Rs. 7000 and Rs. 8000 respectively. These can be sold at Rs. 6000, 8500 and 10500 respectively.

For every two cars of B, he should buy one car of type A or C. If he has Rs. 1,00,000 to invest, what should be buy to maximize his expected gain.

Hint: Let x_1, x_2, x_3 number of cars be purchased of type A, B, C respectively. Gain per car for A, B, C will be Rs. (6000 - 50000). Rs. (85000 - 7000), Rs. (10500 - 8000) respectively. Therefore total expected gain will be

$$Z = 1000x_1 \times 0.7 + 1500x_2 \times 0.8 + 2500x_3 \times 0.6$$

Investment constraints will be given by

$$5000x_1 + 7000(2x_2) \leq 1{,}00{,}000\ and\ 7000\ (2x_2) + 8000x_3 \leq 1{,}00{,}000$$

1.12 Summary

Operation research is that field of study, which tells how to conduct and carry the operations or sequence of functions involved to achieve a target. It is a tool which helps in deciding what to do, when to do, how to do in a cost and time effective manner. Today O.R. has found its necessity in every field of science be it aviation, telecommunication, computer applications or manufacturing processes. Application of operation research starts' with formulation of a problem i.e. mathematical model for the problem. For this the conditions and information available to us is converted into mathematical models. Solution of these equations (models) provide the answer to the problem.

1.13 Further Readings

- Hadley, G. (1969). Linear Programming. Addison Welley, Reading, USA.
- Mustafi, C.K. (1988). Operations Research Methods and Practice. Wiley Eastern Limited, New Delhi.

Unit - 2 Graphical Method to Solve LPP

Structure

2.1 Introduction

2.2 Objectives

2.3 Graphical Solution to Linear Programming Problem

2.4 A Worked Example

2.5 Key Words

2.6 Examples/ Answers

2.7 Summary

2.8 Further Readings

2.1 Introduction

Let us begin with an example:

Example 2.1

A certain glass company produces high-quality glass products, including windows and glass doors. It has three plants. Aluminum frames and hardware are made in plant 1, wood frames are made in Plant 2, and Plant 3 produces the glass and assembles the products.

Because of declining earning top management has decide to restructure the company's product line. Unprofitable products are being discontinued, releasing production capacity to launch two new products having large sales potential:

Product 1: An 8-foot glass door with aluminum framing

Product 2: A 4 * 6 foot double hung wood framed window

Product 1 requires some of the production capacity in Plants 1 and 3 but none in Plant 2. Product 2 needs only Plants 2 and 3. The marketing division has concluded that the company could sell as much of either product as could be produced by these plants. However, because both products would be competing for the same production capacity in Plant 3, it is not clear which mix of the two products would be most profitable. Therefore, an OR team has been formed to study this question.

The OR team began by having discussion with upper management to identify management's objective for the study. These discussion led to developing the following definition of the problem: available in the three plants. (Each product will be produced in batches of 20, so the production rate is defined as the number of batches produced per week.) Any combination of production rates that satisfies these restrictions is permitted, including producing none of one product and as much as possible of the other.

The OR team also identified the data that needed to be gathered:

1. Number of hours of production time available per week in each plant for these products. (Most of the time in these plants already is committed to current production the available capacity for the new products is quite limited.)

2. Number of hours of production time used in each plant for each batch produced of each new product.

3. Profit per batch produce of each new product. (profit per batch produce was chosen as an appropriate measure after the team concluded that the incremental profit from each additional batch produced would be roughly constant regardless of the total number of batches produced. Because no substantial costs will be incurred to initiate the production and marketing of these new products the total profit from each one is approximately this profit per batch produced times the number of batches produced.)

Obtaining reasonable estimates of these quantities required enlisting the help of key personal in various units of the company. Staff in the manufacturing division provided the data in the first category above. Developing estimates for the second category of data required some analysis by the manufacturing engineers involved in designing the production process for the new products. By analyzing cost data from these same engineer-, and the marketing division along with a pricing decision from the marketing division the accounting department developed estimates for the third category.

Table A
Summarizes The Data Gathered

Plant	Production Time per Batch Hours Product		Production Time Available per week, Hours
	1	2	
1	1	0	4
2	0	2	12

3	3	2	18
Profit per batch	$ 3,000	$ 5,000	

The OR team immediately recognized that this was a linear programming problem of the classic product mix type, and the team next undertook the formulation of the corresponding mathematical model.

Formulation as a Linear Programming Problem:

To formulate the mathematical (linear programming) model for this problem, let

x_1 = number of batches of product I produced per week

x_2 = number of batches of products II produced per week

Z = total profit per week (in thousands of dollars) from producing these two product,

Thus x and x_2 are the decision variables for the model. Using the bottom row of Table A we obtain

$$Z= 3x_1 + 5x_2.$$

The objective is to choose the values of x_1 and x_2 so as to maximize $Z= 3x_1 + 5x_2$ subject to the restrictions imposed on their values by the limited production capacities available in the three plants. Table A indicates that each batch of product 1 produced per week uses I hour of production time per week in Plant 1, whereas only 4 hour per week are available. This restriction is expressed mathematically by the inequality $x_1 \leq 4$. Similarly plant 2 imposes the restriction that $2x_2 \leq 12$. The number of hours of production time used per week in Plant 3 by choosing x and x2 as the new products production rates would be $3x_1 + 2x_2$. Therefore, the mathematical statement of the plant 3 restriction is $3x_1 + 2x_2 \leq 18$. Finally, since production rates cannot be negative, it is necessary to restrict the decision variables to be nonnegative $x_1 \geq 0$ and $x_2 \geq 0$.

To summarize in the mathematical language of linear programming, the problem is to choose values of x, and x_2 so as to

Maximize $Z= 3x_1 + 5x_2,$

Subject to the restrictions $x_1 \leq 4$

$2x_2 \leq 12$

$3x_1 + 2x_2 \leq 18$

(Notice how the layout of the coefficient of x and x_2 in this linear programming model essentially duplicates the information summarized in Table A)

Graphical Solution

This very small problem has only two decision variables (x_1, x_2) and therefore only two dimensions, so a graphical procedure can be used to solve it. This procedure involves constructing a two dimensional graph with x and x_2 as the axes. The first step is to identify the values of (x_1, x_2) that are permitted by the restrictions. This is done by drawing each line that borders the range of permissible values for one restriction. To being, note that the no negativity restrictions $x_1 \geq 0$ and $x_2 \geq 0$ require (x_1, x_2) to lie on the positive side of the axes (including actually on either axis), i.e., in the first quadrant. Next, observe that the restriction $x_1 \leq 4$ means that (x_1, x_2) cannot lie to the right of the line $x_1 = 4$. These results are shown in Fig 2.1 where the shaded are contains the only value of (x_1, x_2) that are still allowed.

In the similar fashion the restriction $2x_2 \leq 12$ (or, equivalently, $x_2 \leq 6$) implies that the line $2x_2 = 12$ should be added to the boundary of the permissible region. The final restriction, $3x_1 + 2x_2 \leq 18$, requires plotting the points (x_1, x_2) such that $3x, + 2x_2 = 18$ (another line) to complete the boundary. (Note that the points such that $3x_1 + 2x_2 = 18$, so this is the limiting line above which points do not satisfy the inequality). The resulting region of permissible values of (x_1, x_2), called the feasible region, is shown in fig 2.2

The final step is to pick out the point in this feasible region that maximizes the values of $Z= 3x_1 + 5x_2$. To discover how to perform this step efficiently begin by trial and error. Try for example, $Z=10= 3x_1 + 5x_2$ to see if these are in the permissible region any values of (x_1, x_2) that yield a value of Z as large as 10. By drawing the line $3x_1 + 5x_2 =10$ (see Fig. 2.3) you can see that there are many points on this line that lie within the region. Having gained perspective by trying this arbitrarily chosen value of Z = 10, we should next try a larger arbitrary value of Z, say $Z = 20 = 3x_1 + 5x_2$. Fig. 2.3 reveals that a segment of the line $3x_1 + 5x_2 = 20$ lies within the region, so that the maximum permissible value of Z must be at least 20.

Now notice in Fig. 2.3 that the two lines just constructed are parallel. This is no coincidence, since a line constructed in this way has the form $Z = 3x_1 + 5x_2$ for the chosen values of Z, which implies that $5x_2 = 3x_1 + Z$ or equivalently,

$$x_2 = \left(-\frac{3}{5}\right)x_1 + \left(\frac{1}{5}\right)Z$$

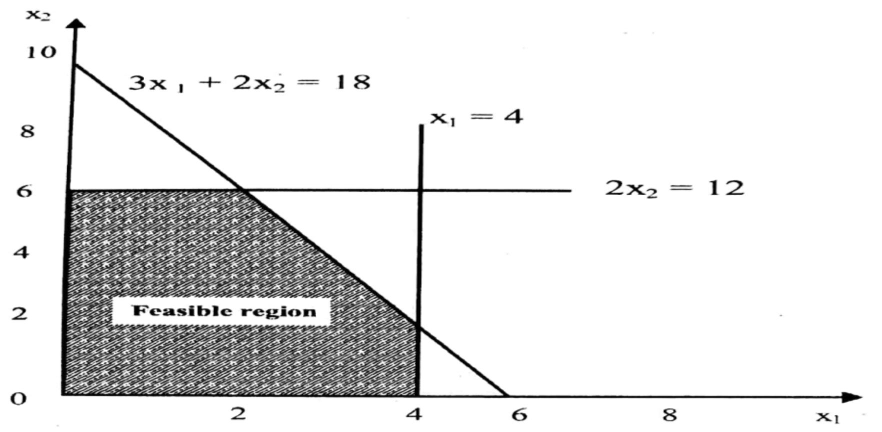
This last equation, called the slope – intercept form of the objective function, demonstrates that the slope of the line is -3/5, whereas the intercept of the line with the x_2 axis is 1/5 Z. The fact that the slope is fixed at -3/5 means that all lines constructed in this way are parallel.

Again, comparing the $10 = 3x_1 + 5x_2$ and $20 = 3x_1 + 5x_2$ lines in Fig. 2.3 we note that the line giving a larger value of Z (Z=20) is farther up and away from the origin than the other line (Z=10). This fact also is implied by the slope intercept form of the objective function, which indicates that the intercept with the x_1, axis increases when the value chosen for Z is increased.

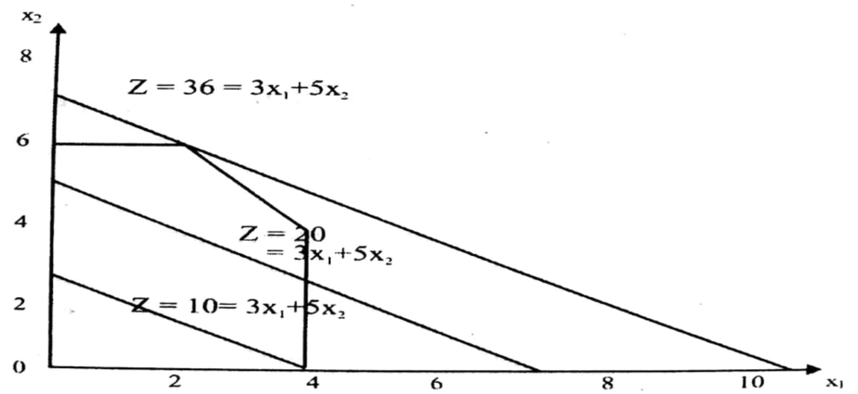
These observations imply that our trial-and- error procedure for constructing lines in Fig. 2.3 involves nothing more than drawing a family of parallel lines containing at least one point in the feasible region and selecting the line that corresponds to the largest value of Z. Figure 2.3 shows that this line passes through the point (2,6), indicating that the optimal solution is $x_1 = 2$ and $x_2=6$. The equation of this line is $3x_1 + 5x_2 = 3(2) + 5(6) = 36 = Z$, indicating that the optimal value of Z is Z = 36. The point (2,6) lies at the intersection of the two lines $2x_2 = 12$ and $3x_1 + 2x_2 = 18$ shown in Fig. 2.2, so that this point can be calculated algebraically as the simultaneous solution of these two equations.

Having seen the trial-and- error procedure for finding the optimal point (2,6) you now can streamline this approach for other problems. Rather than draw several parallel lines, it is sufficient to form a single line with a ruler to establish the slope. Then move the ruler with fixed slope through the feasible region in the direction of improving Z. (when the objective is to minimize Z, move the ruler in the direction that decreases Z.) Stop moving the ruler at the last instant that it still passes through a point in this region. This point is the desired optimal solution.

This procedure often is referred to as the graphical method for linear programming. It can be used to solve any linear programming problem with two decision variables. With considerable difficulty, it is possible to extend the method to three decision variables but not more than three.



(Fig. 2.2)



(Fig. 2.3)

Common Terminology for Linear Programming

Prototype Example	**General Problem**
Production capacities of plants	Resources
3 plants	m resources
Production of Products	Activities
2 products	n activities
Production rate of product j, x	Level of activities j, xi
Profit Z	Overall measure of performance Z

A feasible solution is a solution for which all the constraints are satisfied.

An infeasible solution is a solution for which at least one constraint is violated.

In the example, the points (2,3) and (4,1) in fig. 2 are feasible solutions, while the points(-1,3) and (4,4) are infeasible solutions.

The **feasible region** is the collection of all feasible solutions.

The feasible region in the example is the entire shaded area in Fig 2.2

It is possible for a problem to have **no feasible solutions.**

An **optimal solution** is a feasible solution that has the most favorable value if the objective function is to be maximized, whereas it is smallest value if the objective function is to be minimized.

Most problems will have just one optimal solution. However it is possible to have more than one.

Another possibility is that a problem has **no optimal solutions.** This occurs only if 1) it has no feasible solutions or (2) the constraints do not prevent improving the value of the objective function (Z) indefinitely in the favorable direction (positive or negative).

A corner-point feasible (CPF**) solution** that lies at a comer of the feasible region.

The details of these solution are discussed later.

2.2 Objectives

After reading this unit you must be able to understand:

- Problem of linear programming
- Feasible and basic feasible solutions
- Graphical method of solutions of LPP

Thus the graphical solution can be summarized as following:

2.3 Graphical Solution to Linear Programming Problem

Simple linear programming problems of decision variables can be easily solved by graphical method. The outlines of graphical procedure are as follows :

Step 1: Consider each in equality constant as equation.

Step2: Plot each on the graph as each one will geometrically represent a straight line.

Step 3: Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality- constraint corresponding to that line is '≤' then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '≥' sign, the region above the line in the first quadrant is shaded. The points lying in common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step 4: Choose the convenient value of Z (say =0) and plot the objective function line.

Step 5: Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6: Read the coordinates of the extreme point(s) selected in step (5), and find the maximum or minimum (as the case may be) value of z. The following examples will make the outlined graphical procedure clear.

2.4 A Worked-out Example

Example 2.2: Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs Re. 1 per week to feed. I have only Rs. 80 to spend for hens, how many of each kid should I buy to give a profit of more than Rs. 6 per week, assuming that I cannot house more than 20 hens.

Solution: ***Formulation as L.P.P.:*** Let x_1 be the number of old hens and x_2 the number of young hens to be bought.

Since old lay 3 eggs per week and young ones lay 5 eggs per week, the total number of eggs obtained per week will be = $3x_1 \times 5x_2$.

Consequently the cost of each eggs being 30 paise, the total gain will be = Rs. $0.30\ (3x_1 + 5x_2)$.

Total expenditure for feeding (x_1+x_2) hens at the rate of Re. 1 each will be Rs. 1 (x_1+x_2).

Thus total profit z earned per week will be z = total gain – total expenditure

Or $z= 0.30 (3x_1+5x_2)- (x_1+x_2)$ or $z = 0.50x_2- 0.10x_1$ (objective)

Since old hens can be bought at Rs. 2 each and young ones at Rs. 5 each and there only Rs. 80 available for purchasing hens, the constraint is: $2x_1 + 5x_2 \leq 80$.

Also, since it is not possible to house more than 20 hens at a time $x_1+ x_2 \leq 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function z is to be maximized. Thus there is no need to add one more constraint, i.e. $0.5x_2 - 0.1x_1 \geq 6$.

Again it is not possible to purchase negative quality of hens therefore $x_1 \geq 0$, $x_2 \geq 0$ (see fig 2.1).

Finally, the problem becomes:

Find x_1 and x_2 (real numbers) so as to maximize the profit function

$Z= 0.50x_2 – 0.10x_1$

Subject to the constraints:

$2x_1 + 5x_2 \leq 80$, $x_1+x_2 \leq 20$, and $x_1, x_2 \geq 0$.

Graphical Solution: Plot the straight lines $2x_1 + 5x_2 = 80$, $x_1+x_2 = 20$ on the graph and shade the feasible region (in squares) as shown in the figure (Fig. 2.4).

The feasible region is OBEC. The coordinates of the extreme points of the feasible region are:

O = (0,0), C = (20, 0), B = (0, 16),

E = (20/3, 40/3)

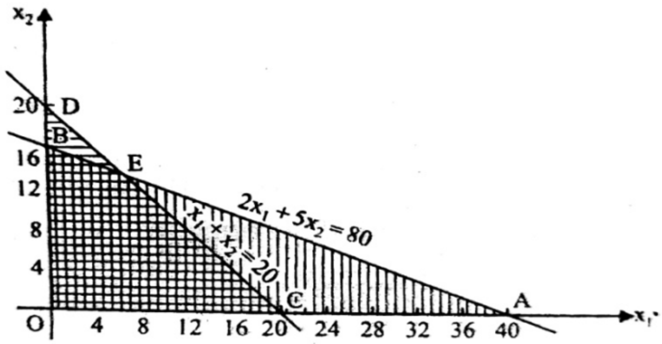


Fig. 2.4

The value of z at these vertices are:

$Z_o = 0$, $Z_c = 0.50 \times 0 - 0.10 \times 20 = -2$,

$Z_B = 0.50 \times 16 - 0.10 \times 0 = 8$,

Z_i; $= 0.50 \times 40/3 - 01.0 \times 20/3 = 6$.

Since the maximum value of z is Rs. 8 which occurs at the point B = (0,16) the solution to the given problem is $x_1 = 0$, $x_2 = 16$, max. Z = Rs. 8

Hence only 16 young hens I should buy in order to get maximum profit of Rs. 8 (which is > 6).

2.5 Key Words

1. **Solution to LPP:** Any set $x = (x_1, x_2, \ldots x_{n+m})$ of variables is called a solution to LP problem, if it satisfies a set of constraints.

2. **Feasible Solution (FS):** Any set $x = (x_1, x_2, \ldots x_{n+m})$ of variables is called a feasible solution of LP problem, if it satisfies a set of constraints and non- negativity restriction also.

3. **Basic Solution (BS):** A basic solution to the set of constraints is a solution obtained by setting any n variables (among m + 11 variables) equal to zero and solving for remaining m variables, provided the determinant of the coefficients of these m variables is non-zero.

The number of basic solution thus obtained will be at the most $^{m+n}C_m = \frac{(m+n)!}{n!m!}$, which is the number of combinations of n+m things taken m at a time.

4. **Basic Feasible Solution (BFS):** A basic feasible solution is a basic solution which also satisfies the non-negativity restrictions, that is all basic variables are non-negative.

5. **Optimal Basic Feasible Solution (BFS):** A basic feasible solution is said to be optimum, if it also optimizes (maximizes or minimizes) the objective function.

6. **Unbounded Solution:** If the value of the objective function Z can be increased or decreased indefinitely, such solutions are called unbounded solutions.

2.6 Examples/ Answers

P-2.1 Maximize $z= 2x_1 +3x_2$; s.t. $x_1 + x_2 \leq 1$, $3x_1 +x_2 \leq 4$: x_1, $x_2 \geq 0$.

[Hint: Vertical of the feasible region are: (0, 0), (1,0), (0,1)]

P-2.3 Maximize $z= 5x_1 +7x_2$; s.t. $x_1 + x_2 \leq 4$, $3x_1 +8x_2 \leq 24$, $10x_1 +7x_2 \leq 35$: x_1, $x_2 \geq 0$.

[**Hint**: Vertical of the feasible region are: (0, 0), (7/2 ,0), (7/3, 5/3 (8/5, 12/5) and(0, 3)]

[**Here**, $x_1= 8/5$, $x_2= 12/5$, max $z= 124/5$]

P-2.3 Maximize $z = 3x_1 +4x_2$; s.t. $x_1 - x_2 \leq -1$, $-x_1 +x_2 \leq 0$: x_1, $x_2 \geq 0$.

[The problem has no solution]

2.7 Summary

Geometric properties of LP problems, which are observed while solving them graphically, are summarized below:

1. The region of feasible solutions has an important property which is called the convexity property in geometry, provided the feasible solution of the problem exists.
2. The boundaries of the regions are lines or planes.
3. There are corners or extreme points on the boundary, and there are edges joining various corners.

4. The objective function can be represented by a line or a plane for any fixed value of z.

5. At least one corner of the region of feasible solutions will be an optimal solution whenever the maximum or minimum value of z is finite.

6. If the optimal solution is not unique, there are points other than corners that are optimal but in any case at least one corner is optimal.

7. The different situations are fount when the objective function can be made arbitrarily large. Of course, no corner is optimal in that case.

2.8 Further Readings

- Hadley, G. (1969). Linear Programming. Addison Welley, Reading, USA.
- Mustafi, C.K. (1988). Operations Research Methods and Practice. Wiley Eastern Limited, New Delhi.

U.P. Rajarshi Tandon Open
University, Prayagraj

DECSTAT – 109
Operation Research

Block: 2 Simplex Method of Solving LPP

Unit – 3 : Simplex Method 42 - 70

Unit – 4 : Duality Problem in LPP 71 - 78

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics, D. D. U. Gorakhpur University, Gorakhpur.

Dr. Shruti
Member-Secretary
School of Sciences, U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Shubra Katara **Writer**
Department of Statistics, Bareily College, Bareily

Prof. V. K. Sehgal **Reviewer**
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Prof. R. C. Yadava **Editor**
Department of Statistics, Banaras Hindu University, Varanasi

Dr. Shruti **Course/ SLM Coordinator**
School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109 **OPERATION RESEARCH**
©UPRTOU
First Edition: *March 2008* (Published with the support of the Distance Education Council, New Delhi)
Second Edition: ***July 2021***
ISBN :

Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2021.
Printed By: K.C.Printing & Allied Works, Panchwati, Mathura -281003.

Block & Units Introduction

The ***Block - 2 – Simplex Method of Solving LPP,*** of operation research deals with Simplex Method of solving LPP and the Principles of Duality in the LPP. It consists of two units.

Unit – 3 – Simplex Method describes the simplex method of solving LPP with illustrations.

Unit – 4 – Duality Problem in LPP, gives the principle of duality in LPP along with simple problems based on duality theorem.

At the end of every block/unit the summary, self assessment questions and further readings are given.

Unit-3 Simplex Method

Structure

3.1 Introduction
3.2 Objectives
3.3 Principle of simplex method
3.4 Computations aspect of simplex method
3.5 Simplex method with several decision variables
3.6 Two phase and M-method
3.7 Multiple, unbounded solutions and infeasible problems
3.8 Sensitivity Analysis
3.9 Key Words
3.10 Summary
3.11 Further Readings

3.1 Introduction

Graphical solution to LPP problem is suitable for two variables only. In case our study consist of more number of variables, we will apply a more general method, called simplex analysis to solve our problem. The simplex method provides an algorithm, which is based on fundamental theorem of linear programming.

As we have seen earlier, the graphical method gives the solution in the form of a feasible region. Since the feasible region consists of infinite number of points, i.e. solutions, its hard to obtain an optimum solution. Even though the number of basic feasible solutions is limited, a great effort is required is finding all basic feasible solutions and select the one which optimizes the objective. In such a situations, we find that Simplex Method is very suitable for solving linear programming problems. The method through an iterative progress progressively approaches and ultimately reaches to the maximum or minimum value of the objective function. The method also helps the decision maker to identify the redundant constraints an unbounded solutions, multiple solution and an infeasible problem.

3.2 Objectives

After reading this chapter you must be able to understand:

- Simplex method of solving a LPP

- Simplex computation
- Two phase and M-method of computation
- Sensitivity analysis

3.3 Principle of Simplex Method

We explain the principle of the Simplex method with the help of the two variables linear programming problem.

Example 1.1

Maximize $50x_1 + 60x_2$

Subject to:

$$2x_1 + x_2 \leq 300$$

$$3x_1 + 4x_2 \leq 509$$

$$4x_1 + 7x_2 \leq 812$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution:

We introduce new variables $x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$

So that the constraints become equations

$$2x_1 + x_2 + x_3 = 300$$

$$3x_1 + 4x_2 + x_4 = 509$$

$$4x_1 + 7x_2 + x_5 = 812 \ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

The variables x_3, x_4, x_5 are known as slack variables corresponding to the three constraints. The system of equations now has five variables (including the slack variables) and three equations.

Basic Solution:

In the system of equations as presented above we may equate any two variables to zero. The system then consists of three equations with three variables. If this system of three equations with three variables is solvable such a solution is known as a basic solution.

In the example considered above suppose we take $x_1 = 0$, $x_2 = 0$. The solution of the system with remaining three variables is $x_3 = 300$, $x_4 = 509$, $x_5 = 812$. This is a basic solution of the system. The variables x_3, x_4 and x_5 are known as basic variables while the variables x_1, x_2 are known as non basic variables (variables which are equated to zero).

In the general case, if the number of constraints of the linear programming problem is m and number of variables (including the slack variables) is n then there are at most ${}^nC_{n-m} = {}^nC_m$ basic solutions.

Basic Feasible Solution

A basic solution of a linear programming problem is a basic feasible solution if it is feasible, i.e. all the variables are non-negative. The solution $x_3 = 300$, $x_4 = 509$, $x_5 = 812$ is a basic feasible solution of the problem. Again, if the number of constraints is m and the number of variables (including the slack variables) is n the maximum number of basic feasible solution is ${}^nC_{n-m} = {}^nC_m$.

The following result (Hadley, 1969) will help you to identify the extreme points of the convex set of feasible solution analytically.

Every basic feasible solution of the problem is an extreme point of the convex set of feasible solutions and every point is a basic feasible solution of the set of constraints.

When several variables are present in a linear programming problem it is not possible to identify the extreme points geometrically. But we can identify them through the basic feasible solutions. Since of the basic feasible solution will maximize or minimize the objective function, we can carry out this search starting from one basic feasible solution to another. The simplex method provides a systematic search so that the objective function increases (in the case of maximization) progressively until the basic feasible solution has been identified where the objective function is maximized. The computational aspect of the simplex method is presented in the next section.

3.4 Computations Aspect of Simplex Method

Basic Solutions and Basic Variables

Let $\underset{m\times n}{A}\ \underset{n\times 1}{x} = \underset{m\times n}{b}$ $(m \leq n)$ be a system of non-homogeneous linear equations and rank of matrix A is m. Choose a non-singular sub matrix B of order m x n (i.e. |B| = determinant B $\neq 0$) in A. Then variables whose columns are included in B are called basic variables and remaining (n-m) variables are known as non basic variables. Solution of A $\underline{x}$ = $\underline{b}$ obtained by putting non-basic variables equal to zero and solving the resulting system of equations in A x = b is known as basic solution. One may have more than one basic solution in a problem.

In a basic solution all basic variable are non-zero, then it is known as non degenerate basic solution otherwise a degenerate basic solution.

The problem given by Ex. 1.1 given by (1.1) will be

$A\,\underline{x} = \underline{b}$

Or

$$\begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 4 & 7 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 300 \\ 609 \\ 812 \end{pmatrix}$$

Where the rank of A is p(A) = 3. We have taken initially B as

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; and \; B\,\underline{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

When rank of B is p(B) = 3.

We again consider the linear programming problem

Maximize $50x_1 + 60x_2$

Subject to:

$$2x_1 + x_2 + x_3 = 300$$

$$3x_1 + 4x_2 + x_4 = 509$$

$$4x_1 + 7x_2 + x_5 = 812$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

The slack variables provide a basic feasible solution to start the simplex computation. This is also known as initial basic feasible solution. If z denote the profit the z = 0 corresponding to this basic feasible solution. We denote by C_B the coefficient of the basic variables in the objective function and X_B the numerical values of the basic variables. The numerical values of the basic variables are $X_{B1} = 300$, $X_{B2} = 509$, $X_{B3} = 812$. The profit $z = 50x_1 + 60x_2$ can be also expressed as $z - 50x_1 - 60x_2 = 0$. The computation starts with the first simplex table as indicated below:

Table 1

C_B	Basic	C_j	50	60	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5
0	x_3	300	2	1	1	0	0
0	x_4	509	3	4	0	1	0
0	x_5	812	4	7	0	0	1
	$Z_j - C_j$		-50	-60	0	0	0

The coefficients of the basic variables in the objective function are $C_{B1} = C_{B2} = C_{B3} = 0$. The topmost row of Table 1 indicates the coefficient of the variables x_1, x_2, x_3, x_4 *and* x_5 in the objective function respectively. The column under x_1 present the coefficient of x_1 in the three equations. The remaining columns have also been formed in similar manner.

On examining the profit equations $z = 50x_1 + 60x_2$ you may observe that if either x_1 or x_2 which is currently non basic is included as a basic variable the profit will increase. Since the coefficient of x_2 is numerically higher we choose x_2 to be included as a basic variable in the next iteration. An equivalent criterion of choosing a new basic variable can be obtained from the last row of table 1 (corresponding to z). Since the entry corresponding to x_2 is smaller between the two negative values x_2 will be included as a basic variable in the next iteration. However with three constraints there can only be three basic variables. Thus by making x_2, a basic variable one of the existing basic variables will become non- basic. You may identify this variable using the following line of argument.

Form the first equation

$$2x_1 + x_3 = 300 - x_2$$

But $x_1 = 0$. Hence in order that $x_3 \geq 0$

$$300 - x_2 \geq 0 \text{ i.e. } x_2 \leq 300$$

Similar computation form the second and the third equation lead to

$$x_2 \le \frac{509}{4}, \qquad x_2 \le \frac{812}{7} = 116$$

$$Thus\ x_2 = Min\left(\frac{300}{1}, \frac{509}{4}, \frac{812}{7}\right) = 116$$

If $x_2 = 116$, from the third equations you may observe that

$7x_2 + x_5 = 812$ i.e. $x_5 = 0$

Thus the variable x_5 becomes non-basic in the next iteration. The revised values of the other two basic variables are

$x_3 = 300 - x_2 = 184$

$x_4 = 509 - 4 \times 116 = 45$

Referring back to Table 1, we obtain the elements of the next Table (Table 2) using the following rules:

1) In the z row we locate the quantities, which are negative. If all the quantities are positive, the inclusion of any non-basic variable will not increase the value of the objective function. Hence the present solution maximizes the objective function. If there are more than one negative values we choose the variable as a basic variable corresponding to which the z value is least as this is likely to increase the profit most.

2) Let x_j be the incoming basic variable and the corresponding elements of the j^{th} column be denoted by y_{1j}, y_{2j} and y_{3j}. if the present values of basic variables are x_{B1}, x_{B2} and x_{B3} respectively, then we compute $\min\left[\frac{x_{B1}}{y_{1j}}, \frac{x_{B2}}{y_{2j}}, \frac{x_{B3}}{y_{3j}}\right]$ for $y_{ij} > 0$, $y_{2j} > 0$, $y_{3j} > 0$. You may note that if any $y_{ij} \geq 0$, this need not be included in the comparison. If the minimum occurs corresponding to $\frac{x_{Br}}{y_{rj}}$ then the r^{th} variable win become non basic in the next iteration.

3) Table 2 is computed from Table 1 using the following rules.

a) The revised basic variables are x_3, x_4 and x_2. Accordingly, we make $C_{B1} = 0$, $C_{B2} = 0$, $C_{B3} = 60$.

b) As x_2 is the incoming basic variable we make the coefficient of x_2 one by dividing each element of row 3 by 7. Thus the numerical value of the element corresponding to x_1 is 4/7, corresponding to x_5 is 1/7 in Table 2.

c) The incoming basic variable should appear only in the third row. So we multiply the third row to Table 2 by 1 and subtract it from the first row of

Table 1 element by element. Thus the element corresponding to x_2 in the first row of Table 2 is zero. The element corresponding to x_1 is

$$2 - 1 \times \frac{1}{7} = \frac{10}{7}$$

$$\textit{the element correspond to } x_5 \textit{ is}$$

$$0 - 1 \times \frac{1}{7} = -\frac{1}{7}$$

In this way we obtain the elements of the first and the second row in Table 2. The numerical values of the basic variables in Table 2 can also be computed in a similar manner.

Let C_{B1}, C_{B2} and C_{B3} be the coefficient of the basic variables in the objective. Function. For example in Table 2 $C_{B1} = 0$, $C_{B2} = 0$, $C_{B3} = 60$. Suppose corresponding to a variable j, the quantity zj is defined as $z_j = C_{B1}$, $Y_1 + C_{B2}$, $Y_{2j} + C_{B3}\ y_{3j}$. Then the final row (z-row) can also expressed as $z_j - c_j$. For example

$$z_i - c_1 = \frac{10}{7} \times \frac{5}{7} \times 0 + 60 \times \frac{4}{7} - 50 = -\frac{100}{7}$$

$$z_5 - c_5 = -\frac{1}{7} \times 0 - \frac{4}{7} \times 0 + \frac{1}{7} \times 60 - 0 = \frac{60}{7}$$

1) We now apply rule 1 to Table 2. The only negative $z_j - c_j$ is $z_1 - c_1 = -100/7$. Hence x_1 should be made a basic variable at the next iteration.
2) We compute the minimum of the ratios

$$Min\left[\frac{184}{10}, \frac{45}{5}, \frac{116}{4}\right]$$

$$= Min\left[\frac{644}{5}, 63, 203\right] = 63$$

Since this minimum occurs corresponding to x_4 it becomes a non basic variables in next iteration.

3) Table 3 is computed from Table 2 using the rules (a), (b) and (c) as described before

Table 2

C_B	Basic	C_1	50	60	0	0	0
	Variables	X_B	X_1	X_2	X_3	X_4	X_5

0	X_3	184	10/7	0	1	0	-1/7
0	X_4	45	5/7	0	0	1	-4/7
60	X_5	116	4/7	1	0	0	1/7
	z_j-c_j		-100/7	0	0	0	-60/7

Table 3

C_B	Basic	C_1	50	60	0	0	0
	Variables	X_B	X_1	X_2	X_3	X_4	X_5
0	X_3	94	0	0	1	-2	1
50	X_1	63	1	0	0	7/5	-4/5
60	X_2	80	0	1	0	-4/5	3/6
	z_j-c_j		0	0	0	22	-4

1) Z_5-c_5 < 0 Hence x_5 should be made a basic variable in the next iteration.
2) We compute the minimum of the ratios.

$$Min\left[\frac{94}{1},\frac{80}{3/5}\right] = 94$$

Note that since y_{25} <o, the corresponding ratio is not taken for comparison. The variable x3 becomes non basic at the next iteration.

3) Table 4 is computed from Table 3 following the usual steps.

Table 4

C_B	Basic	C_1	50	60	0	0	0
	Variables	X_B	X_1	X_2	X_3	X_4	X_5
0	X_5	94	0	0	1	-2	1
50	X_1	691/5	1	0	4/5	-1/5	0
60	X_2	118/5	0	1	-3/5	2/5	0
	z_j-c_j		0	0	4	14	0

Since $z_j - c_j \geq 0$ for all j, the objective function cannot be improved any further. Hence the objective function is maximized for $x_1 = \frac{691}{5}\ and\ x_2 = \frac{118}{5}$. The maximum value of the objective function is 8326.

3.5 Simplex Method with Several Decision Variables

Computational procedure explained in the previous section can be readily extended to linear to linear programming problems with more than two decision variables. This is illustrated with the help of the following example.

Example 1.2: The products A, B and C are produced in three machine centers X, Y and Z. Each product involves operation of each of the machine centers. The time required for each operations for unit amount of each product is given below. 100, 77 and 80 hours are available at machine centers X, Y and Z respectively. the profit per unit of A, B and C is Rs. 3 and Rs. 1 respectively.

Products	Machine Centres			Profit per unit
	X	Y	Z	
A	10	7	2	Rs. 12

B	2	3	4	Rs. 3
C	1	2	1	Rs. 1
Available hours	100	77	80	

Find out a suitable product mix so as to maximize the profit.

Solution:

The linear programming formulation of the product mix problem is an follows:

Maximize $12x_1 + 3x_2 + x_3$

Subject to:

$$10x_1 + 2x_2 + x_3 \leq 100$$

$$7x_1 + 3x_2 + 2x_3 \leq 77$$

$$2x_1 + 4x_2 + x_3 \leq 80$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

We introduce slack variables x_4, x_5 and x_6 to make inequalities equations. Thus the problem can be stated as Maximize $12x_1 + 3x_2 + x_3$

Subject to:

$$10x_1 + 2x_2 + x_3 + x_4 \leq 100$$

$$7x_1 + 3x_2 + 2x_3 + x_5 \leq 77$$

$$2x_1 + 4x_2 + x_3 + x_6 \leq 80$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$$

The first simplex Table can be obtained in a straight forward manner from the equations. We observe that the basic variables are x_4, x_5 and x_6. Therefore, $C_{B1} = C_{B2} = C_{B3} = 0$.

Table 1

C_B	Basic	C_j	12	3	1	0	0	0

	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	100	10	2	1	1	0	0
0	x_5	77	7	3	2	0	1	0
0	x_6	80	2	4	1	0	0	0
	$Z_j - C_j$		-12	-3	-1	0	0	0

1) Z_j-C_j = -12 is the smallest negative value. Hence x_1 should be made a basic variable at the next iteration.
2) We compute the minimum of the ratios

$$Min\left[\frac{100}{10},\frac{77}{7},\frac{80}{2}\right] = 10$$

The variable x_4 corresponding to which minimum occurs is made a non-basic variables

3) Table 2 is computed from Table 1 using the following rules.
   a) The revised basic variables are x_1, x_5 and x_6. Accordingly, we make $C_{B1} = 0$, $C_{B2} = 0$, $C_{B3} = 0$.
   b) As x_1 is the incoming basic variable we make the coefficient of x_1 one by dividing each element of row 1 by 10. Thus the numerical value of the element corresponding to x_2 is 2/10, corresponding to x_3 is 1/10 and so on in Table 2.
   c) The incoming basic variable should appear only in the first row. So we multiply the first row to table 2 by 7 and subtract it from the second row of Table 1 element by element. Thus the element corresponding to x_1 in the first row of Table 2 is zero. The element corresponding to x_2 is

   $$3 - 7 \times \frac{2}{10} = \frac{7}{10}$$

In this way we obtain the elements of the second and third row in Table 2. The computation of the numerical values of the basic variables in Table is made in a similar manner.

Table 2

C_B	Basic	C_j	12	3	1	0	0	0

	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
12	x_1	10	1	1/5	1/10	1/10	0	0
0	x_5	7	0	16/10	13/10	- 7/10	1	0
0	x_6	60	0	18/5	-1/5	-1/5	0	1
	$Z_j - C_j$		0	-3/5	1/5	6/5	0	0

1) $Z_2 - C_2 = -\frac{3}{5}$ Hence x_2 should be made a basic variable at the next iteration.
2) We compute minimum of the ratios

$$Min\left[\frac{10}{1}, \frac{7}{16}, \frac{60}{18}\right]$$

$$= Min\left[50, \frac{70}{16}, \frac{300}{18}\right] = \frac{70}{16}$$

Hence the variables x_5 will be a non variable in the next iteration.

3) Table 3 computed from Table following the lines indicated in a, b and c.

Table 3

C_B	Basic	C_j	12	3	1	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	x_2	35/8	0	1	13/16	-7/16	5/8	0
0	x_6	177/4	0	0	-17/8	11/8	-9/4	1
	$Z_j - C_j$		0	0	11/16	15/16	3/8	0

As all $z_j - c_j \geq 0$ the present solution $x_1 = \frac{73}{8}, x_2 = \frac{35}{8}\ and\ x_6 = \frac{177}{4}$ maximizes the value of the objective function. The maximum value of the objective function is $12 \times \frac{73}{8} + 3 \times \frac{35}{8} = \frac{981}{8}$.

3.6 Two Phase and M-Method

The simplex method illustrated in the last two sections was applied to linear programming problem with less than or equal to type constraints. As a result we could introduce slack variables, which provided an initial basic feasible solution of the problems may also be characterized by the presence of both less than or equal to type or greater than of equal to type constraints. It may also contain some equations. Thus it is not always possible to obtain an initial basic feasible solution using slack variables.

Two methods are available to solve linear programming by simplex method in such cases. These methods will be explained with the help of numerical examples.

Two Phase Method

We illustrate the two phase method with the help of the problem presented in activity 5 of Unit 3.

Example 1.3

$$Minimize\ 12.5x_1 + 14.5x_2$$

Subject to:

$$x_1 + x_2 \geq 2000$$

$$0.4x_1 + 0.75x_2 \geq 1000$$

$$0.075x_1 + 0.1x_2 \geq 200$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution

Although the objective function $12.5x_1 + 14.5x_2$ is to be minimized, the values of x_1 and x_2 which minimized this objective function are also the values which maximize the revised objective function $-12.5x_1 - 14.5x_2$.

The second and the constraint are multiplied by 100 and 1000 respectively for computational convenience. Thus the linear programming problem can be expressed as

Minimize $-12.5x_1 - 14.5x_2$.

Subject to:

$$x_1 + x_2 \geq 2000$$

$$40x_1 + 75x_2 \geq 100000$$

$$75x_1 + 100x_2 \geq 200000$$

$$x_1 \geq 0, x_2 \geq 0$$

We convert the first two inequalities by introduction surplus variables x_3 and x_4 respectively. The third constraints is changed into an equation by introducing a slack variable x_5. Thus linear programming problem can be expressed as

Minimize $-12.5x_1 - 14.5x_2 = -\frac{25}{2}x_1 - \frac{29}{2}x_2$

Subject to:

$$x_1 + x_2 - x_3 \geq 2000$$

$$40x_1 + 75x_2 - x_4 \geq 100000$$

$$75x_1 + 100x_2 + x_5 \geq 200000$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

Although surplus variables can convert greater than or equal to type constraints into equations they are unable to provide initial basic variables to start the simplex computation. We introduce two additional variables x_6 and x_7 known as artificial variables to facilitate the computation of an initial basic solution. The computation is carried out in two phases.

Phase I

In this phase we consider the following linear programming problem

Maximize $- x_6 - x_7$

Subject to:

$$x_1 + x_2 - x_3 + x_6 \geq 2000$$

$$40x_1 + 75x_2 - x_4 + x_7 \geq 100000$$

$$75x_1 + 100x_2 + x_5 \geq 200000$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_6 \geq 0, x_7 \geq 0$$

An initial basic feasible solution of the problem is given by $x_6 = 2000$, $x_7 = 100000$, $x_5 = 200000$. As the minimum value of the Phase I objective function is zero at the end of the Phase I computation both x_6 and x_7 become zero.

Phase II

The basic feasible solution at the end of Phase I computation is used as the initial basic feasible of the problem. The original objective function is introduced in Phase II computation and the usual simplex procedure is used to solve the problem.

Phase I Computation

Table 1

C_B	Basic	C_j	0	0	0	0	0	-1	-1
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-1	x_6	2000	1	1	-1	0	0	1	0
-1	x_7	100000	40	75	0	-1	0	0	1
0	x_5	200000	75	100	0	0	1	0	0
		$Z_j - C_j$	-41	-76	1	1	0	0	0

x_2 becomes a basic variable and x_7 becomes a non-basic variable in the next iteration. It is no longer considered for re-entry.

Table 2

C_B	Basic	C_j	0	0	0	0	0	-1
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6

-1	x_6	$\frac{2000}{3}$	$\frac{7}{15}$	0	-1	$\frac{1}{75}$	0	1
0	x_2	$\frac{4000}{3}$	$\frac{8}{15}$	1	0	$-\frac{1}{75}$	0	0
0	x_5	$\frac{200000}{3}$	$\frac{65}{3}$	0	0	$\frac{4}{3}$	1	0
		$Z_j - C_j$	$-\frac{1}{15}$	0	1	$-\frac{7}{15}$	0	0

x_1 becomes a basic variable and x_6 becomes a non basic variable in the next iteration. It is no longer considered for re- entry.

Table 3

C_B	Basic	C_j	0	0	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5
$-\frac{25}{2}$	x_1	$\frac{10000}{7}$	1	0	$-\frac{15}{7}$	$\frac{1}{35}$	0
$-\frac{29}{2}$	x_2	$\frac{4000}{7}$	0	1	$\frac{8}{7}$	$-\frac{1}{35}$	0
0	x_5	$\frac{250000}{7}$	0	0	$\frac{325}{7}$	$\frac{16}{21}$	1
		$Z_j - C_j$	0	0	0	0	0

The Phase I computation is computation is complete at this stage. Both the artificial variables have been removed from the basis. We have also found a basic feasible solution of the problem, namely, $x_1 = \frac{10000}{7}, x_2 = \frac{4000}{7}, x_5 = \frac{250000}{7}$. In phase II computation we use the actual objective function of the problem.

Phase II Computation

Table 1

C_B	Basic	C_j	0	0	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5
$-\frac{25}{2}$	x_1	$\frac{10000}{7}$	1	0	$-\frac{15}{7}$	$\frac{1}{35}$	0
$-\frac{29}{2}$	x_2	$\frac{4000}{7}$	0	1	$\frac{8}{7}$	$-\frac{1}{35}$	0
0	x_5	$\frac{250000}{7}$	0	0	$\frac{325}{7}$	$\frac{5}{7}$	1
		$Z_j - C_j$	0	0	$\frac{143}{14}$	$\frac{2}{5}$	0

As all $z_j - c_j \geq 0$ the current solution maximizes the revised objective function. Hence the solution of the problem is given by $x_1 = \frac{10000}{7}, x_2 = \frac{4000}{7} = 571\frac{3}{7}$. The minimum value of the objective function is 26142. The problem has been solved earlier by graphical method in Unit 3.

M-Method

The M-method also uses artificial variables for locating an initial basic feasible solution. We illustrate this with the help of the previous example.

$Maximize -\frac{25}{2}x_1 - -\frac{29}{2}x_2$

Subject to:

$$x_1 + x_2 - x_3 = 2000$$

$$40x_1 + 75x_2 - x_4 = 100000$$

$$75x_1 + 100x_2 + x_5 = 200000$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0$$

We introduce artificial variables to the first and the second constraint respectively. The objective function is revised using a large positive number

M. Thus instead of the original linear programming problem the following linear programming problem is considered.

$$Maximize -\frac{25}{2}x_1 - \frac{29}{2}x_2 - M(x_6 + x_7)$$

Subject to:

$$x_1 + x_2 - x_3 + x_6 = 2000$$

$$40x_1 + 75x_2 - x_4 + x_7 = 100000$$

$$75x_1 + 100x_2 + x_5 = 200000$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_7 \geq 0, x_7 \geq 0$$

The coefficient of the artificial variables in the objective function are large negative numbers. As the objective function is to be maximized in the optimum or optimum or optimal solution (where the objective function is maximized) the artificial variables will be zero. The basic variables of the optimal solution are therefore variables other than artificial variables and hence is a basic solution of the original problem. The successive simplex Tables are given below:

Table 1

C_B	Basic	C_j	$\frac{25}{2}$	$\frac{29}{2}$	0	0	0	-M	-M
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-M	x_6	2000	1	0	-1	0	0	1	0
-M	x_7	100000	40	1	0	-1	0	0	1
0	x_5	200000	75	0	0	0	1	0	0
		$Z_j - C_j$	$-41M + \frac{25}{2}$	$-76M + \frac{29}{2}$	M	M	0	0	0

As M is a large positive number the coefficient of M in the row would decide the incoming basic variable. As -76M < -41M, x_2 becomes a basic variable in the next iteration re-placing x_7. The variable x_7 being an artificial variable it is not considered for re-entry as a basic variable.

Table 2

C_B	Basic	C_j	$\frac{25}{2}$	$\frac{29}{2}$	0	0	0	-M
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
-M	x_6	$\frac{2000}{3}$	1	0	-1	$\frac{1}{75}$	0	1
$-\frac{29}{2}$	x_2	$\frac{4000}{3}$	0	1	0	$-\frac{1}{75}$	0	0
0	x_5	$\frac{200000}{3}$	0	0	0	$\frac{4}{3}$	1	0
		$Z_j - C_j$	$-\frac{7}{15M}+\frac{143}{30}$	0	M	$\frac{M}{75}+\frac{29}{150}$	0	0

x_1 becomes a basic variable and x_6. The variable x6 being an artificial variable is not considered for re- entry as a basic variable.

Table 3

C_B	Basic	C_j	$-\frac{25}{2}$	$-\frac{29}{2}$	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5
$-\frac{25}{2}$	x_1	$\frac{10000}{7}$	1	0	$-\frac{15}{7}$	$\frac{1}{35}$	0
$-\frac{29}{2}$	x_2	$\frac{4000}{7}$	0	1	$\frac{8}{7}$	$-\frac{1}{35}$	0
0	x_5	$\frac{250000}{7}$	0	0	$\frac{325}{7}$	$\frac{16}{21}$	1

		$Z_j - C_j$	0	0	$\frac{143}{14}$	$\frac{2}{35}$	0

Hence the optimum solution of the problem is $x_1 = \frac{10000}{7} = 1428\frac{4}{7}, x_2 = \frac{4000}{7} = 571\frac{3}{7}$ with the minimum value of the objective function being $26142\frac{6}{7}$.

3.7 Multiple, Unbounded Solution and Infeasible Problems

The simplex method can identify multiple solution of a linear programming problem. If a problem possesses an unbounded solution is also located in course of simplex computation. If a linear programming problem is infeasible it is revealed by simplex computation. We illustrate these applications of simplex method with the help of a number of examples.

Example 1.4: We consider the linear programming problem

Maximize $2000x_1 + 3000x_2$
Subject to :

$$6x_1 + 9x_2 \le 100$$

$$2x_1 + x_2 \le 20$$

$$x_1 \ge 0, x_2 \ge 0$$

Solution:

After introducing slack variables $x_3 \ge 0, x_4 \ge 0$ the inequalities can be converted into equations as follows

$$6x_1 + 9x_2 + x_3 = 100$$

$$2x_1 + x_2 + x_4 = 20$$

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0$$

The successive tables of simplex computation are shown below:

Table 1

C_B	Basic	C_j	2000	3000	0	0
	Variables	X_B	x_1	x_2	x_3	x_4

0	x_3	100	6	9	1	0
0	x_4	20	2	1	0	1
	$Z_j - C_j$		-2000	-3000	0	0

Table 2

C_B	Basic	C_j	2000	3000	0	0
	Variables	X_B	x_1	x_2	x_3	x_4
0	x_2	100/9	2/3	1	1/9	0
0	x_4	80/9	4/3	0	-1/9	1
	$Z_j - C_j$		0	0	3000/9	0

Since $Z_j - C_j \geq 0$ for all the variables, $x_1 = 0, x_2 = 100/9$ is an optimum solution of the problem. The maximum value of the objective function is 100000/3. However, $Z_j - C_j$ the value corresponding to the non-basic variable x_1 is also zero. This indicates that there is more than one optimum solution of the problem. In order to compute the value of the alternative optimum solution we introduce x_1 as a basic variable replacing x_4. The subsequent computation is presented in the next Table.

C_B	Basic	C_j	2000	3000	0	0
	Variables	X_B	x_1	x_2	x_3	x_4
0	x_2	20/3	0	1	1/6	1/2
0	x_1	20/3	1	0	-1/12	3/4
	$Z_j - C_j$		0	0	1000/3	3000

Thus $x_1 = 20/3, x_2 = 20/3$ also maximize the objective function. The maximum value as in previous solution is 100000/3.

Example 1.5: Consider the linear programming problem

Maximize $5x_1 + 4x_2$

Subject to:

$$x_1 \leq 7$$
$$x_1 - x_2 \leq 8$$
$$x_1 \geq 0, x_2 \geq 0$$

Solution:

After introducing slack variables $x_3 \geq 0, x_4 \geq 0$ the corresponding equations are

$$x_1 + x_3 = 7$$
$$x_1 - x_2 + x_4 = 8$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

The successive simplex iterations are shown below:

Table 1

C_B	Basic	C_j	5	4	0	0
	Variables	X_B	x_1	x_2	x_3	x_4
0	x_3	7	1	0	1	0
0	x_4	8	1	-1	0	1
	$Z_j - C_j$		-5	-4	0	0

Table 2

C_B	Basic	C_j	2000	3000	0	0

	Variables	X_B	x_1	x_2	x_3	x_4
5	x_1	7	1	0	1	0
0	x_4	1	0	-1	-1	1
	$Z_j - C_j$		0	-4	5	0

$Z_2 - C_2 \geq 0$ indicates x_2 should be introduced as a basic variable in the next iteration. However, both $y_{12} \leq 0, y_{22} \leq 0$. Thus it is not possible to proceed with the simplex computation any further as you cannot decide which variable will be non-basic at the next iteration. This is the criterion for unbounded solution.

If in the course of simplex computation $Z_j - C_j < 0$ *but* $y_{ij} \leq 0$ for all i then the problem has no finite solution.

Intuitively you may observe that the variable x_2 in reality is unconstrained and can be increased arbitrarily. This is why the solution is unbounded.

Example 1.6: We consider the linear programming problem formulated in Unit 3, Section 6.

Minimize $200x_1 + 300x_2$

Subject to:

$$2x_1 + 3x_2 \geq 1200$$

$$x_1 + x_2 \leq 400$$

$$2x_1 + 3/2x_2 \geq 900$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution:

After converting the minimum problem into a maximization problem and introducing slack, surplus, artificial variables the problem can be presented as

Maximize $200x_1$ - $300x_2$

Subject to:

$$2x_1 + 3x_2 - x_3 + x_6 = 1200$$

$$x_1 + x_2 + x_4 = 400$$

$$2x_1 + 3/2x_2 - x_5 + x_7 = 900$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0, x_7 \geq 0, x_7 \geq 0$$

The variables x_6 and x_7 are artificial variables. We use two phase method to solve this problem. In phase I, we use the objective function:

Maximize – x_6 – x_7

Along with the constraints given above. The successive simplex computations are given below:

Table 1

C_B	Basic	C_j	0	0	0	0	0	-1	-1
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
-1	x_6	1200	2	3	-1	0	0	1	0
0	x_4	400	1	1	0	1	0	0	0
-1	x_7	900	2	3/2	0	0	-1	0	1
	$Z_j - C_j$		-4	$-\frac{9}{2}$	1	0	1	0	0

Table 2

C_B	Basic	C_j	0	0	0	0	0	-1

	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_7
0	x_2	400	2/3	1	-1/3	0	0	0
0	x_4	0	1/3	0	1/3	1	0	0
-1	x_7	300	1	0	½	0	-1	1
	$Z_j - C_j$		-1	0	-1/2	0	1	0

Table 3

C_B	Basic	C_j	0	0	0	0	0	-1
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	x_6	400	0	1	-1	-2	0	0
0	x_4	0	1	0	1	3	0	0
-1	x_7	300	0	0	-1/2	-3	-1	1
	$Z_j - C_j$		0	0	1/2	3	1	0

Thus $Z_j - C_j \geq 0$ for all the variables but the artificial variable x_7 is still a basic variable. This indicates that the problem has no feasible solution.

If n course of simplex computation by two phase method one or more artificial variables remain basic variables at the end of phase I computation, the problem has no feasible solution.

3.8 Sensitivity Analysis

After a linear programming problem has been solved, it is useful to study the effect of change in the parameters of the problem on the current optimal solution. Some typical situation are the impact of the changes in the profit or cost in the objective function in the current solution or an increase or decrease in the resource level in the present composition of the product mix. Such an investigation can be carried out from the final simplex Table and is known as sensitivity analysis or post optimality analysis. We shall illustrate this method the help of an example.

Example 1.7 : We consider the linear programming problem introduced in section 1.5

Maximize $12x_1 + 3x_2 + x_3$

Subject to:

$$10x_1 + 2x_2 + x_3 \leq 100$$
$$7x_1 + 3x_2 + 2x_3 \leq 20$$
$$2x_1 + 4x_2 + x_3 \leq 80$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The final simplex Table presenting the optimum solution (x_4, x_5, x_6 being the slack variables) is presented below:

Table 3

C_B	Basic	C_j	12	3	1	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
12	x_1	73/8	1	0	1/16	3/16	-1/8	0
3	x_2	35/8	0	1	13/16	-7/16	5/8	0
0	x_6	177/4	0	0	-17/8	11/8	-9/4	1
	$Z_j - C_j$		0	0	11/16	15/16	3/8	0

Change in the Profit coefficient

i) **Non Basic Variable**

The variable x_3 non-basic. if its profit level is increased to C_3 then the solution remained unchanged sq long as

$Z_3 - C_3 \geq 0$

i.e. $Z_3 - C_3 + C_3 - C_3 \geq 0$

i.e. $C_3 \leq C_3 + Z_3 - C_3 = 1\frac{11}{16}$

if $C_3 > 1\frac{11}{16}$then the present solution is no longer optimum. A new round of simplex computation is to be performed in which x_3 becomes a basic variable. However, if $\bar{C}_3 < 1\frac{11}{16}$, the present solution remains optimum.

ii) Basic Variable

In this case, the changes can be both positive and negative, as in either case the current solution may become non-optimal. Let us consider the basic variable x_1. Let us define by $\Delta_1 = \bar{C}_1 - 12$ as the change (both positive and negative) in the profit coefficient 12. We divide each Z_j-C_j value of non basic variable by the corresponding coefficient in the x1 row which is denote by $\frac{Z_j - C_j}{y_{ij}}$ .Then the basic variables remain unchanged so long as (Mustafi, 1988)

$$minimum\left[\frac{Z_j - C_j}{y_{ij}}; y_{ij} > 0\right] \leq \Delta_1 \leq Minimum\left[\frac{Z_j - C_j}{-y_{ij}}; y_{ij} > 0\right]$$

Referring to the final simplex Table, we observer that corresponding to the non-basic variables x_3 and x_5, $y_{13} = -\frac{1}{16}, y_{15} = -\frac{1}{8}$

Hence,

$$Minimum\left[\frac{Z_j - C_j}{y_{ij}}; y_{ij} > 0\right]$$

$$= Minimum\left[\frac{\frac{11}{16}}{\frac{1}{16}}, \frac{\frac{3}{8}}{\frac{1}{8}}\right]$$

$$= Minimum\,(11,3) = 3.$$

Corresponding to the non-basic variables x_4, $y_{14} > 0$.

Hence

$$Minimum\left[\frac{Z_j - C_j}{y_{ij}}; y_{ij} > 0\right] = \frac{15/16}{3/16} = 5$$

Hence

$$-5 \leq \bar{C}_1 - 12 \leq 3\ i.e.\ 7 \leq \bar{C}_1 \leq 15.$$

Thus the optimal solution is insensitive so long as the changed profit coefficient $\bar{C}_1$ is between Rs. 7 and Rs. 15 although the present coefficient is Rs. 12. Of course the value of the objective function has to be revised after introducing the change value $\bar{C}_1$.

3.9 Key Words

A Slack Variable corresponding to a less than or equal to type constraint is non negative variable introduced to convert the constraint into an equation.

A Basic Solution of a system of m equations and n variables (m < n) is a solution where at least n-m variables are zero.

A Basic Feasible Solution of a system of m equations and n variables (m < n) is a solution where m variables are non-negative and n-m variables are zero.

A Basic Variable of a feasible solution has a non-negative value

A Non Basic Variable of a basic feasible solution has a value equal to zero.

A Surplus Variable corresponding to greater than or equal to type constraint is a non-negative variable introduced to convert the constraint into an equation.

An Artificial Variable is a non-negative variable introduced to facilitate the computation of an initial basic feasible solutions.

The Optimum Solution of linear programming problem is the solution where the objective function is maximized or minimized.

The Sensitivity Analysis of a linear programming problem is a study of the effect of changes of the profit or resource level on the solution.

3.10 Summary

The simplex method is the appropriate method for solving a linear programming problem with more than two decision variables. For less than or equal to type constraints slack variables are introduced to make inequalities equations. A particular type of solution known as a basic feasible solution is important for simplex computation. Every basic feasible solution is an extreme point of the convex set of feasible solutions and vice versa. A basic feasible solution of a system with m equations and n variables has m non-negative variables known as basic variables and n-m variables with value zero known as non-basic variables. We can always find a basic feasible solution with the help of the slack variables. The objective function is maximized or minimized at one of the basic feasible solutions. Starting with the initial basic feasible solution obtained from the slack variables the simplex method improves the value of objective function step by step by bringing in an new basic variable and making one of the present basic variables non basic. The selection of the new basic variable and the omission of a current basic variable solution improves the value of the objective function. The iterative procedure stops when it is no longer possible to obtain a better value of the objective function than the present one. The existing basic feasible solution is the optimum solution of the problem which maximizes or minimizes the objective function as the case may be.

When one or more of the constraints are greater than or equal to type surplus variables are introduced to make inequalities equation. Surplus variables, however, cannot be used td obtain an initial basic feasible solution. If some of the constraints are greater than or equal to type or equations artificial variables are used to initiate the simplex computation. Two methods, namely, Two Phase Method and M-method are available to solve linear programming problems in these cases. The simplex method can identify multiple or unbounded solutions and infeasible problems.

The simplex method also provides a mean for carrying out sensitivity or post optimality analysis of the problem. It is possible to study the effect of change in profit contribution for a particular product without solving the problem all over again. The effect of change in various resource levels can also be ascertained by making a few additional calculations.

3.11 Further Readings

- Hadley, G. (1969). Linear Programming. Addison Welley Reading, USA.
- Mustafi, C.K., (1988) Operations Research Methods and Practice, Wiley Eastern Limited, New Delhi.

Unit-4 Duality Problem in LPP

Structure

4.1 Introduction

4.2 Objectives

4.3 Dual linear programming problem

4.4 Formulation of a dual problem with example

4.5 Key Words

4.6 Summary

4.7 Further Readings

4.1 Introduction

One of the most important discoveries in the early development of linear programming was the concept of duality and its division into important branches. Every linear programming problem has another linear programming problem associated with it. The original problem is called the primal while the one associated with it is called the dual problem. The relationship between the primal and dual problems is actually a very intimate and important one.

4.2 Objectives

After going through this unit you must be able to understand:

- Dual Problem
- Concept of Duality
- Formulation of a Dual Problem
- Properties of Dual Problem

4.3 Dual Linear Programming Problem

Every linear programming problem is associated with another linear programming problem known as its dual problem. The original problem in this context is known as the primal problem. The formulation of the dual linear programming problem (also sometimes referred to as the concept of duality) is substantially helpful to our understanding of linear programming. The variables of the dual linear programming problem also known as dual variables have important economic interpretations, which can be used by a decision maker for planning his resources. Under certain circumstances the dual problem is easier to solve than the primal problem. The solution of the dual problem leads to the solution of the primal problem and thus efficient computational techniques can be developed through the concept of duality. Finally in the problems of competitive strategy solution of both the primal and the dual problem is necessary to understand the problem fully.

We introduce the concept of duality with the help of the product mix problem introduced in Unit-1 of section 1.5.

Example 1: Three products A, B, C are produced in three machine centers x, y, z. Each product involves operation of each of the machine centers. The time required for each operation on various products in indicated in the following Table. 100, 77 and 80 hours are only available at machine centers x, y and z respectively. The profit per unit of A, B and C is Rs. 12, Rs. 3 and Re 1 respectively.

Table Showing the Data of the Product Mix Problem

Products	**Machine Centers**			**Profit per unit**
	X	**Y**	**Z**	
A	10	7	2	Rs. 12
B	2	3	4	Rs. 3
C	1	2	1	Re 1
Available hours	100	77	80	

Solution

The linear programming formulation or the primal problem is given by

Maximize $12x_1 + 3x_2 + x_3$

Subject to:

$$10x_1 + 2x_2 + x_3 \leq 100$$

$$7x_1 + 3x_2 + 2x_3 \leq 77$$

$$2x_1 + 4x_2 + x_3 \leq 80$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

After introducing the slack variables $x_4 \geq 0, x_5 \geq 0, x_6 \geq 0$ and using simplex method we obtain the optimum solutions as $x_1 = 73/8$, $x_2 = 35/8$ and the maximum value of the objective function as 981/8. The final Simplex Table is given below:

Table 3

C_B	Basic	C_j	12	3	1	0	0	0
	Variables	X_B	x_1	x_2	x_3	x_4	x_5	x_6
12	x_1	73/8	1	0	-1/16	3/16	-1/8	0
3	x_2	35/8	0	1	13/16	-7/16	5/8	0
0	x_6	177/4	0	0	-17/8	11/8	-9/4	1
	$Z_j - C_j$		0	0	11/16	15/16	3/8	0

Suppose a prospective investor is planning to purchase the resources x, y, z. What offer is he going to make to manufactures? Let us assume that W_1, W_2 and W_3 are the offers made per hour of machine time x,y, arid z respectively. Then these prices W_1, W_2, W_3 must satisfy the following conditions

a) $W_1 \geq 0, W_2 \geq 0, W_3 \geq 0$.
b) Assuming that the prospective investor is behaving in a rational manner he would try to bargain as much as possible and hence the total amount payable to the manufacturer would be title as possible. This leads to the condition:

$$\text{Minimize } 100W_1 + 77W_2 + 80W_3$$

c) The total amount offered by the prospective investor to the three resources required to produced one unit of each product must be at least as high as the profit earned by the manufacturer per unit. Since these resources enable the manufacturer to earn the specified profit corresponding to the product he would not like to sell it for anything less assuring he is behaving rationally. This leads to inequalities.

$$10W_1 + 7W_2 + 2W_3 \geq 12$$

$$2W_1 + 3W_2 + 4W_3 \geq 3$$

$$W_1 + 2W_2 + W_3 \geq 1$$

We have, thus a linear programming problem to ascertain the values of the variables. The variables are known as dual variables. The primal problem presented in this example (i) considers maximization of the objective function (ii) has less than or equal to type constraints, and (iii) has non-negativity constraints on the variables. Such a problem is known as a primal problem in the standard form.

4.4 Formulation of A Dual Problem

If the primal problem is in the standard form the dual problem can be formulated using the following rules.

1) The number of constraints in the primal problem is equal to the number of dual variables. The number of constraints in the dual problem is equal to the number of variables in the primal problem.

2) The primal problem is a maximization problem the dual problem is a minimization problem.

3) The profit coefficients of the primal problem appear on the right hand side of the constraints of the dual problem.

4) The primal problem has less than or equal to type constraints while the dual problem has greater than or equal to type constraints.

5) The coefficient of the constraints of the primal problem which appear from left to right are placed from top to bottom in the constraints of the dual problem and vice versa.

It is easy to verify these rules with respect to the example discussed before.

Properties of the Dual Problem

1) if the primal problem is in the standard form the solution of the dual problem can be obtained from the Z_j-C_j values of the slack variables in the final simplex Table.

Example: In the example discussed previously the variables x_4, x_5, x_6 are slack variables. Hence the solution of the dual problem is $w_1 = z_4 - c_4 = 15/16$, $w_2 = 3/8$, $w_3 = 0$.

2) The maximum value of the objective function of the primal problem is the minimum value of the objective function of the dual problem.

Example: The maximum value of the objective of the primal problem is 981/8. The minimum value of the objective function of the dual problem is

$$100 \times \frac{15}{16} + 77 \times \frac{3}{8} + 80 \times 0 = \frac{981}{8}$$

The result has an important practical implication. If the problem is analyzed by both the manufacturer and the investor then neither of the two can outmaneuver the other.

Shadow Price

The shadow price of a resource is the unit price that is equal to the increase in profit to be realized by one additional unit of the resources.

Example: The maximum value of the objective function can be expressed as

$$100 \times \frac{15}{16} + 77 \times \frac{3}{8} + 80 \times 0$$

If the first type of resource is increased by one unit the maximum profit will increase by 15/16 which is the value of the first dual variable in the optimum solution.

Thus the dual variable is also referred to as the shadow price or imputed price of a resource. This is the highest price the manufacturer would be willing to pay for the resource. The shadow price of the third resource is zero as there is already an unutilized amount; profit is not increased by more of it until the current supply is totally exhausted.

3) Suppose the number of constraints and variables in the primal problem in m and n respectively. The number of constraints and variables in the dual problem is, therefore n and m respectively. Suppose the slack variables in the primal are denoted by $y_1, y_2, \ldots y_n$ and the surplus variables in the dual problem are denoted by $z_1, z_2, \ldots z_n$.

a) In the optimum solution

if $x_i > 0$ $z_i = 0$ $i = 1, 2, \ldots n$.

if $z_i > 0$ $x_i = 0$ $i = 1, 2, \ldots n$.

if $w_i > 0$ $y_i = 0$ $i = 1, 2, \ldots n$.

if $y_i > 0$ $w_i = 0$ $i = 1, 2, \ldots\ldots\ldots\ldots n.$

b) If a solution of the primal and the corresponding dual problem satisfy the above conditions then it must be an optimum solution.

This result is commonly referred to as the complementary slackness conditions.

Referring to the final simplex Table of the problem discussed before we observe m = 3, n=3. In the optimum solution

$x_1 = 73/8$	$x_2 = 35/8$	$x_3 = 0$
$z_1 = 0$	$z_2 = 0$	$z_3 = 0$
$y_1 = 0$	$y_2 = 0$	$y_3 = 177/4$
$w_1 = 15/16$	$w_2 = 3/8$	$w_3 = 0$

Thus the complementary slackness condition is satisfied.

4) If the primal problem is in non-standard form, the structure of the dual problem remains unchanged. However, if a constraint is greater than equal to type, the corresponding dual variable is negative or zero. If a constraint in the primal problem is equal to type, the corresponding dual variable is unrestricted in sign (may be positive or negative).

Example 2: Consider the primal linear programming problem

Maximize $12x_1 + 15x_2 + 9x_3$

Subject to:

$$8x_1 + 16x_2 + 12x_3 \leq 250$$

$$4x_1 + 8x_2 + 10x_3 \geq 80$$

$$7x_1 + 9x_2 + 8x_3 = 105$$

$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$$

The corresponding dual problem is

Minimize $250w_1 + 80w_2 + 105w_3$

Subject to:

$$8W_1 + 4W_2 + 7W_3 \geq 12$$

$$16W_1 + 8W_2 + 9W_3 \geq 15$$

$$12W_1 + 10W_2 + 8W_3 = 9$$

$$w_1 \geq 0, w_2 \geq 0, x_3 \; unrestricted \; in \; sign.$$

4.5 Key Words

The Dual Problem corresponding to a linear programming problem is another linear programming problem formulated from the parameters of the original problem.

The Primal Problem is the original linear programming problem.

The Dual Variables are the variables of the dual linear programming problem.

The Shadow Price of a resource is the change in the optimum value of the objective functions per unit increase of the resource.

4.6 Summary

Every linear programming problem has an accompanying linear programming problem known as a dual problem. The variables of the dual problem are known as dual variables. The dual variables have an economic interpretations, which can be used by management for planning its resources. The solutions of the dual problem can be obtained from the simplex computation of the original problem. The solution has a number of important properties, which can also be helpful for computational purposes.

4.7 Further Readings

- Hadley, G. (1969). Linear Programming. Addison Welley Reading, USA.
- Mustafi, C.K., (1988) Operations Research Methods and Practice, Wiley Eastern Limited, New Delhi.

U.P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109
Operation Research

Block: 3 Transportation Problem and Assignment Problem

Unit – 5	**:**	**Representation of Transportation Problem & Assignment Problem as Linear Programming Problem**	**82 - 92**
Unit – 6	**:**	**Different Methods of Finding Initial Feasible Solution of a Transportation Problem**	**93 - 107**
Unit – 7	**:**	**Solution of Assignment Problem with using Hungarian Method**	**108 - 118**

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics, D. D. U. Gorakhpur University, Gorakhpur.

Dr. Shruti
Member-Secretary
School of Sciences, U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Shubra Katara **Writer**
Department of Statistics, Bareily College, Bareily

Prof. V. K. Sehgal **Reviewer**
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Prof. R. C. Yadava **Editor**
Department of Statistics, Banaras Hindu University, Varanasi

Dr. Shruti **Course/ SLM Coordinator**
School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109 **OPERATION RESEARCH**
©UPRTOU
First Edition: *March 2008* (Published with the support of the Distance Education Council, New Delhi)
Second Edition: ***July 2021***
ISBN :

 Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2021.
Printed By: K.C.Printing & Allied Works, Panchwati, Mathura -281003.

Block & Units Introduction

The ***Block - 3 – Transportation Problem and Assignment Problem,*** deals with transportation problem (TP) and assignment problem (AP). It contains three units.

Unit – 5 – Representation of Transportation Problem & Assignment Problem as linear Programming Problem, deals with representation of these problems as a special case of linear programming problem.

Unit – 6 – Different Methods of Finding Initial Feasible Solution of a Transportation Problem, describes the different methods of finding initial feasible solution to TP and MODI methods of finding optional solution of TP.

Unit – 7 – Solution of Assignment Problem with using Hungarian Method, gives the solution of AP using Hungarian method.

At the end of every block/unit the summary, self assessment questions and further readings are given.

Unit-5: Representation of Transportation Problem & Assignment Problem as Linear Programming Problem

Structure

5.1 Introduction of T.P. and A.P.
5.2 Objectives
5.3 Transportation Problem as L.P.P
5.4 Non-degenerate Transportation Problem
5.5 Balanced Transportation Problem
5.6 Assignment Problem and L.P.P.
5.7 Balance Assignment Problem
5.8 Self Assessment Exercises
5.9 Answers
5.10 Summary
5.11 Further Readings

5.1 Introduction

The transportation problem is a particular class of linear programming problem where the objective is to minimize the cost of distributing produce from number of sources or origins to a number of destinations. The special features of transportation are illustrated with the help of the following example:

5.2 Objectives

After studying this unit you should be able to:

- Formulate a transportation problem as a linear programming problem
- Identify the characteristics of a transportation problem
- Formulate an assignment problem as….a linear programming problem
- Identify the characteristics of an assignment problem

5.3 Representation of Transportation Problem as LPP

Example 1: A multi-plant company has three manufacturing plants A, B and C and dispatches his product to four different retail shops 1,2,3 and 4. The table below shows the capacities of the three plants, the quantities of products required at the various retail shops and the cost of transporting one unit of the product from each of three plants to each of the four retail shops

Plants	Retail Shops				Availability
	1	2	3	4	
A	19	30	50	10	7
B	70	30	40	60	9
C	40	08	70	20	18
Requirement	5	8	7	14	34

The table usually referred to as Transportation Table provides the basic data regarding the transportation problem. The availability of the product at plants A,B and C is 7, 9, and 18 respectively. The requirement at retail shops 1,2,3 and 4 is 5,8,7, and 14 respectively. The quantities inside the bordered rectangle are known as until transportation costs. The cost of transportation of one until from plant A to retail 1 is 19 plant A to retail shop 2 is 30 and so on. A transportation problem can be formulated as a linear programming problem as follows:

Let

X_{11}= Amount of a product to be transported from plant A to retail shop 1

X_{12} = Amount of a product to be transported from plant A to retail shop 2

X_{34} = Amount of a product to be transported from plant C to retail shop 4

Let the until transportation cost be denoted by C_{11}, C_{12},……….. C_{34}, i.e. C_{11} = 19, C_{12} = 20, and so on. Let the availabilities in three plants be denoted by a_1 = 7, a_2=9 and a_3 = 18. The requirements of the retail shops are b_1=5, b_2=8, b_3=7 and b_4=14.

Then the transportation problem can be formulated as

$$Minimize\ Z = C_{11}x_{11} + C_{12}x_{12} + ------ C_{34}x_{34}$$

Subject to

$$x_{11} + x_{12} + x_{13} + x_{14} = a_1$$

$$x_{21} + x_{22} + x_{23} + x_{24} = a_2$$

$$x_{31} + x_{32} + x_{33} + x_{34} = a_3$$

$$x_{11} + x_{21} + x_{31} = b_1$$

$$x_{12} + x_{22} + x_{32} = b_2$$

$$x_{13} + x_{23} + x_{33} = b_3$$

$$x_{14} + x_{24} + x_{34} = b_4$$

$$x_{11} \geq 0, x_{12} \geq 0 - --, x_{34} \geq 0.$$

The problem has 7(=3+4) constraints and 12 (=3*4) variables one of the constraints can be eliminated since $a_1+a_2+a_3 = b_1+b_2+b_3+b_4$. Thus the problem has in fact six constraints and twelve variables.

In general the transportation problem is stated as follows:

Let a_1, a_2, ----, a_i, ----, a_m be the quantities of a product available at m places called or sources and let b_1, b_2, ----, b_j, ----, b_n be the quantities of the same product required at other n places called destinations Assume that the total quantity available is equal to the total required quantity, i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$$

Suppose c_{ij} is the cost of transportation of a unit from i^{th} (i=1,2,---,m) to the j^{th} destination (j= 1,2,-----, n). The problem is to determine x_{ij} the quantity of a product which is to be transported from the i^{th} origin to j^{th} destination in such a way that the total transportation cost is minimum.

The transportation table is given as follows:

Origins	Destination						Available
	D_1	D_2	….	D_j	….	D_n	
O_1	c_{11}	c_{12}	…	c_{1j}	…	c_{1n}	a_1
O_2	c_{21}	c_{22}	…	c_{2j}	…	c_{2n}	a_2
.	.	.		.		.	.
.	.	.	…	.	…	.	.

O_i	c_{i1}	c_{i2}	…	c_{ij}	…	c_{in}	a_i
.	.	.		.		.	.
.	.	.	…	.	…	.	.
O_m	c_{m1}	c_{m2}	…	c_{mj}	…	c_{mn}	a_m
Requirement	b_1	b_2	…	b_j	…	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$

Mathematically, transportation problem is stated as

Determine x_{ij} which

$$Minimize\ Z = C_{11}x_{11} + C_{12}x_{12} + ------ C_{mn}x_{mn} \ \ldots\ldots\ldots\ldots\ldots (i)$$

Such that

$$\sum_{j=1}^{n} x_{ij} = a_i; i = 1,2,---,m \qquad (ii)$$

$$\sum_{i=1}^{m} x_{ij} = b_j; j = 1,2,---,m \qquad (iii)$$

$$and\ \ x_{ji} \ \geq\ 0 \qquad (iv)$$

In this problem, Z is the total transportation cost, called objective function given by equation (i). The equation (ii) is the supply restriction of j^{th} source and the equation (iii) is the demand required at j^{th} destination. The quantity of a product can not be negative, so non negative restriction given by equation (iv) is imposed on quantities.

The mathematical form of a transportation problem looks like a linear programming with objective function CD, set of constraints (ii) and (iii) and non negative restriction (iv). Thus a transportation problem is a special case of linear programming problem. A transportation problem with m origins and n destinations has m + n -1 constraints and m n variables.

5.4 Non Degenerate Transportation Problem

A feasible solution of a transportation problem is that which satisfies the row and column sum restrictions and non-negative restrictions. In general, it is stated the a transportation problem with m origins and n destinations has m + n-1 constraints. Thus any basic feasible solution has at most non-zero xij (occupied cells). If a basic feasible solution of a transportation problem consisted of exactly m +n-1 non zero xij, the problem is said to be non degenerate. On the other hand if a basic feasible solution of transportation problem has less than m + n-1 non zero xij, the problem is said to be a degenerate transportations problem.

5.5 Balanced Transportation Problem

If the total availability at the origin is equal to the total requirement at the destinations i.e., if$\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j$ the problem is known as a balanced transportation problem; otherwise it is called as an unbalanced transportation problem.

The transportation problem given in example 1 is a balanced transportation problem as $\sum_i a_i = \sum_j b_j = 34$

A balanced transportation problem always has a feasible solution. In the next unit a number of procedures will be presented to derive an initial feasible of the problem.

Activity 1:

Fill up the blanks:

i) The objective of a transportation problem is to …………………..the transportation cost.

ii) If a transportation problem has 5 origins and 4 destinations, the number of variable is ………………….and the number of constraints is ……………

iii) One of the constraints of transportation problem is eliminated, since the total availability is ………………to the total requirement.

iv) A transportation problem is known as an unbalanced transportation problem if

……………………………………………………………………..

v) A transportation problem with m origins and n destinations is said to degenerate transportation problem if

……………………………………..

5.6 Assignment Problem as LPP

The general form of the assignment problem is stated as follows:

Given n facilities (men or machines), n jobs and the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimized (maximized or minimized): Different examples identical with the assignment problem are given below:

A department head may have four subordinates and four jobs to done. He may like to know which job should be assigned to which subordinate so that all these task can be accomplished in the shortest possible time.

A truck rental services may have an empty truck in each of the five cities ,1,2,3,4, and 5 and needs an empty truck in each of the five cities 6,7,8,9 and 10. He would like to ascertain the assignment of trucks to various cities so as to minimize the total distance traveled.

A marketing manager may have five salesmen and five sales districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates the sales per month for each salesman in each district. He may assign a particular salesman to a particular district. He may assign a particular salesman to a particular district in such a way that will result in maximum sales.

The assignment problem is similar to the transportation problem, stated as follows:

Suppose c_{ij} is the measure of effectiveness when i^{th} jobs (i= 1,2,---,n) is assigned to j^{th} facility (j=1,2,---,n). It is also assumed that the overall measure of effectiveness is to be minimized (such that the total cost of doing all jobs). Her jobs represent sources and facilities represent destinations. The cost matrix $[c_{ij}]$ is given below:

	Facility						Total
	1	2	….	…	….	n	
1	c_{11}	c_{12}	…	…	…	c_{1n}	1
2	c_{21}	c_{22}	…	…	…	c_{2n}	1

.	.	.		.		.	.
Jobs	.	.	…	.	…	.	.
.	.	.		.		.	.
.							
n	C_{n1}	c_{n2}	…	…	…	C_{nn}	1
Total	1	1	…	…	…	1	n

Mathematically, an assignment problem can be stated follows:

Since a particular facility can be assigned to only one job and a particular job can be assigned to only one facility. We define

$x_{ij} = 1,$ if i^{th} job is assigned to j^{th} facility

$=o,$ if i^{th} job is not assigned to j^{th} facility

Then,

$$x_{i1} + x_{i2} + ------- + x_{in} = 1; i = 1,2,---,n$$

$$x_{1j} + x_{2j} + ------- + x_{nj} = 1; i = 1,2,---,n$$

The objective function is

$$Minimize\ Z = C_{11}x_{11} + C_{12}x_{12} + ------- C_{mn}x_{mn}$$

$$x_{ji} \geq 0$$

Thus the structure of an assignment problem is identical with that of a transpiration problem with n origins and n destinations. The assignment problem is a special case of transportation problem with m = n and $a_i = b_j = 1$.

5.7 Balanced Assignment Problem

If the number of facility is equal to the number of jobs, the assignment problem is said to be balanced; otherwise it is known as unbalanced assignment problem. If the number of jobs is less than the number of facility, we introduce one or more dummy jobs is less as required of zero duration time/ cost to make the assignment problem balanced. Likewise, if the number of facilities is less that the number of jobs, we introduced one or more dummy facility with duration time/cost zero to make the assignment problem balanced.

In unit 3, we will discuss the solution of balanced assignment problem only.

Activity 2:

Fill up the blanks:

i) The assignment problem with n persons and n jobs is to assign each person to job in such a way that the --------------------- is optimized.

ii) The assignment problem is a special cases of transportation problem when

………………………………………………………………………

iii) An assignment problem is said to be unbalanced if ……………………………………………………………………… …………………

5.8 Summary

Transportation problem is a special type of linear programming problem. In the most general form, a transportation problem has a number of origins and a number of destinations. A certain quantity of a particular product is available at each origin and each destination has a certain requirement of the same product. The transportation problem is to determine x_{ij}, the amount of the quantity which is to be transported from is minimum subject to the availability constrains and the requirement constrains.

The basic feasible solution of a non-degenerate transportation problem with m origins and n destinations has m+n-1 non zero x. (occupied cells). If the basic feasible solution has less than m +n-1 non zero x_{ij}, the transportation problem is said to be degenerate.

In balanced transportation problem the total availability at the origins is equal to the total requirements at the destinations.

If the total availability is not equal to the total requirements, transportations problem is said to be unbalanced.

The assignment problem consider the allocation of a number of jobs to a number of facilities (men or machines) so that the total completion time or cost is minimized. If the times of compilation or the costs corresponding to every assignment is written down in a matrix form it is referred to as a cost matrix. If number of men is the same as the number of jobs the assignment problem is said to balanced if the number of jobs is different from the number of men the assignment problem said to be unbalanced

5.9 Key Words

The origin of a transportation problem is the location for which the number of units of a units of a product is dispatched.

The destination of a transportation problem is the location to which its number of units of the same product is transported.

The unit transportation cost is the cost of transporting one unit of the product from an origin to a destination.

A balanced transportation problem is a transportation problem when the total availability at the origins is equal to the total requirements at the destinations.

An unbalanced transportations problem is a transportation problem where the total availability at the origins is different from the total requirement at the destinations

A non degenerate transportation problem with m origins and n destinations has a basic feasible solution is equal to m+n-1 positive basic variables.

A degenerate Transportation problem with m origins and n destinations has a basic feasible solution is less than m+n-1 positive basic variables.

An assignment problem is a special type of linear programming problem where the objective is to minimize the cost or the time of completing a number of jobs by a number of persons.

A cost matrix is a form of matrix in which the times of completion or the costs corresponding to every assignment are written.

A balanced assignment problem is an assignment problem where the number of persons is equal to the number of jobs.

A unbalanced assignment problem is an assignment problem where the number of persons is not equal to the number of jobs.

5.10 Exercises

E-1) Describe the transportation problem with its general mathematical formulation.

E-2) Show that a transportation problem is a special type of linear programming problem.

E-3) What are the characteristics of a transportation problem/

E-4) What is meant by non-degenerate basic feasible solution of a transportation problem?

E-5) What is meant by unbalanced transportation problem?

E-6) What is assignment problem? Give two applications.

E-7) Give the mathematical formulation of an assignment problem. How does it differ from a transportation problem.

E-8) Explain the conceptual justification that an assignment problem can be views as linear programming problem.

E-9) What is meant by unbalanced assignment problem:

5.11 Answers

Activity 1

i) Minimize
ii) 20,8
iii) Equal
iv) Total availability at the origins is equal to the total requirement at the destinations.
v) It has less than m+n-1 non zero x_{ij} (occupied cells).

Activity 2

i) One and only one, measure of effectiveness.
ii) The number of origins is equal to the number of destinations and each availability and requirement has value one.
iii) The number of facilities is the different from the number of jobs.

5.12 Further Readings

- Ackoff, R.L. and Sasieni, M.W., Fundamentals of Operations Research, John Wiley & Sons. New Delhi, (1968).
- Taha, H. A.: Operations Research- An Introduction; Maxwell Macmilan, New York.
- Sharma, J.K. Operations Research, Theory and Applications, Macmillan India limited, New Delhi, (2003)

Unit-6: Different Methods of Finding Initial Feasible Solution of a Transportation Problem

Structure

6.1 Introduction
6.2 Objectives
6.3 Basic Feasible Solution of a Transportation Problem
   6.3.1 North-West Corner rule
   6.3.2 Matrix Minimum Method
   6.3.3 Vogel's Approximation Method (VAM)
6.4 Modified Distribution Method (MODI)
6.5 Maximization in a Transportation Problem
6.6 Summary
6.7 Key Words
6.8 Self Assessment Exercises
6.9 Answers
6.10 Further Readings

6.1 Introduction

This unit describes the different methods of finding initial feasible solution to Transportation Problem and MODI methods of finding optional solution of Transportation Problem.

6.2 Objectives

After studying this unit you should be able to:

- Locate a basic feasible solution of a transportation problem by various methods.
- Ascertain minimum transportation cost schedule by Modified Distribution (MODI) method.
- Discuss suitable method when the problem is to maximize the objective function instead of minimizing it.

6.3 Basic Feasible Solution of a Transportation Problem

In the previous unit, it is stated that a transportation problem with m origins and n destinations has m + n - 1 constraints. Thus any basic feasible solution has at most m + n - 1 non zero x, basic solutions.

A feasible solution is that which satisfies the row and column sum restrictions and non negative restrictions. A balanced transportation problem always has a feasible solution.

The following methods are available for the computation of an initial basic feasible solution.

6.3.1 The North West Corner Rule

In this method, we start from the North West (Upper left) corner cell, i.e., (1,1) cell and allocated the maximum possible amount here completely disregarding the transportation cost. The method is illustrated with the help of the following example:

Example 2.1: Consider a manufacture who operates three factories and dispatches his product to four different warehouses. The capacities of the three factories, the quantity of products required at the various warehouses and the cost of transporting one unit of the product from each of three factories to each of the four warehouses are presented as follows:

Factory	Warehouses				Capacity
	W_1	W_2	W_3	W_4	
F_1	1	2	1	4	30
F_2	3	3	2	1	50
F_3	4	2	5	9	20
Requirement	20	40	30	10	100

By applying the North West Corner rule to the above transportation, we obtain $x_{11} = 20$ which is min ($a_1 = 30$, $b_1 = 20$). Eliminating the first column as the requirement of the warehouse has been filled, the reduced transportation table becomes

Factory	Warehouses			Capacity
	W_1	W_2	W_3	

F_1	2	1	4	10
F_2	3	2	1	50
F_3	2	5	6	20
Requirement	40	30	10	80

The value of a1 is reduced to 10 in the revised transportation table as 20 units have already been supplied to warehouse W_1 from factory F_1. We now allocate 10 = min (a_1 = 10, b_2 = 40) units to the North West Corner of the revised transportation table. Thus $x_{22} = 10$. Elimination the first row as the first factory is unable to supply any more, the reduced transportation table becomes

Factory	Warehouses			Capacity
	W_2	W_3	W_4	
F_2	3	2	1	50
F_3	2	5	9	20
Requirement	30	30	10	70

The value of b_2 is reduced to 30 in the revised transportation table as 10 units have already have been supplied to warehouse W_2 from factory F_2. We now allocate 30 = min (a_2,b_2) to the North West Corner of the revised transportation table. Thus $x_{22} = 30$.

Proceeding in this way we obtain $x_{23} = 20$, $x_{33} = 10$ and $x_{34} = 10$. The resulting feasible solution is as shown in the following table:

	W_1	W_2	W_3	W_4	a_i
F_1	20	10			30
F_2		30	20		50

F_3			10	10	20
b_j	20	40	30	10	100

The corresponding transportation cost is given by

1*20 + 2*10 + 3*30 + 2*20 + 5*10 + 9*10 = 310.

It is noted that as soon as a value x_{ij} is obtained, a row or a column is eliminated from further consideration. The last of x_{ij} eliminated both a row and a column.

Hence a feasible solution obtained by the North West corner rule can have at most m + n - 1 positive x_{ij} if the transportation problem has m origins and n destinations. Thus the solution is a feasible solution. In the present problem, we have a basic solution with six non zero x", as m = 3, n = 4. Thus it has non degenerate basic feasible solution.

6.3.2 Matrix Minimum Method

We examine the row and column corresponding to which cost c_{ij} is minimum in the entire transportation table. If these are two or more minimum costs, then we should select the row and the column corresponding to the **lower numbered** row. If they appear in the same row we should select the lower numbered column.

We choose the value of the corresponding x as much as possible subject to availability and requirement constraints. A row (column) is dropped which fulfills availability (requirement) constraints. The same procedure is repeated with the reduced transportation cost matrix. The method is illustrated with the help of the transportation problem presented in the example 2.1.

We observe that $c_{11} = c_{13} = c_{24} = 1$, which is the minimum transportation cost in the entire transportation table. We select $c_{11} = 1$. Hence $x_{11} = 20$ and the first column is eliminated from any further allocation. The reduced transportation matrix is

Factory	Warehouses			Capacity
	W_2	W_3	W_4	

F_2	2	1	4	10
F_3	3	2	1	50
F_4	2	5	9	20
Requirement	40	30	10	80

The value of a1 is reduced to 10 in the revised transportation table. $C_{13} = C_{24} = 1$ is the minimum transportation cost in the reduced transportation table. We select $c_{13} = 1$. Hence $x_{13} = 10$ and the first row is eliminated from any further allocation. The reduced transportation matrix is

Factory	Warehouses			Capacity
	W_2	W_3	W_4	
F_2	3	2	1	50
F_3	2	5	9	20
Requirement	40	20	10	70

The value of b_3 is reduced to 20 in the revised transportation table $c_{24} = 1$ is the minimum transportation cost in the revised transportation table. So $x_{24} = 10$. Proceeding in this way we observe that $x_{23} = 20$, $x_{32} = 20$ and $x_{24} = 10$. Thus the final feasible solution is as shown below:

	W_1	W_2	W_3	W_4	a_i
F_1	20		10		30
F_2		20	20	10	50
F_3		20			20

b_j	20	40	30	10	100

The transportation coat associated with the basic feasible solution obtained by the matrix minimum method is

$$1*20 + 1*10 + 3*20 + 2*20 + 1*10 + 2*10 = 180.$$

The minimum transportation cost obtained by this method is much lower than the corresponding cost of the basic feasible solution obtained by using North West corner rule. This is to be expected as the matrix minimum method take into account the minimum unit transportation cost while choosing the values of the basic variables.

6.3.3 Vogel Approximation Method (VAM)

Vogel Approximation Method for finding a basic feasible solution involved the following steps:

(i) We determine the penalty for each row and column of the given transportation table. The penalties are calculated for each row (column) by subtracting the lowest cost element in that row (column) from the next lowest cost element in the same row (column0.

(ii) We select the row or column which has the largest penalty among all the rows and columns. If the penalties corresponding to two or more rows or columns are equal, we select the topmost row and the extreme left column.

(iii)If c_{ij} is the minimum cost in the row or column corresponding to the largest penalty, we choose the value of the corresponding x_{ij} as much as possible subject to the row and column constraints. If min (a_i, b_j) occurs at i^{th} row then i^{th} row is eliminated; otherwise j^{th} row is eliminated.

The steps (ii) and (iii) are now performed on the reduced matrix until all the basic variables have been obtained.

Consider the transportation problem given in the example 2.1. The penalty for the various rows and columns is given as

	W_1	W_2	W_3	W_4	a_i	Penalty
F_1	1	2	1	4	30	1

F_2	3	3	2	1	50	1
F_3	4	2	5	9	20	2
b_j	20	40	30	10	100	
Penalty	2	1	1	3		

The highest penalty occurs in the fourth column. The minimum c_{ij} in this column is $c_{24} = 1$. Hence $x_{24}= \min(a_2, b_4) = 10$ and the fourth column is eliminated. The value of a_2 is reduced to 40. The penalty for various rows and column of the reduced cost matrix is

	W_1	W_2	W_3	a_i	Penalty
F_1	1	2	1	30	1
F_2	3	3	2	40	1
F_3	4	2	5	20	2
b_j	20	40	30	90	
Penalty	2	1	1		

The highest penalty occurs in the third row and the first column. We arbitrarily select the third row. The minimum c_{ij} in the row is $c_{32} = 20$. Hence $x_{32} = 20$ and the third row is eliminated. Proceeding in this way, we have $x_{11} = 20$, $x_{23} = 20$, $x_{22}= 20$. Thus the feasible solution is as shown below:

	W_1	W_2	W_3	W_4	a_i
F_1	20		10		30
F_2		20	20	10	50
F_3		20			20

b_j	20	40	30	10	100

The transportation cost corresponding to feasible solution obtained is

$$1*20 + 1*10 + 3*20 + 2*20 + 1*10 + 2*10 = 180.$$

The VAM provides a basic feasible solution whose cost is quite close to minimum transportation cost.

Activity 1.

Three plants (A,B,C) supply the requirement of five districts (E,F,G,H,I). The availability at the plants the requirement of the districts and the unit transportation costs are given in the following table:

Plants	Districts					Availability
	E	F	G	H	I	
A	1	9	13	36	51	50
B	24	12	16	20	1	100
C	14	33	1	23	26	150
Requirement	100	70	50	40	40	300

Find an initial basic feasible solution of the transportation problem by using

i) North West Corner Rule
………………………………………………………………………………
…

ii) Matrix Minimum Method
………………………………………………………………………………
…

iii) Vogel Approximation Method
………………………………………………………………………………
…

6.4 Modified Distribution (MODI) Method

The modified distribution method, also known as MODI method or U-V. Method provides a minimum cost solution (optimum solution) to the transportation problem. The following steps are involved in the modified distribution method;

1) Obtains a basic feasible solution of the transportation problem using one of three methods descried in the previous section.

2) Introduce a dual variables corresponding to the row constraints and column constraints. For the transportation with m origins and n destinations, there will be m + n dual variables. The variable corresponding to the row constraints are denoted by u_j (i=1,2,-----,m) while the dual variables corresponding to column constraints are denoted by vi (i=1,2,-----,n). The values of the dual variables should be determined such that

$$u_i + v_j = c_{ij} \; if \; x_{ij} > 0$$

3) One of the dual variables$u_i = 0 \; or \; v_j = 0$ is chosen arbitrarity. Generally, we choose that $u_i = 0 \; or \; v_j = 0$ for which the corresponding row or column has the maximum number of $x_{ij} => 0$.

4) Calculate cell evaluations

$$\Delta_{ij} = c_{ij} - (u_i + v_j)$$

for all those cells which $x_{ij} = 0$ i.e. for unoccupied cells. If all $\Delta_{ij} \geq 0$ the corresponding solution of the transportation problem is optimum.

If one or more of $\Delta_{ij} \geq 0$, the solution is not optimum. In this case as new basic feasible solution if formed. For this purpose, we choose the cell for which Δ_{ij} is least and allocate in this cell as much a possible subject to the row and the column constraints. The allocation of a number of adjacent cell are adjusted so that a basic variables become non basic.

5) A new set of dual variables are computed and entire procedures is repeated until we get the optimum solution.

Let us consider the transportation problem in the example 2.1 with a basic feasible solution obtained by North West Corner Rule.

	W_1	W_2	W_3	W_4	a_i
F_1	1	2	1	4	30
F_2	3	3	2	1	50
F_3	4	2	5	9	20
b_j	20	40	30	10	100

1) The initial variable feasible solution by North West Corner rule is

$$x_{11}= 20,\ x_{22} = 10,\ x_{23} = 20,\ x_{33}= 10,\ x_{34}=10$$

2) The dual variables u_1, u_2, u_3 *and* v_1, v_2, v_3, v_4 can be computed for the cells where $x_{ij} \geq 0$ i.e. for the occupied cells from the following equations:

$$u_1 + v_1 = 1, u_1 + v_2 = 2, u_2 + v_2 = 3, u_2 + v_3 = 2, u_3 + v_3 = 5, u_3 + v_4 = 9$$

3) The dual variable u_1, u_2 *and* u_3 occurs equal number of times (2) in the equation, we choose arbitrary $u_1 = 0$. The value of the dual variables are

$$u_1 = 0,, u_2 = 1, \qquad u_3 = 4, v_2 = 2, v_3 = 1, v_4 = 5$$

We now calculate cell evaluations $\Delta_{ij}= c_{ij} - (u_i + v_j)$ for all the cells where $x_{ij} = 0$ i.e. for unoccupied cells. There are four negative values-1, -5, -1, -4. The most negative value -5 occurs to the cell (2,4).

Thus in the next iteration x_{24} will be a basic variable changing one of the present basic variable non basic. We observe that for allocating one unit in cell (2,4) we have to reduce one unit in cell (2,3) and (3,4) and increase one unit in cell (3,3). The maximum value that can be allocated to cell (2,4) is 10 subject to the row and column constraints. The revised basic feasible solution (second basic feasible solution) is

$x_{11}= 20, x_{12} = 10, x_{22} = 30, x_{23}= 10, x_{24}= 10, x_{33} = 30$

The dual variables u_1, u_2, u_3 *and* v_1, v_2, v_3, v_4 can be computed for the second basic feasible solution from the following equations;

$$u_1 + v_1 = 1, u_1 + v_2 = 2, u_2 + v_2 = 3, u_2 + v_3 = 2, u_2 + v_4 = 1, u_3 + v_4 = 5$$

The new set of dual variables choosing arbitrarily $u_2 = 0$ as it occurs most often in the equations is

$$u_1 = -1,, u_2 = 0, u_3 = 3, v_1 = 2, v_2 = 3, v_3 = 2, v_4 = 1$$

The most negative value (-4) of cell evolutions Δ_{ij} occurs to the cell (3,2). Thus in the next iteration X_{32} will be a basic variable changing one the second basic variable non-basic. The maximum value that can be allocated to cell (3,2) is 20 subject to the row and column constraints. The next revised basic feasible solution (third basic feasible solution) is

$x_{11}= 20, x_{12} = 10, x_{22} = 10, x_{23}= 30, x_{24}= 10, x_{32} = 20.$

It can be verified that the new set of dual variables satisfies the optimality condition. The corresponding transportation cost is

1*20 + 2*10 + 3*10 + 2*30 + 1*10 + 2*20 = 180.

Which is less than the transportation cost corresponding to the initial basic feasible solution.

Activity 2:

1) Compute the dual variables of the third iteration in the above example and verify that the solution presented is the optimum solution.

   ………………………………………………………………………………
   …

2) Verify that the initial basic feasible solution provided by Vogel approximation is the optimum solutions

   ………………………………………………………………………………
   …

6.5 Maximization in a Transportation Problem

There are certain types of transportation problems where the objective function is to be maximized instead of being minimized. These problems can be converted into the usual minimization problem by subtracting all elements of the cost matrix from the highest elements of the matrix. The other procedure is the same as that of minimization problem.

6.6 Summary

The $x_{ij} \geq 0$ which satisfy the row and column restrictions is known as basic feasible solution. A transportation problem with m origin and n destinations has at most m+n-1 non zero x_{ij}. If one or more basic variables are zero solution is said to degenerate. A number of techniques are available for computing and initial basic feasible solution of a transportation problem. These are North West Corner Rule, Matrix Minimum Method and Vogel's Approximation Method (VAM). The last two methods give much better solution than the first one. Optimum solution of a transportation can obtained from Modified Distribution (MODI) Method.

6.7 Key Words

A Basic Feasible Solution is the non negative values of the amount of the consignment that satisfy the row and column restrictions.

The North West Corner Rule is a method of computing a basic feasible solution of a transportation problem where the basic variables are selected from the North West Corner, i.e. top left corner.

The Matrix Minimum Method is a method of computing a basic feasible solution of a transportation problem where the basic variables are chosen according to the minimum unit cost of transportation.

The Vogel's Approximation Method (VAM) is an iterative procedures of computing a basic feasible solution by considering row and column penalties of the transportation problem.

The Modified Distribution Method (MODI) is the method of computing optimum solution of a transportation problem.

6.8 Self Assessment Exercises

E-10 Consider the following transportation with the following unit transportation costs, availability and requirements

Origin	Destination	Available

	1	2	3	4	
1	19	30	50	10	7
2	70	30	40	60	9
3	40	08	70	20	18
Requirement	5	8	7	14	34

Find a initial basic feasible solute of the problem by (i) North West Corner Rule (ii) Matrix minimum method (iii) VAM. Compute the corresponding transportation costs.

E-11 Find the optimum solution in case of (iii) of the transportation problem given in exercise 1.

E-12 A manufacturer has distribution centers at locations A, 13 and C. These locations have availability of 200, 160 and 90 units of his product, respectively. His retail outlets at 0, E and F require 180, 120 and 150 units respectively. The transportation cost (in rupees) per unit between each centre outlet is given below:

	D	E	F	Availability
A	16	20	12	200
B	14	08	18	160
C	26	24	16	90
Requirement	180	120	150	450

Determine the optimum distribution to minimize the cost of transportation.

E-13 A transportation has three origins and four destinations. The unit costs of transportation, availability at the origins and the requirement at the destinations are given below:

Origin	Destination	Available

	D_1	D_2	D_3	D_4	
O_1	6	3	5	4	22
O_2	5	9	2	7	15
O_3	5	7	8	6	8
Requirement	7	12	17	9	45

The present allocation is follows:

O_1 to D_2 12; O_1 to D_3 1; O_1 to D_4 9; O_2 to D_3 15; O_3 to D_1 15; O_3 to D_1 7 and O_3 to D_3 2. Check if the allocation is optimum? If not find an optimum solution.

6.9 Answers

Activity 1

i) North West Corner Rule

$x_{11} = 50$, $x_{21} = 50$, $x_{22} = 50$, $x_{32} = 20$, $x_{34} = 40$, $x_{35} = 40$

Minimum cost 4520.

ii) Matrix Minimum Method

$x_{11} = 50$, $x_{33} = 50$, $x_{22} = 60$, $x_{31} = 50$, $x_{32} = 10$, $x_{34} = 40$

Minimum cost 2810.

iii) Vogel's Approximation Method

$x_{11} = 50$, $x_{22} = 60$, $x_{31} = 50$, $x_{25} = 40$, $x_{32} = 10$, $x_{33} = 50$, $x_{34} = 40$

Minimum cost 2810.

Activity 2

1) $u_1 + v_1 = 1, u_1 + v_2 = 2, u_2 + v_2 = 3, u_2 + v_3 = 2, u_2 + v_4 = 1, u_3 + v_2 = 2, u_1 = -1, u_2 = 0, u_3 = -1, v_1 = 2, v_2 = 2, v_3 = 2, v_4 = 1. All\ \Delta_{ij} = c_{ij} - (u_i + v_j) \geq 0, the\ solution\ is\ optimum.$

2) $u_1 + v_1 = 1, u_1 + v_3 = 1, u_2 + v_2 = 3, u_2 + v_3 = 2, u_2 + v_4 = 1, u_3 + v_2 = 2, u_1 = -1, u_2 = 0, u_3 = -1, v_1 = 2, v_2 = 3, v_3 = 2, v_4 = 1. All\ \Delta_{ij} = \geq 0, the\ solution\ is\ optimum.$

Self Assessment Exercises:

i) North West Corner Rule $x_{11} = 5$, $x_{12} = 2$, $x_{22} = 6$, $x_{23} = 3$, $x_{33} = 4$, $x_{34} = 14$. Minimum transportation cost 1015.

ii) Matrix Minimum Method $x_{14} = 7$, $x_{21} = 2$, $x_{23} = 7$, $x_{31} = 3$, $x_{32} = 8$, $x_{34} = 7$. Minimum transportation cost 779.

iii) VAM $x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$ Minimum transportation cost 779

2) $x_{11} = 5$, $x_{14} = 2$, $x_{22} = 2$, $x_{23} = 7$, $x_{32} = 6$, $x_{34} = 12$, Minimum transportation cost 743.

3) $x_{11} = 140$, $x_{13} = 60$, $x_{21} = 40$, $x_{22} = 120$, $x_{33} = 90$, Minimum transportation cost Rs. 5920.00

4) The given allocation is not optimum. Optimum solution is $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$, Minimum transportation cost 149.

6.10 Further Readings

- Ackoff, R.L. and Sasieni, M.W., Fundamentals of Operations Research, John Wiley & Sons. New Delhi, (1968).
- Taha, H. A.: Operations Research- An Introduction; Maxwell Macmilan, New York.
- Mustafi, C.K., Operations Research Methods and Practice, Wiley Eastern Limited, New Delhi, 1988.
- Sharma, J.K. Operations Research, Theory and Applications, Macmillan India limited, New Delhi, (2003)

Unit-7: Solution of Assignment Problem with using Hungarian Method

Structure:

7.1 Introduction
7.2 Objectives
7.3 Solution of an Assignment Problem
7.3.1 Hungarian Method
7.4 Maximization in an Assignment Problem
7.5 Key Words
7.6 Self Assessment Exercises
7.7 Answers
7.8 Summary
7.9 Further Readings

7.1 Introduction

This unit gives the solution of AP using Hungarian method.

7.2 Objectives

After studying this unit you should be able to:

- Use Hungarian method for solving an assignment problem.
- Modify the assignment problem when the objective function is to be maximized.

7.3 Solution of an Assignment Problem

The assignment problem with n facilities and n jobs has n! possible assignment. One way of finding an optimum assignment is to write first all n! possible arrangements and then evaluate its measure of effectiveness (total time. Total distance or total cost) for each arrangement and select the assignment. Which optimize the measure of effectiveness. The method leads to a computation problem for a large assignment. For moderate value of n. say n = 10 the possible number of arrangement is 3628800.

The basic result (Mustafi, 1988) on which the solution of an assignment problem is based can be stated as follows:

If a constant is added to every element of row and / or a column of the cost matrix of an assignment problem the resulting assignment problem has the same optimum solution as the original problem and vice-versa.

This result can be used in two different ways to solve the assignment problem. If an assignment problem has some negative cost elements, we may convert it into an equivalent assignment problem by adding a suitable large constant to the elements of the relevant row or column to make all the cost elements are non-negative. Next we look for a feasible solution which has zero assignment cost after adding suitable constants to the elements of various rows and column. Since it has been assumed the all the cost elements are non-negative, this assignment must be optimum. Based on this principle a computational technique known as Hungarian Method is developed which is discussed below:

7.3.1 Hungarian Method

The method consists of following steps when the objective function is that of minimization type.

1) Find the smallest element in each row of the cost matrix of the given problem. Subtract this smallest element from each element of that row. Therefore there will be at least one zero in each row of this matrix known as the first reduced cost matrix.

2) Find the smallest element in each column of the reduced cost matrix. Subtract this smallest cost element from each element in that column. As a result of this each row and column now has at least one zero value in the second reduced cost matrix.

3) Determine an assignment as follows:

   i) Select a single zero value cell in each row or column that has not be assigned or eliminated and box that zero as an assigned cell.

   ii) For every zero that has assigned, cross out al other zeros in the same row and or column.

   iii) If there are two or more zeros in a row and / or column and one can not be chosen by inspection, choose the assigned zero cell arbitrarily.

   iv) The above process may be continued until every cell containing zero is either assigned (boxed) or crossed out.

4) An optimal assignment is reached, if the number of assigned cells is equal to the number of rows (columns). If a cell containing zero was chosen arbitrary, there may be an alternative optimum. If an optimum solution does not exist, go to step 5.

5) Draw a set of horizontal and vertical lines to cover all the zeros in the reduced cost matrix obtained in step 3, by using following procedure:

i) Mark check (√) to those rows where no assignment has been made.

ii) Examine the checked (√) rows. If any assigned zero occurs in those rows, check (√) the respective columns that contain those assigned zeros.

iii) Examine the checked (√) the respective rows that contains those assigned zeros.

iv) The process may be continued until no more zeros or columns can be checked.

v) Draw lines through all unchecked rows and through all checked columns.

6) Examine those elements that are not covered by any line. Choose the smallest of these elements and subtract this smallest from all those elements which are not covered a line. Add this smallest element to every element that lies at the intersection of two lines.

7) Repeat steps 3 to 6 until an optimal solution is obtained.

Example 1: Five men are available to perform five different jobs. Only one man can work on anyone job. The time (in hours) each will take to perform each job is given in the time matrix:

Jobs

	1	2	3	4	5
A	10	5	13	15	16
	3	9	18	13	6
	10	7	2	2	2

	B
Men	C
	D
	E

The objective is to assign men to job such the total man-hours is a minimum.

Solution:

Step 1: Subtract the smallest element of a row from each element of that row, we get the first reduced time matrix

Table 1

Jobs

		1	2	3	4	5
	A	5	0	8	10	11
	B	0	6	15	10	3
Men	C	8	5	0	0	0
	D	0	4	2	0	5
	E	3	5	6	0	8

Step 2: Subtract the smallest element of a column from each element of that column in the above reduced matrix, the second reduced time matrix

Table 2

Jobs

1	2	3	4	5
5	[0]	8	10	11
[0]	6	15	10	3

A

B

Men C

D

E

The Second reduced matrix is that same as that of the first reduced matrix as each column contains zero element.

Step 3: Determine an Assignment

Examine all the rows starting A one –by-one until a row containing only one zero element is located. Row A of above table 2 has only one zero in the cell (A, 2), Box (assign) this zero. No other zero exists in the boxed column. Row B has only one zero in the cell (B,1), Box this zero. A zero in the boxed column in crossed out. Row E has only one zero in the cell (E,4). Box this zero. All zeros in the boxed column are crossed out.

We now examine each column starting from column 1. The column 3 has one zero in the cell (C,3), Box this zero. Crossed out zeros in the boxed row. All zeros in the table now are either boxed (assigned) or crossed out (eliminated) as shown in Table2.

Step 4: The solution obtained in step 2 is not optimal, because we are able to make four Assignments when five are required.

Step 5: Cover all the zeros of table 2 with four lines, since four assignment are made. Mark (√) row D, since it has no assignment then mark (√), columns 1 and 4 since row D has zero elements in these columns. We then mark (√) row Band E, since column 1 and 4 has an assigned zero in row Band E respectively.

Notice that on other rows and columns can be marked. We may draw four lines through unchecked rows A and C checked columns 1 and 4. This shows in table 3

Table 3

1 2 3 4 5

A	5	[0]	8	10	11	
B	[0]	6	15	10	3	√
C	8	5	[0]	0	0	
D	0	4	2	0	5	√
E	8	5	6	[0]	8	√

Step 6: Develop the new revised matrix. Examine those elements that are not covered by a line in Table 3. Select the smallest elements. This is 2 (two), by subtracting 2 from the uncovered elements and adding it to elements 5,10,8 and 0 at cells (A,1), A,4), (C,1) and (C,4) respectively that lie at the intersection of two lines, we get the new revised matrix as shown in Table 4.

Table 4

	1	2	3	4	5
A	5	7	8	12	11
B	0	4	13	10	1
C	10	5	0	2	0
D	0	2	0	0	3
E	3	3	4	0	6

Step 7: Go to step 3 and determine as assignment. Examine each row in Table 4, we find that rows A, Band E contain only one zero at cell (A,2), (B,1), and (E,4) respectively. Box the zero at cell (A,2). The column 2 has no zero. Box the zero at cell (B,1) and cross out other zero in the column 1. Box the zero at cell (E,4) and cross out other zero in the column 1. Box the zero at cell (E,4) and cross out other zero in column 4.

Now, examine each column, we find that column 5 has only one zero at cell (C,5). Box this zero and cross out other zero in row C. Again examine row D has zero at cell (D,3). Box this zero. All zeros in the table now are either assigned or cr0ssed out as shown in table 5.

Table 5

	1	2	3	4	5
A	7	[0]	8	12	11
	[0]	4	13	10	1
	10	5	[0]	2	[0]

B

C

D

E

Since the number of assignments is equal to the number of rows (columns), the assignment in Table 5 is optimal. The optimal assignment among men and job is

$$A \rightarrow 2, B \rightarrow 1, C \rightarrow 5, D \rightarrow 3, E \rightarrow 4.$$

The minimum total time (in hours) is 5+3+2+9+4= 23.

Activity 1:

A metal shop has five jobs to be done has five machines to do them. The assignment of jobs to machines must be done on a one to one basis. The cost of processing each job on any machine is given below:

Unit of Cost

		Jobs			
		I	II	III	IV
Machines	M_1	20	25	22	28
	M_2	15	18	23	17
	M_3	19	17	21	24
	M_4	25	23	24	24

Find the assignment which minimizes the total cost of the work.

7.4 Maximization in an Assignment Problem

There are problems where certain facilities have to be assigned to a number of jobs so as to maximize the overall performance of the assignment. These problems can be converted into a minimization problem in the following way. Locate the largest element in the given assignment table and then subtract all the elements of the table from the largest element. The revised assignment problem so obtained can be solved by using ***Hungarian method***. The procedure is illustrated with the help of the following example.

Example 2: A marketing manager has five salesman and five sales districts. Considering the capabilities of the salesman and the nature of the districts, the marketing manager estimate that sales per month (in thousand rupees) for each salesman in each district would be as follows:

Districts

		D_1	D_2	D_3	D_4	D_5
Salesman	A	5	11	10	12	4
	B	2	4	6	3	5
	C	3	12	5	14	6
	D	6	14	4	11	7
	E	7	9	8	12	5

Find the assignment of salesman to districts that will result in maximum sales.

Solution: This is maximization assignment problem. The largest element in the assignment Table is 14. We subtract each element from 14. The revised assignment Table is given below:

Districts

		D_1	D_2	D_3	D_4	D_5
Salesman	A	9	3	4	2	10
	B	12	10	8	11	9
	C	11	2	9	0	8
	D	8	0	10	3	7
	E	7	5	6	2	9

We now apply Hungarian Method to obtain the minimum sales assignment of the revised problem. The solution is

$$Salesman\ 1 \rightarrow District\ D_3$$

$$Salesman\ 2 \rightarrow District\ D_5$$

$$Salesman\ 3 \rightarrow District\ D_4$$

$$Salesman\ 4 \rightarrow District\ D_2$$

$$Salesman\ 5 \rightarrow District\ D_1$$

The maximum sales of assignment is 50

Activity 2: Verify the solution of the assignment problem given in Example 2.

7.5 Key Words

Hungarian Method is a technique to find out optimum solution of a assignment problem.

7.6 Self Assessment Exercises

E-14 One car is available at each of the stations 1,2,3,4,5 and one car is required at each of the stations 6,7,8,9, and 10. The distances between the various stations are given in matrix below. How should the cars be dispatched so as to minimize the total mileage covered?

	6	7	8	9	10
1	15	21	6	4	9
2	3	40	21	10	7
3	9	6	5	8	10
4	14	8	6	9	3
5	21	18	18	7	4

E-15 There are four jobs to be assigned one each to four machines and the associated cost matrix is as follows.

		Machines			
		A	B	C	D
Jobs	1	41	72	39	52
	2	22	29	49	65
	3	27	39	60	51
	4	45	50	48	52

How should the jobs be assigned to the machines as to minimize the total cost of assignment.

E-16 A team of 5 horses and 5 riders has entered a jumping show contest. The number of penalty points to be expected when each rider rides any hours shown below:

		R_1	R_2	R_3	R_4	R_5
	H_1	5	3	4	7	1
	H_2	2	1	7	6	5
Horse	H_3	4	1	5	2	4
	H_4	6	8	1	2	3
	H_5	4	2	5	7	1

How should the horses be allocated to the riders sp as to minimize the penalty points.

1) $1 \to 9, 2 \to 6, 3 \to 7, 4 \to 8, 5 \to 10. Minimum\ Distance\ 23\ miles$
2) $1 \to C, 2 \to B, 3 \to A, 4 \to D. Minimum\ cost\ 147$
3) $H_1 \to R_5, H_2 \to R_1, H_3 \to R_4, H_4 \to R_3, H_5 \to R_2, . Minimum\ points\ 8.$

7.7 Answers

Activity 1.

(i) $M_1 \to I, M_2 \to IV, M_3 \to II, M_4 \to III$

(ii) $M_1 \to III, M_2 \to I, M_3 \to II, M_4 \to IV. Minimum\ Cost\ 78.$

7.8 Summary

Although assignment problem can be formulated as linear programming problem, it is solved by a special technique known as Hungarian Method because of its special feature. The Hungarian method is based on the principle that if a constant is added to every element of a row and / or a column of cost matrix the optimum solution of the resulting assignment problem is the same as the original problem and vice-versa. The original cost matrix can be reduced to another cost matrix by assign constants to the elements of rows and columns where the total completion time or the total cost of an assignment is zero. As the optimum solution remains uncharged, the optimum solution of the resulting assignment after this reduction is also the optimum solution of the original problem.

7.9 Further Readings

- Ackoff, R.L. and Sasieni, M.W., Fundamentals of Operations Research, John Wiley & Sons. New Delhi, (1968).

- Taha, H. A.: Operations Research- An Introduction; Maxwell Macmilan, New York.
- Mustafi, C.K., Operations Research Methods and Practice, Wiley Eastern Limited, New Delhi, 1988.
- Sharma, J.K. Operations Research, Theory and Applications, Macmillan India limited, New Delhi, (2003)

U.P. Rajarshi Tandon Open
University, Prayagraj

DECSTAT – 109
Operation Research

Block: 4 Theory of Games

Unit – 8	:	Basic Concepts of Games Theory	122 - 133
Unit – 9	:	Dominance Rule, Equivalence of Rectangular Games with Linear Programming	134 - 152

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Department of Statistics, University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics, D. D. U. Gorakhpur University, Gorakhpur.

Dr. Shruti
Member-Secretary
School of Sciences, U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Shubra Katara **Writer**
Department of Statistics, Bareily College, Bareily

Prof. V. K. Sehgal **Reviewer**
Department of Mathematical Sciences and Computer Applications
Bundelkhand University, Jhansi

Prof. R. C. Yadava **Editor**
Department of Statistics, Banaras Hindu University, Varanasi

Dr. Shruti **Course/ SLM Coordinator**
School of Sciences, U. P. Rajarshi Tandon Open University, Prayagraj

DECSTAT – 109 **OPERATION RESEARCH**
©UPRTOU
First Edition: *March 2008* (Published with the support of the Distance Education Council, New Delhi)
Second Edition: ***July 2021***
ISBN :

 Printed and Published by Dr. Arun Kumar Gupta Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2021.
Printed By: K.C.Printing & Allied Works, Panchwati, Mathura -281003.

Block & Units Introduction

The ***Block - 4 – Theory of Games,*** is the last block of this SLM ,with its two units, which covers a very important concept known as theory of games. The game theory is a type of decision theory where all possible alternatives available to the opponent are considered before taking a decision. J. Von Neumann, the renowned mathematician, gave the minimax (maximin) criterion to carry out the analysis of the problems in game theory. It is very important to understand the principles of game theory as these concepts give the algorithm to handler the actual problems of decision theory in competitive situations in industry or elsewhere.

Unit – 8 – Basic Concepts of Game Theory, deals with introduction to game theory, two-person zero-sum game, different types of strategies and games with or without a saddle point.

Unit – 6 – Dominance Rule, Equivalence of Rectangular Games with Linear Programming, deals with the theories of dominance rule, solution methods of games without saddle point and equivalence of rectangular games with linear programming.

At the end of every unit the summary, self assessment questions and further readings are given.

Unit-8: Basic Concept of Game Theory

Structure

8.1 Introduction
8.2 Objectives
8.3 A Game, Pure and Mixed strategies
8.4 Two-Person Zero-sum, game
8.5 Pay-off Matrix
8.6 Games with saddle point
8.7 Games without saddle point and mixed strategies
8.8 Methods of solving game problems
8.9 Exercises
8.10 Summary
8.11 Further Readings

8.1 Introduction

Many decisions are taken in a competitive situation in which the outcome depends not on those decisions alone but rather on the interaction between the decision maker and that of a competitor. The term 'game' now includes not only pleasurable activities of this kind, but also much earnest competitive situations of war and peace. It was not coincidental that the classic work on the theory of games was first published during the Second World War. Many competitive situations are still, too, complex for the theory in its present state of development to solve. Other methods of which war games are long established examples and business games of more recent origins have been used. The availability of computers has allowed increasingly large scale operations to be represented with great realism. The theory of games has been developed alongside, gaming techniques and knowledge of the concepts involved especially the importance of the role of change help of clarify the issues in many decision making process.

8.2 Objectives

After studying this unit, you should be able to know what is

- A game
- A two person zero sum game
- Saddle point of a game
- Maxmine and minimax principles

- Methods of solving a problem of games theory using maximin and minimax principle.
- Concept of mixed strategies

8.3 A Game, Pure and Mixed Strategies

A competitive situations is called a game if it has the following properties;

1. There are finite number of participants called players.
2. Each players has finite number of strategies available
3. Every game results in an outcome.

Number of Players: If a game involves only two players (Competitors), then it is called a **two-person game.** However if the number of players are more than two say n, the game is known as **n- person game.**

Sum of Gains and Losses: If a game involves only gains that exactly the losses to another players, such that sum of gains and losses equals zero, then the game is said to be **zero-sum game** otherwise it is said to be **non-zero game**.

Strategy: The strategy for a players is the list of all possible actions (or moves or courses of action) that he will take for every pay off (outcome) that might arise. It is assumed that the rules governing the choices are known in advance to players and are expressed in terms of numerical values. Here it is not necessary that players have definite information about each other's strategies.

A pure Strategy is a decision rule always to select a particular course of action.

If a player is guessing as to which activity is to be selected by the other players then a probabilities situation is obtained and the objective function is to maximize the ones gains or minimize losses. A mixed strategy is a selection among pure strategies with fixed probabilities.

8.4 Two-Person Sum-Zero Game

A game with only two players (say players A and players B) is called a two-person zero-sum game if the losses of one player are equivalent to the gains of other the sum of their net gains is zero. It is also known as rectangular game.

Basis Assumptions of the Game:

1. Each players has available to him a finite number of possible courses of action. The list may not be same for each player.
2. Player A attempts of maximize gains and player B minimizes losses.
3. The decision of both players are made individually prior to the start of the play with no communication between them.
4. The decisions are made simultaneously and also announced simultaneously so that neither player has an advantages resulting from direct knowledge of the other player's decision.
5. Both the players know not only the possible payoffs to themselves but also of each other.

8.5 Pay-off Matrix

In a two person game, say with player A and players B, the payoff matrix can be formed by adopting following rules:

(i) Let the player A has m activities and player B has n activities

(ii) We can from a payoff matrix for player A as well as payoff matrix for player B.

(iii) Row designations for each matrix are activities available to players A.

(iv) In the same player B's activities are column designations in both matrices.

(v) Let the V_{ij} be the entry corresponding to ith row and jth column in the payoff matrix for A then V_{ij} will be payment made to a when A chosen activity i and B chooses activity j. This payment may be positive or negative as the case may be. In other words, when two players are playing gain of one person is loss of other person. So depending upon the activity one chooses he may gain or loose. Whenever one gains it is a positive payment and vice versa.

(vi) In a two-person zero-sum game the sum of corresponding entries in payoff matrices of A and B is zero. In other words the cell entry V_{ij} in player B's payoff matrix will be negative of the corresponding cell entry in Player A's payoff matrix.

Following is the way payoff matrices of players A and B will look:-

The Player A's Payoff Matrix

Player A ↓ **Player B** →

	1	2	…..	j	…	n
1	V_{ij}	V_{12}	……	V_{1j}		V_{1n}
2	V_{21}	V_{22}	…….	V_{2j}	…	V_{2n}
:	:	:	:	:	:	:
i	V_{i1}	V_{i2}	……	V_{ij}	…	V_{in}
:	:	:	:		:	
m	V_{m1}	V_{m2}	…..	V_{mj}	…	V_{mn}

The Player B's Payoff Matrix

Player A ↓ **Player B** →

	1	2	…..	j	…	n
1	$-V_{ij}$	$-V_{12}$	……	$- V_{1j}$	$- V_{i2}$	$-V_{1n}$
2	$-V_{21}$	$-V_{22}$	…….	$- V_{2j}$	$- V_{i2}$	$-V_{2n}$
:	:	:	:	:	:	:
i	$- V_{i1}$	$- V_{i2}$	……	$- V_{i1}$	$- V_{i2}$	$-V_{in}$
:	:	:	:		:	
m	$- V_{m1}$	$- V_{m2}$	…..	$- V_{mj}$	$- V_{i2}$	$-V_{mn}$

Example 1: Consider a coin throwing game between two players. After a throw each player selects a side i.e. head (H) or tail (T). if the outcomes match (H, H or T.T.), A wins Re 1 from B, otherwise B wins Re 1 from A. Thus the payoff matrix for A and B will be.

Payoff Matrix for A

	B

A		H	T
	H	+1	-1
	T	-1	+1

Payoff Matrix for A

	B		
		H	T
A	H	-1	+1
	T	+1	-1

This is an example of two person zero-sum game. *The payoff matrix of B follows from payoff matrix of A and therefore it is customary to write the payoff matrix of A only.*

8.6 Games with Saddle Point

Before describing a saddle point we explain the minimax (maximin) principle

Minimax and Maximin Principle:

The Minimax and Maximin Principle states that if a player lists the worst possible outcomes of all his strategies he will choose that strategy to be most suitable for him which corresponds to the best of these worst outcomes. Such a strategy is called an optimal strategy.

Consider the payoff matrix to a game which represents payoff of player A. Now the objective of the study is to know how these players must select their respective strategies so that they may optimize their payoff. Such a decision making criterion is referred to as the minimax-maximin principle.

Step 1: Select the minimum element in each row and enclose it in a rectangle

(□)

Step 2: Select the maximum element in each column and enclose it in a circle

(○)

Step 3: Find out the element which is enclosed by the rectangle as well as the circle. Such element is the value of the game and that position is called as the **Saddle point**.

The saddle point of a game is point in payoff matrix such it has got minimum payoff in the rows and maximum payoff in its columns.

Value of the Game: The payoff at saddle point (r, s) is called the value v of game and it is equal to the maximin ($\bar{v}$) or the minimax ($\bar{v}$) of the game.

A game is said to be a fair if $\bar{v} = \underline{v} = 0$
A game is said to be strictly determinable if $\bar{v} = v = v$.
A saddle point of the payoff matrix is also referred as the **Equilibrium of the Game.**

Example 2: Solve the game whose payoff matrix is given by

Player B

Player A

	I	II	III	IV	V
I	-2	0	0	5	3
II	3	2	1	2	2
III	-4	-3	0	-2	6
IV	5	3	-4	2	-6

Step 1: The first step is to select the row minimum and enclose it in rectangle

Step 2: The second step is select the column maximum and enclose it in a circle

Player A	Player B					
		I	II	III	IV	V
	I	-2	0	0	5	3
	II	3	2	1	2	2
	III	-4	-3	0	-2	6

	IV	5	3	-4	2	-6

It is clear that saddle point is (I, II) and the value of the game v=I

Player A uses his courses of action II through.

Player B uses his courses of action III through.

The value of the game is I (for player A and -1 for player B.)

Example 3: A company management and the labour union are negotiating a new three year settlement. Each of these has 4 strategies.

I	:	Hard and aggressive bargaining
II	:	Reasoning and logical approach
III	:	Legalistic Strategy
IV	:	Conciliatory approach

The costs to the company are given for every pair of strategy choice.

Company Strategies

		I	II	III	IV
Union Strategies	I	20	15	12	33
	II	25	14	8	10
	III	40	2	10	5
	IV	-5	4	11	0

What strategy will the two sides adopt? Also, determine the value of the game.

Solution: Select the row minimum and enclose it by a rectang

The select the column maximum and enclose it in a circle.

Union Strategies	Company strategies				
		I	II	III	IV
	I	20	1	12	35
	II	25	14	8	10

	III	40	2	10	5
	IV	-5	4	11	0

Hence the saddle point is (I,III) and the value of the game is 12

Hence the company will always adopt strategy III-Legalistic strategy and union will always adopt strategy 1- Hard and aggressive bargaining.

Example 4: Find the range of values of p and q which will render the entry (2,2) a saddle point of the game

Player B

	B		
Player A	2	4	5
	10	7	q
	4	P	6

Solution: Find ignoring the value of p and q the maximin and minimax values of the payoff matrix.

Player B

	B_1	B_2	B_3
A_1	2	4	5
A_2	10		q
A_3	4	p	

Maximin value $\bar{v}$ = 7= miniax value $\underline{v}$

This impose the condition on p ≤ 7 and on q ≤ 7

Hence the range of p and q will be p ≤ 7 , q ≤ 7.

8.7 Games without Saddle Point and Mixed Strategies

There are some games for which saddle point dose not exist. In such cases both the players must determine an optimal combination of strategies to find a saddle (equilibrium) point. The optimal strategy combination for each players may be determine by assigning to each strategy its probability of being chosen. The strategies so determined are called mixed strategies because they are probabilistic combinations of available choices of strategies.

The value of game obtained by the use of mixed strategies represents least payoff which players A can expect to win and the least which player B can loose.

The expected payoff a player in a game with arbitrary payoff matrix [aij] or order m x n is defined as

$$E(p,q) = \sum_{i-1}^{m} \sum_{i=1}^{m} p_1 a_{ij} q_j$$

$$= P^T AQ \ (in\ matrix\ notation)$$

Where, $P = (p_1\, p_2\, p_m) and\ Q = (p_1\, p_2\, p_n)$ denote the mixed strategies for players A and B respectively. Also, $p_1 + p_2 + \cdots .. + p_m = 1$ and $q_1 + q_2 + \cdots .. + q_n = 1$. A strategy with particular probability a player chooses can be interpreted as the relative frequency with which a strategy is chosen from the number for strategies of the game.

8.8 Method of Solving Game Problems

A mixed strategy game can be solved by different solution methods such as

1. Algebraic method.
2. Analytical or calculus method
3. Matrix method
4. Graphical method, and
5. Linear programming method.

These methods will be discussed in detail in the next unit.

8.9 Exercise

E-1 Find the saddle point and hence solve the following game:

	Player B			
Player A		B_1	B_2	B_3
	A_1	15	2	3
	A_2	6	5	7
	A_3	-7	4	0

E-2 Solve the game whose payoff matrix is given by

	Player B			
Player A		B_1	B_2	B_3
	A_1	15	2	3
	A_2	6	5	7
	A_3	-7	4	0

Find whether a saddle point exists here on not and also obtain the value of the game.

E-3 Find the saddle point and hence solve the following game:

	Player B				
Player A		B_1	B_2	B_3	B_4
	A_1	1	7	3	4
	A_2	5	6	4	5

	A_3	7	2	0	3

E-4 For the following payoff matrix for Player A, determine the optimal strategies for both the player A and B and the value of the game, using maximin principle.

	Player B					
Player A		B_1	B_2	B_3	B_4	B_5
	A_1	3	-1	4	6	7
	A_2	-1	8	2	4	12
	A_3	16	8	6	14	12
	A_4	1	11	-4	2	1

E-5 For what value of 'a' following game will be strictly determinable.

	Player B			
Player A		B_1	B_2	B_3
	A_1	a	6	2
	A_2	-1	a	-7
	A_3	-2	4	a

8.10 Summary

Game theory is a type of decision theory where all possible alternatives are considered before taking a decision. A game is an activity between two or more persons. Some rules are set for both players and after end of each activity each player either gain something or loose something. A quantitative measure of satisfaction one gets at the end of the day is called a payoff.

Competitive games are classified according to the number of players involved as a two person, three person or n person game.

8.11 Further Readings

- Taha, H. A.: Operations Research- An Introduction; Maxwell Macmilan, New York.
- Hadley, G.: Linear Programming; Addison Wesely.
- Kanti Swaroop et. Al. Introduction to Operation Research; Sultan Chand & Sons, New Delhi.
- P.K. Gupta: Linear Programmign and Theory of Games; Sultan Chand and Sons, New Delhi.

Unit-9: Dominance Rule, Equivalence of Rectangular Games with Linear Programming

Structure

9.1 Introduction
9.2 Objectives
9.3 Rectangular Games without Saddle Point
9.4 Dominance Property of Reducing the Size of the Game
9.5 Solution Methods of Games without Saddle Point
9.6 Equivalence of Rectangular Games with Linear Programming
9.7 Exercises
9.8 Summary
9.9 Further Readings

9.1 Introduction

As studied in earlier unit, if there is no saddle point in a game there is no optimal strategy to solve the game. In this unit this situation of a game is discussed. To solve such games, dominance rule will be discussed. Algebraic method, arithmetic method, graphical method and matrix methods will be explained. Equivalence of rectangular games with Linear Programming Problem will also be discussed.

9.2 Objectives

After reading this unit you will be able to understand

- Rectangular games without saddle point
- Dominance property
- Standard method of solving 2 x 2 game
- Algebraic method
- Matrix method
- Graphical method
- Equivalence of rectangular games with liner programming

9.3 Rectangular Games without Saddle Point

In most situations of a game, the payoff matrix has no saddle point and so the game has no optimal strategies. However the concept of optimal strategy can be applied to all matrix games by introducing a probability with choice of a strategy and mathematical expectation with payoffs.

Let player A chooses a particular activity i such that with probability x_i. The set of activities x = $(x_i, 1 \leq i \leq m)$ of probabilities constitute the strategy of A. Similarly, y= $(y_j, I \leq j \leq n)$ is the set of probabilities showing strategy of B in m x n rectangular game. The vector x = $(x_1, x_2, \ldots\ldots x_m)$ where $x_i \geq 0$, and $x_1 + x_2 + \ldots.. x_m = 1$ is known as the mixed strategy of the player A and on the same lines the vector y= $(y_j, I \leq j \leq n)$ where $y_1 + y_2 + \ldots.. + y_n$ is called the mixed strategy of player B.

The mathematical expectation of the payoff function in a game, whose payoff matrix is $\{v_{ij}\}$ is defined by

$$E\ (x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} (x_i, v_{ij}) y_j$$

Where x and y are the mixed strategies of the players A and B respectively.

The player A chooses x so as to maximize his minimum expectation and the player B should choose y in such a manner that minimizes the player A's greatest expectation. Symbolically

A tries for $\max_x \min_y E(x, y)$

And B tries for $\min_x \max_y E(x, y)$

Strategic Saddle Point: If we select a point (x_0, y_0) such that

$$\max_x \min_y E(x, y) = E(x_0 y_0) = \min_x \max_y E(x, y)$$

Then $(x_0 y_0)$ is called the strategic saddle point of the game with mixed strategies. *It is important to note that such saddle point will always exist.*

9.4 Dominance Property of Reducing the Size of the Game

We can sometimes reduce the size of a game's payoff matrix by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. The concept of dominance is especially useful for the evaluation of two person zero sum games where a saddle point does not exist.

General Rule:

1. If all the elements of a row, say k^{th}, are less than or equal to the corresponding elements of any other row, say r^{th}, then k^{th} row is dominate by the r^{th} row.

2. If all the elements of a column, say i^{th} are greater then or equal to the corresponding elements of any other column say s^{th}, then i^{th} column is dominated by s^{th} column.

3. Omit dominated rows or columns.

4. If some linear combination of some rows m^{th} and n^{th} dominates i^{th} row, i^{th} row will be deleted. Similar argument follows for columns.

Example 2.1: Players A and B play a game in which each has three coins, a five paise (5p), ten paise (10p) and twenty (20p). Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, then a win B's coin but if the sum is even then B wins A's coin. Find the best strategy for player and value of the game.

Solution: The pay off matrix for players A is

Player A	Player B		
	5p : B_1	10p : B_2	20p : B_3
5p : A_1	-5	10	20
10p : A_2	5	-10	-10
20p : A_3	5	-20	-20

It is clear that this game has no saddle point. Therefore, further we try to reduce the size of the given pay off matrix. Note that every element of column B_3 (strategy B_3 for player B) is more than equal to every corresponding elements of column B_2 (strategy B_2 for player B. Evidently the choice of strategy B_3 by the players B will always result in more losses as compared to that of selecting the strategy B. Thus strategy B_3 is inferior to B_2. Hence delete the strategy B_3 from the payoff matrix. The reduced payoff matrix is as:

Player A	Player B	
	B_1	B_2
A_1	-5	10

A_2	5	-10
A_3	5	-20

After column B_3 is deleted, it may be noted that strategy A_3 of player A is dominated by strategy A_2. Since profit due to strategy A_2 is greater than or equal to the profit due to strategy A_3 regardless of which strategy player B selects. Hence strategy A_3 may be deleted from further consideration. Thus the reduced payoff matrix is:

Player A	Player B		Row Minimum	
	B_1	B_2		
A_1	-5	10	-5	$\leftarrow Maximin$
A_2	5	-10	-10	
Column Maximum	5	10		
	$\uparrow Minimax$			

As shown in the reduced 2x2 matrix, the maximin value ($\bar{v} = 5$) is not equal to the minimax value (v=5). Hence there is no saddle point and one cannot determine the point of equilibrium. For this type of game situation, it is possible to obtain a solution by applying the concept of mixed strategies.

The optimal strategies for player A may be:

A_1 with probability $p_1 = \frac{1}{2}$

A_2 with probability $p_2 = \frac{1}{2}$

A_3 with probability $p_3 = 0$.

And value of the game is zero i.e. v= 0

Similarly optimal strategies for player B may be:

B_1 with probability $q_1 = \frac{1}{2}$

B_2 with probability $q_2 = \frac{1}{2}$

B$_3$ with probability $q_3 = 0$.

The value of the game remains zero i.e. v= 0.

9.5 Solution Method of Games without Saddle Point

(1) Standard Method

For any 2 * 2 two person zero-sum game without any saddle point having the payoff matrix for player A as

	B_1	B_2
A_1	a_{11}	a_{21}
A_2	a_{21}	a_{22}

The optimum mixed strategies

Where

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 & p_2 \end{bmatrix} \; and \; S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$p_1 = \frac{a_{22} - a_{21}}{(a_{21} + a_{22}) - (a_{12} + a_{12})} \; and \; p_1 + p_2 = 1$$

$$q_1 = \frac{a_{22} - a_{21}}{(a_{21} + a_{22}) - (a_{12} + a_{12})} \; and \; q_1 + q_2 = 1$$

And the value of game is

$$v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} + a_{21}) - (a_{12} + a_{21})}$$

(2) Algebraic Method

This method can be used to determine probability value by using different strategies by players A and B.

This method becomes quite lengthy when number of strategies for different strategies for both players are large. Consider a game where payoff matrix is

$$A = \left(a_{ij}\right)_{mzn}$$

Let (p_1, p_2, p_3) and (q_1, ….q_n) be the probabilities with which players A and B adopt their mixed strategies (A_1,…….A_m) and (B_1, B_2……B_n) respectively. A_1, A_2……., A_m are the rows and B_1, B_2,…….,B_n are columns of payoff matrix A. If v is the value of the game, then expected gain to player A when player B selects strategies B_1, B_2,…..B_n are by one can be determined as:

Since player A is the gainer and expects at least v as gain, we have

$$a_{11}p_1 + a_{12}p_2 + \cdots \dots \dots . + a_{1m}p_m \geq v$$

$$a_{21}p_1 + a_{22}p_2 + \cdots \dots \dots . + a_{2m}p_m \geq v$$

…………………………………………

…………………………………………

$$a_{n1}p_1 + a_{n2}p_2 + \cdots \dots \dots . + a_{nm}p_m \geq v$$

Where, $p_1 p_2 + \cdots \dots . + p_m = 1 \; and \;\; p_i \geq 0 \; for \; all \; i.$

Similarly, the expected loss to player B when player A adopts strategies A_1,……A_m one by one can be determined. Since player B is the looser player, therefore we have

$$a_{11}p_1 + a_{12}q_2 + \cdots \dots \dots . + a_{n1}q_n \geq v$$

$$a_{12}p_1 + a_{22}p_2 + \cdots \dots \dots . + a_{n2}p_n \geq v$$

…………………………………………

…………………………………………

$$a_{1m}q_1 + a_{2m}q_2 + \cdots \dots \dots . + a_{nm}q_n \geq v$$

Where, $q_1 + \cdots \dots . + q_n = 1 \; and \;\; q_j \geq 0 \; for \; all \; j.$

Now, to get the value of p_i's and q_i's from above inequalities these inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations so obtained is inconsistent, then at least one of the inequalities must hold as strict inequality. The solution can now be obtained only by applying trial and error method.

(3) Arithmetic Method

Arithmetic method provides as easy technique for obtaining the optimum strategies for each player in (2 x 2) games without saddle point. The method consists of the following steps:

Step 1: Find the difference of two numbers in column 1 and put it under the column II neglecting the negative sign if occurs.

Step 2: Find the difference of two numbers in Column II, and put it under the column I, neglecting the negative sign if occurs.

Step 3: Repeat the above two steps for the rows also.

The value thus obtained are called the oddments.

These are the frequencies with which the players must use their courses of action in their optimum strategies.

Remarks: The arithmetic method (also known as short cut method) should not be used to solve a 2 x2 game having saddle point because the method yields incorrect answer in this case.

(4) Matrix Method

If the pay off matrix of a game is square matrix, then optimal strategy mixture as well as value of the game may be obtained by the matrix method. The solution of a two person zero-sum game with mixed strategies with square payoff matrix may be found by using the following formulae:

$$Player\ A's \quad optimal\ stragety = \frac{[1 \quad 1]P_{adj}}{[1 \quad 1]P_{adj}\begin{bmatrix}1\\1\end{bmatrix}}$$

$$Player\ B's \quad optimal\ stragety = \frac{[1 \quad 1]P_{cof}}{[1 \quad 1]P_{adj}\begin{bmatrix}1\\1\end{bmatrix}}$$

Value of the game

= (player A's optimal strategies) * (pay off matrix p_{ij}) * (player B's optimal strategies)

here p_{adj} is adjoint matrix, P_{cof} is cofactor matrix Player A's optimal strategies are in the form of a row vector and B's optimal strategies are in the form of a column vector.

However, in rare cases, this method can be used for finding a solution of a game with size more than 2 x 2. The solution may violate the non-negativity condition of probabilities, that is $p_i \geq 0, q_j \geq 0$ although the requirement $p_1+.....+p_m = 1$ or $q_1 +......q_n = 1$ is satisfied. P stands for pay off matrix.

(5) Graphical method (for 2 x n or for m x 2 Games)

The graphical method is useful for the game where the payoff matrix is of the size 2 x 2 or m x 2, that is the game with mixed strategies that has only undominated pure strategies for one of the players in the two person zero sum game. Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Therefore if one player has only two strategies, the other will also use the same number of strategies. Hence this method is useful in finding out which of the two strategies can be used. Consider the 2 x n pay off matrix of a game without a saddle point.

Player A	Player B B_1	B_2 ………… B_n	Probability
A_1	a_{11}	a_{12} ……………… a_{1n}	p_1
A_2	a_{21}	a_{22} …………… a_{2n}	p_2
Probability	q_1	q_2 ……………… q_n	

Player A has two mixed strategies A_1 and A_2 with probability of their selection p_1 and p_2 respectively such that $p_1 + p_2 = 1$ and $p_1 \geq 0, p_2 \geq 0$. New for each of the pure strategies available to B, expected pay off for player A would be as follows:

B's pure move	**A's expected pay off (E_i (P))**
B_1	$a_{11}p_1 + a_{21}p_2$
B_2	$a_{12}p_1 + a_{21}p_2$
B_n	$a_{1n}p_1 + a_{2n}p_2$

The player B would like to choose that pure move for which expected payoff of player A is minimum. Let us denote this minimum expected pay off for A by

$$v = \underset{i}{\text{Min}}\{a_{1i}p_1 + a_{2i}p_2\}; \qquad i = 1,2, \dots, n$$

The objective of player A is to selected p_1 and hence p_2 in such a way that v is as large as possible. This may be done by plotting the straight lines.

$$E_i(P) = a_{1i}p_1 + a_{2i}p_2$$

$$= (a_{1i} - a_{2i})\, p_1 + p_{2i} \text{ as linear function of } p_1$$

The highest point on the lower boundary of these lines will give maximum expected payoff among the minimum expected payoffs on the lower boundary (lower envelop) and the optimum value of probability p_1 and p_2.

Similarly the two strategies of player B corresponding to those lines which pass through the maximum point can be determined. It helps in reducing the size of the game to (2 x 2), which can be solved as explained earlier.

The (m x 2) games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lowest point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q_1 and q_2.

9.6 Equivalence of Rectangular Games with Linear Programming

A two person zero-sum game can also be solved by linear programming approach. The advantage of using linear programming technique is that it solves with mixed strategy of any size.

To illustrate the connection between game problems and linear programming let us consider (mxn) payoff matrix (a_{ij}) for player A. Moreover let.

$$S_A = \begin{bmatrix} A_1 \ldots\ldots\ldots\ldots & A_m \\ p_1 \ldots\ldots\ldots & p_m \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 \ldots\ldots & B_n \\ q_1 \ldots\ldots & q_n \end{bmatrix}$$

Be the optimum strategies for player A and Player B respectively. Then

$$\sum_{i=1}^{m} p_i = \sum_{j=1}^{n} q_j = 1$$

Then, the expected gains g_i (i =,…..,n) of player A against B's pure strategies will be

$$g_i = a_{11}p_1 + a_{12}p_2 + \cdots \ldots\ldots + a_{1m}p_m$$

$$g_2 = a_{21}p_1 + a_{22}p_2 + \cdots \ldots \ldots . + a_{2m}p_m$$

$$g_m = a_{n1}p_1 + a_{n2}p_2 + \cdots \ldots \ldots . + a_{nm}p_m$$

And the expected loss 1_j (j=1,…n) of player B against A's pure strategies will be

$$1_i = a_{11}q_1 + a_{12}q_2 + \cdots \ldots \ldots . + a_{1n}p_m$$

$$1_j = a_{21}q_1 + a_{22}q_2 + \cdots \ldots \ldots . + a_{2n}p_n$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .$$

$$1_n = a_{m1}q_1 + a_{m2}q_2 + \cdots \ldots \ldots . + a_{mn}p_m$$

The objective of player A is to select P_i (i=1,…m) such that he can maximize his minimum excepted gain the player B desires to select q_j(j= 1,2,….,n) that will minimize his maximum expected loss

Now to obtain values of probability P_i, the value of the game to player A for all strategies by player B must be at least equal to v. Thus to maximize the minimum expected gains, it is necessary that

$$a_{11}p_1 + a_{21}p_2 + \cdots \ldots \ldots + \cdots . + a_{m1}p_m \geq v$$

$$a_{12}p_1 + a_{22}p_2 + \cdots \ldots \ldots + \cdots . + a_{m2}p_m \geq v$$

....

....

$$a_{1n}p_1 + a_{2n}p_2 + \cdots \ldots \ldots + \cdots . + a_{mn}p_m \geq v$$

Where, $p_1 + p_2 + p_m = 1\,;\;\; p_i \geq 0 \; for \; all \; i.$

Divide both sides of the m inequalities and equation by v, the division is valid as long as v > o. Incase, v < 0, the direction of inequality constraints must be reversed. But if v = 0 division would be meaningless. In this case a constant can be added to all entries of the matrix ensuring that the value of the game (v) for the revised matrix becomes more than zero. After optimal solution is obtained, the true value of the game is calculated by subtracting the same constant value.

$$Let \;\; x_i = \frac{p_i}{v} = (> 0). \quad Then, we \; have$$

$$a_{11}\frac{p_1}{v} + a_{21}\frac{p_2}{v} + \cdots \ldots \ldots + \cdots . + a_{m1}\frac{p_m}{v} \geq 1$$

$$a_{12}\frac{p_1}{v} + a_{22}\frac{p_2}{v} + \cdots \dots \dots + \cdots . + a_{m2}\frac{p_m}{v} \geq 1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{1n}\frac{p_1}{v} + a_{2n}\frac{p_2}{v} + \cdots \dots \dots + \cdots . + a_{mn}\frac{p_m}{v} \geq 1$$

$$and, \frac{p_1}{v} + \frac{p_2}{v} + \cdots .. \frac{p_m}{v} = \frac{1}{v}$$

Since the objective of player A is to maximize the value of the game, i.e., v which in turn is equivalent to minimizing $\frac{1}{v}$; the resulting linear programming problem can be stated as

$$Minimize\ Z_p\left(=\frac{1}{v}\right) = x_1 + x_2 \dots . + x_m$$

Subject to the constraints

$$a_{11}x_1 + a_{21}x_2 + \cdots \dots \dots + \cdots . + a_{m1}x_m \geq 1$$

$$a_{12}x_1 + a_{22}x_2 + \cdots \dots \dots + \cdots . + a_{m2}x_m \geq 1$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$a_{1n}x_1 + a_{2n}x_2 + \cdots \dots \dots + \cdots . + a_{mn}x_m \geq 1$$

Where

$$x_i = \frac{p_i}{v} \geq 0 \qquad \forall i = 1,2, \dots \dots m.$$

Similarly, player B has a similar problem with inequalities of the constraints reversed, i.e. minimize the expected loss. Since minimization programming problem can be stated as:

$$Maximize\ Z_q\left(=\frac{1}{v}\right) = y_1 + y_2 \dots . + y_n$$

Subject to the constraints

$$a_{11}y_1 + a_{12}y_2 + \cdots \dots \dots + \cdots . + a_{1n}y_n \leq 1$$

$$a_{21}y_1 + a_{22}y_2 + \cdots \dots \dots + \cdots . + a_{2n}y_n \leq 1$$

……………………………………………

……………………………………………

$$a_{m1}y_1 + a_{m2}y_2 + \cdots \dots \dots + \cdots . + a_{mn}y_n \le 1$$

Where

$$y_j = \frac{q_j}{v} \ge 0 \qquad \forall j = 1,2, \dots \dots , n.$$

It may be noted that the LP problem for player B is the dual of LP problem for player A and vice versa. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since for both the players $Z_p = Z_q$, the expected gain to player A in the game will exactly equal to the expected loss to player B.

Example: For the following payoff matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

Player A	Player B		
	B_1	B_2	B_3
A_1	1	-1	3
A_2	3	5	-3
A_3	6	2	-2

Solution:

The first step is to find out the saddle point (if any) in the pay off matrix as shown below:

Player A	Player B			Row Minimum
	B_1	B_2	B_3	
A_1	1	-1	3	-1 ← Maximin

A_2	3	5	-3	-3
A_3	6	2	-2	-2
Column Maximum	6	5	(3) ← Minimax	

The given game payoff matrix does not have a saddle point. Since the maximin value is -1, therefore, it is possible that the value of the game (v) may be negative or zero, because $-1 < v < 1$. Thus a constant which is at lest equal to the negative of maximin value, i.e. more than -1, is added to all the elements of pay off maxtrix. Thus, adding a constant 4 to all the elements of the pay off matrix, the payoff matrix becomes:

Player A	Player B			Probability
	B_1	B_2	B_3	
A_1	5	3	7	p_1
A_2	7	9	1	p_2
A_3	10	6	2	p_3
Probability	q_1	q_2	q_3	

Let p_i (i= 1,2,3) and q_j (j=1,2,3) be the probabilities of selecting strategies A_i (i= 1,2,3,) and B_j (j=1,2,3) by players A and B respectively.

The expected gain for player A will be as follows:

$5p_1 + 7p_2 + 10p_3 \geq v$ (if B chooses strategy B_1)

$3p_1 + 9p_2 + 6p_3 \geq v$ (if B chooses strategy B_2)

$7p_1 + 7p_2 + 2p_3 \geq v$ (if B chooses strategy B_3)

When $p_1 \geq 0$ (i= 1,2,3) and $p_1 + p_2 + p_3 = 1$

Dividing each inequality by v we get

$$5\frac{p_1}{v} + 7\frac{p_2}{v} + 10\frac{p_m}{v} \geq 1$$

$$3\frac{p_1}{v} + 9\frac{p_2}{v} + 6\frac{p_m}{v} \geq 1$$

$$7\frac{p_1}{v} + \frac{p_2}{v} + 2\frac{p_m}{v} \geq 1$$

$$\frac{p_1}{v} + \frac{p_2}{v} + \frac{p_m}{v} = \frac{1}{v}$$

In order to simplify, we define new variables:

$$x_1 = \frac{p_1}{v}, x_2 = \frac{p_2}{v}, x_3 = \frac{p_m}{v}$$

The problem for player A, then becomes,

$Minimize\ z_p \left(= \frac{1}{v}\right) = x_1 + x_2 + x_3$

Subject to the constraints

$$5x_1 + 7x_2 + 10x_3 \geq 1$$

$$3x_1 + 9x_2 + 6x_3 \geq 1$$

$$7x_1 + x_2 + 2x_3 \geq 1$$

$$and\ x_1, x_2, x_3 \geq 0$$

Player B's objective is to minimize his expected loss which can be reduced-to minimizing the value of the game v. hence, the problem of player B can be expressed as follows:

$Maximize\ z_q \left(= \frac{1}{v}\right) = y_1 + y_2 + y_3$

Subject to the constraints

$$5y_1 + 3y_2 + 7y_3 \leq 1$$

$$7y_1 + 9y_2 + y_3 \leq 1$$

$$10y_1 + 6y_2 + 2y_3 \leq 1$$

$$and\ y_1, y_2, y_3 \geq 0$$

$$Where \qquad y_1 = \frac{q_1}{v}, y_2 = \frac{q_2}{v} \text{ and } y_3 = \frac{q_3}{v}$$

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables s_1, s_2, s_3 to convert the three inequalities of equalities. The problem becomes

$$Maximize\ z_q = y_1 + y_2 + y_3 + 0.s_1 + 0.s_2 + 0.s_3 \text{ subject to the constraints}$$

$$5y_1 + 3y_2 + 7y_3 + s_1 = 1$$

$$7y_1 + 9y_2 + y_3 + s_2 = 1$$

$$10y_1 + 6y_2 + 2y_3 + s_3 = 1$$

$$and\ y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$$

The initial solution is shown in the following table;

Initial Solution

		C_j	1	1	1	0	0	0	
Unit cost (C_B)	Variables in Basis (B)	Solution values y_B(=b)	y_1	y_2	y_3	s_1	s_2	s_3	Mini Ratio y_B/y_1
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	10	6	2	0	0	1	1/10
		Z_q=0	-1	-1	-1	0	0	0	Z_j-C_j

Proceeding with usual simplex method, the optimal solution is as follows:

Optimal Solution

		C_j	1	1	1	0	0	0	
Unit cost (C_B)	Variables in Basis (B)	Solution values y_B(=b)	y_1	y_2	y_3	s_1	s_2	s_3	
1	y_3	1/10	2/5	0	1	3/20	-1/10	0	
1	y_2	1/10	11/15	1	0	-1/60	7/60	0	
1	s_3	1/5	24/5	0	0	-1/5	-3/5	0	
		Z_q=1/5	2/15	0	0	2/15	1/15	0	$Z_j - C_j$

The optimal solution (mixed strategies) for player

B is : $y_1 = 0$, $y_2 = 1/10$, $y_3 = 1/10$ and expected value of the game

= $1/Z_q$ - constant

= 5 – 4

= 1

These solutions may be converted back into the original variables:

$$if\,\frac{1}{v} = \frac{1}{5}, \qquad then\; v = 5$$

$$y_1 = \frac{q_1}{v}, \qquad then\; q_1 = y_1 \times v = 0$$

$$y_2 = \frac{q_2}{v}, \qquad then\ q_2 = y_2 \times v = 0$$

$$= \frac{1}{10} \times 5 = \frac{1}{2}$$

$$y_3 = \frac{q_3}{v}, \qquad then\ q_3 = y_3 \times v \ = \frac{1}{10} \times 5 = \frac{1}{2}$$

The optimal strategies for player A are obtained from (Z_j – C_j) row of the optimal solution

$$x_1 = \frac{2}{15}, \qquad x_2 = \frac{1}{15}\ \ and\ x_3 = 0$$

Hence,

$$p_1 = x_1 \times v = \frac{2}{3}$$

$$p_2 = x_2 \times v = \frac{1}{3}$$

$$p_3 = x_3 \times v = 0$$

Hence the probabilities of using strategies by both the players are;

$$Player\ A = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$$

$$Player\ B = \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

And the value of the original game is v= 1.

9.7 Exercises

E-1 Use graphical method to reduce the following games and hence solve them.

$$\begin{array}{c} \qquad B \\ (a)\quad A\begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

$$B$$

$$(b)\quad A\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ -7 & -9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix}$$

E-2 Two companies are competing for the same product. The different strategies are given in the following payoff matrix. What are the best strategy for both the companies?

$$\begin{array}{cc} & Company\ B \\ Company\ A & \begin{bmatrix} 3 & -3 & 4 \\ -1 & 1 & -3 \end{bmatrix} \end{array}$$

E-3 Obtained the optimal strategies for both persons and the value of the game for the following game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$$

E-4 In the following 3 x 3 game find the optimal strategies and the value of game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 3 & -2 & 4 \\ -1 & 4 & 2 \\ 2 & 2 & 6 \end{bmatrix} \end{array}$$

E-5 Use concept of dominance to reduce the size of the matrix and hence solve the game.

$$\begin{array}{cc} & B \\ A & \begin{bmatrix} 1 & 8 & 3 \\ 6 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

9.8 Summary

In a concise and systematic manner we can say that a two person zero sum game or a rectangular game can be solved as follows:

(i) First of all search for a saddle point; if it exists the problem is readily solved.

(ii) If there is no saddle point use the concept of dominance to reduce the size of the payoff matrix of game. Delete the dominated rows or columns and further check it for dominance.

(iii) If the size of the reduced matrix becomes 2 x 2 it can be solved either by arithmetic algebraic method.

(iv) If the size of matrix becomes 2 x n or m x 2, solve it by graphical method to reduce it to 2 x 2 and then solve it by arithmetic or algebraic method.

(v) If the reduced size becomes 3 x 3 or higher, then method of linear programming can be to solve it.

9.9 Further Readings

- Taha, H. A.: Operations Research- An Introduction; Maxwell Macmilan, New York.
- Hadley, G.: Linear Programming; Addison Wesely.
- Kanti Swaroop et. Al. Introduction to Operation Research; Sultan Chand & Sons, New Delhi.
- P.K. Gupta: Linear Programmign and Theory of Games; Sultan Chand and Sons, New Delhi.