U.P. Rajarshi Tandon Open University, Prayagraj

PGSTAT – 115/ MASTAT – 115
Actuarial Statistics

Block: 1 Probability Models & Life Tables

Unit – 1 : Basic Concepts

Unit – 2 : Utility theory

Unit – 3 : Survival Distributions and life table

Unit – 4 : Multiple life functions

Unit – 5 : Application of multiple Decrement Theory

Block: 2 Insurance & Annuities

Unit – 6 : Fundamentals of computation of Interest rate

Unit – 7 : Life insurance

Unit – 8 : Life annuities

Unit – 9 : Net premiums

Unit – 10 : Net premium reserves

Unit – 11 : Some practical considerations

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Blocks & Units Introduction

The present SLM on ***Actuarial Statistics*** consists of eleven units with four blocks.

The ***Block - 1 – Probability Models,*** is the first block, which is divided into three units, deals with theory of linear programming problem and Non-linear programming problem especially quadratic programming problem.

The ***Unit - 1*** **– Basic Concepts deals with** Introductory Statistics and Insurance Applications: Discrete, continuous and mixed probability distributions. Insurance applications, sum of random variables.

In ***Unit – 2 -*** Utility theory deals with its Introduction, Utility functions, Expected utility Criterionof insurance ,Types of Utility Functions.

In ***Unit – 3 -*** Survival Distributions and life table :Life table and its relation with survival function, examples, assumptions for fractional ages, some analytical laws of mortality, select and ultimate tables, curtate future lifetime, force of mortality.

In ***Unit – 4 -*** Multiple life functions: Introduction, Joint Distribution of Future life time ,joint life and last survivor status, insurance and annuity benefits through multiple life functions evaluation for special mortality law.

In ***Unit – 5 -*** Application of multiple Decrement Theory :Multiple decrement models, deterministic and random survivorship groups, associated single decrement tables, central rates of multiple decrement, net single premiums and their numerical evaluations

The ***Block - 2*** **– Insurance and Annuities** is the second block with Six units.

In ***Unit – 6 –*** Fundamentals of computation of Interest rate:Principles of compound interest. Nominal and effective rates of interest and discount, force of interest and discount, compound interest, accumulation factor, continuous compounding

Unit – 7 – Life insurance: Insurance payable at the moment of death and at the end of the year of death-level benefit insurance, endowment insurance, diferred insurance and varying benefit insurance, recursions, commutation functions.

Unit – 8 – Life annuities: Single payment, continuous life annuities, discrete life annuities, life annuities with monthly payments, commutation functions, varying annuities, recursions, complete annuities-immediate and apportionable annuities-due.

In ***Unit – 9 –*** Net premiums: Continuous and discrete premiums, true monthly payment premiums, apporionable premiums, commutation functions, accumulation type benefits. Payment premiums, apportionable premiums, commutation functions, accumulation type benefits.

Unit – 10 - Net premium reserves: Continuous and discrete net premium reserve, reserves on a semicontinuous basis, reserves based on true monthly premiums, reserves on an apportionable or discounted continuous basis, reserves at fractional durations, allocations of loss to policy years, recursive formulas and differential equations for reserves, commutation functions.

Unit – 11 - Some practical considerations: Premiums that include expenses-general expenses types of expenses, per policy expenses. Claim amount distributions, approximating the individual model, stop-loss insurance..

At the end of every block/unit the summary, self assessment questions and further readings are given.

Utility Theory

Introduction: Utility Theory is used particular in economics & insurance. In general It is used for decision making & is based on defining special functions that are called utility functions.

Utility Functions: A utility function can be described as a function which measures the value or utility that an individual or institution attached to the monetary amount x. This function satisfies the following conditions

$i)\ \mu'(x) > 0$

$ii)\ \mu''(x) < 0$

Interpretations of the two conditions

Condition I: It says that u is an increasing function of x i.e. an individual whose utility function is u prefers amount y to amount z if, y>z, i.e. the individual prefers more money to less.

Condition II: It says that u is a concave function. This means that as the wealth of individual increase in wealth. For eg. An increase in wealth to 1000 is worthless to the individual if his wealth is 20,00,000 compared to the case when the independent wealth is 10,00,000.

An individual whose utility function satisfies condition (i) & (ii) is said to be risk averse, This leads to definition of coefficient of

Coefficient of Risk aversion: An individual whose utility function $\mu(x)$satisfies $\mu'(x) > 0$ & $\mu''(x) < 0$is risk average with coefficient of risk aversion being given as:

$$\gamma(x) = \frac{-\mu''(x)}{\mu'(x)}$$

Expected utility criterion (EVC)=

Decision making using a utility function is based on EUC. This criterion says that a decision maker should calculate the expected utility of resulting wealth under each course of action & then select that course of action that gives that greatest value for expected utility of resulting wealth.

If two course of action, yield the same expected utility of resulting wealth, then, the decision maker has no preference between there two course of action, so mathematically

Let us consider an investor with utility function u. Suppose he is choosing between two investments which will lead to random net gains of x_1 *and* x_2 respectively.

Further, suppose that the investor has current wealth w so that result of investing in investment I is $w + x_i; i = 1,2$. Then, under the expected utility criterion, the investor would choose Investment 1 over Investment 2 if and only if

$$E[\mu(w + x_1)] > E[\mu(w + x_2)]$$

Further, the investor would be indifferent between the two investments if

$$E[\mu(w + x_1)] > E[\mu(w + x_2)]$$

Example: Suppose that an investor has a utility function given as

$\mu\ (x) = -\exp\ [-0.002x]$

i) Check whether thi utility function satisfies the necessary conditions
ii) Is the individual risk averse. If yes calculate the coefficient of risk aversion.
iii) Suppose that the investor comes across two investments that lead to gains of x_1f and x_2 respectively such that

$X_1 \sim N(10^4, 500^2)$

$X_2 \sim N(1.1 \times 10^4, 2000^2)$

Which of these two investments will the investor prefers (assuming hi wealth w).

Soultion:-

$\mu'\ (x) = 0.002 \text{ exp } [-0.002x]$

$\mu''\ (x) = -4 \times 10^6 \text{ exp } [-0.002x]$

Coeff. Of risk aversion = $\frac{-\mu''(x)}{\mu'(x)} = 2 \times 10^{-3} = 0.002.$

$E[\mu(w + x_1)] = -E[\text{exp } \{-0.002(w + x_1)\}]$

$= -exp\{-0.002w\}E[exp\{-0.002x_1\}]$

$= -exp\{-0.002w\}exp\left\{-0.002 \times 10^4 + \frac{1}{2}0.002^2 \times 500\right\}$

$= -exp\{-0.002w\}exp\{-19.5\}$

Where the third line from the fact that the expectation in the second line is M_{x_1} . Similarly,

$E[\mu(w + x_1)] = -exp\{-0.002w\}exp[u(w + x_1)]$

Hence, the investor prefer investment 1 as $E[u(w + x_1)]$is greater than $E[u(w + x_2)]$.

EVC may lead to an outcome i.e., inconsistent with other criterion: for eg . $E(x_1) = 10^4$

$E(x_2) = 1.1 \times 10^4$

A investor can choose the investment that gives the greater net gain and this may be the faulty decision.

Suppose that an investor make decision based on utility function given as

$\mu\ (x) = \beta\ logx\ ; x > 0, \beta > 0$

i) Does it satisfy the condition of a utility function?
ii) What is the risk condition of the investor? (Coeff. of risk aversion).

iii) Suppose he has a wealth B & has an option available in therms of beying shares of companies I, i=1,2,3…..,n.

Which course of action will the investor prefer based on EUC?

Sol:- i) $\mu'(x) = \beta/x$

$\mu''(x) = \beta/x^2$

ii) coefficient = $\beta/x^2 \;\; /\beta/x = \frac{1}{x}\; risk\ averse$

iii) $E[U(\beta\, x_1)] = E[\beta\; log(\beta x_i)]$

The investor will prefers the share of company i to that of company j, by EUC if,

$E[U(\beta x_i)] > E\left[U\left(\beta x_j\right)\right]$

$= E[\beta\; log(\beta x_i)] > E\left[\beta log\left(\beta x_j\right)\right]$

$= E[\log\beta + log x_i] > E\left[\log\beta + log x_j\right]$

$= E[log x_i] > E\left[log x_j\right]$

This choice of investment does not have to do anything with the initial wealth he possesses, this is because of choosing the logarithmic function as the utility function.

So, one must be very careful in making the right choice about the utility function.

An important result of utility function:-

Suppose that a utility function v is defined is terms of a utility function by

$j(x) = a\;\mu(x) + b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$ for constant a & b, s.t. a > 0. Then decision made under EUC will be the same under j as they are u.

Proof: Suppose a per EUC we prefer an investment x_1 over x_2, so that

$$E[\partial(w+x_1)] > E[\partial(w+x_2)]$$

$$= E[a\mu(w+x_1)+b] > E[a\mu(w+x_2)+b]$$

$$= aE[\mu(w+x_1)]+b > aE\ [\mu(w+x_1)]+b$$

Which is true if

$$E[U(w+x_1)] > E[U(w+x_2)]$$

Remarks:

i) If means necessary (N) sufficient (s) which holds.
ii) Necessary means you start with result = condition
iii) S means you start with condition = result.
iv) If only n then condition may hold but result may not hold.
v) If only S then condition may not hold but result may hold.

<u>Jenen's Inequality:-</u>

It is a well known result is probability theory & it has though important applications in actuarial science. Jensen's Inequality starts that if u is a concave function then.

$E[\mu(x)] \leq \mu[E\{x\}] \dots \dots \dots \dots \dots \dots \dots \dots \dots .(1)$

Provided there quantities exists

<u>Remark</u>;- if u is a convex function then.

$E[\mu(x)] \geq \mu[E\{x\}] \dots \dots \dots \dots \dots \dots \dots \dots \dots .(2)$

<u>Proof:</u> We prove Jensen's Inequality on the assumption that there is a <u>Tayeor Series expansion</u> of u about the point a.

Thus, writing a Tayer series expansion with a reminder term as

$$\mu(x) = \mu(a) + u'(a)(x-a) + \frac{1}{2}u''(z)(x-a)^2 \ldots\ldots\ldots\ldots\ldots (3)$$

Where,

Z lies between a & x. Now if the function is concave we know that

$$u''(z) < 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

So, from (3), we have

$$\mu(x) \le \mu(a) + u'(a)(x-a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Now replace x by X is (5) & put a = E[x] then we get

$$u(x) \le u(E[x]) + u'(E[x])(x - E[x]) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Take expectation on the both sides, we have

$$E[u(x)] \le U\ [E(x)]$$

Hence proved

Application of Jensen's Inequality to Insurance: Appropriate premium levels for insurance cover from the view point of both an individual & insurer maximum premium given by insure – $P \ge E(x)$ Minimum Premium charged by company (insurer) - $\Pi \ge E(x)$.

Result – 1:- The maximum premium that an individual is prepared to pay is atteast equal to the expected less, i.e.

Proof: Consider first an individual whose wealth is w. suppose that the individual can obtain completes pursuance protector against a random loss x. then the maximum premium that the individual is prepared it pay for this protection is P, where U(W-P) = E [U(W-X)] ----------(1)

Reason for (1) to be true

(i) Expected utility criterion (EUC)

(ii) We know $U'(x) > 0$, so for any premium $\bar{P} < p, u(w - \bar{p}) > U(w - P)$.

Now by Jensen's inequality

$E\ [U(W - x)] \leq UE[(w - x)] - \ U\ [w - E[x]] - - - - - - - - -(2)$

Now from (1) & (2)

$U(W - P) \leq U(W - E\ [x]) - - - - - - - - - - - - - -(3)$

As it is an increasing function

(3) = W-P $\leq W - E\ [x]$

=$P \geq E\ [x]$

<u>Result (2):</u> The insurer requires a premium π that is at least equal to the expected loss i.e. $\pi \geq E[x]$.

<u>Proof:</u> Suppose that an insurer has a utility function 0. Let this wealth be w.

Now, he provides complet insurance protection an individual against a random loss x. Suppose the minimum acceptable premium for him is π.

Now according to EUC

$ə(w) = E[ə(w + \pi - X)] - - - - - - - - (1)$

Because the insurer is choosing between offering & net offering insurance.

Also, we can see that as 0 is an increasing function for any premium $\bar{\Pi} > \boldsymbol{\Pi}$

$E[\pi(w + \bar{\pi} - x)] > E[\pi(w + \pi - X)] - - - - - - - - (2)$

Now, applying Jensen's Inequality is the RHs of eg (10

We have

$$E[\partial(w+\bar{\pi}-x)] \leq \partial\,[E(w+\pi-X)] = \partial\,(w+\pi-E[x]) - - - -(3)$$

Using (1) & (3)

$\partial(w) \leq \partial(w+\pi-E[x]) - - - - - - - - (4)$

As ∂ is an increasing function

$= W \leq w+\pi-\boldsymbol{E}[\boldsymbol{x}]$

$= \pi \geq \boldsymbol{E}[\boldsymbol{x}] - - - - - - (\mathbf{5})$

<u>Types of Utility Function</u>

It is possible to construct a utility function by assigning different value to different levels of wealth. e.g. An individual might set U(0) = 0, U (10) = 5; U (20) = 8 & so on ….. (learly, it more practical to assign value through a suitable mathematical function now consider some mathematical function's which may be regarded as having suitable forms to be used as utility function:-

(1) Exponential
(2) Quadratic
(3) Logarithmic
(4) Fractional Power

1)**Exponential Utility Function: (EUF)**

A utility function of the form

$\mu(x) = -\exp\{-\beta x\} - - - - - - (1)$ where, $\beta > 0$

Is called an exponential utility function.

Properties:

(1) Suppose that an individual has wealth w. suppose he has a choice between n course of action such that the i^{th} course of action will result is a wealth $w+x_i$; i= 1,2,……,n. Then

by EUC the individual would calculate $E[\mu(w+w_i)]\forall i=1,2,\dots\dots,n$ & would choose say course of action j, if

$E[\mu(w+x_j)] > E[u(w+x_j)]$

$= E\ [-exp\{-\beta(w+x_j)\}] > E[-exp\{-\beta(w+x_j)\}]$

$= E[exp\{-\beta x_j\}] < E[exp\{-\beta x_j\}]$

Which is independent of initial wealth w. this is an important property of EUF.

(2) The maximum premium P that an individual with utility function.

$\mu(x) = \ -\exp\ \{-\beta x\}$

Would be prepared to pay for insurance against the random loss x.

$P = \beta^{-1}\log M_x(\beta)$

Proof: By EUC

$\mu(w-p) = E\ [\mu(w-x)] \dots\dots\dots(1)$

$= -\exp\{-\beta\,(w-p)\} = E[-\exp\{-\beta(w-x)\}]$

$= \ -\exp\{-\beta(w-p)\} = \ -E[\exp\{\beta(w-x)\}]$

$= \ -\exp\{-\beta w\}.\exp\{\beta p\} = -\exp\{-\beta w\}\,.\,E[\exp\{\beta x\}]$

$\exp\{\beta p\} = E[\exp\ \{\beta x\}$

$= M_x(\beta)$

$\beta p = logM_x(\beta)$

$p = \beta^{-1}log\ M_x(\beta)$

Example:- Show that the maximum premium p, that an individual with utility function $\mu(x) = -\exp\{-\beta x\}$ is prepared to pay for complete insurance cover against a random loss x. where $x \sim N(\mu.\sigma^2)$ is an increasing function of β and explain this result.

Soultion: $\mu(x) = -\exp(-\beta x)$ $M_x(\beta) = exp\left\{\mu\beta + \frac{1}{2}\beta^2\right\}$.

$$\mu'(x) = \frac{1}{\beta}\exp(-\beta x)$$

$$\mu''(x) = -\frac{1}{\beta^2}\exp(-\beta x)$$

$$\gamma(x) = -\frac{\mu''(x)}{\mu'(x)} = \beta$$

$$P = \beta^{-1}loge[M_x(\beta)] = \beta^{-1}log\left[exp\left\{\beta\mu + \frac{1}{2}\beta^2\sigma^{-2}\right\}\right]$$

$$= \mu + \frac{1}{2}\sigma^2\beta(\beta = increasing\ function)$$

Coefficient of risk aversion is an increasing function of & it is independent of x. Thus the higher the risk aversion of an investor, the higher would be the value of & this would mean that the investor would be ready to pay a maximum premium (p).

Example: An individual it facing a random loss, x, where $x \sim \gamma$ (2,0.01), and can obtain complete insurance cover against this loss fcl a premium of 208. The individual makes decision on the bases of an exponential utility function with parameter 0.001. is the individual prepared to insure for this premium.

<u>Solution:</u> $\mu(x) = -\exp(-0.001x)$

$$\mu'(x) = -\frac{1}{0.001}\exp(-0.001x)$$

$$\mu'(x) = -10^6\exp(-0.001x)$$

$$\gamma(x) = -\frac{\mu''(x)}{\mu'(x)} = 10^3$$

$$P = (0.001)^{-1} log\left[\left(1-\frac{\beta}{\gamma}\right)^{-\alpha}\right]$$

$$= \left[(0.001)^{-1} log\left[\left(1-\frac{0.001}{0.01}\right)^{-2}\right]\right]$$

$$= 210.72 = P = maximum\ premium\ he\ is\ willing\ to\ pay$$

$$Since \quad 208 < P$$

$$= 208 < 210.72$$

(2) Quadratic Utility Function (QUF)

A utility function of the form

$\mu(x) = \ x - x - \beta x^2 \quad for\ x < \frac{1}{\alpha\beta}; \quad \beta > 0.$ is called a Quadratic utility function.

$$\mu'(x) = 1 - 2\beta x > 0 = x < \frac{1}{2\beta}$$

$$\mu''(x) = -2\beta < 0\ ; \quad \beta > 0.$$

The biggest drawback of this utility function is that we can’t use it for problems where random outcomes are distributed on $(-\infty, \infty)$.

Under EUC; we will prefer outcome x_j to outcome x_i if;

$$E\left[\mu\left(W + x_j\right)\right] > E[\mu(w + x_i)]$$

$$\mu\left(W + x_j\right) = W + x_j - \beta\left(W + x_j\right)^2$$

$$\mu(W + x_i) = W + x_i - \beta(W + x_i)^2$$

$E\left[U\left(W+x_j\right)\right] > E[\mu(w+x_i)]$

$= E\left[x_j\right] - \beta E\left[x_j^2\right] - 2\beta w E\left[x_j\right] > E[x_i] - \beta E[x_i^2] - 2\beta w E[x_i]$

$= (1-2\beta w)E\left[x_j\right] - \beta E\left[x_j^2\right] > (1-2\beta w)\ E[x_i] - \beta E[x_i^2]$

Thus, the result is based only on first 2 (raw) moments of x_i & x_j and also depends on w (i.e. the initial wealth)

Example: An individual whose wealth is w has choice between investment 1 and 2, which will result is wealth of $w+x_1$ and $w+ x_2$ respectively, where $E[x_i]=10$, $v[x_1]=2$ and $E\ [x_2]=10.1$. The individual makes decision on the basis of a quadratic utility function with parameter β = 0.002. for what range of values for $v[x_2]$ will the individual choose investment 1 when W=200? Assume that $P_r(w+x_i < 250) = 1\ for\ i = 1\ and\ 2.$

Solution: The individual will choose investment 1 if $E[\mu(W+x_1)] > E[\mu(w+x_2)]$ or equivalently

$E[200+x_1-\beta(200+x_1^2)] > E[200+x_2-\beta(200+x_2)^2].$ where = 0.002. After some straight forward algebra, this condition becomes

$E[x_1](1-400\beta) - \beta E[x_1^2] > E[x_2](1-400\beta) - \beta E[x_2^2]\ or$

$E[x_2^2] > (E[x_2]-E[x_1])(\beta^{-1}-400\beta) + E[x_1^2] = 112$

Which is equivalent to $V[x_2] > 9.99$

Example: An insurer is considering offering complete insurance cover against a random loss x, where. E[x]= V[x]= 100 and Pr (x>0)=1. The insurer adopts the utility function $\mu(x) = x - 0.001x^2$ for decision making purposes. Calculate the minimum premium that the insurer would accept for this insurance cover when the insurer's wealth, w, is (a) 100, (b) 200 and (c0 300.

Solution: The minimum premium, , is given by $\mu(w) = E[\mu(w+\pi-x)]$ so when W = 100, we have μ (100)=90

$$= E[100+\pi-x-0.001(1000+\pi^2)-2(100+\pi)x+x^2]$$

$$= 100+\pi-E[x]-0.001(100+\pi)^2-2(100+\pi)E[x]+E[x^2]$$

This simplifies to

$$\pi^2 - 1{,}000\pi + 90.100 = 0$$

Which gives π = 100.13. Similarly, when W = 200, we find that π = 100.17 and when W=300, π = 100.25. We note the π increases as W increase and the this is an undesirable property as we would expect that as the insurer's wealth increases the insurer should be better placed to absorb random losses and hence should be able to reduce the minimum acceptable premium.

(3) Logarithmic Utility Function: A utility function of the form $\mu(x) = \beta \log x \ for \ x > 0, \beta > 0$ is called a logarithmic utility function.

Constraints: As $\mu(x)$is defined only for positive value of x, the utility function is unsuitable for use in situations where outcomes could lead to negative wealth.

(1) Check that it is a properly defined utility function.

$$\mu(x) = \beta \log \ \ x$$

$$\mu'(x) = \frac{\beta}{x}$$

$$\mu''(x) = \frac{\beta}{x^2}$$

$$\gamma(x) = \frac{\mu''(x)}{\mu'(x)} = \frac{1}{x} = coefficient\ of\ risk\ aversion.$$

$$\gamma'(x) = \frac{1}{x^2}\ decreasing\ function \frac{\theta}{x}\ (as\ wealth\ function)$$

This means that risk aversion is a decreasing j^n of wealth i.e. as the wealth increase, risk aversion would from other utility function. Therefore one must be cautious about the logarithmic utility j^n.

Example: An investor who makes decisions on the basis of a logarithmic utility function is considering investing is shares of one n companies. The investor has wealth B, and investment in shares of company I will result of wealth Bx_i for {=1,2,……,n. show that the investment decision is independent of B.

Solution: The investor prefers the shares of company I to these of company I if,

$$E[\mu(Bx_i)] > E[\mu(Bx_j)]$$

Now

$$E[\mu(Bx_i)] = E[\beta \; log(Bx_i)] = \beta E[logB]\beta E[logx_i]$$

So the investor prefers the share of company I to those of company j if and only if

$E[logx_i] > E[logx_j]$ independent of B.

(4) Fractional power utility function: A utility function of the form $\mu(x) = x^{\beta}, \;\; x > 0 \;\&\; 0 < \beta < 1.$ is called a fractional power utility function.

As with a logarithmic utility function $\mu(x)$, is defined only for positive x and so its applications are limited in the same way as for a logarithmic utility function.

To check it is a properly utility function

$$\mu(x) = x^{\beta}$$

$$\mu'(x) = \beta x^{\beta-1} = \frac{\beta}{x^{1-\beta}} > 0$$

$$\mu''(x) = \beta(\beta-1)x^{\beta-2} = -\frac{\beta(1-\beta)}{x^{2-\beta}} < 0$$

$$\gamma(x) = \frac{-\mu''(x)}{\mu'(x)} = \frac{\beta(1-\beta)x^{\beta-2}}{\beta x^{\beta-1}} = (1-\beta)x^{-1} = \frac{1-\beta}{x} > 0$$

Example: An individual is facing a random loss x, that is uniformly distributed on (0,200). The individual can buy partial insurance cover against this loss under which the individual would pay y = min (x, 100) so that the individual would pay the loss in fall if the loss was less than 100 and would pay 100 otherwise. The individual makes decisions using the utility function $\mu(x) = x^{2/5}$. is the individual prepared to pay 80 for this partial insurance cover if the individual's wealth is 300?

Solution: $x \sim \mu(0,200),\ \ x = amount\ of\ loss\ \mu(x) = x^{2/5}$

$y = amount\ paid\ by\ (insurer)\ investment$

$$Y = \begin{cases} x, & x < 100, \quad y = \min(x,100) \\ 100, & x \geq 100, \end{cases}$$

$$f_x(y) = \begin{cases} fx^{(x)}, & x < 100 \\ 1 - fx^{(x)}, & x \geq 100 \end{cases}$$

Z= amount paid by reinsurer

$$Z = \begin{cases} 0, & x < 100, \\ x - 100, & x \geq 100, \end{cases}$$

$$\begin{cases} P = max.premium\ investor\ ready\ topay \\ \quad p = \beta^{-1} \log M_x(\beta) \end{cases}$$

(We can't use this here be fore their p gives full insurance cover)

If he does not by partial insurance, then he would pay the amount x in full:- he would left with (300-x). if he buys partial insurance with 80← premium & will also pay y→ he would left – (300-80-y).

Then by EUC

$E[U(300-x)] > E[U(300-80-y)]$

$E[U(300-x)] = E\left[U(300-x)^{2/5}\right]$

$$= \int_{300}^{200} (300-x)^{2/5}\ \frac{1}{200}\,dx$$

Let 30-x=y = dx= dy

$$= \int_{300}^{100} y^{2/5}\ \frac{1}{200}(-dy)$$

$$= \frac{1}{200}\left[\frac{y^{7/5}}{7/5}\right]_{100}^{200} = 8.237$$

$E[U(300-80-y)] = E\left[(300-80-y)^{2/5}\right]$

$$= \int_{0}^{200} (220-y)^{\frac{2}{5}} f_x(y)\,dy$$

$$\int_{0}^{100} (220-x)^{2/5}\ \frac{1}{200}\,dx + \int_{100}^{200} (120)^{215}\,dx$$

$$= \frac{1}{200}\left(\frac{-5}{7}(220-x)^{7/5}\right)\int_{0}^{140} +100 \times 120^{2/5}$$

=7.280

Hence the individual is not prepared to pay 80 for this partial insurance cover.

Insurance and Utility

Suppose a decision makers owns a property that may be damaged or destroyed in the next accounting period. The amount of the loss, which may be 0, is random variable denoted by x. we assume, that the distribution of x is known. The E[x] , the loss in the next period, may be interpreted as the long-term average less if the experiment of exposing the property to damage may be observed under identical conditions a great many times. It is clear that this long term set of trials could not be performed by an individual decision maker. This illustration shows the utility theory for the purpose of gaining insights into the economic role of insurance.

Suppose that an insurance organization (insurer) was established to help reduce the financial consequences of the damaged or destruction of property. The insurer would issue contracts (policies) that would promise to pay the owner of a property a defined amount equal to or less than the financial loss if the property were damaged or destroyed during the period of the polity. The contingent payment linded to the amount of the loss is called a claim payment. In return for the promise contained in the policy, the owner of the property (insured) pays a consideration (premium).

The amount of the premium payment is determined after an economic decision principle has been adapted by each of the insurer and insured. An opportunity exists for a mutually advantageous insurance policy when the premium for the policy set by the insurer is less than the maximum amount that the property owner is willing to pay for insurance.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this case, the insurer would adopt the expected value principle in setting its premium, the insurer would set its basic price for full insurance coverage as the expected loss, E[x] = μ. In this context μ is called the pure on net premium for the 1-period insurance policy. To provide for expenses, taxes, and profit and for some security against adverse loss experience, the insurance system would decide to set the premium for the policy by loading, adding to, the pure premium. For insurance, the loaded premium denoted by H, might be given by.

$$H = (1+a)\,\mu + c. \quad a > 0,\ c > 0$$

In this expression the quantity $a\mu$ can be viewed as being associated with expenses that wary with expected losses and with the risk that claims experience will deviate from expected. The constant C provides for expected expenses that do not vary with losses .

We now apply utility theory to the decision problem faced by the owner of the property subject to loss. The property owner has a utility of wealth function (w) , where wealth w is measured in monetary terms. The owner faces a possible loss due to random events that may damage he property. The distribution of the random loss x is assumed to be known. The owner will be indifferent between paying an amount G to the insurer who will assume the random financial loss and assuming the risk himself. This situation can be stated as

$$\mu(w-G) = E[\mu(w-x)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The r.h.s. of (1) represents the expected utility of not buying insurance when the owner's current wealth is w. The l.h.s. of (1) represents the expected utility of paying G for incomplete financial protection.

If the owner has an increasing linear utility function that is, $\mu(w) = bw + d$ with b > 0. The owner will be adopting, the expected value principle . in this case the owner prefers, or is indifferent to, the insurance when

$$\mu(w-G) = b(w-G)+d \geq E[\mu(w-x)] = E[b(w-x)+d], b(w-G)+d \geq b(w-\mu)+d$$

$$G \leq \mu$$

That is, if the owner has an increasing linear utility function, the premium payments that will make the owner prefer, of be indifferent to, complete insurance absence of a subsidy, an insurer, over the long term must charge more than its expected losses. Therefore, in this case, there seems to be little opportunity for a mutually advantageous insurances contract. If an insurance contract it is result the insurer must charge a premium in excess of expected losses and expenses to avoid a bias toward insufficient income. The property owner then cannot we a linear utility function.

The preferences of a decisions maker must satisfy certain consistency requirements to ensure the existences of a utility function. Although these requirements were into listed they do not include any specifications that would force a utility function to be linear, quadratic, exponential, logarithmic, or any other particular form. In fact each of these named functions might serve as a utility function for some decision maker or they might be spliced together to reflect some other decision maker's preferences.

Nevertheless, it seems natural to assume that u(w) is an increasing function. “more is better”. In addition it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is the idea of decreasing marginal utility is economics. We know that $\Delta^2 \mu(w) \leq 0$. if these ideas are extended to smoother functions the two properties suggested by observation are $\mu'(w) > 0 \; and \; \mu''(w) < 0$. The second inequality indicates that u(w) is a strictly concave downward function .

In discussing insurance decisions using strictly concave downward utility functions, we will make use of one form of Jensen’s inequalities. There inequalities states that for a random variable x and functions u(w),

$$if \;\mu''(w) < 0, then\; E[\mu(x)] \leq \mu(E[x]), \dots \dots \dots \dots \dots . . (2)$$

$$if \;\mu''(w) > 0, then\; E[\mu(x)] \geq \mu(E[x]), \dots \dots \dots \dots \dots . . (3)$$

Jensen’s inequalities require the existence of the two expected values.

There is a figure

Proof of eq^n (2)

If $E[x] = \mu$ exists, one considers the line tangent to $\mu(w)$,

$$y = \mu(\mu) + \mu'(\mu)(w - \mu),$$

At the point $\mu.\mu(\mu)$. Because of the strictly concave characteristic of the graph of u(w) will be below the tangent line, that is

$$\mu(w) \leq u(\mu)(w-\mu) - - - - - - - - - - - - - (4)$$

For all values of w. if w is replaced by the random variable x and the expectation is taken on each side of the inequality has several applications in actuarial mathematics. Let us apply Jensen's inequality (2) to the decision maker's insurance problem as formulated in (1). We will assume that the decision maker's preferences are such that $\mu'(w) > 0 \ and \ \mu''(w) < 0$. Applying Jensen's inequality to (1), we have.

$$\mu(w-G) = \ E[\mu(w-x)] \leq u(w-\mu) - - - - - - - - - - (5)$$

Because $\mu'(w) > 0, u(w)$is an increasing function. Therefore (5) implies that $w - G \leq w - \mu, or\ G \geq \mu\ with\ G > \mu$ unless x is a constant. In economic terms, we have found that $\mu'(w) > 0 \ and \ \mu''(w) < 0$ if the decision maker will pay an amount greater than the expected loss for insurance. If G is atleast equal to the premium set by the insurer, there is an opportunity for a mutually advantageous insurance policy.

Formally we say a decision maker with utility function is risk averse if and only if, $\mu''(w) < 0$.

We now employ a general utility function for the insurer. We let u_i (w) denote the current wealth of the insurer measured in monetary terms. Then the minimum acceptable premium H for assuming random loss X, from the viewpoint of the insurer may be determined from (6);

$$\mu_i(w_i) = E[U_i(w_i + H - X)] ------(6)$$

The l.h.s. of (6) is the utility attached to the insurer's current position. The r.h.s. is the expected utility associated with collecting premium H and paying random loss X. In other words the insurer is indifferent between the current position and providing insurances for X at premium H. If the insurer's utility function is such that $\mu_i(w) > 0, u_I''(w) < 0$we can use Jenson's inequality (2) along with (6) to obtain.

$$\mu_i(w_i) = E[U_i(w_i + H - X)] \leq U_i(w_i + H - \mu)$$

Following the same line of reasoning displayed in connects with (5). We can conclude that $H \geq \mu$. If G as determined by the decision maker by solving (5) is such that $G \geq H \geq \mu$.an insurances contract is possible that is the expected utility of neither party to the contracts is decreased.

A utility function is based on the decision maker's preferences for various distractions of outcomes. An insurer need not be an individual. It may be a partnership, corporation, or government agency. In this situation the determination of $\mu_i(w)$ the insurer utility function may be a rather complicated matter. For example, if the insurer is a corporation, one of management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflating attitudes towards risk among the groups of stockholders.

Several elementary function are used to illustrate properties of utility functions.

An exponential utility function is of the form $\mu(w) = -e^{-\alpha w}$ for all w and for a fixed x>0. And has several attractive features.

First $\mu'(w) = \alpha \; - e^{-\alpha w} \; > 0$

Second $\mu'^{\prime(w)} = -\alpha^2 \; - e^{-\alpha w} \; < 0$

Therefore may serve as the utility function of a risk-averse individual third finding

$E[-e^{-\alpha x}] = -E[-e^{-\alpha x}] = -M_x(-\alpha)$

is essentially the same as finding the m.g.f. of x. in this expression.

$M_x(\pm) = E[e^{tx}]$

denotes the m.g.f. of x. Fourth insurance premium do not depends on the wealth of the decision maker. This statement is verified for the insured by substituting the exponential utility function into (1) that is,

$-e^{-\alpha(w-G)} = E\left[-e^{-\alpha(w-x)}\right]$

$e^{\alpha G} = M_x(\alpha)$

$G = \dfrac{logM_x(\alpha)}{\alpha}$

and G does not depends on W.

The verification you the insurer is done by substituting the exponential utility function with parameter α_I into (6)

$-e^{-\alpha_I W_I} = E\left[-e^{-\alpha_I(W_I+H-X}\right]$

$-e^{-\alpha_I W_I} = -e^{-\alpha_I(W_I+H)}M_x(\alpha_I)$

$H = \frac{\log M_x(\alpha_I)}{\alpha_I}$.

Example: A decision maker's utility function is given by $(w) = e^{-5w}$. the decision has tow random economic prospects (gains) available. The outcome of the first denoted by x. has normal distribution with mean 5 and variance 2. Hence forth, a statement about a normal distribution with mean μ and variance σ^2 ill be abbreviated as $N(\mu, \sigma^2)$. The second prospect. Denoted by Y, is distributed as N(6.25).Which prospect will be preferred.

Solution: We have

$$E[u(x)] = E[-e^{-5x}]$$

$$= -M_x(-5) = -e^{\left[-5(5)+\frac{(5^2)(2)}{2}\right]}$$

$$= -L$$

and

$$E[u(Y)] = E[-e^{-sy}]$$

$$= -M_x(-5) = -e^{\left[-5(6)+\frac{(5^2)(2.5)}{2}\right]}$$

$$= -e^{1.25}$$

Therefore

$$E[u(x)] = -L > E[u(x)] = -e^{1.25}$$

and the distribution X is preferred to the distribution of Y.

* The family of fractional power utility function is given by

$$u(w) = w^r, \qquad w > 0.0 < 1 < 1.$$

A member of this family might represent the preferences of a risk averse decision maker since

$$u'(w) = \gamma h^{r-1} > 0 \; and \; u''(w) = \gamma (r-1)^{r-2} < 0$$

In this family premiums depends on the wealth of the decision maker in a manner that may be sufficiently realistic is many situations.

Example: A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of W-10 and faces random loss X with a uniform distribution on (0,10). What is the maximum amount this decision maker will pay for complete insurance against the random loss?

Solution: Substituting into (1), we have

$$\sqrt{10 - G} = E\left[\sqrt{10 - X}\right]$$

$$= \int_0^{10} \sqrt{10 - X}\, 10^{-1}$$

$$= \frac{-2(10 - x)^{\frac{3}{2}}}{3(10)}$$

$$= \frac{2}{3}\sqrt{10}$$

$$G = 5.556$$

The decision maker is risk averse and has u'(w)>0. Following the discussion of (5), we would expect G>E[x]. and in this example G= 5.5556>E[x]=5.

*The family of quadratic utility function is given by

$$u(w) = w - \alpha w^2,\ \ w < (2\alpha)^{-1}, \alpha > 0$$

A member of this family represent the preferences of a risk- averse decision maker since $u''(w) = -2\alpha.$ while a quadratic utility function is convenient because decision depends only on the first two moments of the distributions of outcomes under considerations there are certain consequences of its use that strike some people as being unreasonable Example given below illustrate one of there consequences.

Example:- A decision maker's utility of wealth function is given by u(w) = w - 0. 0/w^2, w < 50.

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount C with probability 1-p. for the values of w, c, and p exhibited in the table the decision maker will pay for complete insurance assume $C \leq w < 50.$

Solution: For the facts stated. (1) becomes

$$u(w - G) = p\,u(w) + (1 - p)u(w - c),$$

$$(w - g) - 0.01\,(w - G)^2 = p\left(w - \frac{0.0}{w^2}\right) + (1 - \mathrm{p})[(\mathrm{w} - \mathrm{c}) - 0.01(\mathrm{w} - \mathrm{c})^2]$$

For given values of w, p and c this expression becomes a quadratic equation. Two solutions are shown.

Wealth	Loss	Probability	Insurance Premium
W	c	p	G
10	10	0.5	5.2p
20	10	0.5	5.37

Insurance Applications

The functions that are natural in the context of reinsurance. Throughout we let X denote the amount of a claim and let X have distribution function F. Further we assume that all claim amounts are non-negative quantities, so that F(x) = 0 for x < 0 and assume that X is a continuous random variable, with density function f.

A reinsurance arrangement is an agreement between an insurer and a reinsurer under which claims that occur in a fixed period of time (e.g. one year) are split between the insurer and the reinsurer in an agreed manner. Thus, the insurer is effectively insuring part of a risk with a reinsurer and of course, pays a premium to the reinsurer for this cover. One effected of reinsurer is that it reduces the variability of claim payments by the insurer.

Proportional Reinsurance

Under a proportional reinsurance arrangement, the insurer pays a fixed proportion, say, of each claim that occurs during the period of the reinsurance arrangement. The remaining proportion (1-a), of each claim is paid by the reinsurer.

Let Y denote the part of a claim paid by the insurer under this proportional reinsurance arrangement and let z denote the part paid by the reinsurer. In terms of random variables Y= aX and Z= (1-a) X, and trivally Y+Z = X. thus the random variables Y and Z both scale transformation of the random variable X.

Suppose X is continuous then calculate

(i) c.d.f. of Y
(ii) c.d.f. of Z
(iii)p.d.f. of y

(iv)p.d.f. of z

$(i)\quad cdf\ of\ y - F_y\,(y) = P\,(aX \le y) = P\,\left(X\ \le \frac{y}{a}\right)\,G_y(y) =\ F_x(y/a)$

$(ii)\quad cdf\ of\ Z - F_z\,(Z) = P\,(Z \le z) = P\big((1-q) \times \le z\big) = P\,\left(X\ \le \frac{z}{(1-a)}\right)\ =\ H_z(Z)$

$(iii) pdf\ y - g_y\ = \frac{d}{dy}\,G_y(y)\ =\ f_x(y/a)\,.\frac{1}{a}$

$(iv) pdf\ Z - h_z\ = \frac{d}{dz}\,H_z(z) =\ f_x\left(\frac{Z}{1-a}\right)\,y\left(\frac{1}{1-a}\right)$

Example (1) Let $X \sim \Upsilon(\alpha,\lambda)$ what is the distribution of ax?

Solution:- As

$$f_{(x)} = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

It follows that the density function of ax is

$$\frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{a^{\alpha}\Gamma(\alpha)}$$

Thus the distribution of ax is $\Upsilon(\alpha, \lambda/a)$.

Example:- Let $X \sim LN\ (\mu,\sigma)$. What is the distribution of ax?

Solution:- As

$$f_{(x)} = \frac{1}{x\sigma\sqrt{2\Lambda}} exp\ \left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}$$

It follows that the density function of ax is

$$\frac{1}{x\sigma\sqrt{2\Lambda}} exp\ \left\{-\frac{(\log x - log\,a - \mu)^2}{2\sigma^2}\right\}$$

Thus the distribution of ax is LN $(\mu + \log a,\ \sigma)$.

Excess of Loss Reinsurance:- Under an excess of reinsurance arrangement a claim is shared between the insurer and the reinsurer only if the claim exceeds a fixed amount called the retention level. Otherwise, the insurer pays that claim in full. Let M denote the retention level, an let Y and Z denote the amounts paid by the insurer and the reinsurer respectively under this reinsurance arrangement Mathematically,

This arranged can be represented as the insurer pay y = min (X, M) and the reinsurer pays Z= max (0,X-M), with Y+Z= X.

The Insurer's Position:-

Let F_y be the distribution of Y. Then is follows from the definition of Y that

$$F_y(x) = \begin{cases} F(x) & for \quad x < M \\ 1 & for \quad x \geq M \end{cases}$$

Thus, the distribution of y is mixed with a dentition function f(x) for 0 < x < M, and a mass of probability at M, with $P_r(Y = M) = 1 - F(M)$.

As Y is a function of X, the moment of Y can be calculated from.

$$E[Y^n] = \int_0^\infty (\min(x, m))^n \, f(x)dx.$$

and this integral can be split into two parts since min (x, m) equals x for x < M and equals M for x ≥ M. Hence

$$E[Y^n] = \int_0^M x^n f(x)dx + \int_0^\infty M^n f(x)dx$$

$$= \int_0^M x^n f(x)dx + M^n \, (1 - F(M)) - - - - - - - - -(1)$$

In particular,

$$E[Y] = \int_0^M x \, f(x)dx + M \, (1 - F(M)).$$

So that

$$\frac{d}{dm}E[Y] = 1 - F(M) > 0 .$$

Thus as a function of M, E[Y] increase from 0 when M = 0 to E [X] as M → ∞ .

Example (3):- Let F (x) =1- $e^{-\lambda x},\ x \geq o.\ find\ E[y].$

Solution:- We have

$$E[Y] = \int_0^M x.\ \lambda e^{-\lambda x}\,dx + \ M_e{}^{-\lambda M}$$

and integration by parts yields

$$= \lambda \int_0^M x_I.\ e^{-\lambda x}\,dx + M\ e^{-\lambda M}$$

$$= \lambda\left[x\frac{e^{-\lambda x}}{-\lambda}\right]_0^M - \lambda\int_0^M 1.\frac{e^{-\lambda x}}{-\lambda}\,dx + M\quad e^{-\lambda x M}$$

$$= -M\quad e^{-\lambda x M} + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^M + M\quad e^{-\lambda x M}$$

$$= \left[\frac{-e^{-\lambda x}}{\lambda} + \frac{1}{\lambda}\right] = \frac{1}{\lambda}\left[1 - e^{-\lambda x}\ \right].$$

Example:- Let $X{\sim}LN\ (\mu,\sigma).Find\ E[y^n].$

Solution:- Inserting the lognormal density function into the integral in equation (1) we get

$$E[y^n] = \int_0^M x^n.\frac{1}{x\ \sigma\sqrt{2\Lambda}}\exp\left\{-\frac{(\log x-\mu)^2}{2\sigma^2}\right\}dx.$$

To deal with an integral of this type there is a standard substitution namely y = log x. This gives

$$I = \int_{-\infty}^{\log M} \exp\{yn\}\frac{1}{x\ \sigma\sqrt{2\Lambda}}\exp\left\{-\frac{(\log y-\mu)^2}{2\sigma^2}\right\}dy.$$

The technique in evaluating this integral is to write the integral in terms of a normal density function (different to the N (μ, σ^2) density function). To achieve this we apply the technique of "completing the square" in the exponent, as follows:

$$y_n - \frac{(y-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2}[(y-\mu)^2 - 2\sigma^2 y_n]$$

$$= -\frac{1}{2\sigma^2}[(y^2 - 2\mu y + \mu^2) - 2\sigma^2 y_n]$$

$$= -\frac{1}{2\sigma^2}\ [y^2 - 2y\ (\mu + \sigma^2 n) + \mu^2]$$

Nothing that the terms inside the square brackets would give the square of $y - (\mu + \sigma^2 n)$ if the final term $(\mu + \sigma^2 n)^2$ were instead of we can write the exponent as.

$$= -\frac{1}{2\sigma^2}\left[\left(y - (\mu + \sigma^2 n)\right)^2 - (\mu + \sigma^2 n)^2 + \mu^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[\left(y - (\mu + \sigma^2 n)\right)^2 - 2\mu + \sigma^2 n - \sigma^4 n^2\right]$$

$$= \mu n + \frac{1}{2}\sigma^2 n^2 - \frac{1}{2\sigma^2}\left(y - (\mu + \sigma^2 n)\right)^2.$$

Hence,

$$I = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \int_{-\infty}^{logM} \frac{1}{\sigma\sqrt{2\Lambda}}\ exp\left\{-\frac{1}{2\sigma^2}\ \left(y - (\mu + \sigma^2 n)\right)^2\right\}\ dy$$

And as the integrand is the $N\ (\mu + \sigma^2 n, \sigma^2)$ density function.

$$I = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \Phi\left(\frac{\log M - \mu}{\sigma}\right).$$

Finally, using the relationship between normal and lognormal distributions,

$$1 - F(M) = 1 - \Phi\left(\frac{\log M - \mu}{\sigma}\right)$$

So that

$$E[Y^n] = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \Phi\left(\frac{\log M - \mu - \sigma^2 n}{\sigma}\right) + M^n \left(1 - \Phi\left(\frac{\log M - \mu}{\sigma}\right)\right).$$

<u>The Reinsurer's Position</u>:-

From the definition of Z it follows that Z takes the values Zero if $X \le M,$ and takes the values Hence if Fz denotes the distribution function of Z, then Fz (0)= F(M) and for x > 0 , Fz (x) = F (X+M). Thus Fz is a mixed distribution with a mass of probability at 0.

The moments of Z can be found in a similar fashion to these of Y. We have

$$E[Z^n] = \int_0^\infty (0, x - M)^n f(x)dx,$$

and since max (0, x-M) is 0 for $0 \le x \le M$ we have

$$E[Z^n] = \int_0^\infty (x - M)^n f(x)dx - - - - - - - - - - (3)$$

Example 5- $Let\ F(x) = 1 - e^{-\lambda x},\ x \ge 0, Find\ E\ [Z].$

Solution:- Setting n =1 is equation (3) we have

$$E\ [Z] = \int_0^\infty (x - M)\lambda\ e^{-\lambda x} dx$$

$$= \int_0^\infty y\lambda\ e^{-\lambda(y+m)} dy$$

$$= e^{-\lambda M}\ E[X].$$

$$= \frac{1}{\lambda} e^{-\lambda M}$$

Alternatively, the indent E [Z] = E [X] – E [y] yields the answer with E[X] = 1/λ and E [y] given by the solution to example 3.

Example 6- $Let\ F(x) = 1 - e^{-\lambda x},\ x \ge 0, Find\ M_z(t).$

Solution:- By definition $M_Z(t) = E[e^{tz}]$ and as Z = max (0,X-M),

$$M_Z(t) = \int_0^{\infty} e^{t\max(0,2-M)} \lambda e^{-\lambda x}\, dx$$

$$= \int_0^{M} e^{0} \lambda e^{-\lambda x}\, dx + \int_0^{\infty} e^{t(x-M)} \lambda e^{-\lambda x}\, dx$$

$$= 1 - e^{-\lambda M} + \lambda \int_M^{\infty} e^{ty-\lambda(y+M)}\, dy \qquad [y = x - M)$$

$$= 1 - e^{-\lambda M} + \frac{\lambda e^{-\lambda M}}{\lambda - t}.$$

Provided that $t < \lambda$.

The above approach is a slightly artificial way of looking at the reinsurer's position since include zero as a possible "claim amount" for the reinsurer. Alternatively consider the distribution of the non-zero amounts paid by reinsurer. In practice the reinsurer is likely to have information only on these amounts as the insurer is unlikely to inform the reinsurer each time there is a claim whose amount is less than M.

Example-7 Let X have a discrete distribution as follows.

$P_r(x = 100) = 0.6$

$P_r(x = 175) = 0.3$

$P_r(x = 200) = 0.1$

If the insurer effects excess of loss reinsurance with retention level 150, what is the distribution of the non-zero payments made by the reinsurer?

Solution: First we note that the distribution of Z is given by

$P_r(Z = 0) = 0.6$

$P_r(Z = 25) = 0.3$

$P_r(Z = 50) = 0.1$

Now, let W denote the amount of a non-zero payment made by the reinsurer. Then W can take one of the two values: 25 and 50. Since payments of amount 25 are three times as likely as payments of amount 50. We can write the distribution of W as

$P_r(W = 25) = 0.75$

$P_r(W = 50) = 0.25$

The argument in above example can be formalized as follows. Let W denote the amount of a non-zero payment by the reinsurer under an excess of loss reinsurance arrangement with retention level M. The distribution of W is identical to that of z | Z > 0. Hence.

$$P_r(W \leq x)\ P_r(Z < x \mid Z > 0) = P_r(X \leq x + M \mid x > M)$$

From which it follows that

$$P_r(W \leq x) = \frac{P_r(M < X < x + M)}{P_r(X > M)} = \frac{F\ (x + M) - F\ (M)}{1 - F\ (M)} - - - - (4)$$

Differentiation gives the density function of was

$$\frac{F\ (x + M)}{1 - F\ (M)} - - - - - - - (5)$$

Example- 8 Let F (x) = 1-$e^{-\lambda x}, x \geq 0$. What is the distribution of the non-zero claim payments made by the reinsurer?

Solution: By formula (5), the density function is

$$\frac{\lambda\ e^{-\lambda(x+M)}}{e^{-\lambda m}} = e^{-\lambda x}$$

So that the distribution of W is the same as that of X. (This rather surprising result is a consequence of the “memory less” property of the exponential distribution.

Example 9- What is the distribution of the non zero claim payments made by the reinsurer?

Solution: By formula (5), the density function is

$$\frac{\alpha\lambda^{\alpha}}{(\lambda + M + x)^{\alpha+1}} \left(\frac{\lambda + M}{\lambda}\right)^{\lambda} = \frac{\alpha\ (\lambda + M)^{\alpha}}{(\lambda + M + x)^{\alpha+1}}$$

So that the distribution of W is $P_o(\alpha, \lambda + M)$

Policy Excess-

Introduction:-

Insurance policies with a policy excess are very common particularly in motor vehicle insurance. We consider a policy excess arrangement as follows:

Notation: If a policy is issued with an excess of other the insured party pays any loss of amount less than or equal to d in full and pays d on any loss in excess of d. thus if X represents the amount of a loss, when a loss occurs the insured party pays min (X,d) and the insurer pays max (0, X-d)

Comparison With EOL Re!

These quantities are of the same form as the amounts paid by the insurer and the reinsurer when a claim occurs (for the insurer) under an excess of loss reinsurance arrangement. Hence there are no new mathematical consideration involved.

Remark- It is important however, to recognize that x represents the amount of a loss, and not the amount of claim.

Sums of Random Variables

In most insurance application we are interested in distribution of sum of identically and independently distribution random variables.

Example Suppose that an insurer issues n policies and the claim amount from policy i= 1,2,….,n, is a random variable Xi. Then the total amount the insurer pays in claim from these n policies is $S_n = \sum_{i=1}^{n} X_i$

An obious question is the mind of the insurer is to know the distribution of S_n. Whenever the distribution exists is a closed form, we make use one of the following 3 methods:

(i) Moment Generating Function Method
(ii) Convolution
(iii)Recursive Calculation for Discrete random variables.

(i) **Moment Generating Function Method (m.g.f. method)-** This is a very neat way of finding the distribution of S_n. Define M_s to be the moment generating function of X_L . Then

$$M_s(t) = E\left[e^{tS_n}\right] = E\left[e^{t(X_1+X_2+\cdots X_n)}\right]$$

Using independence, it follows that

$$M_s(t) = E\left[e^{tX_1}\right] = E\left[e^{tX_2}\right] \dots \dots \left[e^{tX_n}\right]$$

And as the X_i's are identically distributed

$$M_s(t) = M_s(t)^n$$

Hence, if we can identify $M_s(t)^n$ as the m.g.f. of a distribution we know the distribution of Sn by the uniqueness property of m.g.f.

Example 10-

Let X_1 have a Poisson distribution with parameter λ. what is the distribution of S_n?

Solution: As

$$M_x(t) = \exp\{\lambda(e^t - 1)\}$$

We have

$M_s(t) = \exp\{\lambda n(e^t - 1)\}$

and so S_n has a Poisson distribution with parameter λn.

Example- 11: Let X_1 have an exponential distribution with mean $1/\lambda$. What is the distribution of S_n?

Solution: As

$$M_x(t) = \frac{\lambda}{\lambda - t} \; for\; t < \lambda, we\; have$$

$$M_s(t) = [M_x(t)]^n = \left(\frac{\lambda}{\lambda - t}\right)^n$$

And so S_n has a ϒ (n, λ)distribution

(ii) **Direct Convolution of Distributions**

Direct convolution is a more direct and less elegant method of finding the distribution of S_n us first assume that $\{X_i\}_{i=1}$ are discrete random variable, distribution on the non – negative integers, so that S_n is also distributed on the non-negative integers. Let x be a non-negative integer and consider first the distribution of S_2. The convolution approach to finding Pr ($S_2 \leq x$) consider how the event $\{S_z \leq x\}$ can occur. This event occurs when X_2 takes the value j, where j can be any values less than or equal to x-j, so that their sum is less than or equal to x. Summing over all possible values of j and using the fact that X_1 and x_2 are independent, we have

$$P_r(S_2 \leq x) = \sum_{j=0}^{x} P_r(X_1 \leq x - j)\, P_r(x_2 = j).$$

The same argument can be applied to find $P_r(S_3 \leq x)$ by writing $S_3 = S_2 + X_3$ and by noting that S_2 and X_3 are independent (as $S_2 = X_1 + X_2$). Thus,

$$P_r(S_3 \leq x) = \sum_{j=0}^{x} P_r(X_1 \leq x - j)\, P_r(x_3 = j). ------ (1)$$

The same reasoning gives

$$P_r(S_n = x) = \sum_{j=0}^{x} P_r(S_{n-1} \leq x - j)\, P_r(x_n = j)$$

Now, let F be the distribution function of X_1 and let $f_j = P_r\,(X_1{=}j)$. we define

$$F^{n*}\,(x) = P_r(S_n \leq x)$$

And call the n- fold convolution of the distribution F with itself. Then by equation (1)

$$F^{n*}\,(x) = \sum_{j=0}^{x} F^{(n-1)*}\,(x-j) f_j$$

Now that $F^{1*} = F$ and by convention, se define $F^{0*}(x) = 1$ for $x \geq 0$ with $F^{0*}(x) = 0$ for $x < 0$. Similarly we define $F_x^{n*}(x) = P_r(S_n = x)$ so that

$$F_x^{n*} = \sum_{j=0}^{x} f_{x-j}^{(n-1)*}\;\; f_j$$

with $f^{1*} = f$

when F is a continuous distribution on $(0, \infty)$ with density function f, the analogues of the above result are

$$F^{n*}\,(x) = \int_0^x F^{(n-1)*}\,(x-y) f(y) dy$$

and

$$f^{n*}\,(x) = \int_0^x f^{(n-1)*}\,(x-y) f(y) dy ------(2)$$

These result can be used to find the distribution of S_n directly.

Example-12:- What is the distribution of S_n when $\{Xi\}_{i=1}^{n}$ are independent exponentially distributed random variable, each with mean 1/λ?

Soultion:- Setting n=2 is equation (2), we get

$$F^{2*}(x) = \int_0^x f\,(x-y)f(y)dy$$

$$= \int_0^x \lambda \quad e^{-\lambda(x-y)}\,\lambda\, e^{-\lambda y} dy$$

$$= \lambda^2\ e^{-\lambda x} \int_0^x dy = \lambda^2 x\ \ e^{-\lambda x}$$

So that, S_2 has a $\Upsilon(2,\lambda)$ distribution. Next, setting n = 3 is equation (2), we get

$$f^{3*}(x) = \int_0^x f^{2*}(x-y)f(y)dy$$

$$= \int_0^x f^{2*}(y)f\,(x-y)dy$$

$$= \int_0^x \lambda^2 y\ \ e^{-\lambda y}\,\lambda\, e^{-\lambda(x-y)} dy$$

$$= \frac{1}{2}\,\lambda^3 x^2 e^{-\lambda x}$$

So that the distribution of S_3 is $\Upsilon\ (3,\lambda)$. An inductive argument can now be used to show that you a general value of n, S_n has $\Upsilon\ (n,\lambda)$ a distribution.

In general it is much easier to apply the moment generating function method of find the distribution of S_n.

(iii)Recursive calculation for Discrete Random Variables. In the case when X_i is a discrete random variable, distributed on the non-negative integers, it is possible to calculate the probability function of S_n recursively. Define

$$f_j = P_r(X_i = j)\ and\ g_j = P_r(S_n = j)$$

Each for j= 0, 1, 2,……… We denote the probability generating function of X_1 by P_x so that

$$P_r(x) = \sum_{k=0}^{\infty} Y^j\ f_j$$

and the probability generating function of S_n by P_s, so that

$$P_r(r) = \sum_{k=0}^{\infty} r^k\ g_k$$

Using argument that have previously been applied to moment generating functions, we have

$$P_s(r) = \ P_x(r)^n$$

and differentiation with respect to r gives.

$$P_s' = nP_x(r)^{n-1}P_s'\ (x).$$

When we multiply each side of the above identity by $rP_x(r)$ we get

$$P_x(r)r = \ P_s' = nP_s(r)rP_x'(x)$$

Which can be expected as

$$\sum_{j=0}^{\infty} r^j\ f_j \sum_{k=1}^{\infty} k\ r^k g_k = n\ \sum_{k=0}^{\infty} r^k g_k \sum_{j=1}^{\infty} jr^j f_j -------- (3)$$

To find an expression for g_x, we consider the coefficient of r^x on each side of equation (3), where x is a positive integer. On the left-hand side the coefficient of r^x can be found as follows. For j = 0,1,2,……, x-1, multiply together the coefficient of r^j in the first sum with the coefficient of r^xj is the second sum. Adding these products together gives the coefficient of r^x, namely

$$f_0\ x\ g_x + \ f_1(x-1)g_{x-1} + ---- + f_{x-1}\ g_1 = \sum_{j=0}^{\infty} (x-j)f_j g_{x-j}$$

Similarly, on the right hand side of equal, we have

$$x\ g_x f_0 + \sum_{j=1}^{x-1} (x-j)f_j g_{x-j} = n\ \sum_{j=1}^{x} j\ f_j\ g_{x-j}$$

Which gives (nothing that the sum on the left hand side unaltered when the upper limit of summation is increased to x)

$$g_x = \frac{1}{f_0}\sum_{j=1}^{x}\left((n+1)\,\frac{j}{x} - 1\right) f_j\, g_{x-j} - - - - - (4)$$

The important point about this result is that it gives a recursive method of calculating the probability function $\{g_x\}_{x=0}^{\infty}$. Given the values $\{f_j\}_{j=0}^{\infty}$. We can use the value of go to calculate g1, then the values of g_0 and g_1 to calculate g_2, and so on. The starting value for the recursive calculation is g_0 which is given by f_0^n since S_n takes the value 0 if and only if each x_j, j= 1,2,…….., n1 takes the value 0.

This is very useful result as it permits much more efficient evaluation of the probability function on of S_n than the direct convolution approach of the previous section.

We conclude with three remark about this result:

(i) Computer implementation of formula (4) is necessary especially when n is large. It is however, an easy task to program this formula.

(ii) It is straight forward to adapt this result to the situation when x_1 is distributed on m, m+1, m+2, -----, where m is a positive integer.

(iii)The recursion formula is unstable. That is, it may give numerical should be employed when sense. Thus caution should be employed when applying their formula. however, for most practical purposes, numerical stability is not an issue.

Example- 13:- Let $\{X_i\}_{i=1}^{4}$ be independent and identically distribution random variable with common probabilities function $f_j = P_r(X_1 = j)$ given by

$$f_0 = 0.4, \qquad f_1 = 0.3, f_2 = 0.2, \qquad f_3 = 0.1$$

Let $S_4 = \sum_{i=1}^{4} X_i$ .recursively calculate $P_r\ (S_4 = r)$ for r = 1,2, 3 and 4.

Solution:- The starting value for the recursive calculation is

$$g_0 = P_r\ (S_4 = 0) = f_0^{\ 4} = 0.4^4 = 0.0256$$

Now note that as $f_j = 0$ and for j = 4,5,6…. Equation

(4) can be written with a different upper limit of summation as

$$g_x = \frac{1}{f_0}\sum_{j=1}^{\min(3,x)}\left(\frac{5j}{x} - L\right) f_j g_{x-j}$$

and so

$$g_1 = \frac{1}{f_0}\ 4f_i g_o = 0.0768.$$

$$g_2 = \frac{1}{f_0}\left(\frac{3}{2}f_1 g_1 + 4f_1 g_o\right) = 0.1376$$

$$g_3 = \frac{1}{f_0}\left(\frac{2}{3}f_1 g_2 + \frac{7}{3}f_2 g_1 + 4f_3 g_o\right) = 0.1840$$

$$g_4 = \frac{1}{f_0}\left(\frac{1}{4}f_1 g_3 + \frac{3}{2}f_2 g_2 + \frac{11}{4}f_3 g_{12}\right) = 0.1905$$

Important Discrete Distribution

The Poisson Distribution- When a random variable N has a Poisson distribution with parameter $\lambda > 0$, its probability function is given by

$$P_r(N = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

For x= 0,1,2,------ The moment generating function is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}.e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^x}{2!} + - - - - - -\right\} = e^{-\lambda}.e^{\lambda^{e^t}}$$

$$= e^{\lambda}(e^t - 1) = \exp\ \{\lambda(e^t - 1)\} - - - - - - - - - - - - - (1.1)$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^x . e^{-\lambda} \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda r)^x}{x!} = e^{-\lambda}\, e^{\lambda r}$$

$$= \; e^{\lambda(r-1)} = \exp\,\{\lambda\,(r-1)\}$$

The moments of N can be found from the moment generation function. For example,

$$\mu_r'' = \left[\frac{d^r}{dt^r} M_N(t)\right]$$

$$M_N'(t) = \frac{d}{dt}[exp\{\lambda(e^t - 1)\}]$$

$$= \exp\{\lambda(e^t - 1)\}.\lambda^{e^t} = \lambda^{e^t} M_N(t)$$

$$and$$

$$M_N''(t) = \frac{d^2}{dt^2}[exp\{\lambda(e^t - 1)\}] = \frac{d}{dt}(M_N'(t))$$

$$= \lambda^{e^t} M_N(t) + \lambda^{e^t} . \frac{d}{dt} M_N(t) \;\; [differentation\ by\ par]$$

$$= \lambda e^t M_N(t) + \lambda e^t . \lambda e^t M_N(t)$$

$$M_N''(t) = \lambda e^t M_N(t) + (\lambda e^t)^2 M_N(t)$$

From which it follows that

$$E[N] = \; \lambda\ and\ E\,[N^2] = \lambda + \lambda^2\ so\ that\ V[N] = \; \lambda.$$

We use $P(\lambda)$ the notation to denote a Poisson distribution with parameter λ .

(2) The Binomial Distribution:-

When a random variable N has a binomial distribution with parameter n and q, where n is a positive integer and 0 < q <1. Its probability function is given by

$$P_r(N = r) = \binom{n}{r} q^r (1-q)^{n-x}$$

For x = 0,1,2,--------,n. The moment generating function is

$$M_N(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{r} q^r (1-q)^{n-x} = \sum_{x=0}^{n} \binom{n}{r} (qe^t)^x (1-q)^{n-x}$$

$$= (qe^t + 1 - q)^n$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{n} r^x \binom{n}{r} q^r (1-q)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{r} (qr)^x (1-2)^{n-x} = (qr + (1-q))^n$$

As,

$$M_N'(t) = n(qe^t + 1 - q)^{n-1}.\, qe^t$$

and

$$M_N''(t) = n(n-1)(qe^t + 1 - q)^{n-2}.\, (qe^t)(qe^t) + n(qe^t + 1 - q)^{n-1}.\, qe^t$$

$$= n(n-1)(qe^t + 1 - q)^{n-2}.\, (qe^t)^2 + n(qe^t + 1 - q)^{n-1}.\, qe^t$$

It follows that E[N]= nq, E $[N^2]$= n (n-1) q^2+ nq and V[N] = nq(1-q).

We use the notation B (n,q) to denote a binomial distribution with parameters n and q.

(3) <u>Negative Binomial Distribution</u>:-

When a random variable N has a negative binomial distribution with parameters k>0 and p. where $0 < p < 1$. Its probability function is given by

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x$$

For 0,1,2,------, where q= 1-p. when k is an integer calculation of the probability function is straight forward as the probability function can be expressed in terms of factorials. An alternative method of calculating the probability function, regardless of whether k is an integer, is recursively as

$$P_r(N = x+1) = \frac{k+x}{x+1} q \; P_r(N = x)$$

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x, P_r(N = x+1) = \binom{x+k}{x+1} p^k q^{x+1}$$

$$\frac{P_r(N = x+1)}{P_r(N = x)} = \frac{\binom{x+k}{x+1} p^k q^{x+1}}{\binom{x+k-1}{x} p^k q^x} = \frac{x+k}{x+1} q$$

$$P_r(N = x+1) = \frac{k+x}{x+1} q \; P_r(N = x)$$

For x= 0,1,2----- with starting value $P_r(N = x) = p^k$. The moment generating function can be found by making use of identity.

$$\sum_{x=0}^{\infty} P_r(N = x) = 1 - - - - - - - - - - - - - (1.2)$$

From this it follows that

$$\sum_{0}^{\infty} \binom{k+x-1}{x} (1 - qe^t)^k (qe^t)^x = 1$$

Provided that 0 < qe^t <1. Hence x

$$M_N'(t) = \sum_{x=0}^{\infty} e^{tx} \binom{k+x-1}{x} p^k q^x$$

$$= \frac{p^k}{(1-qet)^k} \sum_{x=0}^{\infty} \binom{k+r-1}{x} (1-qet)^k (qe^t)^x$$

$$= \frac{p^k}{(1-qet)^k} = \left(\frac{1}{1-qe^t}\right)^k$$

Provided that $0 < qe^t < 1$, or equivalently, $t < -\log q$. Similarly, the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^k \binom{k+x-1}{x} p^k q^x = \left(\frac{p}{1-qr}\right)^k$$

Moments of this distribution can be found by differentiating the moment generating function and the mean and variance are given by E{N}= kq/p and V[N]= kq/p^2.

$$M'_N(t) = \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)^k$$

$$M''_N(t) = \frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)^k$$

Equality (1.2) trivially gives

$$\sum_{0}^{\infty} \binom{k+x-1}{x} p^k q^x = 1 - p^k \text{ -------- } (1.3)$$

We use the notation NB (k,p) to denote a negative binomial distribution with parameter k and p.

(4) <u>The Geometric Distribution</u>:- The geometric distribution is a special case of the negative binomial distribution. When the negative binomial parameter k is L, the distribution is called a geometric distribution with parameter p and the probability function is

$$P_r(N = x) = pq^x$$

For x= 0,1,2,------. From above, it follows that E [N] = q/p, V[N] = q/p^2 and

$$M_N(t) = \frac{p}{1 - qe^t} \quad for\, t < \log q.$$

This distribution plays an important role in ruin theory.

Important Continuous Distribution

(1) The Gamma Distribution:-

When a random variable X has a gamma distribution with parameter $\alpha > 0 \; and \, \lambda > 0$ its density is given by

$$f(x) = \frac{\lambda^{\alpha}}{\tau(\alpha)} \; x^{\alpha - 1} \, e^{-\lambda x}$$

For $x \; > \; 0$, where $\tau(\alpha)$is the gamma function defined as

In the special case when α is an integer the distribution is also known as an Erlang distribution, and repeated integration by parts gives the distribution function as

$$f(x) = 1 - \sum_{j=0}^{\alpha - 1} e^{-\lambda x} \; \frac{(\lambda x)^j}{j!}$$

for $x \; \geq \; 0.$ The moments and moments generating function of the gamma distribution can be found by nothing that

$$\int_0^{\infty} f(x) \; dx = 1$$

Yields

$$\int_0^\infty x^{\alpha-1}\, e^{-\lambda x}\ dx = \frac{\tau(\alpha)}{\lambda^\alpha} - - - - - - - - (1.4)$$

The nth moment is

$$E[X^n] = \int_0^\infty x^n\ \frac{\lambda^\alpha x^{\alpha-1}\, e^{-\lambda x}}{\tau(\alpha)}\ dx = \frac{\tau(\alpha)}{\lambda^\alpha} \int_0^\infty x^{n+\alpha-1}\, e^{-\lambda x}\ dx$$

and form identity (1.4) it follows that

$$E[X^n] = \frac{\lambda^\alpha}{\tau(\alpha)} . \frac{\tau(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda^n} - - - - - - - - (1.5)$$

In particular,

n=1

$$E\ [X] = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda} = \frac{\alpha}{\lambda}$$

and, n=2

$$E\ [X^2] = \frac{\tau(\alpha+2)}{\tau(\alpha)\lambda^2} = \frac{\alpha(\alpha+1)}{\lambda^2},$$

So that

$$V[x] = E\ [X^2] - \{E[x]\}^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

We can find the moment generating function in a similar fashion. As

$$M_x(t) = \int_0^\infty e^{tx}\ \frac{\lambda^\alpha x^{\alpha-1}\, e^{-\lambda x}}{\tau(\alpha)}\ dx = \frac{\lambda^\alpha}{\tau(\alpha)} \int_0^\infty x^{\alpha-1}\, e^{-(\lambda-t)x}\ dx - - - - - - -(1.6)$$

application of identity (1.4), gives

$$M_x(t) = \frac{\lambda^\alpha}{\tau(\alpha)} \cdot \frac{\tau(\alpha)}{(\lambda - t)^\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^\alpha ------ (1.7)$$

Note that in identity (1.4) $\lambda > 0$. Hence in order to apply (1.4) to 91.6) we require that $\lambda - t > 0$, so that the m.g.f. exists when $t < \lambda$. u coefficient of skewness of X, which we denoted by Sk[x], is $\frac{2}{\sqrt{\alpha}}$. This follows from the definitiaon of the coefficient of skewness, namely third untral moment divided by standard divided by standard division cubed and the fact that the third untral moment is

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = E[X^3] - \frac{3\alpha}{\lambda} E[X^2] + 2\left(\frac{\alpha}{\lambda}\right)^3$$

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} - \frac{3r}{\lambda} \cdot \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{2\,\alpha^3}{\lambda^3}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\,\alpha^3}{\lambda^3}$$

$$= \frac{2\alpha}{\lambda^3}$$

We use the notation $\gamma(\alpha, \lambda)$ to denote a gamma distribution with parameter α and λ.

(2) <u>The Exponential Distribution</u>:-

The exponential distribution is a special case of gamma distribution. It is just a gamma distribution with parameter x=1. Hence, the exponential distribution with parameter $\lambda > 0$ has density function.

$$f(x) = \lambda e^{-\lambda x}$$

For $x > 0$ and has the distribution function

$$f(x) = 1 - e^{-\lambda x}$$

For $x \geq 0$ form equation (1.5), the nth moment of the distribution is

$$E[X^n] = \frac{n!}{\lambda^n}$$

And from equation (1.7), the moment generating function is

$$M_x(t) = \frac{\lambda}{\lambda - t}, \quad for\ t < \lambda$$

(3) <u>The Pareto Distribution:</u> When a random variable X has a Pareto distribution with parameters x > 0, its density function is given by

$$f(x) = \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}$$

for x>0 integrating this density, we find that the distribution functions is

$$F(x) = 1 - \left(\frac{\lambda}{(\lambda + x)}\right)^\alpha$$

for x ≥ 0. Whenever moments of the distribution exists they can be found from.

$$E[X^n] = \int_0^\infty x^n f(x) dx$$

by integration by parts. However, they can above found individual using the following approach. Since the integral of the density function over (0,∞) equal 1, we have

$$\int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}} dx = 1 \qquad \int_0^\infty \frac{dx}{(\lambda + x)^{\alpha+1}} = \frac{1}{\alpha\lambda^\alpha}$$

an identity which holds provided that x > 0. To find E[X], we can write

$$E[x] = \int_0^\infty xf(x)dx = \int_0^\infty (x + \lambda - \lambda)f(x)dx = \int_0^\infty (x + \lambda)f(x)dx - \lambda$$

and inserting for f, we have

$$E[x] = \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}} . (x + \lambda) dx - \lambda$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^\alpha} dx - \lambda$$

We can evaluate the integral by rewriting the integrand in terms of a α pareto density function with parameters $(\alpha - 1)$ and λ. thus,

$$E[x] = \frac{\alpha\lambda}{(\alpha - 1)} \int_0^\infty \frac{(\alpha - 1)\lambda^{\alpha-1}}{(\lambda + x)^\alpha} dx - \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.8)$$

and since the integral equals 1.

$$E[x] = \frac{\alpha\lambda}{(\alpha - 1)} - \lambda = \frac{\lambda}{\alpha - 1}$$

It is important to note that the integrand in equation (1.8) is a Pareto density function only if α>1, and hence E[x] exists only for α>1. Similarly, we can find E $[x^2]$ from

$$E[x] = \int_0^\infty ((x + \lambda)^2 - 2\lambda x - \lambda^2) f(x) dx$$

$$= \int_0^\infty (x + \lambda)^2 f(x) dx - 2\lambda E[x] - \lambda^2$$

$$= \int_0^\infty (x + \lambda)^2 . \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}} dx - 2\lambda \frac{\lambda}{\alpha - 1} - \lambda^2$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}\, dx - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha - 2)} \int_0^\infty \frac{(\alpha - 2)\lambda^{\alpha-2}}{(\lambda + x)^{\alpha-1}}\, dx - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha - 2)} - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{2\lambda^2}{(\alpha - 1)(\alpha - 2)}, \quad proved\ that\ \alpha > 2.$$

and hence that

$$V[x] = E[x^2] - [E[x]^2]$$

$$= \frac{2\lambda^2}{(\alpha - 1)(\alpha - 2)} - \frac{\lambda^2}{(\alpha - 1)^2}$$

$$V[x] = \frac{\alpha\lambda^2}{(\alpha - 1)^2(\alpha - 2)}$$

We use the notation Pa (α, λ) to denote a Pareto distribution with parameters $\alpha\ and\ \lambda$.

(4) **<u>The Normal Distribution:-</u>**

When a random variable X has a normal distribution with parameter $\mu\ and\ \sigma^2$, its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(x - \mu)^2}{2\sigma^2}\right\}\ for - \infty < x < \infty$$

We use notation N (μ, σ^2) to denote a normal distribution with parameter $\mu\ and\ \sigma^2$. The standard normal distribution has parameter 0 and 1 and its distribution function is denoted by $\emptyset$, where

$$\emptyset(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\Lambda}} \exp\left\{\frac{-Z^2}{2}\right\} dz.$$

A key relationship is that if $X \sim \mu\ (\mu, \sigma^2)\ and\ if\ Z = \frac{(x-\mu)}{6}, then\ Z \sim N(0,1)$.

The moment generating function is

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{\sigma\sqrt{2\Lambda}} \quad \int_{-\infty}^{\infty} e^{tx} \left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{\alpha\Lambda}} \int_{-\infty}^{\infty} exp\,\{t(\mu + \sigma z)\} \exp\left(-\frac{t}{2}\right) dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\,\left\{-\frac{1}{2}(z^2 - 2t\sigma z)\right\} dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\,\left[-\frac{1}{2}\{(z - \sigma t)^2 - \sigma^2 t^2\}\right] dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left\{-\frac{1}{2}(z - \sigma t)^2\right\} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left(-\frac{\mu^2}{2}\right) du$$

$$Hence, M_x(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} - - - - - - - - - - - (1.9)$$

Twice differentiating $M_x(t)$ with respect to t, we get

$$M_X'(t) = (\mu + \sigma^2 t).M_X(t) and\, M_X''(t) = [(\mu + \sigma^2 t) + \sigma^2].M_X(t)$$

$$So\, that, M_X'(0) = \mu \,and\, M_X''(0) = \mu^2 + \sigma^2. Thus,$$

$$E[X] = \mu \,and\, V(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

(5) **<u>The Lognormal Distribution</u>**:- When a random variable X has a lognormal distribution with parameter μ and σ where $-\infty < \mu < \infty \,and\, \sigma > 0,$ its density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(logx - \mu)^2}{2\sigma^2}\right\}$$

For x>0. The distribution function can be obtained by integrating the density function as follows:

$$F(x) = \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{2\Lambda}}\ exp\left\{\frac{(logx - \mu)^2}{2\sigma^2}\right\} dy$$

and the substitution Z= logy yields

$$F(x) = \int_{-\infty}^{logx} \frac{1}{\sigma\sqrt{2\Lambda}}\ exp\left\{-\frac{(z - \mu)^2}{2\sigma^2}\right\} dz.$$

As the integrand is the N (μ, σ^2) density function

$$F(x) = \emptyset\left(\frac{logx - \mu}{\sigma}\right).$$

Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function.

We use the notation LN (μ, σ^2) to denote a lognormal distribution with parameters $and\ \sigma$. From the preceding argument it follows that if X ~ LN (μ, σ^2) , then log X ~ LN (μ, σ^2)

This relationship between normal and lognormal distributions is extremely useful, particularly is deriving moments. If X ~ LN (μ, σ^2) $and\ Y = logX, then$

$$E[X^n] = E[e^{ny}] = M_y(n) = \ exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\}$$

Where the final equality follows by equation (1.9).

Mixed Distribution

The illustrate the idea of a mixed distribution, Let X be exponentially distribution with mean 100, and let the random variable Y be defined by

$$Y = \begin{cases} 0 & if\ X < 20 \\ X - 20 & if\ 20 \le X < 300 \\ 280 & if\ X \ge 300 \end{cases}$$

Figure

Then,

$$P_r(Y = 0) = P_r(X < 20) = t - e^{-0.2} = 0.1813$$

and similarly P_r (Y=280)=0.0498. Thus Y has masses of probability at the point 0 and 280. However in the interval (0,280), the distribution of Y is continuous with for example,

$$P_r(30 < Y \le 100) = P_r(50 < X \le 120) = 0.3053$$

Figure shows that the distribution function, H of Y note that there are jumps at 0 and 280, corresponding to the masses of probability at these points. As the distribution is differentiable in the interval (0, 280), Y has a density function in this interval. Letting h denote the density function of Y, the moments of Y can be found from

$$E[Y^2] = \int_0^{180} x^r h(x)dx + 280^r P_r(Y = 280).$$

It will be convenient to use satisfies integral notation so that we don not have to specify with a distribution is discrete, continuous or mixed. In the notation, we write the rth moment of Y as

$$E[Y^r] = \int_0^{\infty} x^r \, dH(x).$$

More generally if $k(x) = P_r(Z \le x)$ is a mixed distribution on [0,∞] and m is a function, then

$$E[m(Z)] = \int_0^{\infty} m(x) d\, k(x),$$

Where we interpret the integral as

$$\sum_{x_i} m(x_i)\, P_r(Z = x_i) + \int m(x)k(x)dx.$$

Where summation is over the points $\{x_i\}$ at which there is a mass of probability and integration is over the intervals in which K is continuous with density function k.

Practical 6

Aim:- To obtain the probability and moments for given random variables.

Experiment:- Let the random variable X have the distribution function F given by:

$$F_{(x)}(x) = \begin{cases} 0 & X < 20 \\ \dfrac{x+20}{80} & 20 \le X < 40 \\ 1 & X \ge 40 \end{cases}$$

Check the nature of random variable X, Hence calculate

$i)\ \ P\ (X < 30)$

$ii)\ \ P\ (X = 40)$

$iii)\ \ E(X)$

$iv)\ \ V\ (X)$

Theory:-

Calculation:- To check the nature of r.v. x-

$$P(X = 20) = \frac{20 + 20}{80} = 0.5 > 0$$

$$P(X = 40) = 1 - \frac{40 + 20}{80} = 0.25 > 0$$

Points X=20 and X=40 are jumps points. Thus X has a mixed distribution.

$$i)\ \ P\ (X \leq 30) = \frac{30 + 20}{80} = 0.625$$

$$ii)\ \ P\ (X = 40) = 1 - \frac{40 + 20}{80} = 0.25$$

The p d f of the distribution is given as:-

$$f_x(x) = \begin{cases} \frac{1}{80}, & 20 \leq x < 40 \\ 0, & otherwise \end{cases}$$

$$iii)\ \ E(X) = 20 \times 0.5 + \int_{20}^{40} x.\frac{1}{80}dx + 40 \times 0.25$$

$$= 20 + \left[\frac{x^2}{2 \times 80}\right]_{20}^{40} = 20 + \frac{40^2 - 20^2}{160} = 27.5$$

$$E(X^2) = 20^2 \times 0.5 + \int_{20}^{40} \frac{x^2}{80} dx + 40^2 \times 0.25$$

$$= 600 + \left[\frac{x^3}{3 \times 80}\right]_{20}^{40} = 600 + \frac{40^3 - 20^3}{240} = 833.33$$

$$iv)\ V(X) = E[X^2] - \{E[X]\}^2 = 833.33 - (27.5)^2 = 77.083$$

Result:-

$i)\ P(X \le 30) = 0.625$

$ii)\ P(X = 40) = 0.25$

$iii)\ E(X) = 27.5$

$iv)\ V(X) = 77.083$

Important Discrete Distribution

The Poisson Distribution- When a random variable N has a Poisson distribution with parameter λ > 0, its probability function is given by

$$P_r(N = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

For x= 0,1,2,------ The moment generating function is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}.e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left\{1 + \lambda e^t + \frac{(\lambda e^t)^x}{2!} + - - - - - -\right\} = e^{-\lambda}.e^{\lambda^{e^t}}$$

$$= e^{\lambda}(e^t - 1) = \exp\ \{\lambda(e^t - 1)\} - - - - - - - - - - - - - (1.1)$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^x . e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda}\frac{(\lambda r)^x}{x!} = e^{-\lambda}\ e^{\lambda r}$$

$$= \ e^{\lambda(r-1)} = \exp\ \{\lambda\ (r-1)\}$$

The moments of N can be found from the moment generation function. For example,

$$\mu_r'' = \left[\frac{d^r}{dt^r} M_N(t)\right]$$

$$M_N'(t) = \frac{d}{dt}[exp\{\lambda(e^t - 1)\}]$$

$$= \exp\{\lambda(e^t - 1)\}.\lambda^{e^t} = \lambda^{e^t} M_N(t)$$

and

$$M_N''(t) = \frac{d^2}{dt^2}[exp\{\lambda(e^t - 1)\}] = \frac{d}{dt}(M_N'(t))$$

$$= \lambda^{e^t} M_N(t) + \lambda^{e^t}.\frac{d}{dt} M_N(t)\ \ [differentation\ by\ par]$$

$$= \lambda e^t M_N(t) + \lambda e^t.\lambda e^t M_N(t)$$

$$M_N''(t) = \lambda e^t M_N(t) + (\lambda e^t)^2 M_N(t)$$

From which it follows that

$$E[N] = \ \lambda\ and\ E\ [N^2] = \lambda + \lambda^2\ so\ that\ V[N] = \ \lambda.$$

We use $P(\lambda)$ the notation to denote a Poisson distribution with parameter λ .

(2) The Binomial Distribution:-

When a random variable N has a binomial distribution with parameter n and q, where n is a positive integer and 0 < q <1. Its probability function is given by

$$P_r(N = r) = \binom{n}{r} q^r (1-q)^{n-x}$$

For x = 0,1,2,--------,n. The moment generating function is

$$M_N(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{r} q^r (1-q)^{n-x} = \sum_{x=0}^{n} \binom{n}{r} (qe^t)^x (1-q)^{n-x}$$

$$= (qe^t + 1 - q)^n$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{n} r^x \binom{n}{r} q^r (1-q)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{r} (qr)^x (1-2)^{n-x} = (qr + (1-q))^n$$

As,

$$M'_N(t) = n(qe^t + 1 - q)^{n-1}.qe^t$$

and

$$M''_N(t) = n(n-1)(qe^t + 1 - q)^{n-2}.(qe^t)(qe^t) + n(qe^t + 1 - q)^{n-1}.qe^t$$

$$= n(n-1)(qe^t + 1 - q)^{n-2}.(qe^t)^2 + n(qe^t + 1 - q)^{n-1}.qe^t$$

It follows that E[N]= nq, E $[N^2]$= n (n-1) q^2+ nq and V[N] = nq(1-q).

We use the notation B (n,q) to denote a binomial distribution with parameters n and q.

(3) <u>Negative Binomial Distribution</u>:-

When a random variable N has a negative binomial distribution with parameters k>0 and p. where 0 < p < 1. Its probability function is given by

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x$$

For 0,1,2,------, where q= 1-p. when k is an integer calculation of the probability function is straight forward as the probability function can be expressed in terms of factorials. An alternative method of calculating the probability function, regardless of whether k is an integer, is recursively as

$$P_r(N = x+1) = \frac{k+x}{x+1} q \;\; P_r(N = x)$$

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x, P_r(N = x+1) = \binom{x+k}{x+1} p^k q^{x+1}$$

$$\frac{P_r(N = x+1)}{P_r(N = x)} = \frac{\binom{x+k}{x+1} p^k q^{x+1}}{\binom{x+k-1}{x} p^k q^x} = \frac{x+k}{x+1} q$$

$$P_r(N = x+1) = \frac{k+x}{x+1} q \;\; P_r(N = x)$$

For x= 0,1,2----- with starting value $P_r(N = x) = p^k$. The moment generating function can be found by making use of identity.

$$\sum_{x=0}^{\infty} P_r(N = x) = 1 - - - - - - - - - - - - - (1.2)$$

From this it follows that

$$\sum_{0}^{\infty} \binom{k+x-1}{x} (1-qe^t)^k (qe^t)^x = 1$$

Provided that 0 < qe^t <1. Hence x

$$M_N'(t) = \sum_{x=0}^{\infty} e^{tx} \binom{k+x-1}{x} p^k q^x$$

$$= \frac{p^k}{(1-qet)^k} \sum_{x=0}^{\infty} \binom{k+r-1}{x} (1-qet)^k (qe^t)^x$$

$$= \frac{p^k}{(1-qet)^k} = \left(\frac{1}{1-qe^t}\right)^k$$

Provided that $0 < qe^t < 1$, or equivalently, $t < -\log q$. Similarly, the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^k \binom{k+x-1}{x} p^k q^x = \left(\frac{p}{1-qr}\right)^k$$

Moments of this distribution can be found by differentiating the moment generating function and the mean and variance are given by E{N}= kq/p and V[N]= kq/p^2.

$$M_N'(t) = \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)^k$$

$$M_N''(t) = \frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)^k$$

Equality (1.2) trivially gives

$$\sum_{0}^{\infty} \binom{k+x-1}{x} p^k q^x = 1 - p^k \text{ --------- } (1.3)$$

We use the notation NB (k,p) to denote a negative binomial distribution with parameter k and p.

(4) <u>The Geometric Distribution</u>:- The geometric distribution is a special case of the negative binomial distribution. When the negative binomial parameter k is L, the distribution is called a geometric distribution with parameter p and the probability function is

$$P_r(N = x) = pq^x$$

For x= 0,1,2,------. From above, it follows that E [N] = q/p, V[N] = q/p^2 and

$$M_N(t) = \frac{p}{1 - qe^t} \quad for\, t < \log q.$$

This distribution plays an important role in ruin theory.

<u>Important Continuous Distribution</u>

(1) The Gamma Distribution:-

When a random variable X has a gamma distribution with parameter $\alpha > 0$ $and\, \lambda > 0$ its density is given by

$$f(x) = \frac{\lambda^\alpha}{\tau(\alpha)}\; x^{\alpha-1}\, e^{-\lambda x}$$

For $x > 0$, where $\tau(\alpha)$is the gamma function defined as

In the special case when α is an integer the distribution is also known as an Erlang distribution, and repeated integration by parts gives the distribution function as

$$f(x) = 1 - \sum_{j=0}^{\alpha-1} e^{-\lambda x}\; \frac{(\lambda x)^j}{j!}$$

for $x \geq 0$. The moments and moments generating function of the gamma distribution can be found by nothing that

$$\int_0^\infty f(x)\; dx = 1$$

Yields

$$\int_0^\infty \mathrm{x}^{\alpha-1}\,\mathrm{e}^{-\lambda\mathrm{x}}\;dx = \frac{\tau(\alpha)}{\lambda^\alpha} - - - - - - - - (1.4)$$

The n[th] moment is

$$E[X^n] = \int_0^\infty x^n\,\frac{\lambda^\alpha x^{\alpha-1}\,\mathrm{e}^{-\lambda\mathrm{x}}}{\tau(\alpha)}\;dx = \frac{\tau(\alpha)}{\lambda^\alpha}\int_0^\infty \mathrm{x}^{\mathrm{n}+\alpha-1}\,\mathrm{e}^{-\lambda\mathrm{x}}\;dx$$

and form identity (1.4) it follows that

$$E[X^n] = \frac{\lambda^\alpha}{\tau(\alpha)}.\frac{\tau(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda^n} - - - - - - - - (1.5)$$

In particular,

n=1

$$E\,[X] = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda} = \frac{\alpha}{\lambda}$$

and, n=2

$$E\,[X^2] = \frac{\tau(\alpha+2)}{\tau(\alpha)\lambda^2} = \frac{\alpha(\alpha+1)}{\lambda^2},$$

So that

$$V[x] = E\,[X^2] - \{E[x]\}^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

We can find the moment generating function in a similar fashion. As

$$M_x(t) = \int_0^\infty e^{tx}\,\frac{\lambda^\alpha x^{\alpha-1}\,\mathrm{e}^{-\lambda\mathrm{x}}}{\tau(\alpha)}\;dx = \frac{\lambda^\alpha}{\tau(\alpha)}\int_0^\infty \mathrm{x}^{\alpha-1}\,\mathrm{e}^{-(\lambda-\mathrm{t})\mathrm{x}}\;dx - - - - - - -(1.6)$$

application of identity (1.4), gives

$$M_x(t) = \frac{\lambda^\alpha}{\tau(\alpha)} . \frac{\tau(\alpha)}{(\lambda - t)^\alpha} = \left(\frac{\lambda}{\lambda - t}\right)^\alpha - - - - - - (1.7)$$

Note that in identity (1.4) λ > 0. Hence in order to apply (1.4) to 91.6) we require that λ – t > 0, so that the m.g.f. exists when t < λ. u coefficient of skewness of X, which we denoted by Sk[x], is $\frac{2}{\sqrt{\alpha}}$. This follows from the definitiaon of the coefficient of skewness, namely third untral moment divided by standard divided by standard division cubed and the fact that the third untral moment is

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = E[X^3] - \frac{3\alpha}{\lambda} E[X^2] + 2\left(\frac{\alpha}{\lambda}\right)^3$$

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} - \frac{3r}{\lambda} . \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{2\,\alpha^3}{\lambda^3}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\,\alpha^3}{\lambda^3}$$

$$= \frac{2\alpha}{\lambda^3}$$

We use the notation $\gamma(\alpha, \lambda)$ to denote a gamma distribution with parameter α and λ.

(2) <u>The Exponential Distribution</u>:-

The exponential distribution is a special case of gamma distribution. It is just a gamma distribution with parameter x=1. Hence, the exponential distribution with parameter λ > 0 has density function.

$$f(x) = \lambda e^{-\lambda x}$$

For x > 0 and has the distribution function

$$f(x) = 1 - e^{-\lambda x}$$

For $x \geq 0$ form equation (1.5), the nth moment of the distribution is

$$E[X^n] = \frac{n!}{\lambda^n}$$

And from equation (1.7), the moment generating function is

$$M_x(t) = \frac{\lambda}{\lambda - t}, \quad for\ t < \lambda$$

(3) <u>The Pareto Distribution:</u> When a random variable X has a Pareto distribution with parameters x > 0, its density function is given by

$$f(x) = \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}$$

for x>0 integrating this density, we find that the distribution functions is

$$F(x) = 1 - \left(\frac{\lambda}{(\lambda + x)}\right)^\alpha$$

for x $\geq$ 0. Whenever moments of the distribution exists they can be found from.

$$E[X^n] = \int_0^\infty x^n f(x)dx$$

by integration by parts. However, they can above found individual using the following approach. Since the integral of the density function over (0,∞) equal 1, we have

$$\int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}} dx = 1 \qquad \int_0^\infty \frac{dx}{(\lambda + x)^{\alpha+1}} = \frac{1}{\alpha\lambda^\alpha}$$

an identity which holds provided that x > 0. To find E[X], we can write

$$E[x] = \int_0^\infty xf(x)dx = \int_0^\infty (x+\lambda-\lambda)f(x)dx = \int_0^\infty (x+\lambda)f(x)dx - \lambda$$

and inserting for f, we have

$$E[x] = \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}.(x+\lambda)dx - \lambda$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^\alpha}dx - \lambda$$

We can evaluate the integral by rewriting the integrand in terms of a α pareto density function with parameters $(\alpha - 1)$ and λ. thus,

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)} \int_0^\infty \frac{(\alpha-1)\lambda^{\alpha-1}}{(\lambda+x)^\alpha} dx - \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.8)$$

and since the integral equals 1.

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)} - \lambda = \frac{\lambda}{\alpha-1}$$

It is important to note that the integrand in equation (1.8) is a Pareto density function only if α>1, and hence E[x] exists only for α>1. Similarly, we can find E [x^2] from

$$E[x] = \int_0^\infty ((x+\lambda)^2 - 2\lambda x - \lambda^2)f(x)dx$$

$$= \int_0^\infty (x+\lambda)^2 f(x)dx - 2\lambda E[x] - \lambda^2$$

$$= \int_0^\infty (x+\lambda)^2 \cdot \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}\, dx - 2\lambda \frac{\lambda}{\alpha-1} - \lambda^2$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}\, dx - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} \int_0^\infty \frac{(\alpha-2)\lambda^{\alpha-2}}{(\lambda+x)^{\alpha-1}}\, dx - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)}, \quad proved\ that\ \alpha > 2.$$

and hence that

$$V[x] = E[x^2] - [E[x]^2]$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2}$$

$$V[x] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}$$

We use the notation Pa (α, λ) to denote a Pareto distribution with parameters $\alpha\ and\ \lambda$.

(4) **<u>The Normal Distribution:-</u>**

When a random variable X has a normal distribution with parameter $\mu\ and\ \sigma^2$, its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}\ for\ -\infty < x < \infty$$

We use notation N (μ, σ^2) to denote a normal distribution with parameter μ and σ^2. The standard normal distribution has parameter 0 and 1 and its distribution function is denoted by $\emptyset$, where

$$\emptyset(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\Lambda}} \exp\left\{\frac{-Z^2}{2}\right\} dz.$$

A key relationship is that if $X \sim \mu\ (\mu, \sigma^2)\ and\ if\ Z = \frac{(x-\mu)}{6}, then\ Z \sim N(0,1)$.

The moment generating function is

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \frac{1}{\sigma\sqrt{2\Lambda}} \int_{-\infty}^{\infty} e^{tx} \left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{\alpha\Lambda}} \int_{-\infty}^{\infty} exp\, \{t(\mu + \sigma z)\} \exp\left(-\frac{t}{2}\right) dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\, \left\{-\frac{1}{2}(z^2 - 2t\sigma z)\right\} dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\, \left[-\frac{1}{2}\{(z - \sigma t)^2 - \sigma^2 t^2\}\right] dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left\{-\frac{1}{2}(z - \sigma t)^2\right\} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left(-\frac{\mu^2}{2}\right) du$$

$$Hence, M_x(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} - - - - - - - - - - - (1.9)$$

Twice differentiating $M_x(t)$ with respect to t, we get

$M'_X(t) = (\mu + \sigma^2 t).M_X(t) and\, M''_X(t) = [(\mu + \sigma^2 t) + \sigma^2].M_X(t)$

$So\ that, M'_X(0) = \mu\ and\ M''_X(0) = \mu^2 + \sigma^2. Thus,$

$E[X] = \mu\ and\ V(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$

(5) **<u>The Lognormal Distribution</u>**:- When a random variable X has a lognormal distribution with parameter μ and σ where $-\infty < \mu < \infty\ and\ \sigma > 0,$ its density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(logx - \mu)^2}{2\sigma^2}\right\}$$

For x>0. The distribution function can be obtained by integrating the density function as follows:

$$F(x) = \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{2\Lambda}}\ exp\left\{\frac{(logx - \mu)^2}{2\sigma^2}\right\} dy$$

and the substitution Z= logy yields

$$F(x) = \int_{-\infty}^{logx} \frac{1}{\sigma\sqrt{2\Lambda}}\ exp\left\{-\frac{(z - \mu)^2}{2\sigma^2}\right\} dz.$$

As the integrand is the N (μ, σ^2) density function

$$F(x) = \emptyset\ \left(\frac{logx - \mu}{\sigma}\right).$$

Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function.

We use the notation LN (μ, σ^2) to denote a lognormal distribution with parameters $and\ \sigma$. From the preceding argument it follows that if X ~ LN (μ, σ^2) , then log X ~ LN (μ, σ^2)

This relationship between normal and lognormal distributions is extremely useful, particularly is deriving moments. If X ~ LN (μ, σ^2) $and\ Y = logX, then$

$$E[X^n] = E[e^{ny}] = M_y(n) = \ exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\}$$

Where the final equality follows by equation (1.9).

Mixed Distribution

The illustrate the idea of a mixed distribution, Let X be exponentially distribution with mean 100, and let the random variable Y be defined by

$$Y = \begin{cases} 0 & if\ X < 20 \\ X - 20 & if\ 20 \le X < 300 \\ 280 & if\ X \ge 300 \end{cases}$$

Figure

Then,

$$P_r(Y = 0) = \ P_r(X < 20) = t - e^{-0.2} = 0.1813$$

and similarly P_r (Y=280)=0.0498. Thus Y has masses of probability at the point 0 and 280. However in the interval (0,280), the distribution of Y is continuous with for example,

$$P_r(30 < Y \le 100) = \ P_r(50 < X \le 120) = 0.3053$$

Figure shows that the distribution function, H of Y note that there are jumps at 0 and 280, corresponding to the masses of probability at these points. As the distribution is differentiable in the interval (0, 280), Y has a density function in this interval. Letting h denote the density function of Y, the moments of Y can be found from

$$E[Y^2] = \int_0^{180} x^r h(x)dx + 280^r P_r(Y = 280).$$

It will be convenient to use satisfies integral notation so that we don not have to specify with a distribution is discrete, continuous or mixed. In the notation, we write the r[th] moment of Y as

$$E[Y^r] = \int_0^{\infty} x^r \, dH(x).$$

More generally if $k(x) = P_r(Z \leq x)$ is a mixed distribution on [0,∞] and m is a function, then

$$E[m(Z)] = \int_0^{\infty} m(x)d\,k(x),$$

Where we interpret the integral as

$$\sum_{x_i} m(x_i)\, P_r(Z = x_i) + \int m(x)k(x)dx.$$

Where summation is over the points $\{x_i\}$ at which there is a mass of probability and integration is over the intervals in which K is continuous with density function k.

Practical 6

Aim:- To obtain the probability and moments for given random variables.

Experiment:- Let the random variable X have the distribution function F given by:

$$F_{(x)}(x) = \begin{cases} 0 & X < 20 \\ \dfrac{x+20}{80} & 20 \leq X < 40 \\ 1 & X \geq 40 \end{cases}$$

Check the nature of random variable X, Hence calculate

$i)\ P\ (X < 30)$

$ii)\ P\ (X = 40)$

$iii)\ E(X)$

$iv)\ V\ (X)$

Theory:-

Calculation:- To check the nature of r.v. x-

$$P(X = 20) = \frac{20 + 20}{80} = 0.5 > 0$$

$$P(X = 40) = 1 - \frac{40 + 20}{80} = 0.25 > 0$$

Points X=20 and X=40 are jumps points. Thus X has a mixed distribution.

$$i)\ P\ (X \leq 30) = \frac{30 + 20}{80} = 0.625$$

$$ii)\ P\ (X = 40) = 1 - \frac{40 + 20}{80} = 0.25$$

The p d f of the distribution is given as:-

$$f_x(x) = \begin{cases} \frac{1}{80}, & 20 \leq x < 40 \\ 0, & otherwise \end{cases}$$

$$iii)\ E(X) = 20 \times 0.5 + \int_{20}^{40} x.\frac{1}{80} dx + 40 \times 0.25$$

$$= 20 + \left[\frac{x^2}{2 \times 80}\right]_{20}^{40} = 20 + \frac{40^2 - 20^2}{160} = 27.5$$

$$E(X^2) = 20^2 \times 0.5 + \int_{20}^{40} \frac{x^2}{80} dx + 40^2 \times 0.25$$

$$= 600 + \left[\frac{x^3}{3 \times 80}\right]_{20}^{40} = 600 + \frac{40^3 - 20^3}{240} = 833.33$$

$$iv)\ V(X) = E[X^2] - \{E[X]\}^2 = 833.33 - (27.5)^2 = 77.083$$

Result:-

$i)\ P(X \le 30) = 0.625$

$ii)\ P(X = 40) = 0.25$

$iii)\ E(X) = 27.5$

$iv)\ V(X) = 77.083$

Important Discrete Distribution

The Poisson Distribution- When a random variable N has a Poisson distribution with parameter λ > 0, its probability function is given by

$$P_r(N = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

For x= 0,1,2,------ The moment generating function is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}.e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^x}{2!} + - - - - - -\right\} = e^{-\lambda}.e^{\lambda^{e^t}}$$

$$= e^{\lambda}(e^t - 1) = \exp\ \{\lambda(e^t - 1)\} - - - - - - - - - - - - - (1.1)$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^x.e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda}\frac{(\lambda r)^x}{x!} = e^{-\lambda}\ e^{\lambda r}$$

$$= \ e^{\lambda(r-1)} = \exp\ \{\lambda\ (r-1)\}$$

The moments of N can be found from the moment generation function. For example,

$$\mu_r'' = \left[\frac{d^r}{dt^r}M_N(t)\right]$$

$$M_N'(t) = \frac{d}{dt}[exp\{\lambda(e^t - 1)\}]$$

$$= \exp\{\lambda(e^t - 1)\}.\lambda^{e^t} = \lambda^{e^t}M_N(t)$$

and

$$M_N''(t) = \frac{d^2}{dt^2}[exp\{\lambda(e^t - 1)\}] = \frac{d}{dt}(M_N'(t))$$

$$= \lambda^{e^t}M_N(t) + \lambda^{e^t}.\frac{d}{dt}M_N(t)\ \ [differentation\ by\ par]$$

$$= \lambda e^t M_N(t) + \lambda e^t.\lambda e^t M_N(t)$$

$$M_N''(t) = \lambda e^t M_N(t) + (\lambda e^t)^2 M_N(t)$$

From which it follows that

$E[N] = \lambda \; and \; E\,[N^2] = \lambda + \lambda^2 \; so \; that \; V[N] = \lambda.$

We use $P(\lambda)$ the notation to denote a Poisson distribution with parameter λ .

(2) The Binomial Distribution:-

When a random variable N has a binomial distribution with parameter n and q, where n is a positive integer and 0 < q <1. Its probability function is given by

$$P_r(N = r) = \binom{n}{r} q^r (1-q)^{n-x}$$

For x = 0,1,2,--------,n. The moment generating function is

$$M_N(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{r} q^r (1-q)^{n-x} = \sum_{x=0}^{n} \binom{n}{r} (qe^t)^x (1-q)^{n-x}$$

$$= (qe^t + 1 - q)^n$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{n} r^x \binom{n}{r} q^r (1-q)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{r} (qr)^x (1-2)^{n-x} = (qr + (1-q))^n$$

As,

$$M_N'(t) = n(qe^t + 1 - q)^{n-1}.\, qe^t$$

and

$$M_N''(t) = n(n-1)(qe^t + 1 - q)^{n-2}.\, (qe^t)(qe^t) + n(qe^t + 1 - q)^{n-1}.\, qe^t$$

$$= n(n-1)(qe^t + 1 - q)^{n-2}.\, (qe^t)^2 + n(qe^t + 1 - q)^{n-1}.\, qe^t$$

It follows that E[N]= nq, E [N^2]= n (n-1) q^2+ nq and V[N] = nq(1-q).

We use the notation B (n,q) to denote a binomial distribution with parameters n and q.

(3) Negative Binomial Distribution:-

When a random variable N has a negative binomial distribution with parameters k>0 and p. where 0 < p < 1. Its probability function is given by

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x$$

For 0,1,2,------, where q= 1-p. when k is an integer calculation of the probability function is straight forward as the probability function can be expressed in terms of factorials. An alternative method of calculating the probability function, regardless of whether k is an integer, is recursively as

$$P_r(N = x+1) = \frac{k+x}{x+1} q \; P_r(N = x)$$

$$P_r(N = x) = \binom{x+k-1}{x} p^k q^x, P_r(N = x+1) = \binom{x+k}{x+1} p^k q^{x+1}$$

$$\frac{P_r(N = x+1)}{P_r(N = x)} = \frac{\binom{x+k}{x+1} p^k q^{x+1}}{\binom{x+k-1}{x} p^k q^x} = \frac{x+k}{x+1} q$$

$$P_r(N = x+1) = \frac{k+x}{x+1} q \; P_r(N = x)$$

For x= 0,1,2----- with starting value $P_r(N = x) = p^k$. The moment generating function can be found by making use of identity.

$$\sum_{x=0}^{\infty} P_r(N = x) = 1 - - - - - - - - - - - - - (1.2)$$

From this it follows that

$$\sum_{0}^{\infty}\binom{k+x-1}{x}(1-qe^{t})^{k}(qe^{t})^{x}=1$$

Provided that 0 < qe^t <1. Hence x

$$M_N'(t)=\sum_{x=0}^{\infty}e^{tx}\binom{k+x-1}{x}p^{k}q^{x}$$

$$=\frac{p^k}{(1-qet)^k}\sum_{x=0}^{\infty}\binom{k+r-1}{x}(1-qet)^{k}(qe^{t})^{x}$$

$$=\frac{p^k}{(1-qet)^k}=\left(\frac{1}{1-qe^{t}}\right)^{k}$$

Provided that 0 < qe^t < 1, or equivalently, t < -logq. Similarly, the probability generating function is

$$P_N(r)=\sum_{x=0}^{\infty}r^{k}\binom{k+x-1}{x}p^{k}q^{x}=\left(\frac{p}{1-qr}\right)^{k}$$

Moments of this distribution can be found by differentiating the moment generating function and the mean and variance are given by E{N}= kq/p and V[N]= kq/p^2.

$$M_N'(t)=\frac{d}{dt}\left(\frac{p}{1-qe^{t}}\right)^{k}$$

$$M_N''(t)=\frac{d^2}{dt^2}\left(\frac{p}{1-qe^{t}}\right)^{k}$$

Equality (1.2) trivially gives

$$\sum_{0}^{\infty}\binom{k+x-1}{x}p^k q^x = 1-p^k \text{ --------- } (1.3)$$

We use the notation NB (k,p) to denote a negative binomial distribution with parameter k and p.

(4) <u>The Geometric Distribution</u>:- The geometric distribution is a special case of the negative binomial distribution. When the negative binomial parameter k is L, the distribution is called a geometric distribution with parameter p and the probability function is

$$P_r(N = x) = pq^x$$

For x= 0,1,2,------. From above, it follows that E [N] = q/p, V[N] = q/p^2 and

$$M_N(t) = \frac{p}{1-qe^t} \quad for\, t < \log q.$$

This distribution plays an important role in ruin theory.

<u>Important Continuous Distribution</u>

(1) The Gamma Distribution:-

When a random variable X has a gamma distribution with parameter $\alpha > 0$ $and\, \lambda > 0$ its density is given by

$$\mathrm{f(x)} = \frac{\lambda^\alpha}{\tau(\alpha)}\ \mathrm{x}^{\alpha-1}\ \mathrm{e}^{-\lambda \mathrm{x}}$$

For $x\ >\ 0$, where $\tau(\alpha)$is the gamma function defined as

In the special case when α is an integer the distribution is also known as an Erlang distribution, and repeated integration by parts gives the distribution function as

$$f(x) = 1 - \sum_{j=0}^{\alpha-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!}$$

for $x \geq 0.$ The moments and moments generating function of the gamma distribution can be found by nothing that

$$\int_0^{\infty} f(x)\ dx = 1$$

Yields

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x}\ dx = \frac{\tau(\alpha)}{\lambda^{\alpha}} - - - - - - - - (1.4)$$

The n^th moment is

$$E[X^n] = \int_0^{\infty} x^n \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\tau(\alpha)}\ dx = \frac{\tau(\alpha)}{\lambda^{\alpha}} \int_0^{\infty} x^{n+\alpha-1} e^{-\lambda x}\ dx$$

and form identity (1.4) it follows that

$$E[X^n] = \frac{\lambda^{\alpha}}{\tau(\alpha)} \cdot \frac{\tau(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda^n} - - - - - - - - (1.5)$$

In particular,

n=1

$$E\ [X] = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda} = \frac{\alpha}{\lambda}$$

and, n=2

$$E\ [X^2] = \frac{\tau(\alpha+2)}{\tau(\alpha)\lambda^2} = \frac{\alpha(\alpha+1)}{\lambda^2},$$

So that

$$V[x] = E\ [X^2] - \{E[x]\}^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

We can find the moment generating function in a similar fashion. As

$$M_x(t) = \int_0^\infty e^{tx}\ \frac{\lambda^\alpha x^{\alpha-1}\ \mathrm{e}^{-\lambda x}}{\tau(\alpha)}\ dx = \frac{\lambda^\alpha}{\tau(\alpha)} \int_0^\infty \mathrm{x}^{\alpha-1}\ \mathrm{e}^{-(\lambda-\mathrm{t})\mathrm{x}}\ dx ------(1.6)$$

application of identity (1.4), gives

$$M_x(t) = \frac{\lambda^\alpha}{\tau(\alpha)} \cdot \frac{\tau(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha ----- (1.7)$$

Note that in identity (1.4) $\lambda > 0$. Hence in order to apply (1.4) to 91.6) we require that $\lambda - t > 0$, so that the m.g.f. exists when $t < \lambda$. u coefficient of skewness of X, which we denoted by Sk[x], is $\frac{2}{\sqrt{\alpha}}$. This follows from the definitiaon of the coefficient of skewness, namely third untral moment divided by standard divided by standard division cubed and the fact that the third untral moment is

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = E[X^3] - \frac{3\alpha}{\lambda}\ E[X^2] + 2\left(\frac{\alpha}{\lambda}\right)^3$$

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} - \frac{3r}{\lambda} \cdot \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{2\ \alpha^3}{\lambda^3}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\ \alpha^3}{\lambda^3}$$

$$= \frac{2\alpha}{\lambda^3}$$

We use the notation $\gamma(\alpha, \lambda)$ to denote a gamma distribution with parameter α and λ.

(2) <u>The Exponential Distribution</u>:-

The exponential distribution is a special case of gamma distribution. It is just a gamma distribution with parameter x=1. Hence, the exponential distribution with parameter λ > 0 has density function.

$$f(x) = \lambda e^{-\lambda x}$$

For x > 0 and has the distribution function

$$f(x) = 1 - e^{-\lambda x}$$

For $x \geq 0$ form equation (1.5), the nth moment of the distribution is

$$E[X^n] = \frac{n!}{\lambda^n}$$

And from equation (1.7), the moment generating function is

$$M_x(t) = \frac{\lambda}{\lambda - t}, \quad for\ t < \lambda$$

(3) <u>The Pareto Distribution:</u> When a random variable X has a Pareto distribution with parameters x > 0, its density function is given by

$$f(x) = \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}$$

for x>0 integrating this density, we find that the distribution functions is

$$F(x) = 1 - \left(\frac{\lambda}{(\lambda + x)}\right)^\alpha$$

for x ≥ 0. Whenever moments of the distribution exists they can be found from.

$$E[X^n] = \int_0^\infty x^n f(x) dx$$

by integration by parts. However, they can above found individual using the following approach. Since the integral of the density function over (0,∞) equal 1, we have

$$\int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}dx = 1 \qquad\qquad \int_0^\infty \frac{dx}{(\lambda+x)^{\alpha+1}} = \frac{1}{\alpha\lambda^\alpha}$$

an identity which holds provided that x > 0. To find E[X], we can write

$$E[x] = \int_0^\infty xf(x)dx = \int_0^\infty (x+\lambda-\lambda)f(x)dx = \int_0^\infty (x+\lambda)f(x)dx - \lambda$$

and inserting for f, we have

$$E[x] = \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}.(x+\lambda)dx - \lambda$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^\alpha}dx - \lambda$$

We can evaluate the integral by rewriting the integrand in terms of a α pareto density function with parameters $(\alpha - 1)$ and λ. thus,

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)} \int_0^\infty \frac{(\alpha-1)\lambda^{\alpha-1}}{(\lambda+x)^\alpha} dx - \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.8)$$

and since the integral equals 1.

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)} - \lambda = \frac{\lambda}{\alpha-1}$$

It is important to note that the integrand in equation (1.8) is a Pareto density function only if α>1, and hence E[x] exists only for α>1. Similarly, we can find E [x^2] from

$$E[x] = \int_0^\infty ((x+\lambda)^2 - 2\lambda x - \lambda^2) f(x) dx$$

$$= \int_0^\infty (x+\lambda)^2 \, f(x) dx - 2\lambda \, E[x] - \lambda^2$$

$$= \int_0^\infty (x+\lambda)^2 \, . \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}} \, dx - 2\lambda \, \frac{\lambda}{\alpha-1} - \lambda^2$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}} \, dx - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} \int_0^\infty \frac{(\alpha-2)\lambda^{\alpha-2}}{(\lambda+x)^{\alpha-1}} \, dx - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} - \frac{2\lambda^2}{\alpha-1} - \lambda^2$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)}, \;\; proved \; that \; \alpha > 2.$$

and hence that

$$V[x] = E[x^2] - [E[x]^2]$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2}$$

$$V[x] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}$$

We use the notation Pa (α, λ) to denote a Pareto distribution with parameters $\alpha \; and \; \lambda$.

(4) **The Normal Distribution:-**

When a random variable X has a normal distribution with parameter $\mu \; and \; \sigma^2$, its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} \; for -\infty < x < \infty$$

We use notation N (μ, σ^2) to denote a normal distribution with parameter $\mu \; and \; \sigma^2$. The standard normal distribution has parameter 0 and 1 and its distribution function is denoted by $\emptyset$, where

$$\emptyset(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\Lambda}} \exp\left\{\frac{-Z^2}{2}\right\} dz.$$

A key relationship is that if $X \sim \mu \; (\mu, \sigma^2) \; and \; if \; Z = \frac{(X-\mu)}{6}, then \; Z \sim N(0,1)$.

The moment generating function is

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) \, dx = \frac{1}{\sigma\sqrt{2\Lambda}} \quad \int_{-\infty}^{\infty} e^{tx} \left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{\alpha\Lambda}} \int_{-\infty}^{\infty} exp \; \{t(\mu + \sigma z)\} \exp\left(-\frac{t}{2}\right) dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp \left\{-\frac{1}{2}(z^2 - 2t\sigma z)\right\} dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp \left[-\frac{1}{2}\{(z-\sigma t)^2 - \sigma^2 t^2\}\right] dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left\{-\frac{1}{2}(z-\sigma t)^2\right\} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp\left(-\frac{\mu^2}{2}\right) du$$

$$Hence, M_x(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} - - - - - - - - - - - -(1.9)$$

Twice differentiating $M_x(t)$ with respect to t, we get

$$M_X'(t) = (\mu + \sigma^2 t).M_X(t) and\ M_X''(t) = [(\mu + \sigma^2 t) + \sigma^2].M_X(t)$$

$$So\ that, M_X'(0) = \mu\ and\ M_X''(0) = \mu^2 + \sigma^2. Thus,$$

$$E[X] = \mu\ and\ V(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

(5) **<u>The Lognormal Distribution</u>**:- When a random variable X has a lognormal distribution with parameter μ and σ where $-\infty < \mu < \infty\ and\ \sigma > 0,$ its density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(logx-\mu)^2}{2\sigma^2}\right\}$$

For x>0. The distribution function can be obtained by integrating the density function as follows:

$$F(x) = \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{2\Lambda}}\ exp\left\{\frac{(logx-\mu)^2}{2\sigma^2}\right\} dy$$

and the substitution Z= logy yields

$$F(x) = \int_{-\infty}^{logx} \frac{1}{\sigma\sqrt{2\Lambda}} \; exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} dz.$$

As the integrand is the N (μ, σ^2) density function

$$F(x) = \emptyset \left(\frac{logx - \mu}{\sigma}\right).$$

Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function.

We use the notation LN (μ, σ^2) to denote a lognormal distribution with parameters $and\ \sigma$. From the preceding argument it follows that if X ~ LN (μ, σ^2) , then log X ~ LN (μ, σ^2)

This relationship between normal and lognormal distributions is extremely useful, particularly is deriving moments. If X ~ LN (μ, σ^2) $and\ Y = logX, then$

$$E[X^n] = E[e^{ny}] = M_y(n) = \; exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\}$$

Where the final equality follows by equation (1.9).

Mixed Distribution

The illustrate the idea of a mixed distribution, Let X be exponentially distribution with mean 100, and let the random variable Y be defined by

$$Y = \begin{cases} 0 & if\ X < 20 \\ X - 20 & if\ 20 \le X < 300 \\ 280 & if\ X \ge 300 \end{cases}$$

Figure

Then,

$$P_r(Y = 0) = P_r(X < 20) = t - e^{-0.2} = 0.1813$$

and similarly P_r (Y=280)=0.0498. Thus Y has masses of probability at the point 0 and 280. However in the interval (0,280), the distribution of Y is continuous with for example,

$$P_r(30 < Y \le 100) = P_r(50 < X \le 120) = 0.3053$$

Figure shows that the distribution function, H of Y note that there are jumps at 0 and 280, corresponding to the masses of probability at these points. As the distribution is differentiable in the interval (0, 280), Y has a density function in this interval. Letting h denote the density function of Y, the moments of Y can be found from

$$E[Y^2] = \int_0^{180} x^r h(x)dx + 280^r P_r(Y = 280).$$

It will be convenient to use satisfies integral notation so that we don not have to specify with a distribution is discrete, continuous or mixed. In the notation, we write the rth moment of Y as

$$E[Y^r] = \int_0^{\infty} x^r \, dH(x).$$

More generally if $k(x) = P_r(Z \leq x)$ is a mixed distribution on [0,∞] and m is a function, then

$$E[m(Z)] = \int_0^{\infty} m(x)d\,k(x),$$

Where we interpret the integral as

$$\sum_{x_i} m(x_i)\, P_r(Z = x_i) + \int m(x)k(x)dx.$$

Where summation is over the points $\{x_i\}$ at which there is a mass of probability and integration is over the intervals in which K is continuous with density function k.

<u>Practical 6</u>

Aim:- To obtain the probability and moments for given random variables.

Experiment:- Let the random variable X have the distribution function F given by:

$$F_{(x)}(x) = \begin{cases} 0 & X < 20 \\ \dfrac{x+20}{80} & 20 \leq X < 40 \\ 1 & X \geq 40 \end{cases}$$

Check the nature of random variable X, Hence calculate

$i)\ P\ (X < 30)$

$ii)\ P\ (X = 40)$

$iii)\ E(X)$

$iv)\ V\ (X)$

Theory:-

<u>**Calculation:-**</u> To check the nature of r.v. x-

$$P(X = 20) = \frac{20 + 20}{80} = 0.5 > 0$$

$$P(X = 40) = 1 - \frac{40 + 20}{80} = 0.25 > 0$$

Points X=20 and X=40 are jumps points. Thus X has a mixed distribution.

$$i)\ P\ (X \leq 30) = \frac{30 + 20}{80} = 0.625$$

$$ii)\ P\ (X = 40) = 1 - \frac{40 + 20}{80} = 0.25$$

The p d f of the distribution is given as:-

$$f_x(x) = \begin{cases} \frac{1}{80}, & 20 \leq x < 40 \\ 0, & otherwise \end{cases}$$

$$iii)\ E(X) = 20 \times 0.5 + \int_{20}^{40} x.\frac{1}{80}dx + 40 \times 0.25$$

$$= 20 + \left[\frac{x^2}{2 \times 80}\right]_{20}^{40} = 20 + \frac{40^2 - 20^2}{160} = 27.5$$

$$E(X^2) = 20^2 \times 0.5 + \int_{20}^{40} \frac{x^2}{80} dx + 40^2 \times 0.25$$

$$= 600 + \left[\frac{x^3}{3 \times 80}\right]_{20}^{40} = 600 + \frac{40^3 - 20^3}{240} = 833.33$$

$iv)\ V(X) = E[X^2] - \{E[X]\}^2 = 833.33 - (27.5)^2 = 77.083$

Result:-

$i)\ P(X \leq 30) = 0.625$

$ii)\ P(X = 40) = 0.25$

$iii)\ E(X) = 27.5$

$iv)\ V(X) = 77.083$

Unit-IV

Life Insurance

Introduction:- Insurance system are establish to reduce the adverse financial impact of some types of random events. Within these systems. Individuals and organizations adopt utility models to represents preferences, stochastic model to represents uncertain financial impact and economic principles to guide pricing. Agreement are reached after analysis of these models.

The models for life insurances designed to reduce the financial impact of the random event of untimely death. Due to the long-term nature of these insurances, the amount of investment earnings up to the time of payment, provide a significant elements of uncertainty. This uncertainty has two types causes:

The unknown rate of earnings over and the unknown length of the investment period. A probability distribution is used to model the uncertainty in regards to the investment period.

The general model is useful in any situations where the size and time of a financial impact can be expressed sobly in terms of the time of the random event.

Insurance Payable at the Month of Death:- The amount and the time of payment of a life insurance benefit depend only on the length of the interval from the issue of the insurance to the death of the insured. Our model will be developed with a benefit function b_t and a discount function. V_t in our model, v_t is the interest discount factor from the time of payment back to the time of policy issue and t is the length of the interval form issue to death. In the case of endowments, t can be greater than or equal to the length of the interval from issue to payment.

For the discount function we assume that the under lying force of interest is deterministic; that is the model does not include a probability distribution for the force of interest. Moreover we usually show the simple formulas resulting from the assumption of the constant as well as a determination force of interest.

We define the present – value function, Zt by

$$Z_t = b_t \, v_t ---------- (1)$$

Thus Z_t is the present value at policy issue, of the benefit payment. The elapsed time from policy issues to the death of the insured's future – lifetime random variable, T = t (x). Thus the present value at policy issue, of the benefit payment is the random variable Z_t. unless the context requires a more elaborate symbol. We denote this r.v. by Z and base the model for the insurance on the equation

$$Z = b_t \, V_t ---------- (2)$$

The random variable Z is an example of a claim random variable. The first step in our of a life insurance will be to define b_t and V_t. The next step is to determine some characteristic of the probability distribution of Z that are consequences of an assured distribution for T.

(1) Level Benefit Insurance:

An n-year term life insurance provides for a payment only if the issued dies within the n-year term of an insurance commencing at issue. If a unit is payable at the moment of death of (x), then

$$b_t = \begin{cases} 1 & t \le n \\ 0 & t > n \end{cases}$$

$$v_t = v_t \qquad t \ge 0$$

$$Z = \begin{cases} v_t & T \le n \\ 0 & T > n \end{cases}$$

These definitions use three conventions.

(i) The future lifetime is a non-negative variable. We define b_t, V_t, and Z only on non-negative value.

(ii) For a t value where b_t is 0, the value of v_t is irrelevant. At these values of t, we adopt definitions of v_t by convenience.

(iii) Unless stated otherwise the force of interest in assumed to be constant.

The expectation of the present-value random variable, Z is called the actuarial present value of the insurance. The expectation of the present value of a set of payments contingent on the occurrence of a set of events is referred to by different terms is different actuarial context. A more exact term, but more cumbersome value of the payments.

The principal symbol for the actuarial present value of an insurance paying a unit benefit in A. The subscript includes the age of the insured life at the time of the calculation.

For the actuarial present value of an insurance on (40), the age might be display as [40], 40 or [20] + 20, for example. [20] + 20 includes the calculations for a 40 years old on the basis of a select table commending at age 20.

The actuarial present value for the n-year term insurance with a unit payable at the moment of death of (x), E [z], is denoted by $A_{x!n}^{-1}$. This can be calculated by recognizing Z as a function of T so that

$$E[z] = E\,[zt] = \int_0^{\infty} Z_t \;\; f_t\;(t)\;\; dt = \int_0^{n} V^t t^{px} \mu_x(t)dt - - - - - - -(3)$$

The f[th] moment of the distribution of Z can be found by

$$E[Z^j] = \int_0^n (v^j)^j\, t^{px}\;\; \mu_x(t)dt$$

$$= \int_0^n e^{-(sj)t}\;\; \mu_x(t)dt.$$

The second integral shows that the j[th] moment of Z is equal to the arterial present value for an n year insurance for a unit amount payable at the moment of death of (x), calculated at a force of interest equal to j times the given force of interest of js.

This property which we call the rule of moments holds generally for insurances paying only a unit amount when the force of interest is deterministic constant or not. More precisely,

$$E\,[z^j]\,@s_t = E\,[z]@\,j\,s_t - - - - - - - - -(4)$$

In addition to the existences of the moments, the sufficient conditions for the rule of moments is $b_t^j\;= b_t\;for\;all\,t\;\geq\;0.$ that is for each t the benefit amount is or 1.

It follows from the rule of moments that

$$Var\,(z) =$$

Where is the actuarial present value for an n-year term insurance for a unit amount calculated at force of interest 2&.

Whole life insurance provides for a payment following the death of the insured at any time in the future. If the payment is to be a unit amount at the moment of death of (x), then

$$b_t\;= 1,\qquad\qquad t\;\geq 0,$$

$$v_t = v_t \qquad t \geq 0$$

$$Z = v_t, \qquad T \geq 0$$

The actuarial present value is

$$\bar{A}_x = E\,[z] = \int_0^{\infty} V^t {}_t p_x \mu_x(t)dt ------(6)$$

For a life selected at x and new age x+h, the expression would be

$$\bar{A}_{x+h} = \int_0^{\infty} V^t {}_t p_{[x]+h}\ \mu_x(h+t)dt$$

Whole life insurance is the limiting case of n-year term insurance as $n \to \infty$.

Example:- The p.d.f of the future lifetime T. for (x) is assumed to be

$$f_T(t) = \begin{cases} \dfrac{1}{80} & 0 \leq t \ \leq 80 \\ 0 & elsewhere \end{cases}$$

At a force of interest. S_1 calculate for Z. the present value random variable for a whole life insurance of unit amount issued to (x):

(a) The actuarial present value
(b) The variance
(c) The 90[th] percentile, $\begin{cases} 0.9 \\ z \end{cases}$

Solution:

(a) $\bar{A}_x = E\,[z] = \int_0^{\infty} V^t {}_t p_x \mu_x(t)dt \ = \ \int_0^{80} e^{-st}\,\frac{1}{80}\,dt \ = \ \frac{1-e^{-80s}}{80\,s} \quad s \neq 0.$

(b) By the rule of moments

$$Var\,(z) = \frac{1-e^{-160s}}{160\,s} - \left(\frac{1-e^{-80s}}{80\,s}\right)^2 \ s \neq 0.$$

(c) For the continuous random variable Z, we have $p\left(z \leq \begin{cases} 0.9 \\ z \end{cases}\right) = 0.9$

Since we have the p.d.f for T and not for z, we proceed by finding the event for T which corresponds to $z \leq \begin{cases} 0.9 \\ z \end{cases}$ Figure, which shows the general relationship between the sample space of T (on the horizontal axis) and the sample space of Z (on the vertical axis). We that. Because z is a strictly decreasing function of T for whole life insurance the percentile from T's distribution that is related to 90th percentile of Z's distribution that at the complementary probability level 0.1. in this example T is uniformly distributed over the interval (0, 80) so $\begin{cases} 0.1 \\ T \end{cases}$ = 8,0 and thus $\begin{cases} 0.9 \\ z \end{cases} = v^{8.0}$

The group is figure can be used to establish relationship between the d.f. and p.d.f. of Z and of T:

Figure

$For\ Z\ \leq 0,\ \ \{Z\ \leq z\}\ is\ the\ null\ event.$

$For\ 0 < z < 1,\ \ \{Z\ \leq z\}\{T\ \geq \log z\ /\ logv\}, and$

$For\ Z\ \geq 1,\ \ \{Z\ \leq z\}\ is\ the\ certain\ event$

Therefore,

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ 1 - F_T(\log z\ /\ logv) & 0 < z < 1 ------(7) \\ 1 & 1 \leq z \end{cases}$$

By differentiation of (7)

$$F_Z(z) = \begin{cases} F_T[\log z) \ / \ (log v] \ [1/(sz)] & 0 < z < 1 \\ 0 & elsewhere \end{cases}$$ ----------------------(8)

For the assumption in example (1) determine

$Z'sd.f.$

$Z's \quad p.d.f.$

$$Solution: - \ @ \, From \, F_T(t) = \begin{cases} \dfrac{t}{80} & 0 \leq t \leq \ 80 \\ 1 & t \ \geq 80 \end{cases}$$

We see that P [T>80] = 0.0 so p [0< z< u^{80}] = 0. 0

Therefore from (7)

$$F_Z(z) = \begin{cases} 0 & z \leq 0 \\ 1 - F_T(\log z \ / \ log v) & 0 < z < 1 \\ 1 & 1 \leq z \end{cases}$$

(b) By differentiation of the d.f. in part (a).

$$F_z(z) = \begin{cases} \left(\dfrac{t}{80}\right)\left(\dfrac{1}{sz}\right) & u^{80} \leq t \leq \ 80 \\ 0 & elsewhere \end{cases}$$

An n - year pure endowment provides for payment at the end the n years if and only if the insured survives least n years from the time of policy issue.

If the amount payable is a unit, then

$$b_t = \begin{cases} 0 & t \leq n. \\ 1 & t > n. \end{cases}$$

$$v_t = v^n \quad t \geq 0$$

$$Z = \begin{cases} 0 & T \leq n \\ v^n & T > n \end{cases}$$

The only amount payable is a unit then

The only element of uncertainty in the pure endowment is whether or not a claim will occur. The size and time of payment, if a claim occurs, are predetermined. In the expression Z = v^n Y. Y is the indicator of the event of survival to age x+n and has the value 0 otherwise. The n-year pure endowment's actuarial present value has two symbols. In an insurance context it is $A^{1}_{x:n]}$ and $_n$Ex in an annuity context.

$$A^{1}_{x:n]} = \quad E\,[Z] = v^n E[Y] = v^n n^p{}_x \; n^q{}_x,$$

and

$$Var\,(z) = v^{2n}\,Var\,(Y) = v^{2n} \;\; n^p{}_x \; n^q{}_x$$

$$= 2_{A^{1}_{x:n]}} - \left(A^{1}_{x:n]}\right)^2 - - - - - - - - (9)$$

An n-year endowment insurance provides for an amount to be payable either following the death of the insured or upon the survival of the insured to the end of the n-year term, whichever occurs first. If the insurance is for a unit amount and the death benefit is payable at the moment of death then

$$b_t = 1 \qquad\qquad t \geq 0$$

$$v_t = \begin{cases} u^t & t \leq n. \\ v^n & t > n. \end{cases}$$

$$Z = \begin{cases} v^T & T \leq n \\ v^n & T > n \end{cases}$$

The actuarial present values is denoted by $A^{1}_{x:n]}$. Since b_t = 1. For the endowment insurance, we have by the rule of moments

$$E\left[Z^i\right] @ \;\& = E\,[Z] \,@\, j\,\xi$$

Moreover,

$$Var\,[Z] = \;\; 2 -_{A^{1}_{x:n]}} - \left(A^{1}_{x:n]}\right)^2 - - - - - - - - (10)$$

This insurance can be viewed as the combination of an n-year term insurance and n-year pure endowment each for a unit amount. Let Z_1, Z_2 and Z_3 denote the present value random variables of the term, the pure endowment, and the endowment insurances, respectively with death benefits payable at the moment of death of (x). From the preceding definitions we have

$$Z_1 = \begin{cases} v^T & T \leq n \\ 0 & T > n \end{cases}$$

$$Z_2 = \begin{cases} 0 & T \leq n \\ v^n & T > n \end{cases}$$

$$Z_3 = \begin{cases} v^T & T \leq n \\ v^n & T > n \end{cases}$$

It follows that

$Z_3 = Z_1 + Z_2$ ……………………………….(11)

and by taking expectations of both sides

$$A^{-}_{x:n]} = A^{-1}_{x:n]} + A^{-2}_{x:n]} --------(12)$$

Can also find the Var (Z_3) by using (11)

$$Var\ (Z_3) = Var\ (Z_1) + Var\ (Z_2) + 2\ Cov(Z_1, Z_2) ------(13)$$

By use of the formula

$$Cov\ (Z_1, Z_2) = E\ [Z_1, Z_2] - E[Z_1]\ E[Z_2] ------(14)$$

and the observation that

$$Z_1, Z_2 = 0$$

For all T, we have

$$Cov(Z_1, Z_2) = -E[Z_1]\, E[Z_2] = -A^{-1}_{x:n]}\, A^{-2}_{x:n]} -----(15)$$

Substituting (5) (9) and (15) into (13) produces a formula for $Var\ (Z_3)$ in terms of actuarial present value for an n-year term insurance and a pure endowment.

Since the actuarial present values are positive, the $Cov(Z_1, Z_2)$ is negative. This is to be anticipated since, of the pair Z_1 and Z_2 one is always zero and the other positive. On the other hand, the correlation coefficient of Z_1 and Z_2 is not -1 since they are not linear function of each other.

(3) Deferred Insurance:

An n-year deferred insurance provides for a benefit following the death of the insured only if the insured dies at least m year following policy issue. The benefit payable and the term of the insurance may be any of those discussed above. For example, an m-year deferred whole life insurance with a unit amount payable at the moment of death has

$$b_t = \begin{cases} 0 & t > m \\ 1 & t \le m. \end{cases}$$

$$v^t \quad = \quad v^n \qquad t > 0,$$

$$Z = \begin{cases} v^T & T > m \\ 0 & T \le m \end{cases}$$

The actuarial present value is denoted by m|$\bar{A}_x$ and is equal to

$$\int_m^\infty v^t \ {}_tP_x Mx|t)\, dt. ------------- (16)$$

(4) **Varying Benefit Insurance:**

The general model given by (1) can be used for analysis in most applications. We have used it with level benefit life insurances. It can also be applied to insurances where the level of the death benefit either increases or decreases in A.P. over all or a part of the term of the insurance. Such insurances are often sold as an additional benefit when a basic insurance provides for the return of periodic premiums at death or when an annuity contract contains a guarantee of sufficient payments to match its initial premium.

An annually increasing whole life insurance providing 1 at the moment of death during the first year, 2at the moment of death in the second year and so on is characterized by the following functions:

$$b_t = [t-1] \qquad t \geq 0$$

$$v_t = v^t \qquad t \geq 0,$$

$$Z = [T+1]\, v^t \qquad T \geq 0$$

Where the L] denote the greatest integer function. The actuarial present value for such an insurance is

$$(I\,\bar{A})_x = E[Z] = \int_0^\infty [\,t+1]^{vt}\; t^{P}{}_x\, M_x(t)dt.$$

The higher order moments are not equal to the actuarial present value at an adjusted force of incres as was the case for insurances with benefit payments equal to 0 or 1. These moments can be calculated directly from their definitions.

The increases is the benefit of the insurance can occur more or less frequently than once per year. For an m-thly increasing whole life insurance the benefit would be 1/m at the moment of death during the first m-th of a year of the term of the insurance, 2/m at the moment of death during the second m^{th} of a year during the term of the insurance and so on increasing by 1/m at m-thly intervals throughout the term of the insurance. For such a whole life insurance the functions are

$$b_t = \frac{[t_m - 1]}{m} \qquad t \geq 0$$

$$v_t = v^t \qquad t \geq 0,$$

$$Z = \frac{v^t [T_m + 1]}{m} \qquad T \geq 0$$

The actuarial present value is

$$\left(I^{(m)} \bar{A}\right)_x = E[z].$$

The limiting case, as m→ ∞ in the mthly increasing whole life insurance is an insurance paying t at the time of death. Its functions are

$$b_t = t, \qquad t \geq 0$$

$$v_t = v^t \qquad t \geq 0,$$

$$Z = T\, v^t \qquad T \geq 0$$

The actuarial present value symbol is $(\bar{I}\,\bar{A})_x$.

This continuously increasing whole life insurance is equivalent to a set of deferred level whole life insurances. This equivalence is shown graphically in Fig. where the region between the line $b_t = t$ and the t- axis represent the insurance over the future lifetime. If the infinitesimal regions are joined in the vertical directions for a fixed t, the total benefit payable at t is obtained. If they are joined in the horizontal direction for a fixe s, an s-year deferred whole life insurance for the level amount ds is obtained.

This equivalent implies that the actuarial present values for the coverage's are equal. The equality can be established as follows:

By definition

$$(\bar{I}\,\bar{A})_x = \int_0^{\infty} t^{vt}\ {}_tP_x\, M_x(t)dt.$$

and interpreting t in the integrand as the integral from zero to t in figure, we have

$$(\bar{I}\,\bar{A})_x = \int_0^{\infty} \left(\int_0^t ds\right) v^t\ {}_tP_x\, M_x(t)dt.$$

If we interchange the order of integration and for each s value, integrate on t from s to x, we have

$$(\bar{I}\,\bar{A})_x = \int_0^{\infty}\int_s^{\infty} v^t\ {}_tP_x\, M_x(t)dt.$$

$$= \int_0^{\infty} s|\,\bar{A}_x\ ds$$

By (16)

If, for any of these m-thly increasing life insurances the benefit is payable only if death occurs within a term of n years the insurances is an mthly increasing n-year term life insurance.

Complementary to the annually increasing n-year term life insurance is the annually decreasing n-year term life insurance providing n at the moment of death during the first year, n-1 at the moment of death during the second year and so on with coverage terminating at the end of the nth year. Such as insurance has the following functions:

$$b_t = \begin{cases} n - lt] & t \geq n \\ 0 & t > n, \end{cases}$$

$$v_t = v^t \qquad t > 0,$$

$$Z = \begin{cases} V^T\,(n - LT]) & T \leq n \\ 0 & T > n. \end{cases}$$

The actuarial present value for this insurance is

$$(D\bar{A})^1_{x:n} = \int_0^\infty v^t(n - Lt])\; t^P{}_x\, M_x(t)dt.$$

This insurance is complementary to the annually increasing n-year term insurance in the sense that the sum of their benefit functions is the constant n+1 for the n-year term.

Insurances Payable at the End of the Year of Death

Practice, most benefits are considered payable at the moment of death then earn interest until the payment is actually made. The models were built in term of T, the future lifetime of the insured at policy issue. In most life insurance applications the best information available on the prob. Distribution of T is in the form of a discrete life table. This is the probability of K, the curate future life time of the insured at policy issue, a function of T. We bridge this gap by building models for life insurances in which the size and time of payment of the death benefits depend only on the number of complete year lived by the insured from policy issue up to the time of death. We refer to these insurances simply as payable at the end of the year of death.

Our model is in terms of functions of the curtate future lifetime of the insured. The benefit function, b_{k+1} and the discount function V_{k+1} are respectively the benefit amount payable and the discount factor required for the period from the time of payment back to the time of policy issue when the insured's curtate-future-lifetime is k, that is when the insured dies in year k+1 of insurance. The present value at policy issue of this benefit payment denoted by Z_{k+1} is

$$Z_{k+1} = b_{k+1} \ , v_{k+1} \text{--------------------} (1)$$

Measured from the time of policy issue the insurance year of death is 1 plus the curtate – future- lifetime random variable k. we denote the present value random variable Z_{k+1} by Z.

For an n-year term insurance providing a unit amount at the end of the year of death, we have

$$b_{k+1} = \begin{cases} 1 & k = 0,1,2, \ldots\ldots, n-1 \\ 0 & \quad elsewhere \end{cases}$$

$$bv_{k+1} = v^{k+1}$$

$$Z = \begin{cases} v^{k+1} & \qquad k = 0,1,--,n-1 \\ 0 & \end{cases}$$

The actuarial present value for this insurance is given by

$$A^{1}_{x:n]} = E\,[Z] = \sum_{k=0}^{n-1} v^{k+1}\ p_x\, q_{x+k} ------(2)$$

Note that the International Acturial Notation symbol for the actuarial present value of an insurance payable at the end of the year of death is the symbol for the corresponding insurance payable at the moment of death with the bar removed.

The rule of moments with the appropriate changes is notation, also holds for insurances payable at the end of the year of death. For example for the n-year term insurance above,

$$Var\,(Z) = 2_{A^{1}_{x:n]}} - \left(\,A^{-1}_{x:n]}\right)^2$$

Where

$$2_{A^{1}_{x:n]}} = \sum_{k=0}^{n-1} e^{-2(k+1)} k^{P}{}_{x}\, q_{x+k}.$$

Recursion relations for the term insurance actuarial present values can be derived algebraically from (2)

$$A^{1}_{x:n]} = \sum_{k=0}^{n-1} u^{(k+1)}\, k^{P}{}_{x}\, q_{x+k} = \ uq_x + \sum_{k=1}^{n-1} v^k k^{P_x}\, q_{x+k}$$

$$= v_{qx} + v_{px} \sum_{k=1}^{n-1} v^k k +^{P_{x+1}}\ q_{x+k}$$

$$= v_{qx} + v_{px} \sum_{j=0}^{n-2} v^{j+1}\, j^{P_{x+1}}\, q_{x+1+j}$$

$$= v_{qx} + v_{px}\, A_{x+1:n-1]} -----(3)$$

For (3) to be true at n=1, we define $A^{1}_{x:\overline{n}|}$ = 0.0 for all x.

Recursion Relations:-

We derived the recursion relations for n-year term insurance actuarial present values (3) algebraically. Whereas the relationship will hold for whole life insurance actuarial present values as the limiting case of n-year term insurance as n goes to ∞ . we will establish the whole life insurance relationship independently to illustrate a probabilistic derivation.

Consider A, from its definition E [Z] = E [$V^{k(x)+1}$]. For emphasis we now write this as

$$A_x = \mathrm{E}\,[\mathrm{Z}] \;=\; \mathrm{E}\left[V^{k(x)+1}\right]\mathrm{K(x)} \geq 0],$$

Which is redundant since all of k(x)'s probability is on the non-negative integers. E[Z] can be calculated by considering the event that (x) dies in the first year, that is k (x) = 0, and its complement, that (x) survives the first year, that is k (x) > 1. We can write

$$\mathrm{E}\,[\mathrm{Z}] \;=\; \mathrm{E}\left[V^{k(x)+1}\right]\mathrm{K(x)} \geq 0],$$

One line is left here

In this expression we can readily substitute

$$E\left[v^{k(x)+1} |\, k(x) = 0\right] = v,$$

$$P[k(x) = 0] = q$$

and

$$P[k(x) \geq 1] = p_{x.}$$

To find an expression for the remaining factor, we rewrite it as

$$E\left[v^{k(x)+1} |\, k(x) \geq 1\right] = VE\left[v^{(k(x)-1)+1} |\, k(x) - 1 \geq 0\right]$$

Since, k(x) is the curate future lifetime of (x), given k(x) $\geq$1. k(x)-1 must be the curate future lifetime of (x+1).

If we are willing to use the same probabilities for the conditional distribution of k (x)-1 given k(x) $\geq$1. As we would for a newly considered life age x+1, then we may write.

$$E\left[v^{(k(x)-1)+1} | \, k(x) \geq 0\right] = A_{x+1} ------------(5)$$

and substitute it into (4) to obtain

$$A_x = vq_x + VA_{x+1}px. ------------(6)$$

The assumed equality is (the distribution of the future lifetime of a newly insured life aged x+1)= (the distribution of the future lifetime of a life new age x+1 who was insured 1 year ago), In terms of select tables, the r.h.s. of (5) would be $A_{[x]+1}$. In (6) every x would be [x]. Note that (6) is the same backward recursion formula as (3). That is

$$u(x) = vq_x + vp_x \, u(x+1)$$

It is the starting value that makes the solution the actuarial present value of whole life insurance or of n-year term insurance. We see this same recursion endowment insurance where the starting values are the endowment maturity value.

Analysis of relationship (6) can give more insight into the nature of Ax. After replacement of p_x by 1-q_x and multiplication of both sides by (L+i) l_x, (6) can be rearranged as

$$l_x(l+i)A_{x+1} + dx\,(1 - A_{x+1}) ----(7)$$

For the random survivorship group, this equation has the following interpretation. Together with 1 year's interest, A_x will provide A_{x+1} for all lx lives and an additional L-A_{x+1} for those expected to die within the year. This latter amount for each expected death, that q_x (L-A_{x+1}) set aside for survivors and deaths, the L-A_{x+1} is required only for a death.

Dividing by l_x and then subtracting from both sides of (7), we have $A_x + q_x(L - a_{x+1}) - - - (8)$

In the worlds the difference between the actuarial present values at age x and one later at age x+1 is equal to the annual cost of insurance for the year.

Another expression for A_x can be obtained from (6) by replacing p_x, by $1\text{-}q_x$ multiplying both sides by vt, and rearranging the term to get.

$$V^{x+1}A_{x+1} - V^x A_x = -v^{x+1}q_x(1 - A_{x+1}),$$

or

$$\Delta V^x A_x = -v^{x+1}q_x(1 - A_{x+1}),$$

Summing from x = y to ∞, we obtain

$$-v^y Ay = -\sum_{x=y}^{\infty} v^{x+1}q_x(1 - A_{x+1})$$

and thus

$$Ay = -\sum_{x=y}^{\infty} v^{x+1-y}q_x(1 - A_{x+1})$$

This expression shows that the actuarial present value at y is the present value at y of the annual costs of insurance over the remaining lifetime of the insured.

Thus the function for it are

$$b_{k+L} = L \qquad k = 0,1, - - - - - - ,$$

$$b_{k+L} = \begin{cases} v^{k+1} & k = 0,1 - - - - ,n-1 \\ V^n & n, n+1, - - - - - , \end{cases}$$

$$Z = \begin{cases} v^{k+1} & K = 0,1, - - - - - - - , \\ v^n & K = n, n+1, - - - - - \end{cases}$$

The actuarial present value is

$$A_{x:n]} = \sum_{k=0}^{n-1} v^{k+1} p_x q_{x+k} + v^n \, n^p{}_x. - - - - - (9)$$

The annually increasing whole life insurance, paying k+1 units at the end of insurance year k+1 provided the insured dies in that insurance year has the benefit and discount functions and present value random variable as follows:

$$b_{k+1} = k + 1 \qquad k = 0,1,2, - - - -$$

$$v_{k+1} = v^{k+1} \qquad k = 0,1,2, - - - -$$

$$Z = (k + 1) v^{k+1} \qquad K = 0,1,2, - - - -$$

The actuarial present value is denoted by (IA)x. The annually decreasing n-year term insurance during the n-year period, provides a benefit at the end of the year of death in an amount equal to n-k, where k is the number of complete year lived by the insured since issue. Its functions are

$$b_{k+1} = \begin{cases} n - k & k = 0,1,2, - - - - \\ 0 & k = n, n + 1, - - - -, \end{cases}$$

$$v_{k+1} = v^{k+1} \qquad k = 0,1,2, - - - -$$

$$Z = \begin{cases} (n - k) v^{k+1} & K = 0,1,2, - - -, n - 1 \\ 0 & K = n, n + 1, - - - - \end{cases}$$

The actuarial present –value symbol for this insurance is $(DA)^1_{x:n]}$

The equality of the actuarial present values for the combination of level term insurances and the combination of deferred term insurances can be demonstrated analytically. Thus by definition

$$(DA)^1_{x:n]} = \sum_{k=0}^{n-1}(n-k)v^{k+1}{}_k p_x \; q_{x+k}$$

$$= \sum_{k=0}^{n-1}(n-k)(v^{k+1}{}_k p_x)(\; vq_{x+k})$$

$$= \sum_{k=0}^{n-1}(n-k)k\backslash l^A{}_x --- (10)$$

The total of the column sums

In (10) we can substitute

$$n-k = \sum_{j=0}^{n-k-1}(L)$$

To obtain

$$(DA)^1_{x:n]} = \sum_{k=0}^{n-1}\sum_{j=0}^{n-k-1}(L)v^{k+1}\,{}_k p_x \; q_{x+k}$$

By interchanging the order of summation, we obtain

$$\sum_{k=0}^{n-1}\sum_{j=0}^{n-j-1}(L)v^{k+1}\,{}_k p_x \; q_{x+k}$$

and then by comparing the inner summation to (2) we can write

$$(DA)^1_{x:n]} = \sum_{k=0}^{n-1}A^1_{x:n-f]}$$

Relationship between Insurances Payable at the Moment of Death and the End of the Year of Death:

We begin the study of these relationships with an analysis of the actuarial present value for whole life insurance paying a unit benefit at the moment of death. From (6), we have

$$\bar{A}_x = \int_0^\infty v^t t^{px}\, \mu_x(t)dt = \int_0^1 v^t t^{px}\, \mu_x(t)dt + \int_0^1 v^t t^{px}\, \mu_x(t)dt$$

The change of variable s= t-1 in the second integral gives

$$\bar{A}_x = \int_0^1 v^t t^{px}\, \mu_x(t)dt + v\int_0^\infty v^s \quad s+1^P{}_x\mu_x(s+1)\, ds.----(1)$$

On an aggregate mortality basis

$$s+1^P{}_x\mu_x(s+1) = p_x\; sp_{x+1} \quad \mu_x\,(x+s+1)$$

So the second term of (1) would be $sp_x\, \bar{A}_{x+1}$. On a select mortality basis the second term would be $vp_{[x]}\, \bar{A}_{[x]+1}$. Returning to (1) and using aggregate notation, we have.

$$\bar{A}_x = \int_0^1 v^t t^{px}\, \mu_x(t)dt + vp_{[x]}\, \bar{A}_{[x]+1} = \bar{A}^1{}_{x:1} + vp_x\bar{A}_{x+1} ----(2)$$

The integral is (2) can be expressed in discrete life table functions by adopting one of the assumptions about the form of the mortality functions between integers.

Under the assumption of a uniform distribution of deaths over each year of age

$$t^py\, \mu_y(t) = q_y, \qquad 0 \le t \le 1, \qquad and\; y = 0,1,---.$$

Which can be placed in (2) to obtain

$$\bar{A}_x = q_x \int_0^1 v^t\, dt\; + vp_x\, \bar{A}_{x+1}$$

$$= \frac{i}{8}\ vq_x + vp_x\ \bar{A}_{x+1} ------(3)$$

The domain for this relationship is x=0,1,---w-1, and the starting value is $\bar{A}_w = 0$.

If we multiply both sides of recursion formula (a) by i/s, we have

$$(a)\ \ A_x = \ vq_x + vp_x\ A_{x-1}, \quad x = 0,1,-----w-1,$$

$$\frac{i}{\delta} A_x = \frac{1}{\delta} vq_x + vp_x \left(\frac{1}{\delta} A_{x-1}\right).$$

Since (a) and (3) embody the same recursion formula and have the same domain and the same initial value of 0 at w, (i/&) A_x is the solution for (3), and

$$\bar{A}_x = \frac{i}{\delta} A_x -------------(4)$$

Formula (4) might have been anticipated under the assumption of a uniform distribution of deaths, between integral ages. The effect of the assumptions is to make the unit payable at the moment of death equivalent to a unit payable continuously through the year of death with respect to interest a unit payable continuously over the year is equivalent to i/& the end of the year.

The identity is (4) can be reached using the properties of the future lifetime random variable under the assumption of a uniform distribution of deaths in each year of age. We write T = K+S. we observed there that, under the assumptions of uniform distribution of deaths in each year of age, K and S are independent and S has a uniform distribution over the unit interval. As corollaries to these observations, K+1 and 1-S are also independent and 1-S has a uniform distribution over the unit interval In the identity.

$$\bar{A}_x = E[V^T] = E[V^{k+1}(1+i)^{L-S}]$$

We can use the independent of K+1 and 1-S to calculate the expectation of the product as the product of the expectations.

$$E[V^{k+1}(1+i)^{L-S}] = E[V^{k+1}]\ E[(1+i)^{L-S}] ----(5)$$

The first factor on the r.h.s. is A_x. Since 1-S has the uniform distribution over the unit interval the second factor is

$$E[(1+i)^{L-S}] = \int_0^1 (1+i)^t 1 \; dt = \frac{1}{\delta}$$

Here again we have $\bar{A}_x = (i/\delta)A_x$ under the assumption of uniform distribution of death in each year of age. A similar argument, again based on the assumption of a uniform distribution of deaths in each year of age can be used to show that the actuarial present value of a whole life insurance which pays a unit at end of the m^{th} of a year of death is equal to

$$A_x{}^{(m)} = \frac{i}{i^{(m)}} A_x. ---------(6)$$

We already discussed the assumption that the force of mortality is constant between integral ages. The relationship between the actuarial present values for whole life insurance payable at the moment of death and at the end of the year of death under this assumption is developed. Since the hyperbolic assumpticon implies that the force of mortality decreases over the year of age, it is seldom realistic for human lives. Moreover, it leads to more complicated relationships that we will not develop here.

Next we turn to an analysis of the annually increasing n-year tern insurance payable at the moment of death. For this insurance, the present value random variable is

$$Z = \begin{cases} [LT+1]\, v^T & T < n \\ 0 & T \geq n. \end{cases}$$

Since LT+1 = k+1, we can use the relation T= k+s to obtain

$$Z = \begin{cases} (k+1)v^{k+1}v^{s-1} & T < n \\ 0 & T \geq n. \end{cases}$$

If we let W be the present value random variable for the annually increasing n-year term insurance payable at the end of the year of death,

$$W = \begin{cases}(k+1)v^{k+1} & k = 0,1,----,n-1 \\ 0 & k = n, n+1, --- \end{cases}$$

Then

$$Z = W(1+i)^{L-s}$$

and

$E[Z] = E[W(1+i)^{L-s}]$

Since W is a function of K+1 alone and k+1 and 1-s are independent.

$E[Z] = E[W] = E[(1+i)^{L-s}]$

$= (IA)^{1}_{x:n]} \frac{i}{\delta}.$

These result for the whole life and the increasing term insurance payable at the moment of death. under the assumption of a uniform distribution of deaths over each year of age, are very similar.

$\bar{A}_x = \left(\frac{i}{\delta}\right) A_x$

and

$(IA)^{1}_{x:n]} = \frac{i}{\delta} (IA)^{1}_{x:n]}$

Let us look at the general model to find the basis of the similarities. From (2) {second section}

$$Z= b_T\ V_T \text{--------------(7)}$$

For the two continuous insurance above the conditions used were

$V_T = V^T$ and

b_T was a function of only the integral part of T, the curate future lifetime K.

writing this latter property as , we can write (7) for these insurances as

$$Z = b_{k+1} V^{T}$$

$$= b_{k+1} V^{k+1} (L+i)^{L-S}$$

and

$$E[Z] = E\,[b_{k+1} V^{k+1} (L+i)^{L-S}] - - - - - (8)$$

Under the assumption of a uniform distribution of deaths over each year of age we can infer the independence of K and S and that 1-S also has a uniform distribution. Then we can write (8) as

$$E[Z] = E\,[b_{k+1} V^{k+1}]E\,[(L+i)^{L-S}]$$

$$= E\,[b_{k+1} V^{k+1}]\frac{i}{\delta} - - - - - - (9)$$

For an insurance providing a death benefit at the moment of death that is not a function of K, further analysis is required to express its values in terms of those for an insurance payable at the end of the year of death. Consider the continuously increasing whole life insurance payable at the moment of death.

$$b_t = t \qquad t > 0,$$

$$V_t = V^t \qquad t > 0,$$

$$Z_t = tV^t \qquad t > 0.$$

To find $(\bar{I}\bar{A})_x$, we rewrite

$$Z = (k+s)v^{k+s}$$

$$= (k+1)v^{k+s} - (L-S)v^{k+1}(L+i)^{L-S}$$

$$= (k+1)v^{k+s}(L+i)^{L-S} - v^{k+s}(L-S)(L+i)^{L-S}$$

Now taking expectations under the assumption of a uniform distribution of deaths over each year of age, we have

$$E[Z] = E[(K+1)V^{k+1}]E[(l+i)^{L-S}] - E[V^{k+1}]\, E[(L-S)(L+i)^{L-S}]$$

$$= (IA)_x \frac{i}{\delta} - A_x E\,[(L-S)(L+i)^{L-S}].$$

We can simplify the last factor directly since L-S has a uniform distribution,

$$E\,[(L-S)(L+i)^{L-S}] = \int_0^1 \mu\,(1+i)^{\mu} du = (\bar{D}\bar{S})_{i]} = \frac{L+i}{\delta} - \frac{i}{\delta^2}$$

Thus we can write

$$(\bar{I}\bar{A})_x = \frac{i}{\delta}\left[(IA)_x - \left(\frac{L}{d} - \frac{L}{\delta}A_x\right)\right].$$

Unit-II

<u>Principles of Premium Calculation</u>

Def:- Premium- A premium is the payment that a policy holder makers for a complete or partial insurance cover against a risk.

We now consider:-

Some ways in which premium can be calculated & lay down certain principles for its calculation.

<u>Notation:-</u>

- We denoted π_x by the premium that an insurer change to cover a risk x.
- When we refer to a risk X. what we mean is that the claim from this risk are distributed as the random variable X.
- The premium π_x is then some function of X.

- A rule that assign a numerical value to π_x is referred to as a <u>Premium Calculation Principle</u>.
- Thus a premium principle is of the form where is some function.
- We now consider some desirable properties of premium.

<u>Properties of Premium Principles:</u>

There are many desirable properties for premium calculation principle. We list 5 most important of them.

(i) **<u>Non- negative Loading:</u>** This property requires that we should have $\pi_x \geq E[x]$ i.e. the premium should not be less than the expected claims.

(ii) **<u>Additivity:</u>**- This property requires that if x_1 & x_2 are independent risk, then the premium for the combined risk i.e $\pi_{x_1+x_2}$ should be $\pi_{x_1} + \pi_{x_2}$. if this property is satisfied, then these is no advantage to either an individual or an insurer, in combining risk or splitting them as the total premium does not alter under each course of action.

(iii) **<u>Scale invariance:</u>**- This property requires that if Z= ax. Where a is (+ive), then $\pi_z = a\pi_{x.}$

For eg:- Imagine that the currency of great Britain changes from sterling to euro with 1 pound sterling being converted to a Euros. The, if a British insurer uses a scale invariant premium principle, a premium of 100 pounds sterling would change to 100a Euros.

(iv) **<u>Consistency:-</u>** This property requires that if Y= X+C where C > 0, then, we should have $\pi_y = \pi_x + c$. Thus if the distribution of Y is the distribution of X shifted by C units then the premium for risk Y should be that for risk X. increased by C.

(v) **<u>No Rip-off Property:-</u>** (For upper limit of premium) This property requires that if there is finite max. claim amount of the risk say x_m. Then we should have $\pi_x \leq \pi_m$. if this condition is not satisfied then there is no incentive for an individual to effect insurance i.e.

$$E[x] \leq \pi_x \leq x_m$$

Examples of Premium Principle:

(1) The pure- premium principle: The pure premium principle sets $\pi_x = E(x)$i.e. the pure premium is equal to the insurer's expected claims under the risk. To check if all the properties are satisfied.

(i) $\pi_x \geq E[x] = \pi_x = E[x]$

(ii) $\pi_{x_1+x_2} = E[x_1 + x_2] = E(x_1) + (x_2) = \pi_{x_1} + \pi_{x_2}$

(iii) $if\ x = az.\quad \pi_x\ \ E[az] = aE[z] = \pi_x = a\pi_z$

(iv) $if\ Y = X + C\quad \pi_x = \pi_{y+c} = E[y + c] = E(y) + C = \pi_x = \pi_y + C$

(v) $if\ x \leq x_m$

$E[x] \leq x_m$

$\pi_x = E[x] \leq x_m = \pi_x \leq x_m$

(2) **The Expected value Principle:**

It sets. $\pi_x = (1 + \theta)E[x]$ where $\theta > 0$ is referred to the premium loading factor (imp term). The loading in the premium is thus $\theta E[x]$.

Important Remark about Expected value principle- Clearly this principle is a very simple one but it has a major drawback that it assigns the same premium to all risks with the same mean. However risks with identical means but different variance should have different premium. Checking which properties the expected value principle satisfies:

(i) $\pi_x \geq E[x] = \pi_x = (1 + \theta)E[x] = E[x] + \theta E[x] \geq E[x]$

(ii) $\pi_{x_1+x_2} = (1 + \theta)E[x_1 + x_2] = E(x_1)(1 + \theta) + (x_2)(1 + \theta) = \pi_{x_1} + \pi_{x_2}$

(iii) $if\ x = az.$

$\pi_x\ \ (1 + \theta)E[az] = (1 + \theta)aE[z]$

$= a\pi_z = \pi_z = a\pi_x.$

(iv) $if\ Y = X + C\quad \pi_y$

$= (1 + \theta)E[y] = (1 + \theta)E[x + c]$

$= (1 + \theta)E[x] + C(1 + \theta)$

$$= \pi_x + C(1+\theta) \neq \pi_x + C(untill\, \theta = 0)$$

(v) $x \leq x_m$

$$(1+\theta)x \leq (1+\theta)x_m$$
$$E((1+\theta)x) \leq (1+\theta)x_m$$
$$\pi_x \leq (1+\theta)x_m$$
$$= \pi_x \,\&\, x_m$$
$$\therefore (1+\theta) > 0$$

Rip off Property:- To check whether rip-off property holds we will consider example. Let x be a risk s.t. P(x-b) = 1 s.t. b>0

i.e. X is a degenerate r.v. with its entire mass concentrated at point b. Now, we calculate $\pi_x = (1+\theta)\, b > b\ (\because\ \theta > 0\ )$

$but\ x = b\ \ (or\ i.e.\, xm = b)$

$\pi_x > \pi_m$ that is why no rip off is not being satisfied. This is a counter example, therefore we can say that in general, the no rip off property may not be satisfied by expected value principle.

(3) **<u>The Variance Principle:</u>** The variance principle sets:-

$\pi_x = E[x] + \alpha v\ [X], where\ \alpha > 0.$

Thus the loading in the premium is proportional to v[x]. In this way, this principles removes the drawback of the expected value principle which takes account only of the expected chain.

Checking which properties it satisfies :-

(i) $\pi_x \geq E[x]\ (Non-negavtive\ loading)$

$$E[X] + \alpha v[X] \geq E[X]$$

(ii) $\pi_{x_1+x_2} = E[x_1 + x_2]$

$E[x_1] + E[x_2] + \alpha\ [v(x_1) + v(x_2)]\ (b'oz\ of\ independpent\ of x_1 \,\&\, x_2)$

(iii) $\pi_z = \pi_{ax}$

$= E[aX] + \alpha v[aX] = aE\,[X] + \alpha a^2 v[X]$

$= a[E(x) + \alpha a v(x)] \neq a(E(x) + \alpha v(x))$

(iv) Y= X+C

$\pi_y = \pi_{x+c} \quad = E(X + C) + \alpha v\,(X + C)$

$E(x) + C + \alpha v(x)$

$\pi_y = \pi_x + C.$

(v) $x \ \leq \ x_m$

$\pi_x \leq \ x_m$

$E(x) + \alpha v(x) \leq x_m$

$Eg.\ P(x = 8) = P(x = 12) = 0.5$

E(x)= 4+6= 10

$v(x) = E\,(x^2) - (E(x))^2 = 4$

$\pi_x = 10 + \alpha 4 \quad\ \leq 12, \quad\quad x > 0$

If x =10 then π_x = 50 ≠ 12 or if > 0.5, π_x > xm.

It does not satisfy, no rip off values (property) when α > 05. It satisfy no. rip off property when , $\alpha \leq 0.5$

(4) **The standard deviation Principle:**

Clearly the variance principle does not satisfy the scale in-variance property which is a very desirable property for premium setting & therefore motivated by this fact, we define the std. deviation principle.

Therefore, it sets

$$\pi_x = E[x] + \alpha\sqrt{v(x)} = E[x] + \alpha\, S(x).$$

Checking:

$(i)\ \pi_x \geq E[X]$

$= E[X] + \alpha\, S(x) \geq E\,(x).$

$(ii) \pi_{x_1+x_2} = E[x_1 + x_2] + \alpha\, \sqrt{v(x_1 + x_2)}$

$= E[x_1]E\,[x_2] + \alpha\, \sqrt{v(x_1)v(x_2)}$

$\neq\ \pi_{x_1} + \pi_{x_2}$

$(iii)\ \pi_Z = \pi_{ax}$

$= E\,(ax) + \alpha\sqrt{v(ax)}$

$= aE(x) + a\alpha\sqrt{v(x)}$

$= a\pi_x$

$(iv)\pi_y = \pi_{x+c}$

$= E\,(x + c) + \alpha\, \sqrt{v(x)} + c$

$= E\,(x) + \alpha\, \sqrt{v(x)} + c$

$= \pi_x + C$

$x\ \leq x_m,\quad \pi_x = E(x) + \alpha\sqrt{v(x)}\ \leq\ x_m.$

$= \pi_x \leq x_m$

$if\ P\,(x = 8) =\ P\,(x = 12) =\ 0.5$

$E(x) =\ 10$

$V(x) = 4\ = SD(x) = 2$

$\pi_x = 10 + \alpha\ \ 2 \leq 12$

$when\ \alpha < 1,\ \ \pi_x \leq 12\ \rightarrow No.Rip\ off\ satisfies$

<u>Question:-</u> An insurer thinks about an innovative way of setting up a premium s.t. $\pi_x = V^{-1}\ (E[v(x)])$ where v is a f^n s.t. v^1 (x) > 0 & v" (x) < 0, for x > 0

(i) Calculate $\pi_x \rightarrow$ when $V(x) = x^2$ & x ~ $\gamma(2, 2)$.

(ii) Check which properties does not this property satisfy.

(iii) In case if you feel a particular prop. Is not satisfy then give appropriate counter example.

$$Solution{:}\, -V(x) = x^2\ X{\sim}\gamma(\alpha,\beta)$$

$$E[x] = \frac{\alpha}{\beta} = \frac{2}{2} = 1$$

$$v[x] = \frac{\alpha}{\beta^2} = \frac{2}{4} = \frac{1}{2}.$$

$$E[v(x)] = E(x^2)$$

$$= V(x) + (E(x))^2 = \frac{1}{2} + 1 = \frac{3}{2}$$

$$\pi_x = V(x) = x^2 = v^{-1}(x) = \sqrt{x} = v^{-1}\ (E(x)) = \sqrt{E(x)}$$

$$\pi_x = \sqrt{Ev(x)} \qquad \pi_x = \sqrt{\frac{3}{2}} = \sqrt{1.5} = 1.225$$

Checking:-

$$(i)\ \pi_x = v^{-1}\big[E\big(v(x)\big)\big] = \sqrt{Ev(x)} \geq E(x) = 1.225 \geq 1.$$

$$Since,\ \ E(x^2) \geq \big(E(x)\big)^2 = \sqrt{E(x^2)} \geq E(x)$$

$$(2)\ \ \pi_{(x_1+x_2)} = \sqrt{E[x_1+x_2]^2} = \sqrt{E(x_1^2) + E(x_2^2) + 2E(x_1,x_2)}$$

$\neq \sqrt{E(x_1^2)} + \sqrt{E(x_2^2)}$

(3) $\pi_z = \pi_{ax}$

$= \sqrt{E((ax)^2)}$

$= \sqrt{E(a^2x^2)} = a\sqrt{E(x^2)} = a\pi_x$

(4) $Y = X + C$

$= \bar{\lambda}_y = \pi_{x+c} = \sqrt{E\ (x^2 + c^2 + 2xc)} \neq \sqrt{E(x^2)} + C$

(5) $X \leq x_m$

$= \pi_x = \sqrt{E(x^2)}$

$= \pi_x = \sqrt{6.25} = 7.905$

$x_m = -5$

$= \pi_m > x_m$

$P(x = -5) = P\ (x = -10) = 0.5$
$E(x^2) = 25 \times 0.5 + 100 \times 0.5$
$= 12.5 + 50 = 62.5$

(5) **Principle of Zero Utility:-**

This principle sets the premium π_x to satisfy the equation

$u(w) = E\ [u(W + \pi_x - x)]$.

$Where\ u(x) is\ the\ insurancer's utility\ function\ \&\ W\ is\ the\ insurancer's\ surplus$

Clearly, the utility function u(x) satisfies

$\mu'(x) > 0$

$\mu''(x) > 0$

In general the premium will depend on the insurer's surplus (expect for the case when we have an exponentially utility f^n)

Checking which property does the principle of Zero utility satisfy.

$(i)\pi_x \geq E[x]$

$$E\,[\mu(W \neq \pi_x - x)]$$
$$\leq \mu(w + \pi_x - E_x) \quad (By\ Jensen's\ Inequlity\ u\ is\ a\ convex\ f^n)$$

$$= \ \mu(w) \leq \mu(w + \pi_x - E(x))$$

We know that, $\mu'(x) > 0$

$$\therefore if\ a\ \leq b = \ \mu\,(a) \leq \ \mu\,(b)$$

$$= \ w < w + \pi_x - E[x]$$

$$\pi_x \geq a\ E[x]$$

(ii) Now,

$$\pi_z \ = \ \pi_{ax} = a\pi_x$$

$$E\left(\mu(w + \pi_z - ax)\right) \leq \ \mu(w + \pi_z - aE(x))$$

$$= \ \mu\,(w) \leq \ \mu\left(w + \pi_z - aE(x)\right)$$

$$= \ \pi_x \geq aE(x)$$

$$= a\pi_x \geq aE(x)$$

$\pi_z \geq E(z)$
$\pi_z \geq E(ax)$
$\pi_z \geq aE(x)$

For a counter example

$$X \sim N\,(\mu, \sigma^2)$$

$$\mu(x) = \exp[-\beta x]$$

$$= -\exp[-\beta w] = E\left[-Exp\left(-\beta(w + \pi_x - X)\right)\right]$$

$$= \exp[-\beta w] = \exp(-\beta w)\exp(-\beta\pi_x)E(\exp(\beta x))\exp(\beta\pi_x) = E\,\left(e^{\beta x}\right)$$

$$= \ \pi_x = \beta^{-1}\log\left[E\left(e^{\beta x}\right)\right] - - - - - - - - -(*)$$

$$= \pi_x = \frac{L}{\beta} \log M_x(\beta)$$

$$= \frac{1}{\beta}\ log \left[\exp\left(\beta\mu + \frac{1}{2}\beta^2\sigma^2\right)\right]$$

$$= \pi_x = \mu + \frac{1}{2}\beta\sigma^2 - - - - - - - (A)$$

$$\pi_Z = \frac{1}{\beta} \log M_z(\beta)$$

$$\pi_Z = \frac{1}{\beta} \log\left(e^{\mu a\beta} + \frac{1}{2}\ \sigma^2 a^2\beta^2\right)$$

$$= \ \mu a + \frac{1}{2}\sigma^2 a^2\beta^2 \neq a\pi_x.$$

$$(iii) Y = X + C$$

$$X_Y = \pi_x + C$$

$$\mu\ (w) \leq \mu(w + \pi_x -\ E(x))$$

$$\mu(w) \leq \mu(w + \pi_Y - E(x) + C)$$

$$\mu(w) = E\left[\mu(w + \pi_y - Y)\right] = E\left[\mu(w + \pi_Y - x - c)\right] - - - (1)$$

Also by PZU for the claim X, we have

$$\mu(w) = E[\mu(w + \pi_x - X)] - - - - - - - - - - - - (2)$$

From (1) & (2) on comparing

$$\pi_Y - c = \pi_x$$

$$\pi_Y = \pi_x + c$$

No. Rip off property

We know rip off property will hold if $x \leq x_m \, then \, \pi_x \leq x_m$

Now it is clear that if

$x \leq x_m$

$-x \geq x_m$

$= w + \pi_x - \ x \geq w + \pi_x - x_m ------ (1) \quad \begin{bmatrix} w \geq 0 \\ \pi_Y \geq 0 \end{bmatrix}$

For there by PZU

$\mu(w) = E[\mu(w + \pi_x - X)] ---------(2)$

Now, by the property of expectation if

$X \geq Y \, then, E(x) \geq E[Y] ----(3)$

From (1), (2) & (3)

$\mu(w) = E[\mu(w + \pi_x - x)] \geq E[\mu(w + \pi_x - X_m)] > \mu(w + \pi_x - x_m)$

$(w, \pi_x \,\&\, x_m \, are \, all \, pure \, no.)$

$= \ \mu(w) > \& \ \ \mu''(w) < 0$

$= \ \pi_x \leq w + \pi_x - x_m$

$= \pi_m \ \leq \ x_m$

In general the principle of utility is not additive but the exponential principle is.

From (*)

$\pi_{(x_1 + x_2)} = \beta^{-1} \log E[exp\{\beta(x_1 + x_2)\}]$

$= \beta^{-1} \log E[exp\{\beta x_1\}] E[exp\{\beta x_2\}]$

$= \beta^{-1} \log E[exp\{\beta x_1\}] + \log \beta^{-1} \, E[exp\{\beta x_2\}]$

$= \pi_{x_1} + \pi_{x_2}$

Where, the second live followed by the independence of $x_1 \, and \, x_2$

The principle of zero utility is not scale invariant,

<u>Particular Case of PZU</u>

<u>Exponential Principle:</u>

The exponential principle set:-

$\mu(w) = E[\mu(w + \pi_x - x)] - - - - - - - - - - - - - - (*)$

Where $\mu(w) = -\exp[-\beta w]$ & w is the insurer's surplus. We already know that (*) yields

$\pi_x = \frac{1}{\beta} \log\left(M_x(\beta)\right) - - - - - - - - - - - (**)$

Where $M_x(.)$ is the mgf of x.

1. $\pi_x \geq E(x)$

   $\frac{1}{\beta} \, log\left(M_x(\beta)\right) \geq E(x)$

   $\frac{1}{\beta} \log E\left(e^{\beta x}\right) \geq E(x)$

   $= \pi_x \geq E(x)$

$$e^{\beta x} \geq \beta x$$

$$E\left[e^{\beta x}\right] \geq E(\beta x)$$

$$\log M_x(\beta) \geq \beta E(x)$$

$$= \frac{1}{\beta} \, log M_x(\beta) \geq E(x)$$

2. <u>Consistency</u>

   Y=X+C

   $\pi_Y = \pi_x + C$

   $e^{\beta Y} \geq \beta Y$

   $= e^{\beta(x+c)} \geq \beta(x + c)$

$$\pi_Y = \frac{1}{\beta} log\left(M_y(\beta)\right)$$

$$= \frac{1}{\beta} log E\left(e^{\beta Y}\right)$$

$$= \frac{1}{\beta} log E\left(e^{\beta x + \beta c}\right)$$

$$= \frac{1}{\beta} \log\left(E\left(e^{\beta x}\right). e^{\beta c}\right)$$

$$= \frac{1}{\beta}\left(log E\left[e^{\beta x}\right] + log e^{\beta c}\right)$$

$$= \frac{1}{\beta} log M_x(\beta) + C$$

$$= \pi_x + C$$

3. Additive

$$\pi_{x_1} = \frac{1}{\beta} log M_{x_1}(\beta)$$

$$\pi_{x_2} = \frac{1}{\beta} log M_{x_2}(\beta)$$

$$\pi_{x_1+x_2} = \frac{1}{\beta} log M_{(x_1+x_2)}(\beta)$$

$$= \frac{1}{\beta} log\left(e^{\beta(x_1)}.e^{\beta(x_2)}\right)$$

$$\pi_{(x_1+x_2)} = \frac{1}{\beta} log M_{x_1}(\beta) + \frac{1}{\beta} log M_{x_2}(\beta)$$

4. Scale is invariant

$$Z = ax \quad = \pi_Z = a\pi_x$$

$$\pi_Z = \frac{1}{\beta} log M_z(\beta)$$

$$= \frac{1}{\beta} log\left[E\left(e^{\beta ax}\right)\right]$$

$$= \frac{1}{\beta} log\left[E\left(e^{\beta xa}\right)\right] = \frac{1}{\beta} log M_x(a\beta)$$

$$\neq a \frac{1}{\beta} log M_x(\beta)$$

$$\neq a\pi_x$$

Remark:- Although PZU in general does not satisfy the additive property, its particular case i.e. the exponential principle satisfies this property. This is because in the exponential the premium has a well define form & does not depend on the initial wealth W, while for other utility function like the quadratic utility on initial wealth W & does not have a compact form like for the exponential function.

(6) The Esscher Premium Principle:

The Esscher premium principle sets:-

$$\pi_x = \frac{E[X\,e^{hx}]}{E[e^{hx}]}, where\, h > 0.$$

Infact, we can interpret the Esscher premium as being the pure premium for a risk, $\tilde{X}$ (where by pure premium, we mean premium based on the first principle). For this, we define $\tilde{X}$ based on X as follows:-

Suppose that X is a continuous r.v. on (0,∞) with density function f & we define function g by:-

$$g(x) = \frac{e^{hx} f(x)}{\int_0^\infty e^{hy} f(y)\, dy}$$

Then, g is the density function of a r.v. $\tilde{X}$ which has distribution function G(x) given as:-

$$G(x) = \int_0^x g(w) du = \int_0^x \frac{e^{hx} f(x) du}{\int_0^\infty e^{hy} f(y)\, dy}$$

$$G(x) = \frac{1}{\int_0^\infty e^{hy} f(y)\, dy} \int_0^x e^{hu} f(u)\, du.$$

The distribution function G is known as Esscher transform of F with parameter h, where F (.) is c.d.f. of the r.v. To summarize we are performing the following transformation.

$X \rightarrow \tilde{X}\ (using\, the\, p.d.f.\, of\ X)$

i.e. we transform f(x) to g(x) by weighing f(x) by e^{hx} & density by

$$\int_0^\infty e^{hy} f(y)\, dy$$

This mean that we also transform the c.d.f. of x i.e. F(x) to G(x). since by def. of c.d.f.

$$G(x) = \int_0^x g(u)du = \int_0^x \frac{e^{hu} f(u)\, du}{\int_0^\infty e^{hy} f(y)\, dy}$$

$$= \frac{1}{\int_0^\infty e^{hy} f(y)\, dy} \quad \int_0^x e^{hy} f(y)\, dy$$

i.e.

$$G(x) = \frac{\int_0^x e^{hy} f(y)\, dy}{M_x(h)}$$

Then G is called the Esscher transform of F. Further now the m.g.f. of the r.v. $\tilde{X}$ is given below:-

$$M_{\tilde{X}}(t) = E\left[e^{\tilde{X}t}\right]$$

$$M_{\tilde{X}}(t) = \int_0^\infty e^{zt}, g(z)\, dz - - - - - - (1)$$

Then, using the definition of g(z) in (1), we have

$$M_{\tilde{X}}(t) = \int_0^\infty e^{zt} \left(\frac{e^{hz} f(z)}{\int_0^x e^{hy} f(y)\, dy} \right) dz$$

$$= \frac{1}{M_{\tilde{X}}(h)} \int_0^\infty e^{z(h+t)} f(z) dz$$

$$= \frac{M_x^{(t+h)}}{M_x(h)}$$

i.e. $M_{\tilde{X}}(t) = \frac{M_x{}^{(t+h)}}{M_x(t)}$

i.e., we have also transform the mgf of x into mgf of $\tilde{X}$

Thus, we can see that the premium which is set up by the Esscher principle which is

$$\pi_x = \frac{E[xe^{hx}]}{E[e^{hx}]} \; is\; infact\; \pi_x = E[\tilde{X}]$$

$$E[\tilde{X}] = \int_0^\infty \mu.\, g(u)\, du$$

$$= \int_0^\infty u \left(\frac{e^{hu} f(u)}{\int_0^\infty e^{hy}\, f(y) dy} \right) du$$

$$= E[\tilde{X}] = \frac{E[xe^{hx}]}{E[e^{hx}]}$$

Q (1):- Suppose that X~ $\exp(\lambda)$ calculate F(x) & the Esscher transform of x with parameter h s.t. $h < \lambda$.

Sol:- X~ $\exp(\lambda)$, $\quad f(x) = \lambda e^{-\lambda x}$

$$F(x) = 1 - e^{-\lambda x}$$

Now, Esscher transform of x is then G (x) be dist^n f^n p.d.f. $\tilde{X}$ & g(x) be its p.d.f.

$$g(x) = \frac{e^{hx} f(x)}{\int_0^\infty e^{hx} f(x) dx} = \frac{\lambda\, e^{x(h-\lambda)}}{\lambda \int_0^\infty e^{x(h-\lambda)} dx} = \frac{\lambda e^{-x(h-\lambda)}}{\frac{-\lambda}{(\lambda - h)} [e^{x(h-\lambda)}]_0^\infty}$$

$$= (\lambda - h) e^{-x(h-\lambda)}$$

$\tilde{X} \sim \exp(\lambda - h)$.

$$G(x) = \frac{\int_0^\infty e^{hx} f(x)dx}{M_x(h)} = \frac{\lambda \int_0^\infty e^{x(h-\lambda)}dx}{\lambda/\lambda - h} = \frac{-1}{\lambda - h}\frac{\left[e^{-x(h-\lambda)} - 1\right]}{(\lambda - h)}$$
$$= (\lambda - h)^{-2}\left[1 - e^{-x(h-\lambda)}\right].$$

Q.2:- For the same Question. Suppose λ =1 now calculate the premium using the Esscher principle. Also draw the graph depicting the values of f & g v/s the values of X.

Sol:- $\pi_x = E[\tilde{X}] = \frac{1}{\lambda - h}$

$\lambda - h,\ \ \pi_x = \dfrac{1}{\lambda - h}$

<u>ESSCHER TRANSFORM</u>

Under Esscher transform, we have talked about transforming the c.d.f. F into the c.d.f. G.

Looking more closely at the density transformation, we see that the density g is infact a weighted version of the density function, since we can write

$g(x) = w(x)f(x) - - - - - - - - - - - - - - - - - (1)$

Where,

$w(x) = \dfrac{e^{hx}}{M_x(h)} - - - - - - -(2)$

Also as $h > 0,\ \ w'(x) = \frac{he^{hx}}{M_x(h)} > 0 - - - - - - - - - (3)$

i.e.

W(x) is an increasing f^n of x & so increasing weight attaches as the value of x increases. Now let's under a claim,

$X \sim exp(x)$

Then,

$$M_x(t) = \frac{\lambda}{(\lambda - t)} - - - - - - -(4)$$

$$M_{\tilde{x}}(t) = \frac{\lambda}{(\lambda - h - t)} - - - - - - -(5)$$

Esscher Transform of F is given by

$G(x) = L - exp\{-(\lambda - h)x\}$

Let's take

h= 0,2; λ =1

then

$f(x) = \ e^{-x}\ ; \ \ w(x) = 0.8\ e^{0.2x}$

$g(x) = 0.8\ \ e^{0.2x}, \ \ x > 0$

Now we draw the density f(x) & g (x) on the same graph to comparing purpose.

Figure

Q. Suppose that an insurer faces claim $\sim$ Poisson (λ) calculate

1) The Esscher transform of the c.d.f. of the claim with parameter h.

2) Calculate the premium that the insurer will change based on the Esscher principle.

Sol:-

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad\qquad mgf - e^{u(e^t - 1)}$$

$$g = \frac{e^{hx}f(x)}{\int_0^{\infty} e^{hx}f(x)dx} = \left(e^{\lambda}\lambda\right)^x$$

$$M_{\tilde{X}}(t) = \frac{e^{\mu(e^{t+h}-1)}}{e^{\mu(e^t-1)}} = \frac{e^{\mu(e^t.e^h-1)}}{e^{\mu}(e^t-1)}$$

$$= e^{\mu}(e^t e^h - 1 - e^t + 1)$$

$$= e^{\mu}(e^t e^h - e^t) = e^{\mu}\left(e^t(e^h - 1)\right)$$

$$g(x) = \frac{e^{hx}e^{-\lambda}\lambda^x}{x!\,M_x(h)} = \frac{e^{-\lambda+hx}\lambda^x}{x!\ e^{\lambda(e^h-1)}}$$

$$= \frac{e^{\lambda+hx-\lambda e^h+\lambda}\lambda^x}{x!}$$

$$= \frac{e^{hx-\lambda e^h}.\lambda^x}{x!} = \frac{e^{hx}e^{-\lambda^{e^h}}\lambda^x}{x!}$$

$$= \frac{(\lambda e^h)^x e^{-\lambda^{e^h}}}{x!}$$

$$\therefore \tilde{X} \sim Pois(\lambda^{eh})$$

$$\pi_x = E[\tilde{X}]$$

$$\pi_x = \lambda e^h$$

To check whether Esscher principle satisfies the five properties:-

1. <u>Non-negativity:-</u> $\quad E[\tilde{X}] = \pi_x$

$M_x(0) = E[e^{tx0}] = 1.$

$Case\ 1{:}-h = 0$

$$M_{\tilde{X}}(t) = \frac{M_x(t+h)}{M_x(h)} = \frac{M_x(t)}{M_x(0)} = M_x(t) - - - - - - (1)$$

$$= \pi_x = E[\tilde{X}] = E[X] - - - - - - - - - - - - - - (2)$$

now

Case 2:- h > 0

$$E[\tilde{X}^r] = \frac{d^r}{dt^r} M_{\tilde{X}}(t)\Big]_{t=0} = \left|\frac{d^r}{dt^r}\left[\frac{M_x(t+h)}{M_x(h)}\right]\right|_{t=0} - - - - - - - (3)$$

$$\frac{1}{M_x(h)} = \frac{d^r}{dt^r} M_{\tilde{X}}(t)\Big]_{t=0} = \frac{1}{M_x(h)}\ {M_x}^{(r)}(h) - - - - - - - - - (4)$$

and so,

$$\frac{d}{dh}\ \pi_x = \frac{d}{dh} E[\tilde{X}] = \frac{d}{dh}\frac{M_x'(h)}{M_x(h)}\ \ (put\ r = 1)$$

$$= \frac{{M_x}^{(2)}(h)}{\left(M_x(h)\right)^2} - \left[\frac{M_x'(h)}{M_x(h)}\right]^2 = E[\tilde{X}^2] - \left(E[\tilde{X}^2]\right)^2 = V\left(\tilde{X}\right)$$

$$\therefore \frac{d}{dh}\quad \pi_x \geq 0$$

π_x is a non – decreasing function of h.

$= \pi_x \geq E[X] \qquad\qquad h \geq 0$

2) <u>Consistency:-</u> (Take Y= X+C) Calculate π_Y

$$\pi_x = \frac{E[x\, e^{hx}]}{E[e^{hx}]} \qquad\qquad (by\ esscher\ principle)$$

$$\pi_y = \frac{E[y\, e^{hy}]}{E[e^{hy}]} = \frac{E(x+c)\, e^{h(x+c)}}{E[e^{h(x+c)}]}$$

$$\frac{E\left[x\, e^{h(x+c)} + c\, e^{h(x+c)}\right]}{E[e^{hx}.\, e^{hc}]}$$

$$\frac{e^{hc} E[e^{hx}] + c.\, E[e^{hx}.\, e^{hc}]}{e^{hc} E[e^{hx}]} = \frac{E[xe^{hx}]}{E[e^{hx}]} + c$$

$$\pi_y = \pi_x + c$$

3) <u>Additive Property:</u>

Take Z= $X_1 + X_2$, s.t. x_1 & x_2 are independent.

$$\pi_{x_1+x_2} = \frac{E\left[(x_1 + x_2)\, e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]} = \frac{E\left[x_1\, e^{h(x_1+x_2)} + x_2\, e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]}$$

$$= \frac{E\left[x_1\, e^{h(x_1+x_2)}\right] E\left[x_2\, e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]}$$

$$= \frac{E[x_1 e^{hx_1}].E[e^{x_2 h}] + E[x_2 e^{hx_2}]\, E[e^{hx_1}]}{E[e^{hx_1}]\, E[e^{hx_2}]}$$

$$= \frac{E[x_1 e^{hx_1}]}{E[e^{hx_1}]} + \frac{E[x_2 e^{hx_2}]}{E[e^{hx_2}]}$$

$$= \pi_{x_1} + \pi_{x_2} = \pi_{x_1+x_2}$$

4) <u>Scale Invariance:-</u>

Define u= ax

$$\pi_u = \frac{E[ue^{hu}]}{E[e^{hu}]} = \frac{E[ax\ e^{hax}]}{E[e^{hax}]} \quad (the\ equality\ holds\ when, a = 1)$$

$$a = \frac{E(xe^{hax})}{E(e^{hax})}$$

$$= \pi_u \neq a\pi_x \qquad (a \neq 1)$$

5) <u>No Rip off Property:-</u>

$x \leq x_m$

$E[\tilde{X}] \leq x_m \ \ or\, \pi_x \leq x_m \leftarrow To\, proove$

$x \leq x_m$

$xe^{hx} \leq x_m\, e^{hx}$

$E[xe^{hx}] \leq x_m E[e^{hx}]$

$= \dfrac{E[xe^{hx}]}{E[e^{hx}]} \leq x_m \qquad [e^{hx} \geq 0 = E[e^{hx}] \geq 0]$

$\pi_x \leq x_m$

7. **<u>The Risk- Adjusted Principle:</u>**

Let X be a non negative r.v. with distribution function F. The risk adjusted premium Principle sets:

$$\pi_x = \int_0^\infty [P(X > x)]^{1/f}\, dx = \int_0^\infty [1 - F(x)]^{1/f}\, dx$$

Where, $f \geq 1$ is known as Risk Index.

The basis of this is very similar to that of the Esscher principle is based on a transform called Esscher transform, such that this Esscher transform weight the $dist^n$ of X giving increasing weight to right tail probabilities.

Similarly, the risk adjusted premium is also based on transform defined as follows-

Define the distribution function H of a non-negative r.v. x* by

$1 - H_{x^*}(x) = [1 - F(x)]^{1/f}$

When $f \geq 1$, is the risk- index.

Now, we are actually transforming

$X \to X^*$

By transforming F(.) → H()

Now, by definition of expectation

$$E[X^*] = \int_0^\infty [1 - H(x)]\, dx ------------(1)$$

$$= \int_0^\infty [1 - F(x)]^{1/f}\, dx ------------- (2)$$

$$\pi_x -----------------(3)$$

Quest:- Suppose an incurs claim by exponential (λ) calculate 1) The risk adjusted transform of the dist^n of the claims, 2) The risk adjusted premium π_x.

Sol:- 1) $X \sim \exp(\lambda)$ $\quad F(x) = 1 - e^{-\lambda x}$

$$1 - H(x) = [1 - F(x)]^{\frac{1}{f}}$$

$$= \left[e^{-\lambda x}\right]^{1/f}$$

$$H(x) = 1 - e^{-\lambda x/f}$$

$$= X^* \sim \exp(\lambda/f)$$

2) $E(x^*) = f/\lambda$

$$\pi_x = f/\lambda$$

Quest:- Suppose that an insurer incurs claims according to pareto distribution with parameter α & λ.

1) Find the Risk adjusted transform of $dist^n$ of claims.
2) The Risk adjusted premium π_x.
3) Do you have any condition under which premium will lead do a valid answer. If yes, mention the condition.

Sol:- 1) $F(x) = 1 - \left(\frac{\lambda}{\lambda+x}\right)^{\alpha}$

$$1 - H(x) = [1 - F(x)]^{\frac{1}{f}}$$

$$= \left(\frac{\lambda}{\lambda + x}\right)^{\alpha/f}$$

$$H(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha/f}$$

$$X^* \sim Pareto\left(\frac{\alpha}{f}, \lambda\right)$$

2) $E(x^*) = \int_0^{\infty} \left(\frac{\lambda}{\lambda+x}\right)^{\frac{\alpha}{f}} dx, \;\; \lambda + x = y = dx = dy$

$$= \lambda^{x/f} \int_0^{\infty} (\lambda + x)^{-\alpha/f} \text{ dx}$$

$$= \lambda^{\alpha/f} \int_{\lambda}^{\infty} y^{-\alpha/f} \; dy$$

$$= \lambda^{\alpha/f} \left[\frac{y^{-\frac{\alpha}{f}+1}}{-\frac{\alpha}{f} + 1}\right]_{\lambda}^{\infty}$$

$$= \lambda^{\alpha/f} \left[\frac{-\lambda^{-\frac{\alpha}{f}+1}}{1 - \alpha/f}\right] = \frac{-\lambda^{-\frac{\alpha}{f}} \;\; \lambda^{-\frac{\alpha}{f}}.\lambda}{1 - \alpha/f}$$

$$= \frac{-\lambda}{1-\alpha/f} = \frac{-f\lambda}{f-\alpha} = \frac{f\lambda}{\alpha - f}$$

For π_x to be positive, $f < \alpha$ (provided) $\leftarrow$ condition

General Risk Adjusted Principle

In particular, if the r.v. X is a continuous r.v. with c.d.f. $F_x(x)$, then the transformed r.v. X^* is also a continuous r.v. with c.d.f. H_X^* (y) & pdf

$$h_{x^*}(y) = \frac{1}{f}[1 - F_x(y)]^{\frac{1}{f}-1} * f_x(y) ------(*)$$

This means that just like Esscher transform, the density function of r.v. x* is simply a weight version of the density function of x. The corresponding weights are

$$w(x) = \frac{1}{f}[1 - F(x)]^{\frac{1}{f}-2}$$

Now, we check w(x)as a f^n of X.

$$w'(x) = \frac{1}{f}\left(\frac{1}{f} - 1\right)[1 - F(x)]^{\frac{1}{f}-2}(-f_x(x))(f > 1)$$

∴W(x) is an increasing function of x.

Suppose that is pareto example, α =2, λ=1 & f= 1.5, depict the densites function f & h on the same graph & check whether the statement w(x) is an increasing function of x is justified.

If X~ Pareto (α, λ)

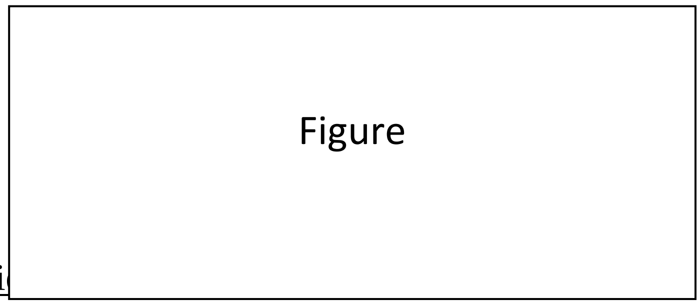
$$f(x) = \frac{(\alpha\lambda)}{(\lambda + x)\alpha + 1} \ (for\ \alpha = 2, \lambda = 1\ \& f = 1.5)$$

$$f(x) = \frac{4}{(1+x)^3} = 4(1+x)^{-3}$$

and we see that x* ~ Pareto $\left(\frac{\alpha}{f}, \lambda\right)$

$$f_x(x) = \frac{\left(\frac{\alpha}{f} . \lambda\right)^{\alpha/f}}{(\lambda + x)^{\frac{\alpha}{f}+1}}$$

$$\frac{1.467}{(1+x)^{2.33}} = 1.467(1+x)^{-2.33}$$



<u>To check which properti</u>

1- $\pi_x \geq E[x]$

We know that

$$E[x] = \int_0^\infty \left(1 - F(x)\right) dx - - - - - - - - -(1)$$

$$= \left(1 - F(x)\right).x]_0^\infty - \int_0^\infty -f(x).x\, dx$$

$$= (0 - 0) + \int_0^\infty f(x)\, x\, dx$$

$$= \int_0^\infty f(x)\, x\, dx$$

Now,

$$\int_0^\infty \left(1 - H(x)\right) dx = \int_0^\infty [1 - F(x)]^{1/\delta}\, dx - - - - - (2)$$

When δ =1

$$\pi_x = \int_0^\infty \left(1 - F(x)\right) dx \; (\delta \geq 1)$$

$$= E[x]$$

$$[1 - F(x)]^{1/\delta} \geq \left(1 - F(x)\right) \qquad \left(from\ (1)\&\ (2)\right)$$

$$\int_0^\infty [1 - F(x)]^{1/\delta} \geq \int_0^\infty \left(1 - F(x)\right) dx$$

$$\pi_x \geq E[x]$$

2- <u>Scale Invariance</u>

Y=ax

$$\pi_y = \int_0^\infty [1 - F(y)]^{1/\delta}\, dy$$

$$= \int_0^\infty [1 - F(ax)]^{1/\delta}\, dx$$

$$= \int_0^\infty [P\ (Y < y)]^{1/\delta}\, dy$$

$$= \int_0^\infty [P\ (ax < y)]^{1/P}\, dy$$

$$= \int_0^\infty [P\ (x < y/a)]^{1/\delta}\, dy$$

$$= \int_0^\infty [1 - F_x\ (Y/a)]^{1/\delta}\, dy$$

$$F(y)$$
$$= P\ (Y < y)$$
$$= P\ (ax < y)$$
$$= P\ (x < y/a)$$
$$= F_x\ (Y/a)$$

$$Let\ \frac{y}{a} = x \quad = dy = adx$$

$$\pi_y = \int_0^{\infty} [1 - F_x\,(x)]^{1/\delta}\; a\;\; dx$$

$$= a\,\pi_x.$$

3- <u>Consistency</u>

Y=x+c

$$\pi_y = \int_0^{\infty} [1 - F(y)]^{1/\delta}\; dy$$

$$= \int_0^{\infty} [P\;(Y > y)]^{1/\delta}\; dy$$

$$= \int_0^{\infty} \big(P\;(x + c > y)\big)^{1/\delta}\; dy$$

$$= \int_0^{\infty} [P\;(x > y - c)]^{1/\delta}\; dy$$

$$y - c = x$$

$$dy = dx$$

$$\pi_y = \int_{-c}^{\infty} \big(P - (x > x)\big)^{1/\delta}\;\; dx$$

$$= P(Y > x) = \begin{cases} 1, & x > c \\ 1 - F_x(y - c), & x \geq c \end{cases}$$

$$\pi_y = \int_0^{c} (1)^{1/\delta}\; dx + \int_c^{\infty} \big(1 - F_x(y - c)\big)\;\; dy \qquad \begin{cases} y - c = x \\ dy = dx \end{cases}$$

$$\pi_y = c + \int_c^{\infty} \big(1 - F_x(x)\big)^{1/\delta}\;\; dx$$

$$\pi_y = c + \pi_x$$

From (*)

$$\int_{-c}^{0} (P.(X > x))^{1/\delta} dx + \int_{0}^{\infty} (P(X > x))^{1/\delta} dx$$

$we\ know\ that$

$P(X > x) = 1 \qquad for \quad x < c$

$1 - F_x(x) = 1\,, \qquad\qquad x < c$

$$= \pi_y = \int_{-c}^{0} 1 \quad dx + \pi_x$$

$\pi_y = C + \pi_x$

4- <u>No Rip –off Property</u>

If $x = x_m$

Then $\pi_y = x_m ----To\ prove$

Where x_m is the maximum value that X can assume.

$$\pi_y = \int_{0}^{\infty} [1 - F(ax)]^{\frac{1}{\delta}} dx$$

$x \geq x_m$

$E(x) \leq x_m$

$$= \int_{0}^{\infty} (1 - F_x(x))\, d_x \leq x_m = \int_{0}^{x_m} (1 - F_x(x))\, dx$$

$$\pi_x = \int_{0}^{x_m} (1 - F_x(x))^{1/\delta} dx ----(1)$$

Further,

$1 - F_x(x) \leq 1$

$$\pi_x = \int_{0}^{x_m} (1 - F_x(x))^{1/\delta} dx \leq \int_{0}^{x_m} 1 \ dx$$

$\pi_x = x_m$

Additive Property

RAP does not satisfy the additive property & to show this we take a simple counter example:

Let x_1 & x_2 be two identical discrete r.v. s.t.

$P(x_1 = 0) = P\,(x_1 = 1) = 0.5$

Let the risk index be $\delta = 2$.

Calculate π_{x_1}, π_{x_2} and $\pi_{x_1+x_2}$

$\pi_{x_1} = E[x_1^*]$

$1 - H(x) = \left(1 - F(x)\right)^{1/\delta}$

$$E[x] = \sum_{x=0}^{1} \left(1 - F(x)\right)^{\frac{1}{\delta}}$$

$= \left(1 - F(0)\right)^{1/\delta} + \left(1 - F(1)\right)^{1/\delta}$

$= (1 - 0.5)^{1/2}$

$= (0.5)^{1/2} = 0.707$

Similarly,

$\pi_{x_2} = (0.5)^{1/2} = 0.707$

$$P(Y \le y) \;=\; \begin{cases} 0.25, & y < 0 \\ 0.75, & 0 \le y < 1 \\ 1 & y < 1 \end{cases}$$

$y = x_1 + x_2 \rightarrow 0{,}1{,}2.$

$$\pi_{x_1+x_2} = 0.5\left(1+3^{1/2}\right) = 1.366.$$

$$\pi_{x_1+x_2} = E(X) = \sum_{x=0}^{2}\left(1-F(x)\right)^{1/\delta} dx.$$

$$= \pi_{x_1} + \pi_{x_2} > \pi_{x_1+x_2}$$

<u>Life Annuities</u>

Introduction:-

A life annuity is a series of payment made continuously or at equal intervals (such as month, quarters, years) while a given life survives. It may be temporary that is, limited to a given term of years, or it may be payable for the whole of life. The payment intervals may commence immediately or, alternatively the annuity may be deferred. Payments may be the annuity may be deferred. Payment intervals due at the beginnings of the payment intervals (annuities-due) or at the ends of such intervals (annuities- immediate).

Here, we study the payment contingent on survival as provided by various forms of life annuities. Life annuity theory is a analogues but brings in survival as a condition for payment. Life annuities pay a major role in life insurance operations. Life insurances are usually purchased by a life annuity of premiums rather than by a single premium. The amount payable at the time of claim may be converted through a settlement option into some form of life annuity for the beneficiary. Some types of life insurance carry this concept even further and instead of featuring a lump sum payable on death, provided stated form of income benefits. Thus for example, there may be a monthly income payable to a surviving spouse or to retived insurance.

Annuities are even more central in pension systems. Infact, a retirement plan can be regarded as a system for purchasing deferred life annuities (payable during retirement by some form of temporary annuity of contributions during active service. The temporary annuity may consist of varying contributions, and valuation of it may take into account not only interest and mortality but other factors such as salary increases and the termination of participation for reasons other than deaths.

Life annuities also have a role in disability and workers compensation insurances. In the case of disability insurance, termination of the annuity benefit by reason of recovery of the disabled insured may need to be considered. For surviving spouse benefits terminate the annuity. The method of obtaining actuarial present value called the current payment technique.

As in the preceding chapter on life insurance unless otherwise state we assume a constant effective annual rate of interest i (or the equivalent constant force of interest &). We also assume aggregate mortality for most of the development and indicate those situations where a select mortality assumption makes a major difference. The actuarial present value symbol. used above introduced below. Expression (6) is an example of the backward recursion formula. Here u(x) = $\bar{a}_x$, $C(x) = \bar{a}_{x:1\rceil}$ and d(x) = vpx. The initial value to use for the whole life annuity is $\bar{a}_w = 0$. There are several ways to evaluate the C(x) term. A simple approach is to use a trapezoid approximation for the integrach

$$\bar{a}_{x:1\rceil} = \int_0^1 v^t t^{p_x}\, dt = \frac{1 + vp_x}{2}.$$

Another approach, based on the assumption of a uniform distribution of deaths within each year of age is examined later.

A relationship familiar from compound interest theory is that

$$L = \delta\, \bar{a}_{t\rceil} + v^t$$

This can be interpreted as indication that a unit invested now will produce annual interest of & invested now will produce annual interest of & payable continuously for t years at which point interest ceases and the investment is repaid. This relationship holds for all values of t this is true for the random variable T:

$$L \pm \delta\, \bar{a}_{T\rceil} + v^t ----------------- (7)$$

Then we taking expectation, we obtain

$$1 = \delta\, \bar{a}_x + \bar{A}_x ----------------- (8)$$

This is subject to the same kind of interpretation as above.

A unit invested now will produce annual interest & payable continuously for as long as (x) survives at the time of death, interest ceases and the investment of 1 is repaid.

To measure on the basis of the assumption in our model the mortality risk in a continuous life annuity we are interest in $Var\left(\bar{a}_{T\rceil}\right)$. We determine

$$Var\left(\bar{a}_{T\rceil}\right) = Var\left(\frac{1-v^T}{\delta}\right) = \frac{Var(v^T)}{\delta^2}$$

$$= \frac{2_{\bar{A}_x} - (\bar{A}_x)^2}{\delta^2} - - - - - - - - - (9)$$

Further, we can observe that since $= \delta\bar{a}_{T\rceil} + v^T, Var\left(\delta\bar{a}_{T\rceil} + v^T\right) = 0.$. thus is no mortality risk for the combination fo a continuous life annuity of δ per year and a life insurance of L payable at the moment of death.

We now turn to temporary and deferred life annuities. The present value of a benefit random variable for an n-year temporary life annuity of 1 per year payable continuously while (x) survives during the next n year is.

$$Y = \begin{cases} \bar{a}_{T\rceil} & 0 \le T < n \\ \bar{a}_{n\rceil} & T \ge n \end{cases} - - - - - - -(10)$$

The distribution of Y in this case is a mixed distribution. In particular the maximum value of Y is limited to $\bar{a}_{n\rceil}$ and there is a positive probability associated with $\bar{a}_{n\rceil}$ of $P\ (T \ge n) = npx.$

The actuarial present value of an n-year temporary life annuity is denoted by $\bar{a}_{x:n\rceil}$ and equals

$$\bar{a}_{x:n\rceil} = E[Y] = \int_0^n \bar{a}_{t\rceil} tp_x\ \mu(x+t)dt + \bar{a}_{n\rceil}\ npx.$$

Integrating by parts gives.

$$\bar{a}_{x:\overline{n}|} = \int_0^n u^t \, t^P{}_x \, dt. - - - - - - - (12)$$

This is the current payment integral for the actuarial present value for the n-year temporary annuity. It can be considered as involving a momentary payment 1 dt made at time t, discounted at interest back to time 0 by multiplying by v^t and further multiplied by $t^p{}_x$ to reflect the probability that a payment is made at time t for times up to time n. No payments are to be made after time n so the probability of such payments is 0.

The same recursion formula as indicated for (6) applies here with u(x) = $\bar{a}_{x:\overline{y-x}|}$ and the same c (x) function which we now recognize as $\bar{a}_{x:\overline{1}|}$. We use here n= y-x. The only thing that needs to be changed is the initial value for which we use u (y) $\bar{a}_{y:\overline{0}|}$ = 0.

Returning to (10), we note that

$$Y = \begin{cases} \bar{a}_{\overline{T}|} = \dfrac{1-z}{\delta} & 0 \le T < n \\ \bar{a}_{\overline{n}|} = \dfrac{1-z}{\delta} & T \ge n \end{cases} - - - - - - -(13)$$

Where,

$$Z = \begin{cases} v^T & 0 \le T < n \\ v^n & T \ge n. \end{cases} - - - - - - (14)$$

In (14), Z is the present value random variable for an n-year endowment insurance. Here

$$E[Y] = \bar{a}_{x:\overline{n}|} = E\left[\frac{1-z}{\delta}\right] = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta^2} - - - - - - (15)$$

and

$$Var\,[Y] = \frac{Var(z)}{\delta^2} = \frac{2_{\bar{A}_{x:n]}} - \bar{A}_{x:n]}{}^2}{\delta^2} ------(16)$$

In terms of annuity values (16) becomes

$$Var\,(Y) = \frac{1 - 2\delta^2 \bar{a}_{x:n]} - \left(1 - \delta \bar{a}_{x:n]}\right)^2}{\delta^2}$$

$$= \frac{2}{\delta}\left(\bar{a}_{x:n]} - 2_{\bar{A}_{x:n]}}\right) - \left(1 - \delta \bar{a}_{x:n]}\right)^2.$$

The analysis for an n-year deferred whole life annuity is similar. The present value random variable Y is defined as

$$Y = \begin{cases} 0 \quad\quad\; = \; \bar{a}_{T]} - \bar{a}_{T]} & 0 \le T < n \\ v^n \, \bar{a}_{T-n]} = \bar{a}_{T]} - \bar{a}_{n]} & T \ge n \end{cases} -------(17)$$

Here the random variable Y can take on a value no larger than $(L/\delta) - \bar{a}_{n]} = v^n/\delta$, and the probability that it takes on a zero value is P(T≤n) = $n^q{}_x$. A typical distribution function is illustrated in the figure.

Figure

Then,

$$n|\bar{a}_x = E[Y] = \int_n^\infty v^n \bar{a}_{\overline{1-n}]}\; {}_t p_x \mu(x+t)\, dt$$

$$= \int_0^\infty v^n \bar{a}_{\overline{s}|} n + sp_x \, \mu(x+n+s)\, ds$$

$$= v^n \, np_x \int_0^\infty \bar{a}_{\overline{s}|} sp_x + n \; \mu(x+n+s)\, ds$$

Which shows that

$n|\bar{a}_x = nE_x \, \bar{a}_{x+n}. - - - - - -(18)$

An alternative development would be to note that, from the definition of Y,

(Y for an n-year deferred whole life annuity)

= (Y for a whole life annuity)- (Y for an n-year temporary life annuity).

Taking expectation gives

$n|\bar{a}_x = \bar{a}_x - \bar{a}_{x:\overline{n}|} - - - - - - -(19)$

Integration by parts can be employed to verify the result gives by the current payment technique. Since the annuity will be paying after time n is x survives, the actuarial present values can be written as

$$n|\bar{a}_x = \int_n^\infty v^t \; {}_t p_x \, dt = \int_n^\infty {}_t E_x \, dt. - - - - - - - (20)$$

To develop the backward recursion formula for deferred annuities with n=y-x>1, we note that we have no term corresponding to the integral for t values between 0 and 1. Thus for $\mu(x) = yx^{\bar{a}_x}$ at ages less than y, c(x) =0, and d (x) = vp_x. For a starting value we would use $\mu(x) = \bar{a}_y$.

One way to calculate the variance of Y for the deferred annuity is the following:

$$Var(Y) = \int_n^{\infty} v^{2n} \left(\bar{a}_{\overline{t-n}|}\right)^2 {}_tp_x \mu(x+t)dt - (n|\bar{a}_x)^2$$

$$= v^{2n} \, {}_np_x \int_0^{\infty} \left(\bar{a}_{\overline{s}|}\right)^2 {}_sp_{x+n} \, \mu(x+n+s)ds - (n|\bar{a}_x)^2$$

and using integration by parts,

$$= v^{2n} \, {}_np_x \int_0^{\infty} 2 \, \bar{a}_{\overline{s}|} \, v^s \, {}_sp_{x+n} \, ds - (n|\bar{a}_x)^2$$

$$= \frac{2}{\delta} v^{2n} \, {}_np_x \int_0^{\infty} (v^s - v^{2s}) \, {}_sp_{x+n} \, ds - (n|\bar{a}_x)^2$$

$$= \frac{2}{\delta} v^{2n} \, {}_np_x (\bar{a}_{x+n} - \bar{a}_{x+n}) - (n|\bar{a}_x)^2 ----(21)$$

We now turn to analysis of an n-year certain and life annuity. This is a whole life annuity with a guarantee of payment for the first n years. The present value of annuity payment is

$$Y = \begin{cases} \bar{a}_{\overline{T}|} & T \leq n \\ \bar{a}_{\overline{T}|} & T > n \end{cases} -------(22)$$

A Typical distribution function is shown in fig.

Figure

Which reflects the mixed nature of the distribution and the minimum value and upper bound of Y. which are $\bar{a}_{n]}$ and $1/\delta$ respectively.

The actuarial present value is denoted by $\bar{a}_{\overline{x:n}]}$. This sym and is adopted to indicate that payment continue until max [T(x),n];

$$\bar{a}_{\overline{x:n}]} = E[Y] = \int_{0}^{n} \bar{a}_{\bar{n}]} t^{p}{}_{x}\, \mu(x+t)\, dt + \int_{n}^{\infty} \bar{a}_{\bar{t}]} t^{p}{}_{x}\, \mu(x+t)\, dt$$

$$= n^{q}{}_{x}\, \bar{a}_{\bar{n}]} + \int_{0}^{\infty} \bar{a}_{\bar{t}]} t^{p}{}_{x} \quad \mu(x+t)\, dt ----(23)$$

Integration by parts can be used to obtain

$$\bar{a}_{\overline{x:n}]} = \bar{a}_{\bar{n}]} + \int_{0}^{\infty} v^{t}\, t^{p}{}_{x}\, dt ------(24)$$

This is the current payment form for the actuarial present value, since at times 0 to n payments is certain whereas for times greater than n payment is made if (x) survives.

Further insight can be obtained by rewriting Y as

$$Y = \begin{cases} \bar{a}_{n]} + 0 & T \leq n \\ \bar{a}_{n]} + \left(\bar{a}_{T]} - \bar{a}_{n]}\right) & T > n \end{cases}$$

Here Y is the sum of a constant $\bar{a}_{n]}$ and the random variable for the n-year deferred annuity. Thus,

$$\bar{a}_{\overline{x:n}]} = \bar{a}_{\bar{n}]} + n|^{\bar{a}_{x}}$$

$$= \bar{a}_{\bar{n}]} + n^{E}{}_{x}\, \bar{a}_{x+n} ----- by\ (10)$$

$$n|\bar{a}_{x} = \bar{a}_{x} - \bar{a}_{x:n]} ------by\ (19)$$

Further more , Since $var\left(y - \bar{a}_{\bar{n}|}\right) - \; var\,(y)$ the variance for the n year certain life is the same as that of the n-year deferred annuity given by (21)

Analogous to the function

$$\bar{S}_{\bar{n}|} = \int_0^n (1+i)^{n-t}\, dt$$

In the theory of interest, we have for life annuities

$$\bar{S}_{x:\bar{n}|} = \frac{\bar{a}_{x:\bar{n}|}}{n^{E}{}_{x}} = \int_0^n \frac{1}{n - t^{E}{}_{x+t}}\, dt - - - - - - - - - - - (26)$$

Repressing the actuarial accumulated value at the end of the term of an n-year temporary life annuity of 1 year payable continuously which (x) survives. Such accumulated value, which is often said to have been accumulated under (or with the benefit of) interest and survivorship, is available at age x+n if (x) survives.

We obtain an expression for $d\,\bar{a}_x/dx$ by differentiating the integral in (4), assuming that the probabilities are derived from an aggregate table

$$\frac{d}{dx}\bar{a}_x = \int_0^\infty v^t \left(\frac{\partial}{\partial x} t^{p}{}_{x}\right) dt = \int_0^\infty v^t\, t^{p}{}_{x}\, [\mu(x) - \mu(x+t)]dt$$

$$= \mu(x)\bar{a}_x - \bar{A}_x = \mu(x)\bar{a}_x - (1 - \delta\, \bar{a}_x).$$

$$\therefore \frac{d}{dx}\, \bar{a}_x = [\mu(x) + \delta]\, \bar{a}_x - 1 - - - - - - - (27)$$

The interpretation is that the actuarial present value changes at a rate that is the sum of the rate of interest income $\delta\, \bar{a}_x$ and the rate of survivorship benefit $(x)\; \bar{a}_x$, less the rate of payment outgo.

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