U.P. Rajarshi Tandon Open University, Prayagraj

SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

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SBSSTAT – 04 **Numerical Methods & Basic Computer Knowledge**
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Blocks & Units Introduction

The present SLM on ***Numerical Methods and Basic Computer Knowledge*** consists of seventeen units with six blocks. Numerical analysis is a branch of mathematics that deals with devising efficient methods for obtaining numerical solutions to difficult Mathematical Problems. Most of the Mathematical problems that arise in science and engineering and applied engineering mathematics are very hard and sometime impossible to solve exactly. Thus, an approximation to a difficult Mathematical problem is very important to make it easier to solve. Due to immense development in the computational technology, numerical approximation has become more popular and modern tool for the research of all subjects of mathematical sciences.

The ***Block - 1 – Solutions of Non- Linear Equations in One Variable,*** is the first block, which is divided into two units; named ***Unit – 1 – Basic Properties of Equations***, and ***Unit – 2 – Solutions of Non-Linear Equations.***

Numerical technique is widely used by scientists and engineers to solve their problems. A major advantage for numerical technique is that a numerical answer can be obtained even when a problem has no analytical solution. However, result from numerical analysis is an approximation, in general, which can be made as accurate as desired. The reliability of the numerical result will depend on an error estimate or bound, therefore the analysis of error and the sources of error in numerical methods is also a critically important part of the study of numerical technique.

The ***Block - 2 – Finite Differences is*** the second block with four units. This block consists of four units regarding, finite differences interpolation with equal intervals, interpolation with unequal and Lagrange’s Interpolation.

In ***Unit – 3 - Finite Differences***; various operators used in finite difference calculus are discussed. The concept of interpolation with equal intervals is given this unit.

Unit - 4 – Interpolation with Equal Intervals; in which Newton’s forward and backward interpolation formulate are discussed.

Unit – 5 - Interpolation with Un-Equal Intervals; is devoted to interpolation with unequal intervals. Newton’s general interpolation formula, divided differences and their properties are discussed in this unit.

Finally in ***Unit – 6 – Lagrange’s Interpolation***; the last unit of this block, Lagrange’s interpolation formula and is applications are given.

The ***Block - 3 – Central Differences,*** is the third block. This block consists of four units regarding, central differences, inverse interpolation, numerical differentiation and numerical integration.

Unit - 7 – Central Difference Interpolation Formulae; deals with the concept of central difference interpolation. Gauss and Bessels's formulae are derived and their applications are successes.

In ***Unit – 8 – Inverse Interpolation***; of the block the problem of inverse interpolation is discussed and various methods for its solution are suggested.

In ***Unit – 9 – Numerical Differentiation***; the concept of numerical differentiation has been defined. Various formulae to solve the problem of numerical differentiation are discussed.

Finally, in ***Unit – 10 - Numerical Integration*** is taken into consideration. Trapezoidal rule, Simpson's rule and Weddle's rule are derived. Euler Maclaurin's summation formula is also given in this unit.

The ***Block - 4 – Solution of Differential Equations,*** is the fourth block. This block consists of two units titled ***Unit - 11 – Numerical Solution of ordinary Differential Equations – I*** and ***Unit – 12 – Numerical Solution of ordinary Differential Equations – II***.

The ***Block - 5 – Computer,*** is the fifth block. This block deals with theory of computer and consists of three units.

Unit – 13 – Introduction to Computer; presents a brief introduction to computers including their historical bolution, generation and classification.

Unit – 14 – Hardware; gives a brief account of hardware in CPU, I/O Devices, Blick diagram and memory organization.

Unit – 15 – System Software; deals with system software, MS-Dos, Files names, Creating, Editing and printing of files, other file management commands etc.

The ***Block - 6 – Basics of Computer Programming,*** is the sixth block. This block includes two units regarding to basics of computer programming and programming languages.

In ***Unit – 16 – Algorithm and Flow Charts***; described the said topics and various example related to these techniques are worked out.

In ***Unit – 17 – Programming Language***; elements ideas related to various programming languages rearranges from machine language to object-oriented programming are discussed.

At the end of every block/unit the summary, self assessment questions and further readings are given.

U.P.RajarshiTandon Open University, Prayagraj

SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 1 Solutions of Non-Linear Equations in One Variable

Unit – 1: Basic Properties of Equations

Unit – 2 : Solutions of Non-Linear Equations

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Block & Units Introduction

The ***Block - 1 – Solutions of Non-Linear Equations in one Variable,*** is the first block. As we know Numerical technique is widely used by scientists and engineers to solve their problems. A majoradvantage for numerical technique is that a numerical answer can be obtained even when aproblem has no analytical solution. However, result from numerical analysis is an approximation,in general, which can be made as accurate as desired. The reliability of the numerical result willdepend on an error estimate or bound, therefore the analysis of error and the sources of error innumerical methods is also a critically important part of the study of numerical technique.

Unit – 1 – Basicproperties of equations; theunit deals with the basic concepts of calculus, properties of equations and roots of equation how the initial approximation is taken and convergence is calculated using various iterative methods.

Unit – 2– Solutions of Non-Linear Equations; gives a brief account of the order of convergence is also calculated which clearly indicates the speed by which approximation of the root could be done.

The end of block/unit the summary, self-assessment questions and further readings are given.

Unit-1: Solutions of Non-Linear Equations in one Variable

Structure

1.1 Introduction

Numerical analysis is a branch of Mathematics that deals with devising efficient methods for obtainingnumerical solutions to difficult Mathematical problems.Most of the Mathematical problems that arise in science and engineering are very hard and sometimeimpossible to solve exactly. Thus, an approximation to a difficult Mathematical problem is very important to make it easier to solve. Due to the immense development in the computational technology,numerical approximation has become more popular and a modern tool for scientists and engineers.

1.2 Objective

In this unit, our objective is to review the basic concepts of calculus, properties of equations and roots of equation how the initial approximation is taken and convergence is calculated using bisection method.

1.3 Review of Calculus

MATHEMATICAL PRELIMINARIES

Theorem 1.1 If$f(x)$ is continuous in $a \leq x \leq b$, and if $f(a)$ and $f(b)$ are of opposite signs, then**,** $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

Theorem 1.2*(Rolle's theorem)* If $f(x)$ is continuous in $a \leq x \leq b$, $f'(x)$ exists in $a < x < b$ and $f(a) = f(b) = 0$, then, there exists at least one value$x$, say ξ, such that $f'(\xi) = 0$, $a < \xi < b$.

Theorem 1.3*(Generalized Rolle's theorem)* Let $f(x)$ be a function which is n times differentiable on [a, b]. If $f(x)$ vanishes at the,$(n + 1)$ distinct points,x_0,x_1, ………, x_n, in (a, b) such that $f^n(\xi) = 0$.

Theorem 1.4*(Intermediate value theorem)*If $f(x)$ be continuous in $[a, b]$ and let k be any number between $f(a)$ and $f(b)$. Then there exists a number ξ in (a, b) such that $f(\xi) = k$

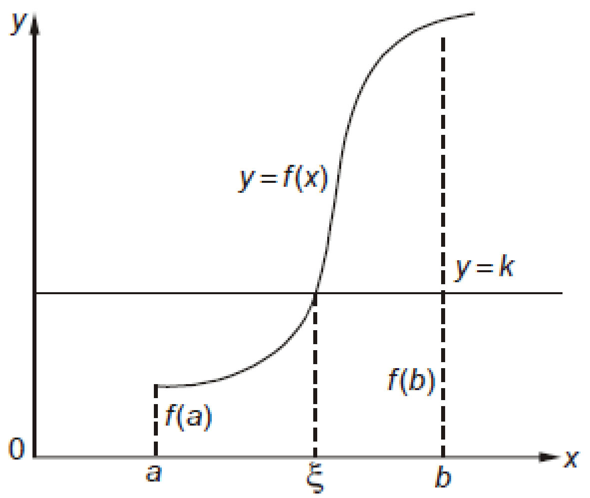


Fig 1 Intermediate value theorem

Theorem 1.5*(Mean-value theorem for derivatives)*If $f(x)$ be continuous in $[a, b]$ and $f'(x)$ exists in (a, b), then there exists at least at least one value of x say ξ, between a and b such that(It is also known as Lagrange's Mean Value Theorem)

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}, \quad a < \xi < b$$

Setting $b = a + h$, this theorem takes the form

$f(a+h) = f(a) + hf'(a+ \theta h)$, $\qquad 0 < \theta < 1.$

Theorem 1.6*(Taylor's series for a function of one variable)*If $f(x)$ is continuous and possesses continuous derivatives of order n in an interval that includes $= a$, then in that interval

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n(x)$$

Where $R_n(x)$, the remainder term, can be expressed in the form

$$R_n(x) = \frac{(x-a)^n}{n!}f^n(\xi), \quad a < \xi < b$$

Theorem 1.7 *(Maclaurin's expansion) It states that*

If $f(x)$ is continuous and possesses derivatives of all order and R_n tends to Zero as n $\rightarrow\infty$ then Maclaurin's theorem becomes the Maclaurin's series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots \frac{x^n}{n!}f^n(0) + \cdots \dots \dots$$

Theorem 1.8 *(Taylor's series for a function of two variables) It states*

$$f(x_1 + \Delta x_1, x_2 + \Delta x_2) = f(x_1, x_2) + \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2$$

$$+\frac{1}{2}[\frac{\partial^2 f}{\partial x^2{}_1}(\Delta x_1)^2 + \frac{\partial^2 f}{\partial x_1 \partial x_2}(\Delta x_1 \Delta x_2) + \frac{\partial^2 f}{\partial x^2{}_2}(\Delta x_2)^2] + \dots\dots.$$

This can easily be generalized.

1.4 Round off Error and Truncation Error

Machine Epsilon

The computer has a finite word length so only a fixed number of digits are stored and used during computation. Hence even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependent and is called machine epsilon.

Error = True value – Approximate value

In any numerical computation, we come across following types of errors:

1. **Inherent errors:** Errors which are already present in the statement of a problem before its solution are called inherent errors. Such errors arise either due to the given data being approximate or due to limitations of mathematical tables, calculators or the digital computer.

Inherent errors can be minimized by taking better data or by using high precision computing aids. Accuracy refers to the number of significant digits in a value e.g. 53.965 is accurate to 5 significant digits.

Precision refers to the number of decimal position or order of magnitude of the last digit in the value e.g. in 53.965, precision in 10^{-3}.

Example: Which of the following numbers have greatest precision?

4.3201, 4.32, 4.320106.

Solution: In 4.3201, precision is 10^{-4}.

In 4.32, precision is 10^{-2}

In 4.320106, precision is 10^{-6}

Hence the number 4.320106 has the greatest precision.

2. **Rounding errors:** They arise from the process of rounding off the numbers during the computation. It is also called procedural error or numerical error. Such errors are unavoidable in most of the calculations due to limitations of computing aids.

   These errors can be reduced however by

   (i) Changing the calculation procedure so as to avoid subtraction of nearly equal numbers or division by a small number

(ii) Retaining at least one more significant digit at each step and rounding off at last step.

Rounding off may be executed in two ways:

(a) **Chopping:** In it, extra digits are dropped by truncation of number. Suppose we are using a computer with a fixed word length of four digits then a number like 12.92364 will be stored as 12.92.

(b) **Symmetric round off:** In it, the last retained significant digit is rounded up by unity if the first discarded digit is ≥ 5 otherwise the last retained digit is unchanged.

3. Truncation errors:

They are caused by using approximate results or on replacing an infinite process by a finite one.

If we are using a decimal computer having a fixed word length of 4 digits, rounding off of 13.658 gives 13.66 whereas truncation gives 13.65.

Note: Truncation error is the difference between a truncated value and the actual value. A truncated quantity is represented by a numeral with a fixed number of allowed digits, with any excess digits "chopped off" (hence the expression "truncated").

As an example of truncation error, consider the speed of light in a vacuum. The official value is 299,792,458 meters per second. In scientific (power-of-10) notation, that quantity is expressed as 2.99792458×10^8. Truncating it to two decimal places yields 2.99×10^8. The truncation error is the difference between the actual value and the truncated value, or 0.00792458×10^8. Expressed properly in scientific notation, it is 7.92458×10^5.

In computing applications, truncation error is the discrepancy that arises from executing a finite number of steps to approximate an infinite process.

Example: Find the truncation error for $e^x at x = \frac{1}{5}$ if

First three terms are retained in expansion

Solution: Error = True value – Approximate value

$$= (1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots) - (1 + x + \frac{x^2}{2!})$$

$$= (\frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots)$$

Put $x = \frac{1}{5}$

$$error = \frac{.008}{6} + \frac{.0016}{24} + \frac{.0032}{120} + \ldots.$$

$$= .0013333 + .0000666 + .0000026 + \cdots \ldots. = .\mathbf{0014025}$$

Check Your Progress

1. Round off the following numbers to two decimal places

   48.21416 and 2.3742

1.5 Some Properties of Equations

Algebraic and Transcendental Equation

Introduction

We have seen that expression of the form

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots \ldots + a_{n-1}x + a_n$$

where $a_i's$ are constant ($a_0 \neq 0$) and n is a positive integer, is called a polynomial in x of degree n, and the equation $f(x) = 0$ is called an algebraic equation of degree n. If $f(x)$ contains some other functions like exponential, trigonometric, logarithmic etc., then $f(x) = 0$ is called a transcendental equation.

For example,

$x^3 - 3x + 6 = 0$, $x^5 - 7x^4 + 3x^2 + 36x - 7 = 0$

are algebraic equations of third and fifth degree, whereas

$x^2 - 3cosx + 1 = 0$, $xe^x - 2 = 0$, $x \log_{10} x = 1.2$etc are transcendental equations. In both the cases, if the coefficients are pure numbers, they are called numerical equations. Here we shall describe some numerical methods for the solution of $f(x) = 0$ where $f(x)$ is algebraic or transcendental or both.

Check Your Progress

1. Give two examples each of algebraic and transcendental equation. What is the difference between the two explain with suitable example.

1.6 Iteration Methods for Finding the Roots (Zero's) of an Equation

Methods for Finding the Root of an Equation

Method for finding the root of an equation can be classified into following two parts:

1. Direct methods
2. Iterative methods

1. Direct methods

In some cases, roots can be found by using direct analytical methods. For example, for a quadratic equation $ax^2 + bx + c = 0$the roots of the equation, obtained by

$$x_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

These are called closed form solution. Similar formulae are also available for cubic and biquadratic polynomial equations but we rarely remember them. For higher order polynomial equations and non-polynomial equations, it is difficult and in many cases impossible, to get closed form solutions. Besides this, when numbers are substituted in available closed form solutions, rounding errors reduce their accuracy.

2. Iterative Methods

These methods, also known as ***trial***and ***error*** methods, are based on the idea of successive approximations, *i.e.*, starting with one or more initial approximations to the value of the root, we obtain the sequence of approximations by repeating a fixed sequence of steps over and over again till we get the solution with reasonable accuracy. These methods generally give only one root at a time.

For the human problem solver, these methods are very cumbersome and time consuming, but on other hand, more natural for use on computers, due to the following reasons:

1. These methods can be concisely expressed as computational algorithms.
2. It is possible to formulate algorithms which can handle class of similar problems. For example, algorithms to solve polynomial equations of degree *n* may be written.
3. Rounding errors are negligible as compared to methods based on closed form solutions.

Order (Or Rate) Of Convergence Of Iterative Methods

Convergence of an iterative method is judged by the order at which the error between successive approximations to the root decreases.

The order of convergence of an iterative method is said to be kth order convergent if *k* is the largest positive real number such that

$$\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$$

Where *A*, is a non-zero finite number called asymptotic error constant and it depends on derivative of *f*(*x*) at an approximate root *x*. e_i and e_{i+1} are the errors in successive approximation.

In other words, the error in any step is proportional to the *k*th power of the error in the previous step. Physically, the *k*th order of convergence means that in each iteration, the number of significant digits in each approximation increases *k* times.

Check Your Progress

1. What does iteration mean and how iterative methods converge after every step.

1.7 Bisection Method

Bisection (Or Bolzano) Method

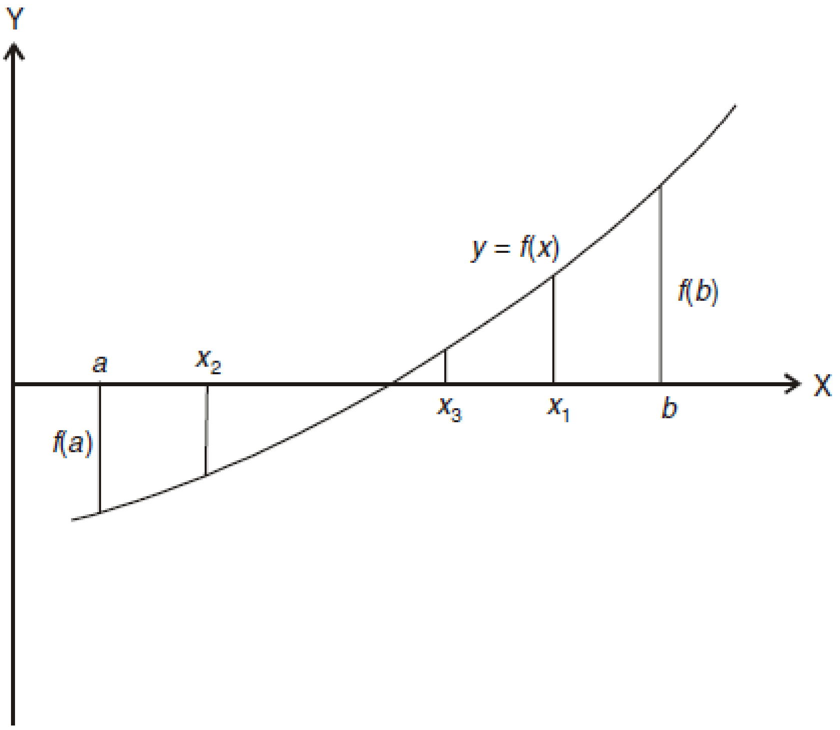
This is one of the simplest iterative methods and is strongly based on the property of intervals. To find a root using this method, let the function *f*(*x*) be continuous between *a* and *b*. For definiteness, let *f*(*a*) be negative and *f*(*b*) be positive.

Then there is a root of *f*(*x*) = 0, lying between *a* and *b*.

Let the first approximation be

$$x_1 = \frac{1}{2}(a+b) \text{ (i.e., average of the ends of the range).}$$

Now of $f(x_1) = 0$ then x_1 is a root of *f*(*x*) = 0. Otherwise, the root will lie between *a* and x_1 or x_1 and *b* depending upon whether $f(x_1)$ is positive or negative.



Then, we bisect the interval and continue the process till the root is found to be desired accuracy. In the above figure, $f(x_1)$ is positive; therefore, the root lies in between a and x_1.

The second approximation to the root now is $x_2 = \frac{1}{2}(a + x_1)$.

If $f(x_2)$is negative as shown in the figure then the root lies in between $x_2$and x_1, and the third approximation to the root is $x_3 = \frac{1}{2}(x_2 + x_1)$and so on.

This method is simple but slowly convergent. It is also called as Bolzano method or Interval halving method.

Procedure for the Bisection Method to Find the Root of The Equation $f(x) = 0$

Step 1 : Choose two initial guess values (approximation) *a* and *b* (where (*a* >*b*))

such that $f(a).f(b) < 0$

Step 2 : Evaluate the mid point x_1 of a and b given by $x_1 = \frac{1}{2}(a + b)$ and also

evaluate $f(x_1)$.

Step 3 : If $f(a).f(x_1) < 0$, then set b = x_1 else set $a = x_1$. Then apply the formula of step 2.

Step 4 : Stop evaluation when the difference of two successive values of x_1 obtained from step 2, is numerically less than the prescribed accuracy.

Order of Convergence of Bisection Method

In Bisection Method, the original interval is divided into half interval in each iteration. If we take mid points of successive intervals to be the approximations of the root, one half of the current interval is the upper bound to the error. Now $f(x_i) = 0$, then x_1 is the root of $f(x)$

In Bisection Method, e_{i+1} = 0.5 e_i or $\frac{e_{i+1}}{e_i^k} = 0.5$

Here e_i and e_{i+1} are the errors i^{th} and $(i+1)^{th}$ iterations respectively. Comparing the above equation with

$$\lim_{i\to\infty}\left|\frac{e_{i+1}}{e_i^k}\right| \le A$$

We get $k = 1$ and A = 0.5. Thus the Bisection Method is first order convergent or linearly convergent.

Example 1: Find the root of the equation $x^3 - x - 1 = 0$ lying between 1 and 2 by bisection method.

Solution Let $f(x) = x^3 - x - 1 = 0$

Since $f(1) = 1^3 - 1 - 1 = -1$, which is negative

A nd$f(2) = 2^3 - 2 - 1 = 5$, which is positive

Therefore, $f(1)$ is negative and $f(2)$ is positive, so at least one real root will lie between 1 and 2.

First Iteration: Now using Bisection Method, we can take first approximation

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

Then $, f(1.5) = (1.5)^3 - 1.5 - 1$

$= 3.375 - 1.5 - 1$ = 0.875

$\therefore f(1.5) > 0$ that is, positive

So root will now lie between 1 and 1.5.

Second Iteration: The Second approximation is given by

$$x_2 = \frac{1 + 1.5}{2} = \frac{2.5}{2} = 1.25$$

Then, $f(1.25) = (1.25)^3 - 1.25 - 1$

$= 1.953 - 2.25$ = - 0.875 < 0

$\therefore f(1.25) < 0$ that is, negative

Therefore, $f(1.5)$ is positive and $f(1.25)$ is negative, so that root will lie between 1.25 and 1.5.

Third Iteration: The third approximation is given by

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$$x_3 = 1.375$$

Then, $f(1.375) = (1.375)^3 - 1.375 - 1$

$$f(1.375) = 0.2246$$

$\therefore f(1.375) > 0$ is, positive

$\therefore$ The required root lies between 1.25 and 1.375

Fourth Iteration: The fourth approximation is given by

$$x_4 = \frac{1.25 + 1.375}{2} = 1.313$$

Then, $f(1.313) = (1.313)^3 - 1.313 - 1$

$$f(1.313) = -\,0.0494$$

Therefore $f(1.313)$ is, negative and $f(1.375)$ is positive. The root lies between 1.313 and 1.375.

Fifth Iteration: The fifth approximation is given by

$$x_5 = \frac{1.313 + 1.375}{2} = 1.344$$

Then, $f(1.344) = (1.344)^3 - 1.344 - 1$

$$f(1.344) = 0.0837$$

Therefore $f(1.313)$ is, negative and $f(1.344)$ is positive. The root lies between 1.313 and 1.344.

Sixth Iteration: The sixth approximation is given by

$$x_6 = \frac{1.313 + 1.344}{2} = 1.329$$

Then, $f(1.329) = (1.329)^3 - 1.329 - 1$

$$f(1.329) = 0.0183$$

Therefore $f(1.313)$ is, negative and $f(1.329)$ is positive. The root lies between 1.313 and 1.329.

Seventh Iteration: The seventh approximation is given by

$$x_7 = \frac{1.313 + 1.329}{2} = 1.321$$

Then, $f(1.321) = (1.321)^3 - 1.321 - 1$

$$f(1.321) = -0.0158$$

Therefore $f(1.321)$ is, negative and $f(1.329)$ is positive. The root lies between 1.321and 1.329.

From the above iterations, the root of $f(x) = x^3 - x - 1 = 0$ up to three places of decimals is **1.325**, which is desired accuracy.

Example 2: Find the root of the equation $x^3 - x - 4 = 0$ lying between 1 and 2 by bisection method.

Solution Given $f(x) = x^3 - x - 4$

We want to find the root lie between 1 and 2

At $a = 1 \Rightarrow f(a) = (1)^3 - 1 - 4$ = - 4 negative

At $b = 2 \Rightarrow f(b) = (2)^3 - 2 - 4$ = 2 positive

This implies that root lies between 1 and 2

First Iteration: Here, $a = 1$ and$b = 2$, $x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$

Now, $f(a) = -4$,$f(b) = 2$. Then, $f(x_1) = (1.5)^3 - 1.5 - 4$ = - 2.125

Since $f(1.5)$ is negative and $f(2)$ is positive.

So root will now lie between 1.5 and 2.

Second Iteration: Here, $x_1 = 1.5$ and$b = 2$, $x_2 = \frac{1.5+2}{2} = 1.75$

Now, $f(x_1) = -2.125$, $f(b) = 2$.

Then, $f(x_2) = (1.75)^3 - 1.75 - 4$ = - 0.39062

Since $f(1.75)$ is negative and $f(2)$ is positive therefore the root lies between 1.75 and 2.

Third Iteration: Here, $x_2 = 1.75$ and $= 2$, $x_3 = \frac{1.75+2}{2} = 1.875$

Now, $f(x_2) = -0.39062$, $f(b) = 2$.

Then, $f(x_3) = (1.875)^3 - 1.875 - 4$ = 0.7169

Since $f(1.75)$ is negative and $f(1.875)$ is positive therefore the root lies between 1.75 and 1.875.

Fourth Iteration: Here, $x_2 = 1.75$ and $x_3 = 1.875$, $x_4 = \frac{1.75+1.875}{2} = 1.8125$

Now, $f(x_2) = -0.39062$, $f(x_3) = 0.71679$

Then, $f(x_4) = (1.8125)^3 - 1.8125 - 4$ = 0.14184

Since $f(1.75)$ is negative and $f(1.8125)$ is positive therefore the root lies between 1.75 and 1.8125.

Fifth Iteration: Here, $x_2 = 1.75$ and $x_4 = 1.8125$, $x_5 = \frac{1.75+1.8125}{2} = 1.78125$

Now, $f(x_2) = -0.39062$, $f(x_4) = 0.14184$

Then, $f(x_5) = (1.78125)^3 - 1.78125 - 4$ = - 0.12960

Since $f(1.78125)$ is negative and $f(1.8125)$ is positive therefore the root lies between 1.75 and 1.8125.

Repeating the process, the successive approximations are

$x_6 = 1.79687$, $x_7 = 1.78906$, $x_8 = 1.79296$, $x_9 = 1.79491$,

$x_{10} = 1.79589$, $x_{11} = 1.79638$, $x_{12} = 1.79613$.

From the above discussion, the value of the root to three decimal places is **1.796**.

Example 3: Using the Bisection Method, find the real root of the equation

$f(x) = 3x - \sqrt{1 + \sin x}$ = 0

Solution: The given equation

$f(x) = 3x - \sqrt{1 + \sin x}$ = 0 is a transcendental equation.

Given $f(x) = 3x - \sqrt{1 + \sin x}$ = 0 --- (1)

Then $f(0) = 0 - \sqrt{1 + \sin 0}$ = -1

And $f(1) = 3 - \sqrt{1 + \sin 1} = 3 - \sqrt{1.8414}$

= 3 – 1.3570 = 1.643 > 0

Thus $f(0)$ is negative and $f(1)$ is positive, therefore, a root lies between 0 and 1.

First Approximation: The first approximation of the root is given by

$$x_1 = \frac{0+1}{2} = 0.5$$

Now, $f(0.5) = 3(0.5) - \sqrt{1 + \sin(0.5)}$

$= 1.5 - \sqrt{1.4794}$ = 1.5 – 1.2163 = 0.2837 > 0

Thus, $f(0.5)$ is positive, while $f(0)$ is negative, therefore, a root lies between 0 and 0.5.

Second Approximation: The second approximation of the root is given by

$$x_2 = \frac{0+0.5}{2} = 0.25$$

Again, $f(0.25) = 3(0.25) - \sqrt{1 + \sin(0.25)}$

$= 0.75 - \sqrt{1.2474}$ = 0.75 – 1.1169 = - 0.3669 < 0

Thus, $f(0.25)$ is negative, while $f(0.5)$ is positive, therefore, a root lies between 0.25 and 0.5.

Third Approximation: The third approximation of the root is given by

$$x_3 = \frac{0.25+0.5}{2} = 0.375$$

Again, $f(0.375) = 3(0.375) - \sqrt{1 + \sin(0.375)}$

$= 1.125 - \sqrt{1.3663}$ = 1.125 – 1.1689 = - 0.0439 < 0

Thus, $f(0.375)$ is negative, while $f(0.5)$ is positive, therefore, a root lies between 0.375and 0.5.

Fourth Approximation: The fourth approximation of the root is given by

$$x_4 = \frac{0.375+0.5}{2} = 0.4375$$

Again, $f(0.4375) = 3(0.4375) - \sqrt{1 + \sin(0.4375)}$

$= 1.3125 - \sqrt{1.4237}$ = 1.3125 – 1.1932 = 0.1193 > 0

Thus, $f(0.4375)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375and 0.4375.

Fifth Approximation: The fifth approximation of the root is given by

$$x_5 = \frac{0.375 + 0.4375}{2} = 0.4063$$

Again, $f(0.4063) = 3(0.4063) - \sqrt{1 + \sin(0.4063)}$

$$= 1.2189 - \sqrt{1.3952} = 1.2189 – 1.1812 = 0.0377 > 0$$

Thus, $f(0.4063)$ is positive, while $f(0.375)$ is negative, therefore, a root lies between 0.375and 0.4063.

Sixth Approximation: The sixth approximation of the root is given by

$$x_6 = \frac{0.375 + 0.4063}{2} = 0.3907$$

Again, $f(0.3907) = 3(0.3907) - \sqrt{1 + \sin(0.3907)}$

$$= 1.1721 - \sqrt{1.3808} = 1.1721 – 1.1751 = - 0.003 < 0$$

Thus, $f(0.3907)$ is negative, while $f(0.4063)$ is positive, therefore, a root lies between 0.3907 and 0.4063.

Seventh approximation: The seventh approximation of the root is given by

$$x_7 = \frac{0.3907 + 0.4063}{2} = 0.3985$$

From the last two observations, i.e. $x_6 = 0.3907$ and $x_7 = 0.3985$ the approximate value of the root up to two places of decimal is given by 0.39. Hence the root is **0.39** approximate

Check Your Progress

1.Find the smallest root of $x^3 - 9x + 1 = 0$using Bisection Method correct to three decimal places. [Ans 0.111]

1.8 Summary

In fact, there is no need of a deeper knowledge of numerical methods and their analysis in most of thecases in order to use some standard softwares as an end user. However, there are at least three reasonsto gain a basic understanding of the theoretical background of numerical methods.

1. Learning different numerical methods and their analysis will make a person more familiar with thetechnique of developing new numerical methods. This is important when the available methods arenot enough or not efficient for a specific problem to be solved.
2. In many circumstances, one has more methods for a given problem. Hence, choosing an appropriatemethod is important for producing an accurate result in lesser time.
3. With a sound background, one can use methods properly (especially when a method has its ownlimitations and/or disadvantages in some specific cases) and, most importantly, one can understandwhat is going wrong when results are not as expected.

1.9 Exercise

Q. 1. If X= 2.536, find the absolute error and relative error when

(i) X is rounded off

(ii) X is truncated to two decimal digits

Q. 2. What do you mean by truncation error? Explain with examples.

Q. 3. Find the real root of $e^x = 3x$ by Bisection Method. [Ans 1.5121375]

Q. 4. Find a root of $x^3 - x - 11 = 0$ Bisection Method correct to three decimal places which lies between 2 and 3. [Ans 2.374]

Q. 5. Compute the root of log x = cos x correct to 2 decimal places using Bisection Method [Ans 1.5121375]

Unit-2: Solutions of Non-Linear Equations

Structure:

2.1 Introduction

Majority of the Mathematical problems that arise in science and engineering are very hard and sometimes impossible to solve exactly manually. Thus, by making an initial approximation and using the iterative methods likefixed point iterative method, Regula –Falsi method and Newton Raphson Method one can calculate the root exactly to the accurate number of places required using the computational tools and techniques.

2.2 Objective

In this unit, our objective is to review the basic concepts of basic iterative methods like fixed point iterative method, Regula –Falsi method and Newton Raphson Method and how their order of convergence is calculated.

2.3 Fixed Point Iteration Method

Iteration Method (Method of Successive Approximation)

This method is also known as the direct substitution method or method of fixed iterations.

To find the root of the equation $f(x) = 0$ by successive approximations, we rewrite the give equation in the form

$x = g(x)$ --- (1)

Now, first we assume the approximate value of root (let x_0) , then substitute it in $g(x)$ to have a first approximation x_1 given by

$$x_1 = g(x_0) \quad \text{--- (2)}$$

Similarly, the second approximation x_2 is given by

$$x_2 = g(x_1) \quad \text{--- (3)}$$

In general, $x_{i+1} = g(x_i)$ --- (4)

Procedure for Iteration Method to Find the Root of the Equation $f(x) = 0$

Step 1 : Take an initial approximation as x_0.

Step 2: Find the next (first) approximation x_1 by usin $x_1 = g(x_0)$

Step 3: Follow the above procedure to find the successive approximations ,

x_{i+1} by using $x_{i+1} = g(x_i)$, $i = 0,1,2,3$ ....

Step 4 : Stop the evaluation where relative error $\leq \varepsilon$, where ε is the prescribed accuracy.

Note 1: The iteration method $x = g(x)$ is convergent if $|g^1(x)| < 1$

Rate of convergence of Iteration Method

Let $f(x) = 0$ be the equation which is being expressed as $x = g(x)$. The iterative formula for solving the equation is

$$x_{i+1} = g(x_i)$$

If a is the root of the equation $x = g(x)$ lying in the interval $]a, b[$, $\alpha = g(\alpha)$.

The iterative formula may also be written as

$$x_{i+1} = g(\, x + \overline{x_i - \alpha)}$$

Then by mean value theorem

$x_{i+1} = g(\alpha) + (x_i - \alpha)g'(c_i)$ where $\alpha < c_i < b$

But $g(\alpha) = \alpha$

$\Rightarrow x_{i+1} = \alpha + (x_i - \alpha)g'(c_i)$

$\Rightarrow x_{i+1} - \alpha = +(x_i - \alpha)g'(c_i)$ -----(1)

Now, if e_{i+1} , e_i are the error for the approximation x_{i+1} and x_i

Therefore, $e_{i+1} = x_{i+1} - \alpha$, $e_i = x_i - \alpha$

Using this in (1), we get

$e_{i+1} = e_i g'(c_i)$

Here $g(x)$ is a continuous function, therefore, it is bounded

$\therefore |g'(c_i)| < k$, where $k \in]a, b[$ is a constant

$\therefore e_{i+1} \leq e_i k$

Or $\frac{e_{i+1}}{e_i} \leq k$

Hence, by definition, the rate of convergence of iteration method is 1. In other words, iteration method converges linearly.

Example 1: Find the root of the equation x $= 0.21 \sin(0.5 + x)$ by iteration method starting with x =0.12.

Solution: Here x $= 0.21 \sin(0.5 + x)$

$\Rightarrow$ f $(x) = 0.21 \sin(0.5 + x)$ --- (1)

Here we observe that $|f(x)| < 1$

$\Rightarrow$

Method of Iteration can be applied.

Now, first approximation of x is given by

$$x^{(1)} = 0.21 \sin(0.5 + 0.12) = 0.21 \sin(0.62)$$
$$= 0.21\,(0.58104) = 0.1220$$

The Second approximation of x is given by

$$x^{(2)} = 0.21 \sin(0.5 + 0.122) = 0.21 \sin(0.622)$$
$$= 0.21\,(0.58299) = 0.12243$$

The third approximation of x is given by

$$x^{(3)} = 0.21 \sin(0.5 + 0.1224) = 0.21 \sin(0.6224)$$
$$= 0.21\,(0.58299) = 0.12243$$

The fourth approximation of x is given by

$$x^{(4)} = 0.21 \sin(0.5 + 0.12243) = 0.21 \sin(0.62243)$$
$$= 0.21\,(0.58301) = 0.12243$$

Here we observe that $x^{(3)} = x^{(4)}$

Hence, the required root is given by $\boldsymbol{x = 0.12243}$

Example 2: Find the root of the equation $f(x) = x^3 + x^2 - 1 = 0$ by using iteration method.

Solution: $\because \quad x^3 + x^2 - 1 = 0$

$\Rightarrow x^2(1 + x) = 1 \Rightarrow x^2 = \frac{1}{1+x}$

$\therefore x = \frac{1}{\sqrt{1+x}} = \emptyset(x)$

Here, $f(0) = -1$ and $f(1) = 1$ so a root lies between 0 and 1.

Now, $x = \frac{1}{\sqrt{1+x}}$ so that,

$$\emptyset(x) = \frac{1}{\sqrt{1+x}}$$

$\therefore \emptyset'(x) = \frac{1}{2(1+x)^{3/2}}$

We have ,$|\emptyset'(x)| < 1$ for $x < 1$

Hence iterative method can be applied

Take, $x_0 = 0.5$, we get

$x_1 = \emptyset(x_0) = \frac{1}{\sqrt{1.5}} = 0.81649$

$x_2 = \emptyset(x_1) = \frac{1}{\sqrt{1.81649}} = 0.74196$

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$$\boldsymbol{x_8 = 0.75487}$$

Example 3: Find the cube root of 15 correct to four significant figures by iterative method.

Solution: Let $x = (15)^{1/3}$ therefore $x^3 - 15 = 0$

$\because \quad 2^3 < 15 < 3^3$

Real root of the equation lies in (2,3). The equation may be written as

$x = \frac{15+20\,x-x^3}{20} = \emptyset(x)$

Now, $\emptyset'(x) = 1 - \frac{3x^3}{20}$ therefore, $|\emptyset'(x)| < 1$

Iterative formula is $x_{i+1} = \frac{15+20\, x_i - x_i^{\,3}}{20}$ -------------(1)

Put $i = 0, x_0 = 2.5$, we get $x_1 = 2.47$

Put $i = 1,\ in\ (1)$, we get $x_2 = 2.466$ (where $x_1 = 2.47$)

Similarly, $x_3 = 2.4661$

Therefore $\sqrt[3]{20}$ correct to 3 decimal places **is 2.466**.

Check Your Progress

2. Solve by iteration method $x^3 + x^2 + 1 = 0$[Ans –0.682327803]

2.4 Chord Methods for Finding Roots- Regula Falsi Method

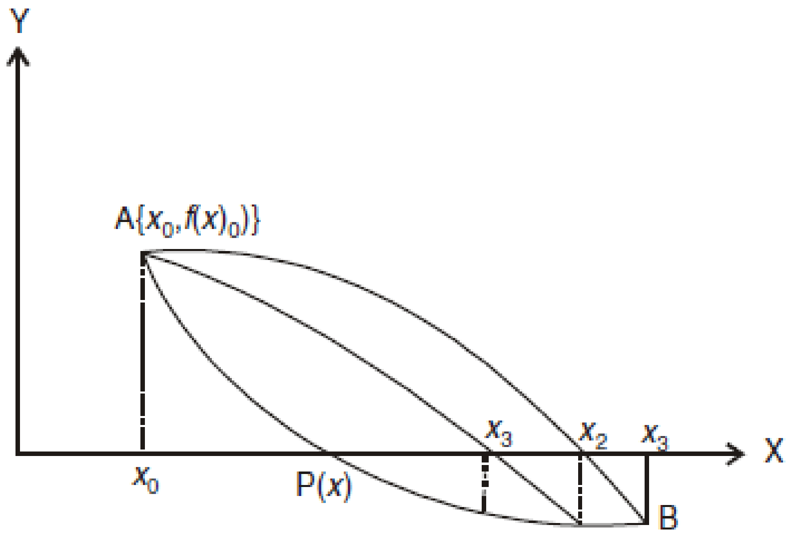
REGULA FALSI METHOD (CHORD METHODS FOR FINDING ROOTS)

This method is essentially same as the bisection method except that instead of bisecting the interval.

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of $y = f(x)$ crosses the X-axis between these two points, a root must lie in between these points.

Consequently, $f(x_0)f(x_1)$< 0. Equation of the chord joining points $\{x_0 , (x_0)\}$ and $\{x_1 , f(x_1)\}$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}\ (x - x_0)$$



The method consists in replacing the curve AB by means of the chord and taking the point of intersection of the chord with X-axis as an approximation to the root.

So the abscissa of the point where chord cuts y = 0 is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

The value of x_2 can also be put in the following form:

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

In general, the $(i+1)th$ approximation to the root is given by

$$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$$

Procedure for the False Position Method to Find the Root of the Equation $f(x) = 0$

Step 1: Choose two initial guess values (approximation) x_0 and x_1 (where $x_1 > x_0$) such that $f(x_0).f(x_1) < 0$

Step 2: Find the next approximation x_2 using the formula

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

and also evaluate $f(x_2)$.

Step 3: If $f(x_2).f(x_1) < 0$, then go to the next step. If not, rename x_0 and x_1 and then go to next step.

Step 4: Evaluate successive approximations using the formula

$x_{i+1} = \frac{x_{i-1} f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})}$, where i =2, 3, 4, ……..

But before applying the formula for x_{i+1}, ensure whether If $f(x_{i-1}).f(x_i) < 0$ if not, rename x_{i-2} and x_{i-1} and proceed.

Step 5: Stop the evaluation when $|x_i - x_{i-1}| < \epsilon$, where ϵ is the prescribed accuracy.

Order (or Rate) of Convergence of Regula Falsi Method

The general iterative formula for False Position Method is given by

$$x_{i+1} = \frac{x_{i-1}f(x_i) - x_i f(x_{i-1})}{f(x_i) - f(x_{i-1})} \quad \text{-------------- (1)}$$

where x_{i-1}, x_i and x_{i+1} are successive approximations to be required root of $f(x) = 0$

The formula given in (1), can also be written as :

$$x_{i+1} = x_i - \frac{(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})} f(x_i) \quad \text{-------------- (2)}$$

Let α be the actual (true) root of $f(x) = 0$, i.e.$f(\alpha) = 0$. If e_{i-1}, e_0 and e_{i+1} are the successive errors in $(i-1)th$, ith and $(i+1)th$ iterations respectively, then

$$e_{i-1} = x_{i-1} - \alpha \text{ , } e_i = x_i - \alpha \text{ , } e_{i+1} = x_{i+1} - \alpha$$

Or $\quad x_{i-1} = \alpha + e_{i-1},\ x_i = \alpha + e_i, x_{i+1} = \alpha + e_{i+1},$

Using these in (2), we obtain

$$\alpha + e_{i+1} = \alpha + e_i - \frac{(e_i - e_{i-1})}{f(\alpha + e_i) - f(\alpha + e_{i-1})} f(\alpha + e_i) \quad \text{------- (3)}$$

Expanding $f(\alpha + e_i)$ and $f(\alpha + e_{i-1})$ in Taylor's series around α , we have

$$e_{i+1} = e_i - \frac{(e_i - e_{i-1})[f(\alpha) + e_i f'(\alpha) + \frac{{e_i}^2}{2} f''(\alpha) + \cdots\ ]}{[f(\alpha) + e_i f'(\alpha) + \frac{{e_i}^2}{2} f''(\alpha) + \cdots\ ] - [f(\alpha) + e_{i-1} f'(\alpha) + \frac{{e_{i-1}}^2}{2} f''(\alpha) + \cdots\ ]}$$

i.e. $\quad e_{i+1} = e_i - \frac{(e_i - e_{i-1})[f(\alpha) + e_i f'(\alpha) + \frac{{e_i}^2}{2} f''(\alpha)\ ]}{[(e_i - e_{i-1}) f'(\alpha) + (\frac{{e_i}^2 - {e_{i-1}}^2}{2}) f''(\alpha)\ ]}$, [on ignoring the higher order terms]

. $\quad e_{i+1} = e_i - \frac{[f(\alpha) + e_i f'(\alpha) + \frac{{e_i}^2}{2} f''(\alpha)\ ]}{[f'(\alpha) + (\frac{e_i + e_{i-1}}{2}) f''(\alpha)\ ]} \qquad$ [since $f(\alpha) = 0$]

. i.e. $\quad e_{i+1} = e_i - \frac{[e_i + \frac{{e_i}^2}{2} \frac{f''(\alpha)}{f'(\alpha)}]}{[1 + \left(\frac{e_i + e_{i-1}}{2}\right) \frac{f''(\alpha)}{f'(\alpha)}]}$

[on dividing the numerator and denominator by $f'(\alpha)$]

i.e. $e_{i+1} = e_i - [e_i + \frac{{e_i}^2}{2} \frac{f''(\alpha)}{f'(\alpha)}][1 + \left(\frac{e_i + e_{i-1}}{2}\right) \frac{f''(\alpha)}{f'(\alpha)}]^{-1}$

i.e. $e_{i+1} = e_i - [e_i + \frac{{e_i}^2}{2}\frac{f''(\alpha)}{f'(\alpha)}][1 - \left(\frac{e_i + e_{i-1}}{2}\right)\frac{f''(\alpha)}{f'(\alpha)}]$

i.e. $e_{i+1} = e_i - [\frac{e_i(e_i + e_{i-1})}{2}\frac{f''(\alpha)}{f'(\alpha)} + \frac{{e_i}^2}{2}\frac{f''(\alpha)}{f'(\alpha)} - \frac{{e_i}^2(e_i + e_{i-1})}{4}\{\frac{f''(\alpha)}{f'(\alpha)}\}^2]$

If e_{i-1} and e_i are very small, then ignoring higher terms we get

$e_{i+1} = e_i e_{i-1} \frac{f''(\alpha)}{2f'(\alpha)}$ ------- (4)

Which can be written as

$e_{i+1} = e_i e_{i-1} M$, where M= $\frac{f''(\alpha)}{2f'(\alpha)}$ and would be a constant ---(5)

In order to find the order of convergence, it is necessary to find a formula of the type

$e_{i+1} = A{e_i}^k with a appropriate value of k.$ ----- (6)

With the help of (6), we can write

$e_{i+1} = A{e_i}^k$ or $e_{i-1} = (\frac{e_i}{A})^{1/k}$

Now substituting the value of e_{i+1} and e_{i-1} in (5), we get

$A{e_i}^k = e_i(\frac{e_i}{A})^{1/k}$M

Or $\quad {e_i}^k =$ M $A^{-\left(1+\frac{1}{k}\right)} . {e_i}^{\left(1+\frac{1}{k}\right)}$ ----- (7)

Comparing the powers of e_i on both sides of (7), we get

$$k = 1 + 1/k$$

Or $\quad k^2 - k - 1 = 0$ ------- (8)

From (8), taking only the positive root, we get $k = 1.618$

By putting the value of $k$in (6), we have

$e_{i+1} = A{e_i}^{1.618}$ or $\frac{e_{i+1}}{{e_i}^{1.618}} = A$

Comparing this with $\lim_{i\to\infty}(\frac{e_{i+1}}{{e_i}^k}) \leq$ A , we see that order (or rate) of convergence of false position method is 1.618.

Example 1 Find the real root of the equation $f(x) = x^3 - 2x - 5 = 0$ by the method of false position up to three places of decimal.

Solution: Given that $f(x) = x^3 - 2x - 5 = 0$

So that $\quad f(2) = (2)^3 - 2(2) - 5 = -1$

and $\quad f(3) = (3)^3 - 2(3) - 5 = 16$

Therefore, a root lies between 2 and 3.

First approximation: Therefore taking, $x_0 = 2$, $x_1 = 3$, $f(x_0)$ = -1, $f(x_1)$ = 16, then by Regula- Falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$= 2 - \frac{3-2}{16+1}(-1) = 2 + \frac{1}{17}$ = 2.0588

Now, $f(x_2) = f(2.0588)$

$$= (2.0588)^3 - 2(2.0588) - 5 = -0.3911$$

Therefore, root lies between 2.0588 and 3.

Second approximation: Therefore taking, $x_0 = 2.0588$, $x_1 = 3$, $f(x_0) = -0.3911$, $f(x_1)$ = 16, then by Regula- Falsi method, we get

$$x_3 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0588 - \frac{3 - 2.0588}{16 + 0.3911}(-0.3911)$$

$= 2.0588 + 0.0225$ =2.0813

Now, $f(x_3) = f(2.0813)$

$$= (2.0813)^3 - 2(2.0813) - 5 = -0.1468$$

Therefore, root lies between 2.0813 and 3.

Third approximation: Therefore taking, $x_0 = 2.0813$, $x_1 = 3$, $f(x_0) = -0.1468$, $f(x_1)$ = 16, then by Regula- Falsi method, we get

$$x_4 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0813 - \frac{3 - 2.0813}{16 + 0.1468}(-0.1468)$$

$= 2.0813 + 0.0084$ =2.0897

Now, $f(x_4) = f(2.0897)$

$$= (2.0897)^3 - 2(2.0897) - 5 = -0.054$$

Therefore, root lies between 2.0897 and 3.

Fourth approximation: Therefore taking, $x_0 = 2.0897$, $x_1 = 3$, $f(x_0) = -0.054$, $f(x_1)$ = 16, then by Regula- Falsi method, we get

$$x_5 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0897 - \frac{3 - 2.0897}{16 + 0.054}(-0.054)$$

$= 2.0588 + 0.0031$ =2.0928

Now, $f(x_5) = f(2.0928)$

$$= (2.0928)^3 - 2(2.0928) - 5 = -0.0195$$

Therefore, root lies between 2.0928 and 3.

Fifth approximation: Therefore taking, $x_0 = 2.0928$, $x_1 = 3$,

$f(x_0) = -0.0195$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_6 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0928 - \frac{3 - 2.0928}{16 + 0.0195}(-0.0195)$$

$= 2.0928 + 0.0011$ =2.0939

Now, $f(x_6) = f(2.0939)$

$$= (2.0939)^3 - 2(2.0939) - 5 = -0.0074$$

Therefore, root lies between 2.0939 and 3.

Sixth approximation: Therefore taking, $x_0 = 2.0939$, $x_1 = 3$,

$f(x_0) = -0.0074$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_7 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0939 - \frac{3 - 2.0939}{16 + 0.0074}(-0.0074)$$

$= 2.0939 + 0.00042$ =2.0943

Now, $f(x_7) = f(2.0943)$

$$= (2.0943)^3 - 2(2.0943) - 5 = -0.0028$$

Therefore, root lies between 2.0943 and 3.

Seventh approximation: Taking, $x_0 = 2.0943$, $x_1 = 3$,

$f(x_0) = -0.0028$, $f(x_1) = 16$, then by Regula- Falsi method, we get

$$x_8 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2.0943 - \frac{3 - 2.0943}{16 + 0.0028}(-0.0028)$$

$= 2.0943 + 0.00016$=2.0945

Therefore, the root is $\mathbf{2.094}$ correct to three decimal places.

Example 2: Using the method of False Position, find the root of equation $f(x) = x^6 - x^4 - x^3 - 1 = 0$ up to four decimal places.

Solution: Let $f(x) = x^6 - x^4 - x^3 - 1$

$$f(1.4) = (1.4)^6 - (1.4)^4 - (1.4)^3 - 1 = -0.056$$

$$f(1.41) = (1.41)^6 - (1.41)^4 - (1.41)^3 - 1 = 0.102$$

Hence the root lies between 1.4 and 1.41.

Using the method of False Position,

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 1.4 - \frac{1.41 - 1.4}{0.102 + 0.056}(-0.056)$$

$= 1.4 + \frac{0.01}{0.158}(0.056)$ =1.4035

Now, $f(1.4035) = (1.4035)^6 - (1.4035)^4 - (1.4035)^3 - 1$

$$f(x_2) = -0.0016(-ve)$$

Hence the root lies between 1.4035 and 1.41.

Using the method of False Position,

$$x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 1.4035 - \frac{1.41 - 1.4035}{0.102 + 0.016}(-0.0016)$$

$= 1.4035 + \frac{0.0065}{0.1036}(0.0016)$ =1.4036

Now, $f(1.4036) = (1.4036)^6 - (1.4036)^4 - (1.4036)^3 - 1$

$$f(x_3) = -0.00003(-ve)$$

Hence the root lies between 1.4036 and 1.41.

Using the method of False Position,

$$x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 1.4036 - \frac{1.41 - 1.4036}{0.102 + 0.00003}(-0.00003)$$

$= 1.4036 + \frac{0.0064}{0.10203}(0.00003)$ =1.4036

Since, x_3 and x_4 are approximately the same upto four places of decimal, hence the required root of the given equation is **1.4036.**

Check Your Progress

1. Find the real root of the equation$x^3 - 2x - 5 = 0$by the method of False Position correct to three decimal places. [Ans 2.094]

2.5 Newton Raphson Method, Order of Convergence.

NEWTON-RAPHSON METHOD (or NEWTON'S METHOD)

This method can be derived from Taylor's series as follows:

Let $f(x) = 0$ be the equation for which we are assuming x_0 be the initial approximation and h be a small corrections to x_0, so that

$$f(x_0 + h) = 0$$

Expanding it by Taylor's series, we get

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \ \ldots\ldots = \ 0$$

Since h is small, we can neglect second and higher degree terms in h and therefore, we get

$$f(x_0) + hf'(x_0) = \ 0$$

From which we have,

$$h = -\frac{f(x_0)}{f'(x_0)}$$

Hence, if x_0 be the initial approximation, then next (or first) approximation x_1 is given by

$x_1 = \ x_0 \ + h = x_0 - \frac{f(x_0)}{f'(x_0)}$

The next and second approximation x_2 is given by

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

In general, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $\quad n = 0, 1, 2, 3, \ldots \ldots$

This formula is well known as Newton-Raphson formula.

The iterative procedure terminates when the relative error for two successive approximations becomes less than or equal to the prescribed tolerance

Procedure for Newton Raphson Method to Find the Root of the Equation $f(x) = 0$

Step 1 : Take the trail solution (initial approximation) as x_0. Find $f(x_0)$ and

$f'(x_0)$.

Step 2 : Find next (first) approximation x_1 by using the formula $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

Step 3 : Follow the above procedure to find the successive approximations x_{n+1}

using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, where n =1, 2, 3, …….

Step 4 : Stop the process when $|x_{n+1} - x_n| < \varepsilon, where \epsilon$ is prescribed accuracy.

Order (or Rate) of Convergence of Newton-Raphson Method

Let α be the actual root of equation $f(x) = 0$ i.e. $f(\alpha) = 0$. Let x_n and x_{n+1} be two successive approximations to the actual root α. If e_n and e_{n+1} are the corresponding errors we have, $x_n = \alpha + e_n$ and $x_{n+1} = \alpha + e_{n+1}$. By Newton-Raphson formula,

$$e_{n+1} = e_n - \frac{f(\alpha + e_n)}{f'(\alpha + e_{n+1})}$$

$$e_{n+1} = e_n - \frac{f(\alpha) + e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2} f'''(\alpha) + \cdots}$$ (By Taylor's Series)

$$e_{n+1} = e_n - \frac{e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \cdots}{f'(\alpha) + e_n f''(\alpha) + \frac{e_n^2}{2} f'''(\alpha) + \cdots}$$ (since $.f(\alpha) = 0$)

$$e_{n+1} = \frac{e_n^2 f''(\alpha)}{2[f'(\alpha) + e_n f''(\alpha)]}$$ (On neglecting high powers of e_n)

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)\{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\}}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \{1 + e_n \frac{f''(\alpha)}{f'(\alpha)}\}^{-1}$$

$$= \frac{e_n^2}{2} \frac{f''(\alpha)}{f'(\alpha)} \{1 - e_n \frac{f''(\alpha)}{f'(\alpha)} + \cdots.\}$$

$$\boldsymbol{\frac{e_{n+1}}{e_n^2} \approx \frac{f''(\alpha)}{2f'(\alpha)}}$$ (Neglecting terms containing powers of e_n)

Hence by definition, the order of convergence of Newton-Raphson method is 2 i.e., Newton-Raphson method is **quadratic convergent.**

This also shows that subsequent error at each step is proportional to the square of the previous error and as such the convergence is quadratic.

Example 1: Find the real root of the equation $x^2 - 5x + 2 = 0$ between 4 and 5 by Newton-Raphson method.

Solution: Let that $f(x) = x^2 - 5x + 2$………….. (1)

Now, $f(4) = 4^2 - 5 \times 4 + 2 = -2$

and $f(5) = 5^2 - 5 \times 5 + 2 = 2$

Therefore, the root lies between 4 and 5

From (1), we get $f'(x) = 2x - 5$ ………….. (2)

Now, Newton-Raphson's method becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{{x_n}^2 - 5x_n + 2}{2x_n - 5}$$

$$x_{n+1} = \frac{{x_n}^2 - 2}{2x_n - 5} n = 0,1,2, \dots \dots ....$$

Let us take x_0=4 to obtain the approximation to the root by putting $n =$ 0,1,2, … … …. into (3), we get

First approximation:

$$x_1 = \frac{{x_0}^2 - 2}{2x_0 - 5} = \frac{4^2 - 2}{2(4) - 5} = \frac{14}{3} = 4.667$$

Second approximation:

The root is given by

$$x_2 = \frac{{x_1}^2 - 2}{2x_1 - 5} = \frac{(4.667)^2 - 2}{2(4.667) - 5} = \frac{19.7781}{4.3334} = 4.5641$$

Third approximation:

The root is given by

$$x_2 = \frac{{x_1}^2 - 2}{2x_1 - 5} = \frac{(4.667)^2 - 2}{2(4.667) - 5} = \frac{19.7781}{4.3334} = 4.5641$$

Fourth approximation:

The root is given by

$$x_4 = \frac{{x_3}^2 - 2}{2x_3 - 5} = \frac{(4.5641)^2 - 2}{2(4.5641) - 5} = \frac{18.8082}{4.1232} = 4.5616$$

Since$x_3 = x_4$, hence the root of the equation is $\mathbf{4.5616}$ correct to four decimal places.

Example 2: Find the real root of the equation $3x = \cos x + 1$ by Newton-Raphson method.

Solution: Let that $f(x) = 3x - \cos x - 1 = 0$ ………….. (1)

So $f(x) = -2$

$f(1) = 3 - cos1 - 1 = 1.4597$

So the root lies between 0 and 1

Let us take $x_0 = 0.6$

From (1) $f'(x) = 3 + \sin x$ ………….. (2)

Therefore the Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Or $$x_{n+1} = \frac{x_n Sin x_n + cos x_n + 1}{3 + Sin x_n}$$ ……………(3)

First approximation:

Putting $n = 0$, in (3) we get first approximation

$$x_1 = \frac{x_0 Sin x_0 + \ cos x_0 \ + \ 1}{3 + Sin x_0}$$

$$= \frac{(0.6) Sin(0.6) + \ cos\ (0.6) + \ 1}{3 + Sin(0.6)}$$

$$= \frac{(0.6) Sin(0.6) + \ cos\ (0.6) + \ 1}{3 + Sin(0.6)}$$

$$= \frac{(0.6)(0.5646) + (0.8253) + \ 1}{3 + (0.5646)}$$

$$= \frac{2.16406}{3.5646}$$

Or

$$x_1 = 0.6071$$

Second approximation:

Putting $n = 1$, in (3) we get second approximation

$$x_2 = \frac{x_1 Sinx_1 + cosx_1 + 1}{3 + Sinx_1}$$

$$= \frac{(0.6071)Sin(0.6071) + cos\,(0.6071) + 1}{3 + Sin(0.6071)}$$

$$= \frac{(0.6071)(0.5705) + (0.8213) + 1}{3 + (0.5705)}$$

$$= \frac{2.1677}{3.5705}$$

Or

$$x_2 = 0.6071$$

Since $x_1 = x_2$Therefore the root as 0.6071 correct to four decimal places.

Check Your Progress

1. Find the cube root of 10 [Ans 2.15466]

2.6 Summary

Numerical analysis includes three parts. The first part of the subject is about the development of a method to a problem. The second part deals with the analysis of the method, which includes the error analysis and the efficiency analysis. Error analysis gives us the understanding of how accurate the result will be if we use the method and the efficiency analysis tells us how fast we can compute the result. The third part of the subject is the development of an efficient algorithm to implement the method as a computer code. A complete knowledge of the subject includes familiarity in all these three parts. Here in this unit we have learnt those mathematical iterative methods which can be used to find the root of transcendental equation upto desired no of accuracy.

2.7 Exercise

Q. 1. Find by iterative method, the real root of the equation $3x - \log_{10} x = 6$ correct to four significant figures. [Ans 0.5671477]

Q. 2. Find by iteration method $\sqrt{30}$ [Ans 5.477225575]

Q. 3. Find the postive root of $xe^{x} = 2$ by method of False Postion. [Ans 0.852605]

Q. 4. Find the real root of the equation $x = \tan x$ using False Position method

[Ans 4.4934]

Q. 5. Find the real root of the equation $x = e^{-x}$ using the Newton-Raphson method.

[Ans 0.5671]

Q. 6. Use Newton-Raphson method to obtain a root, correct to three decimal places of following equation $Sinx = \frac{x}{2}$ [Ans 1.896]

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SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 2 Finite Differences

Course Design Committee

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Block: 2 Finite Differences

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Block & Units Introduction

The ***Block - 2 – Finite Differences is*** the second block with four units. This block consists of four units regarding, finite differences interpolation with equal intervals, interpolation with unequal and Lagrange's Interpolation.

In ***Unit – 3 - Finite Differences***; various operators used in finite difference calculus are discussed. The concept of interpolation with equal intervals is given this unit.

Unit - 4 – Interpolation with Equal Intervals; in which Newton's forward and backward interpolation formulate are discussed.

Unit – 5 - Interpolation with Un-Equal Intervals; is devoted to interpolation with unequal intervals. Newton's general interpolation formula, divided differences and their properties are discussed in this unit.

Finally in ***Unit – 6 – Lagrange's Interpolation***; the last unit of this block, Lagrange's interpolation formula and is applications are given.

At the end of block/unit the summary, self assessment questions and further readings are given.

Unit-3 Finite Differences

Structure

3.1 Introduction

Numerical analysis is a branch of mathematics, which leads to approximate solution be repeated application of four basic operations of Algebra i.e. summation, subtraction, multiplication and division. The knowledge of finite differences is essential for the study of Numerical Analysis and plays an important role in numerical techniques, where tabulated values of the functions are available. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various important formulae. In this section we introduce few basic operators.

3.2 Objectives

After the study of this unit you will be in a position to know about

- Various types of difference operators
- Relation between different operators
- Factorial polynomials and
- How to express a given polynomial in factorial notation

3.3 Forward Difference Operator

Let y = f(x) be any function taking the values y_o, y_1, y_2,…..y_n, which it takes for the equidistant values x_0, x_1, x_2,….x_n of the independent variable x, then y_1- y_0, y_2-y_1,……y_n-y_{n-1} are called the first differences of the function y. They are denoted respectively by $\Delta y_0, \Delta y_1, \ldots\ldots$, etc.

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

$$- - - - - - -$$

$$\Delta y_n = y_n - y_{n-1}$$

The symbol Δ is called difference operator. The differences of the first differences denoted by $\Delta^2 y^0,\ \Delta^2 y^1,\ \ldots\ldots\Delta^2 y_n$ are called second differences, where

$$\Delta^2 y_0 = \Delta[\Delta y_0]$$

$$= \Delta[y_1 - y_0]$$

$$= [\Delta y^1 - \Delta y_0]$$

$$= y_2 - 2y_1 + y_0$$

and

$$\Delta^2 y_1 = \Delta[\Delta y_1]$$

$$= y_3 - 2y_2 + y_1$$

$$- - - -$$

Where Δ^2 is called the second difference operator.

Similarly,

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= y_3 - 3y_2 + 3y_1 - y_1$$

– – – –

In general

$$\Delta^r y_n = \Delta^{r-1} y_{n+1} - \Delta^{r-1} y_n$$

$$= y_{n+r} - \frac{r}{1!} y_{n+r-1} + \frac{r(n-1)}{2!} y_{n+r-1} + \cdots \ldots + (-1)^r y_{n^r}$$

3.3.1 Difference Table

It is a convenient method for displaying the successive differences of a function. The following table is an example to show how the differences formed.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_3$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_4$			
		Δy_4				
x_5	y_5					

The above table is called a diagonal difference table. The first term in the table is y_0. It is called the leading term.

The differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$., are called the leading differences. The differences $\Delta^n y_n$ with a fixed subscript are called forward differences. In forming such a difference table care must be taken to maintain correct sign.

A convenient check may be obtained by noting the sum of the entries in any column equals the differences between the first and the last entries in preceding column.

Another type of difference table called horizontal difference table, which is more compact and convenient, is not discussed here as it is beyond the scope of this book.

3.3.2 Alternative Notation

Let the functions y=f(x) be given at equal spaces of the independent variable x, say at x= a, a+h, a+2h, ….., etc. and the corresponding values of f(a), f(a+h), f(a+2h), ….., etc.

The independent variable x is often called the argument and the corresponding value of the dependent variable is of the function of f(x) at x= a and is denoted by $\Delta f(a)$.

Thus, we have

$$\Delta f(a) = f(a+h) - f(a)$$

Using the above definition we can write

$$\Delta f(a+h) = f(a+h+h) - f(a+h) = f(a+2h) - f(a+h)$$

Similarly,

$$\Delta^2 f(a) = \Delta[\Delta f(a)]$$

$$= \Delta[f(a+h) - f(a)]$$

$$= \Delta f(a+h) - \Delta f(a)$$

$$= f(a+2h) - f(a+h) - [f(a+h) - f(a)]$$

$$= f(a+2h) - 2f(a+h) + f(a)$$

Where Δ^2 is called the second difference of f(x) at x=a.

Note: The operator Δ is called forward difference operator and in general it is defined as

$$\Delta f(x) = f(x+h) - f(x),$$

Where, h is called the interval of differencing. Using the above definition we can write

$$\Delta^2 f(x) = \Delta[\Delta f(x)]$$

$$= \Delta[f(x+h) - f(x)]$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= f(x+2h) - f(x+h) - [f(x+h) - f(x)]$$

$$= f(x+2h) - 2f(x+h) + f(x)$$

Similarly we can write the other higher order differences as $\Delta^3, \Delta^4 \ldots..$, etc. and $\Delta, \Delta^2, \Delta^3, \ldots \ldots., \Delta^n, \ldots..$, are called the forward differences.

The difference table called the forward difference table in the new notation is given below.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$

x_0	$f(x_0)$			
		$\Delta f(x_0)$		
$x_0 + h$	$f(x_0 + h)$		$\Delta^2 f(x_0)$	
		$\Delta f(x_0 + h)$		$\Delta^3 f(x_0)$
$x_0 + 2h$	$f(x_0 + 2h)$		$\Delta^2 f(x_0 + h)$	
		$\Delta f(x_0 + 2h)$		
$x_0 + 3h$	$f(x_0 + 3h)$			

3.3.3 Properties of the Operator Δ

1) If c is a constant then Δc =0.

Proof. Let

$f(x) = c$

$\therefore f(x+h) = c \ (where\ h\ is\ interval\ of\ differenceing)$

$\therefore \ \Delta f(x) = f(x+h) - f(x) = c - c = 0$

⇨$\Delta c = 0$

2) Δ is distributive. i.e., $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

Proof. We have

$\Delta[f(x) \pm g(x)] = [f(x+h) + g(x+h)] - [f(x) + g(x)]$

$= f(x+h) - f(x) + g(x+h) - g(x)$

$= f(x+h) - f(x) + g(x+h) - g(x)$

$= \Delta f(x) + \Delta g(x)$

$Similarly\ , we\ can\ show\ that$

$\Delta[f(x) - g(x)] = \Delta f(x) - \Delta g(x)$

3) If c is a constant then $\Delta[cf(x)] = c\Delta f(x)$

Proof: We have

$$\Delta[cf(x)] = cf(x+h) - cf(x)$$

$$= c[f(x+h) - f(x)]$$

$$= c\Delta f(x)$$

$$\therefore \Delta[cf(x)] = c\Delta f(x).$$

4) If m and n are positive integers then $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

Proof: We have

$$\Delta^m \Delta^n f(x) = (\Delta \times \Delta \times \Delta \ldots . m\ times)(\Delta \times \Delta \times \Delta \ldots . n\ times)\ f(x)$$

$$= [x\Delta \times \Delta \times \Delta(m + n\ times) f(x)]$$

$$= \Delta^{m+n} f(x)$$

Similarly, we can prove the following

5) $\Delta[f_1(x) + f_2(x) + \cdots . f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \Delta f_n(x)$

6) $\Delta[f(x) + g(x)] = f(x) + Ag(x) + g(x)\Delta f(x)$.

7) $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)}$

Note:

1. From the properties (2) and (3) it is clear that Δ is a linear operator.
2. If n is a positive integer $\Delta^n . [\Delta^{-n} f(x)] = f(x)$ and in particular when n =1 we $\Delta[\Delta^{-1} f(x)] = f(x)$ get

Example 3.1 Find $(a)\ \Delta e^{ax}$ $(b)\ \Delta^2 e^x$ $(c)\ \Delta^2 Sin\ x$ $(d)\ \Delta \log x$ $(e) \Delta\ tan^{-1} x$

Solution

(a) $\Delta e^{ax} = e^{a(x+h)} - e^{ax}$

$= e^{ax}(e^{ah} - 1)$

$\Delta e^{ax} = e^{ax}(e^{ah} - 1)$

$(b)\ \Delta^2 e^x = \Delta[\Delta e^x]$

$= \Delta[e^{x+h} - e^x]$

$= \Delta[e^x(e^h - 1)]$

$= (e^h - 1)\Delta e^x$

$= (e^h - 1)^2 e^x$

$\therefore\ \Delta^2 e^x = (e^h - 1)^2 e^2$

$(c)\ \Delta^2 Sin\, x = \sin(x+h) - \sin x$

$$= 2\ Cos\ \left(\frac{x+h+x}{2}\right) sin\ \left(\frac{x+h-x}{2}\right)$$

$$= 2\ Cos\ \left(x+\frac{h}{2}\right) sin\frac{h}{2}$$

$$\therefore\ \Delta \sin x = 2Cos\ \left(x+\frac{h}{2}\right) Sin\ \frac{h}{2}$$

$(d)\ \Delta \log x \quad = \log(x+h) -\ logx$

$$= \log\frac{(x+h)}{x}$$

$$= \log\left[1+\frac{(x+h)}{x}\right]$$

$$\therefore\ \Delta \log x = \log\left[1+\frac{(x+h)}{x}\right]$$

$(e)\Delta\ \ tan^{-1}x = tan^{-1}(x+h) - tan^{-1}x$

$$= tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right]$$

$$= tan^{-1}\left[\frac{h}{1+xh+x^2}\right]$$

Example 3.2 Construct a forward difference table for the following data

X	0	10	20	30
Y	0	0.174	0.347	0.518

Solution:

X	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

Example 3.3 Construct a difference table for y = f(x) = x^3 + 2x + 1 for x = 1, 2, 3, 4, 5.

Solution:

X	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		9		
2	13		12	
		21		6
3	34		18	
		39		6
4	39		24	
		63		
5	136			

3.4 The Operator E

Let y = f(x) be function of x and x, x+h, x+2h, x+3h, …., etc., be the consecutive values of x, then the operator e is defined as

E f(x) = f (x+h)

E is called shift operator, it is also called displacement operator.

Note: E is only a symbol not an algebraic sum.

E^2 f(x) means the operator E is applied twice on f(x), i.e.

E^2 f(x) = f (x+nh)

= E f(x+h)

= f(x+2h)

Similarly,

E^n f(x) = f (x+nh) and

E^{-n} f(x) = f(x-nh)

The operator E has the following properties:

1) $E\left(f_1(x) + f_2(x) + .. + f_n(x)\right) = Ef_1(x) + Ef_2(x) + \ldots + Ef_n(x)$

2) E (cf (x)) = cEf (x) Where c is constant

3) $E^m\left(E^n f(x)\right) = E^n\left(E^m f(x)\right) = E^{m+n} f(x)$

4) If n is positive integer $E^n[E^{-n}f(x)] = f(x)$.

Alternative notation: if y_0, y_1, y_2, ….y_n,….., etc. are consecutive values of the function y= f(x) corresponding to equally spaced values x_0, x_1, x_2,……, x_n etc. of x in alternative notation.

$E\ y_0 = y_1$

$Ey_1 = y_2$

$E^2 y_0 = y_2$

………..

And in general

$E^n\ y_0 = y_n$.

3.4.1 Relation between the Operator E and Δ

From a definition of Δ, we know that

$$\Delta f(x) = f(x + h) - f(x),$$

Where h is the interval of differencing. Using the operator E we can write

$$\Delta f(x) = f(x + h) - f(x)$$

$$= Ef\ (x) - f(x)$$

$$\Delta f(x) = (E - 1)f(x)$$

The above relation can be expressed as an identity.

$$\Delta = E - 1$$

$$i.e.\ E = 1 + \Delta$$

Example 3.4 Prove that $E\Delta = \Delta E$

Proof: $E\Delta\, f(x) = E\big(f(x + h) - f(x)\big)$

$$= Ef\ (x + h) -\ EF\ (x)$$

$$= f(x + 2h) - f(x + h)$$

$$= \Delta f\ (x + h)$$

$$= \Delta E f(x)$$

$$\therefore \quad E\Delta = \Delta E.$$

Example 3.5 Prove that $\Delta \log f(x) = log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$

Solution:

$$f(x+h) = Ef(x)$$

$$= (\Delta + 1)f(x)$$

$$= \Delta f(x) + f(x)$$

$$= \frac{f(x+h)}{f(x)} = \frac{\Delta f(x)}{f(x)} + 1,$$

Applying logarithms on both sides we get

$$log\left[\frac{f(x+h)}{f(x)}\right] = log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

$$= \log f(x+h) - \log f(x) = log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

$$\Delta log f(x) = log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

Example 3.6 Prove that $e^x = \frac{\Delta^2}{E} e^x \frac{Ee^x}{\Delta^2 e^x}$ the interval of differencing being h.

Solution: We know that

$$E\, f(x) = f(x+h)$$

$$Ee^x = e^{x+h}$$

$$and\ \Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$$

$$also\ \ \Delta^2 e^x = e^x.(e^h - 1)^2$$

$$Now\ \left(\frac{\Delta^2}{E}\right)e^x = (\Delta^2 E^{-1})e^x = \Delta^2 e^{x-h}$$

$$e^{-h}(\Delta^2 e^x) = e^{-h} e^x (e^h - 1)^2$$

$$\therefore \quad RHS \;=\; e^{-h}e^{x}(e^{h}-1)\frac{e^{x+h}}{e^{x}(e^{h}-1)^{2}}=e^{x}.$$

Example 3.7: Given $u_0 = 1, u_1 = 11, u_2 = 21, u_3 = 29,\ find\ \Delta^2 u_0$

Solution:

$$\Delta^2 u_0 = (E-1)^4 u_0$$

$=(E^4- {}^4C_1E^3 + {}^4C_2E^2 - {}^4C_3E+1)U_0$

$$= E^4 y_0 - 4E^3u_0 + 6E^2u_0 - 4Eu_0 + u_0$$

$$= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0$$

$$= 29 - 112 + 126 - 44 + 1$$

$$= 0$$

3.5 The Operator D

By denotes the differential coefficient of y with respect to x where $D = \frac{d}{dx}$ we have $Dy = \frac{dy}{dx}$. The n^th^ derivative of y with respect to x is denoted by $D^n y = \frac{d^n y}{dx^n}$.

Relation between the operators Δ, **D and E:** we know that

$$Df(x) = \frac{d}{dx}f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2}f(x) = f''(x)\ \ etc.$$

3.6 Backward Differences

Let y = f(x) be a function given by the values $y_0+y_1+,\dots\dots y_n$ which it takes for the equally spaced values x_0, x_1, x_2, …..x_n of the independent variable x. Then y_1-y_0, y_2- y_1, ……. y_n-y_{n-1} are called the first backward differences of y = f(x). They are denoted by respectively. Thus we have

$$y_1 - y_0 = \nabla y_1$$
$$y_2 - y_1 = \nabla y_2$$
$$y_n - y_{n+1} = \nabla y_{n'}$$

Table 3.1: Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Note: In the above table the differences $\nabla^2 y$ with a fixed subscript i, lie along the diagonal upward sloping.

Alternative notation: Let the function y = f(x) be given at equal spaces of the independent variable x at x = a, a+h, a+2h….. then we define

$$\nabla f(a) = f(a) - f(a-h)$$

Where ∇ is called the backward difference operator, h is called the interval of differencing.

In general we can define.

$$\nabla \mathrm{f(x)} = \mathrm{f(x)} - \mathrm{f(x-h)}.$$

We observe that

$$\nabla \mathrm{f(x+h)} = \mathrm{f(x+h)} - \mathrm{f(x)} = \nabla \mathrm{f(x)}$$

$$\nabla \mathrm{f(x+2h)} = \mathrm{f(x+2h)} - \mathrm{f(x+h)} = \nabla \mathrm{f(x+h)}$$

…………………….

$$\nabla \mathrm{f(x+nh)} = \mathrm{f(x+nh)} - \mathrm{f(x+(n-1)h)}$$

$$= \nabla \mathrm{f(x+(n-1)h)}$$

Similarly we get,

$$\nabla^2 \mathrm{f(x+2h)} = \nabla[[\mathrm{f(x+2h)}]$$

$$= \nabla[\Delta \mathrm{f(x+h)}]$$

$$= \Delta[\Delta \mathrm{f(x)}]$$

$$= \Delta^2 \mathrm{f(x)}$$

$= \nabla^n f(x + nh) = \Delta^n f(x)$

Relation between E and ∇

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x)$$

$$\nabla = 1 - E^{-1}$$

$$\nabla \equiv \frac{E-1}{E}$$

Example 3.7: Prove the following

(a) $(1+\Delta)(1-\nabla) = 1$ (b) $\Delta\nabla = \Delta - \nabla$ (c) $\nabla = E^{-1}\Delta.$

Solution:

$(\boldsymbol{a})$ $(1+\Delta)(1-\nabla)\, f(x) = EE^{-1} f(x)$

$= Ef(x-h)$

$= f(x) = 1f(x)$

$\therefore (1+\Delta)(1-\nabla) \equiv 1$

$(\boldsymbol{b})$ $\Delta\nabla f(x) = (E-1)(1-E^{-1})f(x)$

$= (E-1)[f(x) - f(x-h)]$

$= Ef(x) - f(x) - Ef(x-h) + f(x-h)$

$= f(x+h) - f(x) - f(x) + f(x-h)$

$= [Ef(x) - f(x) + f(x)f(x-h)]$

$= (E-1)f(x) - (1-E^{-1})f(x)$

$= [(E-1) - (1-E^{-1})]f(x)$

$= (\Delta - \nabla)f(x)$

Thus, $\Delta\nabla f(x) = (\Delta - \nabla)f(x)$

That is, $\Delta\nabla = \Delta - \nabla$

$(\boldsymbol{c})\ \ \nabla f(x) = (1 - E^{-1})f(x)$

$= f(x) - f(x - h)$

$and\ \ E^{-1}\Delta f(x) = \ \ E^{-1}[f(x+h) - f(x)]$

$= f((x) - f(x - h) = \nabla$

$\therefore \quad \nabla = E^{-1}\,\Delta.$

3.7 Factorial Polynomial

A factorial polynomial denoted by $x^{(r)}$ is the product of r consecutive factors of which the first factor is x and successive factors

$x^{(r)} = x(x-h)(x-2h)\ldots..[x-(r-1)h]$

$when\ h = 1$

$x^{(r)} = \mathrm{x}(\mathrm{x}-1)(\mathrm{x}-2)\ldots..(\mathrm{x}-\mathrm{r}+1)$

and in particular

$x^{(0)} = 1$

$x^{(1)} = x$

$\Delta x^{(r)} = \ (\mathrm{x}+\mathrm{h})^{(\mathrm{r})} - x^{(r)}$

$= (\mathrm{x}+\mathrm{h})(\mathrm{x}-\mathrm{h})\ldots.(\mathrm{x}+\mathrm{h}) - (\mathrm{x}-1) - (\mathrm{x}-1)\ldots(\mathrm{x}(\mathrm{x}-1)\mathrm{h})$

$= rhx^{(r-1)}$

$\Delta^2 x^{(r)} = \Delta\big(\Delta x^{(r)}\,\big)$

$= \mathrm{r}\ (\mathrm{r}-1)\mathrm{h}^2 x^{(r-2)}$

In general we can write

$\Delta^r x^{(r)} = r(r-1)\ldots.1 \times h^r$

$= h^r r!$

Note:

1. $\Delta^r x^{(r)} = 0$

2. If the interval of differencing is unity then the successive differences of $x^{(r)}$,can be obtained by ordinary successive differential of $x^{(r)}$.

3. If r is a positive integer then

$$x^{(-r)} = \frac{1}{(x+h)(x+2h)\ldots.(x+r)}$$

$and\ if\ r = 1$

$$x^{(-r)} = \frac{1}{(x+1)(x+2)\ldots.(x+r)}$$

3.7.1 To express a given Polynomial in Factorial Notation

A polynomial of degree r can be expressed as a fractional polynomial of the same degree.

Let f(x) be a polynomial of degree which is to be expressed in factorial notation and let

F(x) = $a_0 + a_1x^1 + a_2x^2 + \cdots..a_rx^r$ ………………………. (3.1)

Where a_0, a_r are constants and $a_0 \neq 0$ then

$\Delta f(x) = \Delta[a_0 + a_1x^1 + \cdots..+a_rx^r]$

$\Rightarrow \Delta f(x) = \Delta\left[a_1 + 2a_1x^1 \ldots..+ra_rx^{(r-1)}\right]$

$$\therefore \Delta^2 f(x) = \Delta\left[a_1 + 2a_1x^1 \ldots..+ra_rx^{(r-1)}\right]$$

$\Rightarrow \Delta^2 f(x) = 2a_2 + 2 \times 3a_3x^1 + \cdots.+r(r-1)x^{(r-2)}$

………………………………………………………………………..

$\Delta^r f(x) = a_r r(r-1) \ldots \ldots 2 \times 1x^{(0)}$

$= a_r r!$

Substituting x=0 in the above we get

$f(0) = a_0, \frac{\Delta f(0)}{1!} = a_1, \frac{\Delta^2 f(0)}{2!}x^2 + \cdots \ldots + \frac{\Delta^r f(0)}{r!}x^r$

Putting the values of $a_0, a_1, a_2, \ldots.a_r$ in (1.1) we get

$f(x) = f(0) + \frac{\Delta f(0)}{1!} = x^1, \frac{\Delta^2 f(0)}{2!}x^2 + \cdots \ldots + \frac{\Delta^r f(0)}{r!}x^r$

Example 3.7: If m is a positive integer and interval of difference is 1 prove that

(a) $\Delta^2 x^m = m(m-1)x(m-2)$ $\qquad$ $(b)\ \Delta^2 x^{(-m)} = m(m+1)x(m-2)$

Solution:

$(a)\ \ x^{(m)} = x(x-1)\ldots\ldots[x-(m-1)]$

$\Delta x^{(m)} = [(x+1)x(x+2)\ldots.(x+1-(m-1)] - x(x-1)\ldots.(x-(m-1)]$

$= mx^{(m-1)}$

$\Delta^2 x^{(m)} = \Delta\left[\Delta x^{(m)}\right]$

$= m\Delta x^{(m-1)}$

$= m(m-1)x^{m-2}$

$$x^{(-m)} = \frac{1}{(x+1)(x+2)\ldots.(x+m)}$$

$$\Delta\left[x^{(-m)}\right] = \frac{1}{(x+2)(x+1)\ldots.(x+m+1)} - \frac{1}{(x+1)\ldots..(x+m)}$$

$$= \frac{1}{(x+1)(x+m)}\left[\frac{1}{(x+m+1)} - \frac{1}{(x+1)\ldots(x+m)}\right]$$

$$= m\frac{1}{(x+1)(x+2)\ldots.(x+m+1)}$$

$= (-m)x^{(-m-1)}$

$\Delta^2\left(x^{(-m)}\right) = (-m)(-m-1)x^{(-m-2)}$

$= m(m+1)x^{(-m-2)}$

3.8 Central Differences

The operator δ : We now introduce another operator known as the central difference operator to represent the successive differences of a functional in a more convenient way.

The central difference operator denoted by the symbol δ is defined by

$$y_1 - y_0 = \delta y_{1/2}$$

$$y_2 - y_1 = \delta y_{3/2}$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$n - y_{n-1} = \delta y_{n-1/2}$$

For higher order differences

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_2 - \delta y_1 = \delta^2 y_{3/2}$$

$$\dots \dots \dots \dots \dots \dots.$$

$$\delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2} = \delta^n y_r = \left(E^{1/2} - E^{-1/2}\right)^n y_r$$

In this alternative notation

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

Where h is the interval of differencing. The central difference table can be formed as follows.

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

3.9 Mean Operator

In addition to the Δ, ∇, E *and* δ we define the mean operator (averaging operator) μ as

$$\mu f(x) = \frac{1}{2}\left[\left(x + \frac{1}{2}h\right) f\left(x - \frac{1}{2}h\right)\right]$$

Alternative notation If y = f(x) is a functional notation between the variable x and y then it can also denoted by y = f, or by y = y_x.

Let $y_x, y_{x+h}, y_{x+2h}, \ldots$, etc. denote the values of the dependent variable y = corresponding to the values x, x+h, x+2h,……,etc. of the independent variable then the operators $\Delta, \nabla, E \; and \; \delta$ are defined as

$$\Delta y_x = y_{x+h} - y_x$$

$$\Delta y_x = y_x - y_{x+h}$$

$$\delta y_x = y_{x+1/2h} - y_{x-1/2h}$$

$$\mu = \frac{1}{2}\left(y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h}\right)$$

Where h is the interval of differencing.

Relation between the operators: From the definition we know that

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

$$(\boldsymbol{i}) \qquad \delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

$$= E^{\frac{1}{2}}f(x) - E^{-\frac{1}{2}}f(x)$$

$$= \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)f(x)$$

$$\therefore \;\; \delta \equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

Further,

$$\delta f(x) = E^{-\frac{1}{2}}(E - 1)f(x)$$

$$= E^{-\frac{1}{2}} \; \Delta f(x)$$

$$\therefore \;\; \delta = \; E^{-\frac{1}{2}}\Delta.$$

Note from the above result we get

$$E^{-\frac{1}{2}}\delta = \Delta$$

$$(\boldsymbol{ii}) \qquad \mu f(x) = \frac{1}{2}\left[\left(x + \frac{1}{2}h\right) f\left(x - \frac{1}{2}h\right)\right]$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) f(x)$$

$$\therefore \mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right].$$

$$(\boldsymbol{iii}) \qquad E\nabla f(x) = E[f(x) - f(x-h)]$$

$$= Ef(x) - Ef(x-h)$$

$$= f(x+h) - f(x) = \Delta f(x)$$

$$\therefore \quad E\nabla \equiv \Delta$$

and

$$\nabla E f(x) = \nabla f(x+h)$$

$$= f(x+h) - f(x)$$

$$= \Delta f(x)$$

⇨$\Delta\, E = \Delta$

$$\therefore E\nabla = \Delta E$$

Note: From the above it is clear that operator E and Δ commute and $\Delta, \nabla, \delta\ and\ E\ and\ \mu$ also commute.

3.10 Summary

We define the forward difference operator $\Delta\ as$ Δf(x) = f(x+h)-f(x) and backward difference operator as ∇f(x) = f(x)-f(x-h), where h is the interval of differencing. The operator E is called the sifting operator and is defined as E f(x) = f (x+h). Method to express any function in the form of factorial polynomial is given and is shown that for any polynomial in this form the operation of $\Delta$is equivalent to obtaining its first derivative. Central Deference operator δ and μ are also defined which are used in central difference formulae.

3.11 Exercise

3.1 Show that if $u_x = 2^x\ then\ \Delta u_x = u.$

3.2 Given any function$u_x = x(x-1)(x-2)$ $prove\ that$ $\Delta u_x = 3u(x-1)$

3.3 Prove that $\Delta x^{(n)} = nx^{(n-1)}$ for all integers n.

3.4 Find the function whose first difference is $9x^2 + 11x + 5$.

3.5 Express $3x^2 + 3x + 1$ in factorials. Hence or otherwise find its third difference.

3.6 Evaluate

$$(i)\ \Delta ab^{cx} \qquad (ii)\Delta\left[\frac{2^x}{(x+1)!}\right] \qquad (iii)\ \Delta^2 x^3$$

3.7 Show that $\Delta^n \sin(a+bx) = \left(2\sin\frac{b}{2}\right)^n\ sin\ \left[a + bx + \frac{n}{2}(\pi + b)\right]$

3.8 Evaluate

$$(\boldsymbol{i}) \qquad \Delta^3[(1-x)(1-2x)(1-3x)] \qquad (\boldsymbol{ii}) \qquad \frac{\Delta^2 x^3}{E^2 x^3}$$

$$(\boldsymbol{iii}) \qquad \frac{\Delta^2}{E}\sin(x+h) + \frac{\Delta^3 \sin(x+h)}{E\sin(x+h)}$$

Ans. (3.8) (i) -36 (ii) $\frac{6xh^2}{(x+2h)^2}$ (iii) 2 (Cosh-1) [sin(x+h)+1]

3.12 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit - 4: Interpolation with Equal Intervals

Structure

4.1 Introduction

Meaning of interpolation and extrapolation: In statistical investigation, many a time such situations arise under which it becomes necessary to obtain data for some values, periods or class intervals in addition to data already available. This problem is solved by the technique of interpolation and extrapolation. Interpolation is a statistical process by which the value of any item period is estimated within the limits of items or period given in the series, while in extrapolation the value is estimated for any item or period beyond the limits or periods given in the series. The meaning of interpolation and extrapolation can be explained, suppose, the census figure of India are given for the years 1961, 1971, 1981, 1991 and 2001. Now if we have to estimate the population for any year lying between 1961 and 2001, say fro 1965, 1973 or 1980 the technique for such estimation will be called interpolation. However, if the estimate is required for any year before 1961 or after 2001 the technique of such estimation will be called extrapolation. Thus interpolation consists in evaluation a value, which lies between two extreme points and extrapolation means finding a value that lies outside the two extreme points.

Importance or interpolation and extrapolation: The statistical technique of interpolation and extrapolation are of great practical use in science, engineering economics and business fields. Sometimes it is impossible to collect information for each and every time period due to financial or other practical difficulties or some data may be destroyed due to natural causes like flood, earthquake, fire or due to improper handling the techniques of interpolation and extrapolation are very useful and given an estimate or idea about the phenomenon for any intervening periods. Now let us consider the computation of trajectory of a rocket flight, where we solve the Euler's dynamical equations of motion to compute its position and velocity vectors at specified times during the flight. Under the same conditions, suppose, we require the position and velocity vector at some other intermediate times; we need not compute the trajectory again by solving the dynamical equations. Instead, we can use the best known interpolation technique to get the desired values. The techniques of interpolation are also use in determination of various statistical extrapolation is useful for making future forecast on the basis of past record of events under extrapolation is very important statistical tools but it is should keep in mind that they give only the most likely and that too under the certain assumptions. The accuracy of interpolation depends upon the knowledge of possible fluctions of the phenomenon and phenomenon itself.

The word interpolation denotes the method of computing the value of the function y= f(x) for any given value of the independent variable x when a set of values of y= f(x) for certain values of x are given.

Let y = f(x) which takes the values y_0, y_1, …….. y_n corresponding to the x_0, x_1, x_2………x_n of the independent variable x. If the form of the function is known, we can find any value of y corresponding to any value of x. But in most of the practical problems, the exact from of the function is not known. In such cases the function f(x) is replaced by a simpler function say $\emptyset(x)$ which has the same value as f(x) for x_0, x_1, x_2………x_n. The function $\emptyset(x)$ is called an interpolation function.

4.2 Objectives

After going through this unit you will learn about

- The meaning of interpolation and extrapolation.
- Newton - Gregory Forward Interpolation Formula and its applications.
- Newton - Gregory Backward Interpolation Formula and its applications.

4.3 Missing Values

Let a function y = f(x) be given equally spaced values x_0, x_1, x_2………x_n of the argument and y_0, y_1, y_2….…..y_n denote the corresponding values of the function. If one or more values of y = f(x) are missing we can find the missing values of using the relation between the operators E and Δ.

4.4 Newton - Gregory Forward Interpolation Formula

Let y = f(x) be a function which takes the y_0, y_1, y_2….…..y_n corresponding to the (n+1) values x_0, x_1, x_2………x_n of the independent variable x. Let the values x be equally spaced, i.e.,

$$x_r = x_0 + rh \qquad\qquad r = 0,1,2, \ldots\ldots, h$$

Where h is the interval of differencing let be a polynomial of the nth degree in x taking the same values as y corresponding to x_0, x_1, x_2………x_n i.e., $\emptyset(x)$ represents the continuo function y = f(x) such that $f(x_t), = \emptyset(x_r)$, r = 0, 1, 2, …., n and n and at all other point f(x) = $\emptyset(x)$+ R(x) where R(x) is called the error term (Remainder term) of the interpolation formula. Ignoring the error term let us assume.

$$f(x) \approx \emptyset(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \dots \dots ..$$

$$+a_n(x - x_0)(x - x_1) \dots \dots .. (x - x_{n-1}) \qquad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \quad (4.1)$$

The constants $0, a_1, a_2, \dots \dots . a_n$ can be determined as follows:

Putting x = x_0 in (4.1) we get

$$f(x_0) \approx \emptyset(x_0) = a_0$$

$$\Rightarrow \qquad y_0 = a_0$$

Putting x = x_1 in (4.1) we get

$$f(x_1) \approx \emptyset(x_1) = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$

$$\therefore \qquad y_1 = y_0 + a_1 h$$

$$\Rightarrow \qquad a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Putting x = x_2 in (4.1) we get

$$f(x_2) \approx \emptyset(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x - x_0)(x - x_1)$$

$$\therefore \qquad y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$

$$y_2 = y_0 2(y_1 - y_0) + a_2(2h^2).$$

$$\Rightarrow \qquad a_2 = \frac{y_2 - 2y_1 + y_0}{h} = \frac{\Delta^2 y_0}{2!\, h^2}$$

Similarly by putting x = x_3, x = x_4, ……. x_n in (4.1) we get

$$a_3 = \frac{\Delta^3 y_0}{3!\, h^2}, \qquad a_4 = \frac{\Delta^4 y_0}{4!\, h^2}, \qquad \dots \dots \quad a_n = \frac{\Delta^n y_0}{n!\, h^n}$$

Putting the values of $a_1, a_2, \dots \dots . a_n$ in (4.1), we get

$$f(x) \approx \emptyset(x) = y_0 + \frac{\Delta^2 y_0}{2!\, h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!\, h^2}(x - x_0)(x - x_1)(x - x_2)$$

$$+\cdots..+\frac{\Delta^n y_0}{n!\,h^n}(x-x_0)(x-x_1)(x-x_{n-1}) \qquad (4.2)$$

Writing

$$u=\frac{x-x_0}{h}, we\ get$$

$$x-x_0=uh$$

$$x-x_1=x-x_0+x_0-x_1$$

$$=(x-x_0)-(x_1-x_0)$$

$$=uh-h=(u-1)h$$

Equating (4.2) can be written as

$$f(x_0+uh)=y_0+\Delta y_0+\frac{u(u-1)}{2!}\Delta^2 y_0+\frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots..+\frac{u(u-1)(u-2)\ldots\ldots(u\ldots n+1)}{n!}\Delta^n y_0$$

The above formula is called Newton's forward interpolation formula

Note:

1. Newton forward interpolation formula is used interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table, but the formula contains only those of y which come after the value chosen as y_0.

Example 4.1 Evaluate $y = e^{2x}$ for x = 0.05 using the following table

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.255

Solution: The difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.000	1.0000				
		0.2214			

0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have,

$$x_0 = 0.00, \quad x = 0.05, \quad h = 0.1, \quad u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} = 0.5$$

Using Newton's forward formula

$$f(x_0) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)\ldots\ldots(u\ldots n+1)}{4!}\Delta^4 y_0 + \cdots.$$

$$f(0.05) = 1.0000 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2}(0.0490) + \frac{0.5(0.5-2)(0.5-3)}{6}(0.0109)$$

$$= 1.000 + 0.110 - 0.006125 + 0.000681 - 0.000090$$

$$= 1.105166$$

$\therefore \quad f(0.05) = 1.052$

Example 4.2 The values of sin x are given below for difference values of x. Find the values of sin 32^0.

x	30^0	35^0	40^0	45^0	50^0
y = sin x	0.5000	0.5736	0.6428	0.7071	0.7660

Solution: x = 32^0 is very near to the starting value x_0 = 30^0, by using Newton's forward interpolation formula difference table is

x	y = sin x	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30^0	0.5000				

		0.0736			
35^0	0.5736		-0.0044		
		0.0692		-0.005	
40^0	0.6428		-0.0049		0
		0.0643		-0.005	
45^0	0.7071		-0.0054		
		0.0589			
50^0	0.7660				

$$u = \frac{x - x_0}{h} = \frac{32^0 - 30^0}{5} = 0.4$$

We have $y_0 = 0.5000$, $\Delta y_0 = 0.0736$, $\Delta^2 y_0 = -0.0044$, $\Delta^3 y_0 = -0.005$ putting these values in Newton's forward interpolation formula we get

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots.$$

$$f(32^0) = 0.5000 + 0.4 \times 0.0736 + \frac{0.4(0.4-1)}{2}(-0.0044) + \frac{0.4(0.4-1)(0.4-2)}{4}(-0.0005)$$

$$= f(32^0) = 0.5000 + 002944 + 0.00528 - 0.00032 = 0.529648$$

Example 4.3: In an examination the number of candidates who obtained marks between certain limits were as follows:

Marks	30-40	40-50	50-60	60-70	70-80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution: First of all we construct a cumulative frequency table for the given data.

Upper limits of the intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x Marks	y Cumulative Frequencies	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		4		
		31			
80	190				

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

We have $x_0 = 40$, $x = 45$, $h = 10$, $\Delta y_0 = 42$, $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = -25$, $\Delta^4 y = 37.$

From Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)\dots\dots(u\dots n+1)}{4!}\Delta^4 y_0 + \cdots.$$

$$f(45) = 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(0.5-1)(0.5-2)}{6}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452 = 47.8673 = 48\ (approximately)$$

∴ The number of students who obtained mark less than 45-48, and the number of students who scored between 45 and 50 = 73 – 48 = 25.

4.4 Newton- Gregory Backward Interpolation Formula

Newton's forward interpolation cannot be used for interpolation a value of y near the end of a table of values. For this purpose, we use another formula known as Newton- Gregory backward interpolation formula. It can be derived as follows:

Let y = f(x) be a function which takes the values y_0, y_1,y_2……..y_n corresponding to the values x_0, x_1, x_2………x_n of the independent variable x. Let the values x be equally spaced with h as the interval of differencing, Let,

$$x_r = x_0 + rh \qquad\qquad r = 0, 1, 2, \ldots\ldots, n.$$

Let $\emptyset(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to x_0, x_1, x_2………x_n i.e., $\emptyset(x)$ represents y = f(x) such that $f(x) = \emptyset(x_r)$, r = 0, 1, 2, …. n we may write $\emptyset(x)$as

$$f(x) \approx \emptyset(x) \approx a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \cdots \ldots \ldots$$

$$+a_n(x - x_0)(x - x_{n-1}) \ldots \ldots (x - x_1) \qquad \ldots \ldots \ldots (4.3)$$

Putting x = x_n in (4.3) we get

$$f(x_n) \approx \emptyset(x_n) = a_0$$

$$\Rightarrow \qquad y_n = a_0$$

Putting x = x_{n-1} in (4.3) we get

$$f(x_{n-1}) \approx \emptyset(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow \; y_{n-1} = y_n + a_1(-h)$$

$$\Rightarrow \; a_1 h = y_n + y_{n-1} = \Delta y_n$$

$$\Rightarrow \; a_1 = \frac{\Delta y_n}{1!\, h}$$

Putting x = x_{n-2}, we get

$$f(x_{n-2}) \approx \emptyset(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow \; y_{n-1} = y_n + \left(\frac{y_n - y_{n-1}}{h}\right)(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow \; y_{n-1} = y_n - 2y_n + 2y_{n-2} + (2h^2)a_2$$

$$\Rightarrow \; a_2 = \frac{y_n - 2y_{n-2} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!\, h^2}$$

Similarly by putting x = x_{n-3}, x = x_{n-4}, …….x = x_{n-5} in (4.1) we get

$$a_3 = \frac{\Delta^3 y_n}{3!\,h^3}, \qquad a_4 = \frac{\Delta^4 y_0}{4!\,h^3}, \dots\dots \quad a_n = \frac{\Delta^n y_n}{n!\,h^n}$$

Substituting these values in (4.3) , we get

$$f(x) \approx \emptyset(x) = y_n$$
$$= \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!\,h^2}(x - x_n)(x - x_{n-1})$$
$$+ \frac{\Delta^3 y_n}{3!\,h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2})$$

$$+ \cdots .. + \frac{\nabla^n y_n}{n!\,h^n}(x - x_n)(x - x_{n-1})(x - x_1) \qquad \dots .. (4.4)$$

Writing

$$u = \frac{x - x_n}{h}, we\ get$$

$$x - x_n = uh$$

$$x - x_{n-1} = x - x_n + x_0 - x_{n-1}$$

$$= (uh) + h = (u + 1)h$$

Similarly,

$$x - x_{n-2} = (u + 2)h, \dots .. (x - x_1) = (u + n - 1)h$$

Substituting the above values, we get from (4.4),

$$f(x_n + uh) = y_n + u\nabla y_0 + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+1)}{3!}\nabla^3 y_n$$
$$+ \cdots .. + \frac{u(u+1)(u+2)\dots\dots(u+n-1)}{n!}\nabla^n y_n \qquad \dots\dots\dots\dots\dots.. \quad (4.5)$$

The above formula is known as Newton's backward interpolation formula

Example 4.4 The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees c and P is the percentage of lead in the alloy.

P	40	50	60	70	80	90
t	180	204	226	250	276	304

Find the melting point of the alloy containing 84 percent lead.

Solution: The value of 84 is near the end of the table, therefore we use the Newton's backward interpolation formula. The difference table is

P	t	∇	∇^2	∇^3	∇^4	∇^5
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have

$$x_n = 90, x = 84, h = 10, t_n = y_n = 304, \nabla t_n = \nabla y_n = 28, \nabla^2 y_n = 2,$$

And $\nabla^3 y_n = \nabla^4 y_n = \nabla^5 y_n = 0,$

$$u = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -60$$

From Newton's backward formula

$$f(84) = t_n + u\nabla t_n + \frac{u(u+1)}{2}\nabla^2 t_n + \cdots ..$$

$$f(84) = 304 - 0.6 \times 28 + \frac{(-0.6)(-0.6+1)}{2} 2$$

$$= 304 - 16.8 - 0.24$$

$$= 286.96$$

Example 4.5: Calculate the value of (7.5) for the table.

x	1	2	3	4	5	6	7	8
f(x)	1	8	27	64	125	216	343	512

Solution: 7.5 is near to the end of the table, we use numbers backward formula to find f(7.5).

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

We have

$$x_n = 8, x = 7.5, h = 1, y_n = 512, \nabla y_n = 169, \nabla^2 y_n = 42,$$

And $\nabla^3 y_n = 6,\ \nabla^4 y_n = \nabla^5 y_n = 0,$

$$u = \frac{x - x_n}{h} = \frac{7.5 - 8}{1} = -0.5$$

We get

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+1)}{3!}\nabla^3 y_n + \cdots ..$$

$$f(7.5) = 512 + (00.5)(165) + \frac{-0.5(-0.5+1)}{2}(42) + \frac{-0.5(-0.5+1)(-0.5+2)}{6}(6)$$

$$= 512 - 82.5 - 0.375$$

$= 423.87$

4.6 Summary

Newton Gregory forward and backward differences interpolation formula are used to find out the value of the entry corresponding to any arrangement for which it is not known. Both of these formulae are for the situation when the values in table are at equal interval. If we intended to find the values of the function corresponding to any argument in the end of tabular values, we use backward difference formula.

4.7 Exercise

4.1 Find the missing term in the following table:

x:	16	18	20	22	24	26
y:	39	85	-	151	264	388

4.2 Given the following table, construct difference table and form it estimate y when x = 0.7

X =	0	0.1	0.2	0.3	0.4
Y =	1	1.095	1.179	1.251	1.310

4.3 Apply Gregory Newton backward formula to the following data for finding sun's declination on feb. 12.

Date	1	3	5	
Declination	$-17^0 0'19.0''$	$-16^0 25'22.9''$	$-13^0 49'18.8''$	
Date	7	9	11	13
Declination	$-15^0 12'9.8$	$-14^0 33'39.1''$	$13^0 54'49.8''$	$-13^0 14'45.0''$

Answer:

4.1: 96.4

4.2 : 1.399

4.3: $13^0 34'40.7''$

4.8 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena

2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit-5: Interpolation with Unequal Intervals

Structure

5.1 Introduction

The Operators E, Δ, ∇ and Newton's Gregory Forward and Backward Interpolation formulae can be entries $f(x_0)$, $f(x_1)$, $f(x_2)$, ………$f(x_n)$ are given corresponding to the equidistant arguments x_0, x_1, x_2, ………x_n. It is not always possible that the arguments $x_0 < x_1 < x_2 <$ ………x_n lying in the interval [a, b] are equidistant. In this case we cannot use the operators E, , Δ and, ∇ . Here the divided difference operator Δ is defined and used to obtain Newton's divided difference interpolating polynomial. This polynomial is used for interpolation unequal interpolation.

5.2 Objectives

After going through this unit you should be able to:

- Obtain a divided difference in terms of function values.
- Construct a divided difference table
- Show that the divided difference is independent of the order of its arguments
- Obtain the Newton's divided difference interpolating polynomial for a given data.
- Compute an estimate of f(x) for a given data
- Relate the k^{th} derivative of f(x) with the k^{th} order divided difference from the expression for the error term.

5.3 Divided Differences

Let the function y = f(x) be given at the (n+1) points x_0, x_1, x_2………x_n which need not be equally spaced that is $x_1 - x_0, x_2 - x_1, x_3 - x_2, \dots \dots .. x_n - x_{n-1}$,are not necessary equal. Let $f(x_0)$, $f(x_1)$, $f(x_2)$, ………$f(x_n)$ be the given values (entries) corresponding to arguments x_0, x_1, x_2………x_n. In some places we have written $y_r = f(x_1)$ for r = 0,1,2,3,…….n.

If the arguments were equidistant then the successive differences between values of the entry, without taking into account the corresponding changes in the values of the independent variable, i.e., the arguments are called the forward differences. But if the values of the arguments are given at unequal intervals, then the various differences in entry will be affected by the change in the values of the arguments. The differences in entries defined by taking into account the differences in arguments are called divided differences.

The first order divided difference of f(x) for the arguments x_0, x_1 is defined as

$$f[x_0, x_1] = \frac{f(x_1) - f(x_2)}{x_1 - x_0} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5.1)$$

Here, the ordinary difference of entries $f(x_1) - f(x_0)$ is divided by the difference (x_1-x_0) of the arguments. Hence we call it a divided difference. In fact, it is the ratio of change in f(x) to change in x from x_0 to x_1.

Similarly,

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5.2)$$

$$f[x_2, x_3] = \frac{f(x_3) - f(x_2)}{x_3 - x_2} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5.3)$$

The second order divided differences for the arguments x_0, x_1, x_2……… is defined as

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5.4)$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots . (5.5)$$

………………….

…………………..

The third order divided differences can be defined as

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. (5.6)$$

and so on, nth order divided differences for the arguments x_0, x_1, x_2………x_{n-1} is

$$f[x_0, x_1, x_2 \dots\dots x_n] = \frac{f[x_1, x_2 \dots x_n] - f[x_0, x_1, x_2 \dots . x_{n-1}]}{x_n - x_0} \quad \dots\dots\dots\dots. (5.7)$$

It may be observed that the order of a divided difference is one less than the number of arguments required for its definition.

Another notation for divided differences uses the divided differences operator Δ.

The divided differences as follows;

Divided Difference of first order

$$f[x_0, x_1] = \frac{\Delta}{x_1} f(x_0)$$

$$f[x_1, x_2] = \frac{\Delta}{x_2} f(x_1)$$

Divided Difference of Second order

$$f[x_0, x_1, x_2] = \frac{\Delta^2}{x_1 - x_2} f(x_0)$$

$$f[x_1, x_2, x_3] = \frac{\Delta^2}{x_1 - x_2} f(x_1)$$

Divided Difference of third order

$$f[x_0, x_1, x_2, x_3] = \Delta^3 f(x_2)$$

Divided Difference of order n

$$f[x_0, x_1, x_2, x_3] = \frac{\Delta^n}{x_1, x_2, \dots . x_n} f(x_0)$$

……….(5.8)

5.3.1 Properties of Divided Differences

Theorem 5.1: The differences are symmetrical functions of their arguments.

Or

The divided differences are independent of the order of the arguments.

Proof: The zero[th] order divided difference of f(x) at x_0 is

$$f[x_0] \quad = \quad f(x_0)$$

The first order divided difference of f(x) for arguments $x_0,\ x_1$ is

$$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{f[x_0] - f[x_1]}{x_0 - x_1} = f[x_1, x_0]$$

$$= \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \qquad (5.9i)$$

The second order divided difference of f(x) for arguments x_0, x_1, x_2 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{x_2 - x_0}\left[\left\{\frac{f(x_1)}{x_1 - x_2} + \frac{f(x_2)}{x_1 - x_2}\right\} - \left\{\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0}\right\}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} \qquad (5.9ii)$$

By symmetry of the results in (5.9ii), it is obvious that

$$f[x_0, x_1, x_2] = f[x_0, x_2, x_1] = f[x_1, x_0, x_2] = f[x_1, x_2, x_0] = f[x_2, x_0, x_1] = f[x_2, x_1, x_0]$$

By mathematical induction it can be shown that

$$f[x_0, x_1, x_2 \ldots\ldots x_n] = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots.(x_0 - x_n)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots.(x_1 - x_n)} + \cdots. + \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots.(x_n - x_{n-1})} \qquad (5.9iii)$$

Thus, from (5.9i), (5.9ii) and (5.9iii) it follows that the order of placing the arguments is immaterial in the definition of the divided difference. Thus, f(x_0, x_1, x_2………x_n) is a symmetrical function of x_0, x_1, x_2………x_n for all n.

Hence Proved

Divided Difference Table is the table representing the arguments corresponding entries and the divided differences of all possible orders.

Table 5.1: Divided difference table

Argument	Entry	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$

x	f(x)			
x_1	$f(x_0)$	$f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0}$		
			$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$	
x_1	$f(x_1)$			
		$f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1}$		$f[x_0, x_1, x_2 \ldots\ldots x_n] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_1}$
x_2	$f(x_2)$		$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1}$	
		$f[x_2, x_3] = \frac{f[x_3] - f[x_2]}{x_3 - x_2}$		
x_3	$f(x_3)$			

Here, f(x_0) is called term. $f\left[x_0, x_1, x_{2,}. x_3\right]$ are called leading divided differences of order first, second and third respectively and they lie along the diagonal at f(x_0).

Theorem 5.2: The nth order divided of divided differences of the a polynomial of degree n in x are constant.

Proof: Let f(x)= ax^n be a polynomial of degree n, where n is a positive integer. Then

$$f[x_0, x_1] = \frac{f[x_0] - f[x_1]}{x_0 - x_1} = \frac{a(x_0^n - x_1^n)}{x_0 - x_1} = a[x_0^{n-1} + x_1 x_0^{n-2} + \cdots + x_1^{n-1}]$$

Thus, $f[x_0, x_1]$ is a polynomial of degree (n-1) symmetrical in x_0, x_1 with leading coefficient a.

Now,

$$f[x_0, x_1, x_2] = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \frac{a[x_0^{n-1} + x_1 x_0^{n-2} + \cdots + x_1^{n-1}]}{x_0 - x_2}$$

$$= \frac{a}{x_0 - x_2}[(x_0^{n-1} - x_0^{n-1})x_1(x_0^{n-2} - x_2^{n-2}) + \cdots + x_1^{n-1}(x_0 - x_1)]$$

$$= a[(x_0^{n-2} + x_2 x_0^{n-3} + \cdots + x_2^{n-2}) + x_1(x_0^{n-3} + x_2 x_0^{n-4} + \cdots + x_2^{n-3}) + \cdots \ldots + x_1^{n-1}]$$

Thus, we observe that second order divided difference $f[x_0, x_2, x_1]$ is a polynomial symmetrical in $x_1, x_2, x_3 \ldots \ldots x_n$ with leading coefficient a. The degree of this polynomial is (n-2), so that it is reduced by 2.

Similarly it can be shown that the nth order divided difference $f(x_0, x_1, x_2 \ldots \ldots x_n)$ is a polynomial of degree = n-n=0, which is a constant "a", where a $\neq 0$.

If f(x) is a polynomial of degree n, say,

$$\mathrm{g(x)} = \mathrm{a_0 x^n} + \mathrm{a_1 x^{n-1}} + \mathrm{a_2 x^{n-2}} + \cdots. + \mathrm{a_{n-2} x^2} + \mathrm{a_{n-1} x} + a_n \quad where\ a_0 \neq 0.$$

Then

$$g[x_0] = a\ polynomial\ of\ degree\ n\ in\ x_0.$$

$$g[x_0, x_1] = a\ polynomial\ of\ degree\ (n-1) in\ (x_0, x_1)$$

$$g[x_0, x_1, x_2] = a\ polynomial\ of\ degree\ (n-2) in\ (x_0, x_1, x_2) \ldots \ldots$$

$$g[x_0, x_1, x_2 \ldots . x_k] = a\ polynomial\ of\ degree\ (n-n) = 0\ in\ (x_0, x_1, x_2 \ldots x_n)$$

But the nth divided difference of g(x) is

$$g[x_0, x_1, x_2 \ldots . x_n]$$
$$= z_0[nth\ div.diff.of\ x^n] + a_1[nth\ div.diff.of\ x^{n-1}]$$
$$+ a_2[nth\ div.diff.of\ x^{n-2}] + \cdots . a_n[nth\ div.diff.of\ x^0]$$

$$= a_0.1 + a_1.0 + a_2.0 + \cdots \ldots . + a_n.0$$

$$= a_0(which\ is\ a\ constnat) \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .(5.10)$$

Hence the result.

Corollary:

Thus, we conclude that divided difference of g(x) of order r ≥ (n+1)will be zero.

Now, we shall apply above result to some problems.

Example 5.1: Obtain the third divided difference $f[x_0, x_1, x_2, x_3]$ of the function f(x)= 1/x.

Solution: Here x=x_0, x_1, x_2, x_3 and f(x) = 1/x.

Therefore, first order divided difference f $[x_0, x_1]$ is

$$f[x_0, x_1] = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{(1/x_0) - (1/x_1)}{x_0 - x_1} = -(1/x_0, x_1)$$

The second order divided difference f $[x_0, x_1, x_2]$ is

$$f[x_0, x_1, x_2] = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2} = \frac{\{-(1/x_0, x_1)\} - \{(1/x_1, x_2)\}}{x_0 - x_2} = (-1)^2(1/x_0, x_1, x_2)$$

The third order divided difference $f[x_0, x_1, x_2, x_3]$ is

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_0 - x_3} = \frac{\{(-1)^2(1/x_0, x_1, x_2)\} - \{(-1)^2(1/x_1, x_2, x_3)\}}{x_0 - x_3}$$

$$= (-1)^3(1/x_0, x_1, x_2, x_3) = -\left(\frac{1}{x_0, x_1, x_2}, x_3\right) \qquad Ans.$$

Example 5.2: Prepare the divided difference table for the following data.

x	0	1	2	4	6
f(x)	1	14	15	5	19

Solution: Here

$$x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 4, x_4 = 6.$$

F(0)=1, f[0,1]=13, f[0,1,2]=3, f[0,1,2,4]=1 and f[0,1,2,4,6]=0 are called leading term and leading di

vided differences of first, second, third and fourth order respectively. They lie along the diagonal at f(x_0)=1 as shown by bold italics in the following divided difference table.

Table 5.2: Divided difference table

Argument x	Entry f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{(14-1)}{(1-0)} = 13$			

1	14		$\frac{(1-13)}{(2-0)} = -6$		
		$\frac{(15-14)}{(2-1)} = 1$		$\frac{(-2-(-5))}{(4-0)} = 1$	
2	15		$\frac{(-5-1)}{(4-1)} = -2$		$\frac{(1-1)}{(6-0)} = 0$
		$\frac{(5-15)}{(4-2)} = -5$		$\frac{(3-(-2))}{(6-1)} = 1$	
4	5		$\frac{(7(-5))}{(6-2)} = 3$		
		$\frac{(19-5)}{(6-4)} = 7$			
6	19				

You may attempt the following problems:

P-5.1: If f(x) = x_3, obtain the leading term and the leading divided differences taking arguments x = a, b, c, & d.

P-5.2: Construct the divided difference table for the following data:

x	5	7	11	13	21
f(x)	150	392	1452	2366	9702

P-5.3 Construct the divided difference table for the following data:

x	1	3	4	6	10
f(x)	0	18	57	177	765

P-5.4: Obtain third divided difference of the function $\Psi(x) = x^2 - 2x$ with arguments x = 2, 4, 9, 10.

5.4 Newton's Divided Difference Interpolation Formula

Let $f(x_0), f(x_1), f(x_2), \dots \dots ., f(x_n)$ be (n+1) values (entries) corresponding to the arguments $x_0, x_1, x_2 \dots \dots, x_n$ which are not necessarily equispaced. With these (n+1) values we can fit a polynomial of degree n.

Since the nth divided difference f($x_0, x_1, x_2, \dots . x_n$)would be a constant, therefore, (n+1)th divided difference.

$$f[x, x_0, x_1, x_2, \dots \dots . x_n] = 0 \; for \; x \; e[x_0, x_n] \qquad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots (5.11)$$

Obviously $f[x, x_0, x_1, x_2, \dots \dots . x_n]$ is a polynomial of degree (n+1) in x.

Derivation: From the definition of divided difference

$$f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0}$$

It implies that

$$f(x) = f(x_0) + (x - x_0) f[x, x_0] \qquad \dots \dots \dots \dots \dots \dots \dots (5.12)$$

This is linear Newton's divided difference interpolating polynomial. From the second divided difference

$$f[x, x_0, x_1] = \frac{f(x, x_0) - f(x_0, x_1)}{x - x_1}$$

We get

$$f[x, x_0] = f[x_0, x_1] + (x - x_1) f[x, x_0, x_1]$$

Substituting in Eqn. (3.12), we get

$$f(x) = f(x_0) + (x - x_0)\{f[x_0, x_1] + (x - x_1) f[x, x_0, x_1]\}$$

$$= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_1) f[x, x_0, x_1] \qquad \dots \dots \dots \dots . (5.13)$$

Continuing in this way, we get,

$$\begin{aligned} f(x) = f(x_0) &+ (x - x_0) f[x_0, x_1] + (x - x_1) f[x, x_0, x_1] \\ &+ (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x_3] \\ &+ (x - x_0)(x - x_1)(x - x_2) \dots . . (x - x_{n-1}) f[x_0, x_1, x_2, \dots x_{n-1}, x_n] \\ &+ (x - x_0)(x - x_1)(x - x_2) \dots . . (x - x_n) f[x, x_0, x_1, x_2, \dots x_{n-1}, x_n] \end{aligned}$$

………………(5.14)

Hence, Eqns. (3.11) and (3.14) together yield

$$f(x) = P(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \dots \dots + (x - x_0)(x - x_1)(x - x_2) \dots .. (x - x_{n-1})f[x_0, x_1, x_2, \dots x_{n-1}, x_n]$$

………..(5.15)

Equation (3.15) is known as Newton's divided difference interpolation polynomial

It may also be written as

$$f(x) = P(x) = \sum_{k=0}^{n} f[x_0, x_1, x_2 \dots .. x_k] \prod_{m=0}^{k-1} (x - x_m) \qquad \dots \dots ..(5.16)$$

Example 5.3: Obtain the Newton's divided difference interpolating polynomial of degree less or equal to 4 and use it to evaluate f(6), given.

x	5	7	11	13	21
f(x)	150	392	1452	2366	9702

Solution: Given $x_0 = 5, x_1 = 7, x_2 = 11, x_3 = 13, x_4 = 21$. From the divided difference table, we have

$$f(x_0) = 150, f[x_0, x_1] = 121, f[x, x_0, x_1] = 24, f[x_0, x_1, x_2, x_3] = 1 \text{ and } f[x_0, x_1, x_2, x_3, x_4] = 0.$$

Argument x	Entry f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		

917

21 9702

Therefore, on substituting the values of x_i's and values of leading divided difference in Eqn. (5.15), we obtain

$$f(x) = P_3(x) = 150\ (x-5).121 + (x-5)(x-7).24 + (x-5)(x-7)(x-11).1 + (x-5)(x-7)(x-11)(x-13).0$$
$$= 150 + 121x - 605 + 24x^2 - 288x + 840 + x^3 - 23x^2 + 167\ x - 385 + 0$$
$$= x^3 + x^2$$

Which is Newton's polynomial of degree 3. At x=6.

$$f(6) = 6^3 + 6^2 = 216 + 36 = 252$$

Remarks: if the form of the polynomial is not required then we can directly interpolate the value from Eqn. (5.15). For example, in the present case,

$$f(6) = P_3(6x) = 150\ (6-5).121 + (6-5)(6-7).24 + (6-5)(6-7)(6-11).1 + 0$$

$$= 150 + 121 - 24 + 5 + 0$$

$$= 252$$

5.5 Errors in Interpolating Polynomial

If $P_n(x)$ be the Newton's form of interpolating polynomial of degree less than or equal to n which interpolates f(x) at $x_0, x_1, x_2 \dots\dots x_{n,}$ then the interpolating error $E_n(x)$ of $P_n(x)$ is given by

$$E_n(x) = f(x) - P_n(x) \qquad \dots\dots\dots\dots\dots\dots\dots\dots.(5.17)$$

We know that the interpolating polynomial from $P_n(x)$ coincides with f(x) at $x_0, x_1, x_2 \dots\dots x_n$ and deviates at all the other points in the interval (x_0, x_n)

Let $x^* \varepsilon\ [x_0, x_n]$be any point different from $x_0, x_1, x_2 \dots\dots x_n$.If $P_{n-1}(x)$ is the Newton form of interpolating which interpolates f(x) at $x_0, x_1, x_2 \dots\dots x_n$ and x^*, then

$$P_{n-1}(x^*) = f(x^*) \dots\dots\dots\dots\dots\dots\dots\dots.(5.18)$$

From eqn. (5.16)

$$f(x_{n+1}) = P_n(x) + f[x_0, x_1, x_2 \ldots\ldots x_n, x^*] \prod_{j=0}^{n} (x - x_j)$$

Putting x= x* in the above expression, we get from Eqn. (5.18).

$$f(x^*) = P_{n+1}(x^*) = P_n(x^*) + f[x_0, x_1, x_2 \ldots\ldots x_n, x^*] \prod_{j=0}^{n} (x - x_j)$$

So that the interpolating error at x = x* is :

$$E_n(x^*) = f(x^*) -= P_n(x^*)$$

$$= f[x_0, x_1, x_2 \ldots\ldots x_n, x^*] \prod_{j=0}^{n} (x - x_j) \ldots\ldots\ldots\ldots\ldots\ldots. (5.19)$$

Which is the next term in the Newton's formula.

5.6 Divided Difference and Derivatives

We sate the theorem-3 without proof. It is useful in establishing a relationship between divided difference and the derivative of a function. It also helps us in estimating a useful bound on error.

Theorem 5.3: Let $x_0, x_1, x_2 \ldots\ldots x_n$ be distinct numbers in the interval $[\alpha, \beta]$ and f has continuous derivatives upto order (n+1) in the open interval (α, β). If $P_n(x)$ is the interpolating polynomial of degree less than or equal to n, which interpolates f(x) at the point $x_0, x_1, x_2 \ldots\ldots x_n$ then for each $x \varepsilon [\alpha, \beta]$ there exists a number $\varepsilon(x)$ in (α, β) such that

$$E_n(x) = f(x) - P_n(x) = \frac{f^{n+1}(\xi)}{\{(n+1)!\}} (x^* - x_0)(x^* - x_1) \ldots\ldots (x^* - x_n) \ldots\ldots\ldots (5.20)$$

At x=x*, Eqns. (3.19) & (3.20) yield,

$$E_n(x^*) = f(x^*) - P_n(x^*) = \frac{f^{n+1}(\xi(x^*))}{\{(n+1)!\}} (x^* - x_0)(x^* - x_1) \ldots\ldots (x^* - x_n) \ldots\ldots\ldots (5.20a)$$

$$= f[x_0, x_1, x_2 \ldots\ldots x_n, x^*] \prod_{j=0}^{n} (x^* - x_j)$$

Refer Eqn (5.19)…..(5.20b)

Let $x^* - x_{n+1}$ then on equating (5.20a) & (5.20), we get

$$f[x_0, x_1, x_2 \ldots\ldots x_{n+1}] = \frac{f^n(\xi)}{\{(n+1)\}} = \Delta^{n+1} f(x_0) \qquad (5.21)$$

Further, it can be shown that

$$\xi\varepsilon(\min x_i, \max x_i)$$

The Eqn. (5.21) establishes a relationship between divided difference and the derivatives of the function. This result is stated without proof in Theorem 5.4.

Theorem 5.4: Let f(x) be real valuated function defined on $[\alpha, \beta]$ and n times differentiable in$[\alpha, \beta]$. If $x_0, x_1, x_2 \ldots\ldots x_n$ are (n+1) distinct points in then there exist $\xi\varepsilon(\alpha, \beta)$such that

$$f[x_0, x_1, x_2 \ldots\ldots x_n] = \frac{f^n(\varepsilon)}{\{(n!)\}} = \Delta^n f(x_0) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.22)$$

Corollary 5.1: if f(x) = x^n, then

$$\Delta^n(x_0) = f[x_0, x_1, x_2 \ldots\ldots x_n] = \frac{\{n!\}}{(n!)} = 1 \ldots\ldots\ldots\ldots\ldots (5.23)$$

Corollary 5.2 : For k,n and f(x) = x^k, then

$$f[x_0, x_1, x_2 \ldots\ldots x_n] = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.24)$$

Since

$$\frac{d^n(x^k)}{d(x^n)} = 0$$

Example 5.4: Consider the first divided differences

$$f[x, x_0] = \frac{f(x) - f(x_0)}{x - x_0} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.25a)$$

By mean value theorem

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(\xi)\ where\ x_0 < \xi < x_1 \ldots\ldots\ldots\ldots (5.25b)$$

Eliminating f(x_1) between Eqns. (5.25a) & (5.25b) we obtain

$$f[x, x_0] = f'(\xi) where x_0 < \xi < x_1 \ldots\ldots\ldots\ldots (5.26)$$

Corollary 5.3: Let f(x) be a polynomial in given by

$$f(x) = a_0 + a_1 + a_2x^2 + \cdots \ldots. + a_nx^n (where\ a_n \neq 0)$$

Then, using cor. (5.1) & cor. (5.2), we obtain

$$\Delta^2 f(x) = f[x_0, x_1, x_2 \ldots\ldots x_n] = 0 + a_n \frac{\{n!\}}{(n!)} = a_n \ldots\ldots\ldots\ldots\ldots\ldots.(5.27)$$

All illustration is given below:

Example 5.5: Let f(x) = $3x^4+5x^3-8x^3+7x+14$. Obtain the value of

f[-1, 2, 3, -2]=f[5,8,10,9,4]=f[j,k,l,m,n]

Solution: Here f(x) is a polynomial of degree 4, therefore fourth order divided difference of f(x) with any set of arguments are constant and equal to 3, where 3 is the coefficient of x^4 in f(x).

Thus,

f[-1, 2, 3, -2]=f[5,8,10,9,4]=f[j,k,l,m,n]=3   Ans.

You may now try the following exercises:

E-5.5: obtain the polynomial of the lowest possible degree which assumes the values 15, 12, 3, -21, when x has values 1,2,3,-1 respectively.

E-5.6: The observed values of a function being 72, 168, 120, and 63, respectively at the four positions 9,3,7 and 10 of the independent variable. Obtain the estimate of the value of the function at x= 4 using Newton's interpolation formula.

E-5.7: Obtain the Newton's divided difference polynomial satisfied by (0,2) (1,3) (3,17), (6,158).

E- 5.8: From the following table of values, obtain the Newton's form of interpolating f(n).

n	-1	0	3	6	7
f(n)	3	-6	39	822	1611

Also find the approximation of f(x) at n= 2 and n=4.

E- 5.9: Using Newton's divided difference interpolation formula, obtain f(2), f(8) and f(18) from the following table.

x	4	5	7	9	11	14
f(x)	48	100	294	648	1210	2548

E- 5.10: Use the following data to find Newton's divided difference polynomial which approximates f(x). Hence obtain the value of f(5).

x	0	2	3	4	7	9
f(x)	4	26	58	112	466	922

5.7 Summary

In this unit we have derived Newton's general form of interpolating polynomial. It is used when given abscissa are not necessarily at equal intervals. This form is useful in deriving some other interpolating polynomials (discussed in Unit 6). Concepts of divided differences have been introduced and some of its important properties have been discussed before deriving Newton's general form. The error term has also been derived. The main formulae are listed below.

1. $f[x_0, x_1, x_2 \dots \dots x_n] = \frac{f[x_1,x_2 \dots x_n] - f[x_0,x_1,x_2 \dots . x_{n-1}]}{x_n - x_0}$
2. $f(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_1)f[x, x_0, x_1] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] +$
   $(x - x_0)(x - x_1)(x - x_2) \dots . . (x - x_{n-1})f[x_0, x_1, x_2, \dots x_{n-1}, x_n] + (x - x_0)(x - x_1)(x - x_2) \dots . . (x - x_n)f[x, x_0, x_1, x_2, \dots x_{n-1}, x_n]$
3. $E_n(x) = f[x, x_0, x_1, x_2 \dots . . . x_n] \prod_{j=0}^{n} (x - x_j)$
4. $f[x_0, x_1, x_2 \dots . . x_n] = \frac{f^n\{n!\}}{(n!)} = \Delta^n f(x_0)$ *where* $\xi \varepsilon (\min x_i, \max x_i)$

5.8 Solutions and Answers

E-5.1 Given f(x) = x^3 and x = a, b, c, d,

$\therefore$ *Leading term* $f(a) = a^3$

$$Leading\ first\ div.diff.: f[a, b] = \frac{f(b) - f(a)}{b - a} = \frac{b^3 - a^3}{b - a} = b^2 + ab + a^2$$

$$Similarly\ \ f[b, c] = \frac{f(c) - f(b)}{c - a} = c^2 + bc + b^2$$

$$Leading\ second\ div.diff.: \frac{f(b, c) - f(a, b)}{c - a} = d + c + b$$

$$Similarly\ \ f[b, c, d] = \frac{f(c, d) - f(b, c)}{d - b} = d + c + b$$

$$Leading\ third\ div.diff.: f[a, b, c, d] = \frac{f(b, c, d) - f(a, b, c)}{d - a} = 1$$

$Ans.\ a^3;\ b^2 + ab + a^2; a; 1$

P 5.2 Divided difference table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		
		917			
21	9702				

Figures within brackets show leading term and leading divided differences.

P-5.3 Divided difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1	0				
		9			
3	18		10		
		39		-0.6	
4	57		7		0.1857
		60		1.0714	
6	177		14.5		
		147			
10	765				

P-5.4 $\Psi(x)$ is a polynomial in x and of order 2. Therefore, the second divided differences are constant. Thus the third and higher divided differences with any set of arguments become zero.

Hence, f[2,4,9,10] = 0

P-5.5 We obtain f(1)=15, f(1,2] = -3, f[1,2,3]=-3 and f[1,2,3,-1]=1

Newton's divided difference interpolating polynomial is

$P_3(x)= x^3-9x^2+17x+6$..

P-5.6 We obtain f(9)=72, f[9,3]=-16, f[9,3,7]=2 and f[9,3,7,10]=1

Newton's divided difference interpolating polynomial is

$P_3(x)=72+ (x-9)(-16)+(x-9)(x-3)(02)+(x-9)(x-3)(x-7).1$ and $P_3(4)=177$.

P-5.7 We obtain

f(-1)=3, f[-1,0]=-9, f[-1,0,3]=6, f[-1,0,3,6]=5 and f[-1,0,3,6,7]=1

Newton's form of polynomial is

$f(n)= P_4(n)=3+(-9)(n+1)+6(n+1)n+5(n+1)n(n+3)+(n+1)n(n-3)(n-6)$

which yields f(2) = 6 and f(4)=138

P5.9 $P_3(x)=x^3-x^2$

$F(2)= P_3(2)=4$; $f(8)= P_3(8)=448$ and $f(15)= P_3(15)=3150$

P-5.10 Here, f(0)=4, f[0,2]=11, f[0,2,3]=7 and f[0,2,3,4]=1 and all fourth order divided difference are unity.

Newton's divided difference interpolating polynomial is

$P_3(x) =x^3+2x^2+3x+4$

which gives $f(5)= P_3(5)=194$

5.9 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit-6 Lagrange's Interpolation

Structure

6.1 Introduction

Let f(x) be a real values function of x defined on the interval [a,b]. Suppose that we are given (n+1) values of f(x) as $f(x_0)$, $f(x_1)$, $f(x_2)$, …….$f(x_n)$ at points x_0, x_1, x_2………x_n. respectively which are not necessarily equally spaced.

In this unit we shall show that there exists a polynomial P(x) of degree less than equal to n agrees with the values of f(x) at the given (n+1) distinct points (abscissa). If P(x) is interpolating polynomial, then P(x) must pass through the given (n+1) points (x_i), $f(x_i)$ for i= 01,2,…..,n, so that.

In section 4.2 we shall derive the Lagrange's from of interpolating polynomial having the above property. In section 4.3 we shall derive the general error term in approximating the function f(x) by this interpolating polynomial $P_n(x)$ at a point say x=x*. We shall explore the possibility of calculating a bound error over an interval.

6.2 Objectives

After going through this unit you should be able to

- Find the Lagrange's form of interpolating polynomial which agrees with f(x) at (n+1) distinct abscissa.
- Compute the approximate value of f(x) at a non-tabular point x=x*.
- Computer the error committed in interpolation.

- Find an upper bound of the magnitude of the error.

6.3 Lagrange's Interpolating Polynomial

Suppose we are given (n+1) values $f(x_0)$, $f(x_1)$, $f(x_2)$, …….$f(x_n)$ of the function f(x) for x = $x_0, x_1, x_2 \ldots\ldots\ldots x_n$. Let P(x) be a the required interpolating polynomial that agrees with the given (n+1) values of the function f(x) at the given points x= $x_0, x_1, x_2 \ldots\ldots\ldots x_n$.

One of the possible polynomial can be as under

$$P(x) = A_0(x - x_1)(x - x_2)(x - x_3) \ldots..(x - x_n) + A_1(x - x_0)(x - x_2)(x - x_3) \ldots..(x - x_n) + \cdots \ldots. + + A_n(x - x_0)(x - x_2)(x - x_3) \ldots..(x - x_{n-1}) \ldots\ldots..(6.1)$$

This a polynomial of degree n, since each term of Equation (4.1) is a product of n factors in x. The constants A_0, A_1, A_2,….A_n are unknown and are to be so chosen that f(x) with P(x) at (n+1) points $x_0, x_1, x_2 \ldots\ldots\ldots x_n$.

At x= x_0

$f(x_0) = P(x_0)$

therefore, from Equation (4.1), we get

$$A_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3) \ldots..(x_0 - x_n)}$$

At x= x_1

$f(x_1) = P(x_1)$

therefore, from Equation (4.1), we get

$$A_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3) \ldots..(x_1 - x_n)}$$

Similarly,

$$A_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_2)(x_2 - x_3) \ldots..(x_2 - x_n)}$$

And finally,

$$A_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_2)(x_n - x_3) \ldots..(x_n - x_{n-1})}$$

Putting all these values of A_0, A_1, A_2,….A_n in Eqn. (4.1) we get the Lagrange's interpolating polynomial as

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots..(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots..(x_0-x_n)} + f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)\ldots..(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots..(x_1-x_n)} f(x_1) + \cdots. +$$

$$\frac{(x-x_0)(x-x_2)(x-x_3)\ldots..(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_3)\ldots..(x_n-x_{n-1})} f(x_n) \ldots\ldots\ldots.(6.2)$$

Lagrange's polynomial may alternatively be written as:

$$\frac{f(x)}{(x-x_0)(x-x_1)(x-x_2)\ldots..(x-x_n)}$$

$$= \frac{f(x_0)}{(x-x_0)(x_0-x_1)(x_0-x_2)\ldots..(x_0-x_n)}$$

$$+ \frac{f(x_1)}{(x-x_1)(x_1-x_0)(x_1-x_2)\ldots..(x_1-x_n)} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+ \frac{f(x_n)}{(x-x_n)(x_n-x_0)(x_n-x_1)\ldots..(x_n-x_{n-1})} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.3)$$

Third form of the Lagrange's polynomial is as follows:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots L_n(x)f(x_n)$$

$$= \sum_{k=0}^{n} L_k(x)f(x_k) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(6.4)$$

Where $L_k(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\ldots..(x-x_{k-1})(x-x_{k+1})\ldots\ldots.(x-x_n)}{(x_k-x_0)(x_k-x_{k-1})(x_k-x_{k+1})\ldots\ldots\ldots.(x_k-x_n)} \ldots\ldots\ldots\ldots (6.5)$

for k= 0,1,2,……n.

The polynomial $L_k(x)$, which are of degree less than equal to n, are called Lagrange's interpolation coefficients. It is trivial to show that Lagrange's interpolating polynomial are unique.

Some major advantages of this polynomial are as follows:

- Lagrange's polynomial is simple and by symmetry easy to remember.
- There is no need to construct the divided difference table for its construction and application. We can directly interpolate the unknown value f(x*) for any x=x*.

- There is no need to construct the divided difference table for its construction and application. We can directly interpolate the unknown value f(x*) for any x= x*.
- The coefficients L_k (x)'s are easily determined.
- It can be used for both equal and unequal intervals and the abscissa x_0, x_1, x_2………x_n need not be in order.

Its major disadvantages is that there is always a chance of committing some computational mistake due to a number of positive and negative sings in the numerator and denominator of each term. Secondly, if n is large then evaluation involves a lot of computational work.

The lagrange's form (4.2) shows existence of an interpolating polynomial through the given set of points.

Theorem 6.1: If f(x) is a continuous function defined in the interval [a,b] and the values at $f(x_0)$, $f(x_1)$, $f(x_2)$……..$f(x_n)$ are given for (n+1) points x= x_0, x_1, x_2………x_n then exists an interpolating polynomial $P_n(x)$ of degree less than equal to n which in unique.

Proof: We have already seen that if (n+1) values of f(x) are given at x= x_0, x_1, x_2………x_n distinct points, then there exist a interpolating polynomial; Lagrange's form is one such polynomial.

$$P_n(x) = \frac{(x-x_1)(x-x_2)(x-x_3)\ldots..(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)\ldots..(x_0-x_n)} + f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)(x-x_3)\ldots..+(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\ldots..(x_1-x_n)} f(x_1) + \cdots . +$$

$$\frac{(x-x_0)(x-x_1)(x-x_2)\ldots..(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_3)\ldots..(x_n-x_{n-1})} f(x_n) \ldots\ldots\ldots. (6.6)$$

It agrees at x= x_0, x_1, x_2………x_n since

$P(x_1)$ = $f(x_1)$ for i= 0,1,2,……x_n.

Now let $P_n(x)\ and\ Q_n(x)$ be two distinct interpolating polynomials of degree less than equal to n, which interpolates f(x) at (n+1) distinct points x= x_0, x_1, x_2………x_n.

Let us define

$$h(x) \quad = \quad P_n(x) = Q_n(x) \qquad (6.8)$$

h(x) being the difference of two polynomial of degrees than equal to n, is also a polynomial of degree than equal to n.

From (6.7) and (6.8)

$$h(x_i) = P(x_i) - Q(x_i) = f(x_i) - f(x_i) = 0 \; for \; i = 0,1,2, \ldots\ldots, n.$$

Thus, h(x) has (n+1) distinct zeros.

Lemma: If $P_m(x)$ and $Q_m(x)$ are two polynomials of degrees than equal tom, which agree at (m+1) distinct points $x = x_0, x_1, x_2 \ldots\ldots\ldots x_n$ then $P_m(x) = Q_m(x)$ identically.

It may be conclude that

h(x) = 0 (identically)

or, $P_n(x) = Q_n(x) = 0$

or, $P_n(x) = Q_n(x)$ ……………….(6.9)

Hence proved

This fact leads us to conclude that although the forms of Lagrange's polynomial and Newton's polynomial are different, but they give the identical results.

Let us consider an example. It explain how to construct Lagrange's form of interpolating polynomial and compare it with Newton's form.

Example 6.1: Use the following data to obtain the interpolating polynomial of

(i) Lagrange's form,
(ii) Newton's form

And show that they are identical. Also find out f(3).

x	-1	0	2	5
f(x)	9	5	3	15

Solution:

Part (i) Here

$x_0 = -1$	$x_2 = 0$	$x_3 = 2$	$x_4 = 5$
f(x_0)=9	f(x_1)=5	f(x_2)=3	f(x_3)=15

Therefore, from Eqn. (4.2) the Lagrange's interpolating polynomial is

$$P_3(x) = \frac{(x-0)(x-2)(x-5)}{(-1-0)(-1-2)(-1-5)} + (9) + \frac{[(x-(-1)](x-2)(x-5)}{[0-(-1)(0-2)(0-5)}(5) +$$

$$\frac{[(x-(-0)](x-0)(x-5)}{[2-(1)(2-0)(2-5)}(3)+\quad\frac{[(x-(-1)](x-0)(x-2)}{[5-(-1)(5-0)(5-2)}(15)$$

$$=\frac{x(x-2)(x-5)}{(-1)(-3)(-6)}+(9)+\quad\frac{[(x+1)](x-2)(x-5)}{(1)(-2)(-5)}(5)+$$

$$\frac{(x+1)(x-5)}{(3)(2)(-3)}(3)\quad+\quad\frac{(x+1)x(x-2)}{(6)(5)(3)}(15)$$

$$=\left(-\frac{1}{2}\right)(x^3-7x^2+10x)+\left(\frac{1}{2}\right)(x^3-6x^2+3x+10)$$

$$-\left(\frac{1}{6}\right)(x^3-4x^2+5x)+\left(\frac{1}{6}\right)(x^3-x^2+2x)$$

$$f(x)=P_3(x)=x^2-3x+5$$

…………….(6.10)

Which gives

$$f(3)=P_3(3)=3^2-(3)(3)+5=5$$

Part (ii) For Newton's interpolating polynomial we construct the divided difference table.

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	9			
		$\frac{(5-9)}{\{0-(-1)\}}=-4$		
0	5		$\frac{\{(-1)-(-4)\}}{\{2-(-1)\}}=1$	
		$\frac{(3-5)}{(2-0)}=-1$		0
2	3		$\frac{\{4-(-1)\}}{(5-0)}=1$	
		$\frac{(15-3)}{(5-2)}=4$		
5	15			

Fingers in bold italics in the above table are the leading term and leading divided differences.

Therefore, Newton's interpolating polynomial is given by

$$f(x) = P_3(x) = 9 + [x—1)(-4) + [x—1)(x-0)(1) + [x—1)(x-0)(x-2)(0)$$

$$= 9 - 4(x+1) + x(x+1) + 0$$

$$= 9 - 4x - 4 + x^2 + x$$

$$= x^2 - 3x + 5 \qquad \ldots\ldots\ldots\ldots\ldots\ldots.(6.11)$$

From Eqns. (6.11) & (6.12) we observe that the polynomials $P_3(x)$ and $P_3(x)$ are identical. Although we are given 4 points but eh interpolating polynomials are of degrees 2 in each case, since the second divided differences are constant.

$$f(3) = P_3(x) = 3^2 - (3)(3) + 5 = 5$$

Example 6.2: The mode of a certain frequency curve y= f(x) is expected to be very near to x= 7. Let the frequency function y= f(x) for x= 6.9, 7.0 and 7.5 be equal to 0.36 , 0.40 and 0.30 respectively. Compute the approximate value of the mode and the maximum of f(x).

Solution: Given f(6.9)= 0.36, f(7.0)=0.40 and f(7.5)= 0.30.

$$f(x) = \frac{(x-7.0)(x-7.5)}{(6.9-7.0)(6.9-7.5)} + (0.36) + \frac{(x-6.9)(x-7.5)}{(7.0-6.9)(7.0-7.5)} +$$

$$\frac{(x-6.9)(x-7.0)}{(7.5-6.9)(7.5-7.0)}(0.30)$$

$$= (x-7.0)(x-7.5)(6) + (x-6.9)(x-7.5)(-8) + (x-6.9)(x-7.0)(1)$$

$$= 6x^2 - 87x + 315 - 8x^2 + (115.2)x - 414 + x^2 - (13.9)x + 48.3$$

$$= -x^2 + (14.3)x - (50.7)$$

…………….(6.12)

As the frequency density function.

Now, at the mode of the distribution, f(x) is maximum so that

f' (x) = 0,

and f'' (x) < 0.

From Equation (6.12),

$f'(x) = -2+14.3=0$ gives $x= 7.15$

but

$[f''(x)]_{x=7.15} = -2 < 0.$

Hence the mode of the density is at x = 7.15

Further on putting x= 7.15 in (6.12), we get

$$f_{max}(x) = -(7.15)2+ (4.13)(7.15)-50.7$$

$$= (51.1225)+(102.245)-(50.7)$$

$$= (0.4225)$$

You may try the following exercises:

P-6.1 If f(1) = (-3), f(3)= 9, f(4) = 30 and f(6) = 132, obtain the Lagrange's interpolating polynomial of f(x) and compute f(2) and f(5).

P – 6.2 Use Lagrange's interpolation formula to find the values of y corresponding to x=8 and x= 10 from the following table:

x	5	6	9	11
y=f(x)	12	13	14	16

P-6.3 The mode of a certain frequency curve y= f(x) is expected around x= 9.1 and the values of frequency function f(x) equal to 0.30, 0.35 and 0.25 for x= 8.9. 9.0 and 9.3 respectively. Calculate the mode and maximum of f(x).

P – 6.4: Given f(0)=3, f(1)=6, f(2)= 11, f(3)= 18, f(4)=27. Use Lagrange's formula to get the form of f(x).

P-6.5 Obtain the form of the function y = f(x), using Lagrange's interpolating formula given that

x	0	2	3	6
y=f(x)	648	704	729	792

Also find estimates of f(1) and f(5).

P-6.6 Find the unique polynomial of degree 2 or less such that f(1) = 1 ,f (3)= 37 and f(4) = 61,using the Lagrange's interpolating and the Newton's divided difference interpolation. Also evaluate P(1.5).

6.4 General Error Term of Remainder Term

In deriving the expression for truncation error or remainder term, if the Lagrange's interpolating polynomial $P_n(x)$ be used to interpolate or approximate f(x) then following Rolle's theorem, whose statement (without proof) is given below is helpful

Theorem 6.2: Let g(x) be a function of degree n defined in the closed interval [a,b], which is continuous in the closed interval [a,b] and differentiable in the open interval (a,b) such that g(a) = g(b), then g'(x) vanishes at (n-1) points in (a,b), g" (x) vanishes at (n-2) points in (a,b), ….$g^{(n-1)}$ (x) vanishes at least at one point in (a,b).

Let f(x) be any function which is approximated by means of some polynomial $P_n(x)$ of degree n in x. Suppose that f(x) satisfies all the conditions of Rolle's theorerm. Let $f(x_0)$, f(1),….., $f(x_n)$ be the given values of the function at x= x_0, x_1, x_2………x_n .

$P_n(x)$ is the interpolating polynomial for f(x), therefore,

$$P_n(x_1) = f(x_j) \qquad for\ i = 0,1,2,\dots n.$$

$$Let \quad f(x) \ = P_n(x) + \Psi(x)$$

$$= P_n(x) + g(x)[(x - x_0)(x - x_1)\dots\dots.(x - x_n)] \qquad \dots\dots\dots\dots\dots\dots.(6.13)$$

$$Where \quad \Psi(x) = g(x)(x - x_0)(x - x_1)\dots\dots.(x - x_n)$$

$$Here \quad \Psi(x)\ has\ (n + 1) roots\ at\ \mathrm{x} = \ x_0, x_1, x_2 \dots\dots\dots x_n$$

In order to determine g(x), we define

$$Q(t) = f(t) = P_n(t) - g(t)(t - x_0)(t - x_1)\dots\dots\dots(t - x_n)\dots\dots\dots\dots.(6.14)$$

Now $Q(t)$ vanishes at (n+1) values of t, viz t=$x_0, x_1, x_2 \dots\dots\dots x_n$

In addition, $Q(t) = 0\ for\ t = x,$ using Equation (6.13).

Thus $Q(t)$[p possesses (n+2) real roots $x, x_0, x_1, x_2 \dots\dots\dots x_n. Q(x)$ is continuous function of t and possesses continuous derivatives of all order within the closed interval $[x_0, x_n]$.

Hence, by Rolles theorem,

$$Q^{'}(t)\ has\ at\ least\ (n + 1)\ roots\ lying\ between (x_0, x_n).$$

$$Q''(t) \text{ has at least } (n) \text{ roots lying between} (x_0, x_n).$$

…………………………………………………………..

$Q^{(n+1)}(t)$ *has at least one root*, say t = ξ in the interval (x_0, x_n)

Now from (4.14) and the fact that ${P_n}^{(n+1)}(t)$ = 0, we obtain

$$Q^{(n+1)}(t) = f^{(n+1)}(t) - g(t)\{(n+1)!\} \qquad \dots\dots\dots\dots..(6.15)$$

Since,

$$Q^{(n+1)}(t) \text{ vanishes at the point } t = \xi \text{ where } x_0 < \xi < x_n$$

Therefore,

$$f^{(n+1)}(\xi) = g(t)\{(n+1)!\} \quad for\ x_0 < \xi < x_n$$

This implies for all t,

$$g(t) = \frac{f^{(n+1)}(\xi)}{\{(n+1)!\}} \quad for\ x_0 < \xi < x_n \quad \dots\dots\dots\dots..(6.16)$$

The same result holds for x, if xε (x_0, x_n).

Putting this value of g(x) from Eqn. (6.16) in Eqn (6.13) we find that

$$f(x) = P_n(x) + \frac{f^{(n+1)}(\xi)}{\{(n+1)!\}}(x - x_0)(x - x_1)\dots\dots\dots\dots..(x - x_n) for\ x_0 < \xi < x_n \qquad (6.17)$$

Hence, the required truncation error or remainder term is

$$R_n(x) = f(x) - P_n(x) = \frac{(x - x_0)(x - x_1)\dots\dots\dots\dots..(x - x_n)}{\{(n+1)!\}} f^{(n+1)}(\xi) \quad for\ x_0 < \xi < x_n \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots. \quad (6.18)$$

This is the error term in using Lagrange’s Interpolating polynomial. The Eqn. (6.18) may be written as:

$$E_n(x) = R_n(x) = \frac{f^{(n+1)}(\xi)}{\{(n+1)!\}} \prod_{i=0}^{n} (x - x_1) \qquad (6.19)$$

ξ Depends on the point x* at which error is needed $f^{(n+1)}(\xi)$. is not known, therefore this formula has limited use, we can obtain a bound to the error due to the use of this interpolating polynomial.

If $\left[f^{(n+1)}(\xi)\right] \leq M. then,$

$$|E_n(x)| \leq \frac{M}{\{(n+1)!\}} \prod_{i=0}^{n} (x - x_1) \qquad (6.20)$$

Remarks: The error formula given by Equations (6.18) or (6.19) has a theoretical importance because Lagrange's interpolation polynomial is used in deriving some importance formulae for numerical differentiations quadratures.

Example 6.3: The following table gives the values of f(x) = e*. If we fit Lagrange's interpolating polynomial to the data, obtain the magnitude of maximum possible error in the computed value of e* when x= 0.62.

x	0.61	0.63	0.64	0.67
f(x)=e^x	1.840431	1.877610	1.896481	1.954237

Solution: From Eqn. (6.18), the magnitude of the error associated with the third degree polynomial, is given by

$$|E_n(x)| = \left|(x - x_0)(x - x_1)(x - x_2)(x - x_3)\frac{f^{(4)}(\xi)}{(4)!}\right| \; for \; 0.61 < \xi < 0.67$$

$$= \left|(x - 0.61)(x - 0.63)(x - 0.64)(x - 0.67)\frac{f^{(4)}(\xi)}{(4)!}\right| \ldots\ldots\ldots\ldots\ldots(6.21)$$

Since,

$$\text{f(x) = e}^{x}$$

Therefore,

$$f^{(4)}(4) = e^x \qquad where$$

$$0.61 \leq \; x \leq 0.67 \quad \text{and}$$

$$\text{Max}|\, f^{(4)}(\xi) = e^{0.67} \;=\; 1.954237 \qquad \ldots\ldots(6.22)$$

Substituting Eqn. (6.22) in Eqn. (6.19) and putting x= 0.62, we get the upper bound on the magnitude of the error at x= 0.62 as

$$\left|E_{(0.62)}\right| = \left|\frac{(0.62 - 0.61)(0.62 - 0.63)(0.62 - 0.64)(0.62 - 0.67)(1.954237)}{24}\right|$$

$$= \frac{(0.01)(-0.01)(-0.02)(0.05)(1.954237)}{24}$$

$$= (10^{-7})(0.0814264)$$

$$= 0.00000000814.$$

The actual value of f(0.62)= 1.858928.

You may now try the following exercises:

P-6.7 The following table gives the values of f(x) $=e^x$. Obtain the upper bound of the maximum error at x= 1.25, if Lagrange's interpolating polynomial is used to approximate f(x).

x	1.2	1.3	1.4	1.6
f(x)	3.3201	3.6692	4.0552	4.9530

P-6.8 The values of x and $\log_{10}$ x are (400, 2.6021), (404, 2.6064), (405, 2.6075), (409, 2.6117),. Compute $\log_{10}$ 401 using Lagrange's interpolating polynomial.

P-6.9 Using the following table evaluate $(155)^{1/2}$ by Lagrange's interpolation formula.

x	150	152	154	156
$(x)^{1/2}$	12.247	12.329	12.410	12.490

P-6.10 From the following data estimate the value of f(5) using the Lagrange's interpolating polynomial.

X	0	1	3	4	7
$f(x)$	4	1	43	112	655

P-6.11 In the following table h is the height in above the sea level and p is the barometric pressure. Calculate p when h = 5280.

H	0	4767	6924	10564
p	27	25	23	20

P-6.12: The observed values of a function are respectively 168, 120, 72, and 63 at the four positions 3,7,9, and 10 of the independent variable. Use

(i) Newtons divided difference technique and
(ii) Lagrange's interpolation technique

To obtain the best estimate of the function at the position 6 of the independent variable.

P-6.13 A function takes the values as given in the following table:

x	0	1	3	4
$f(x)$	5	14	41	98

Obtain the value of f(x) when x = 2.5

P- 6.14 obtain an estimate of x (correct to one decimal place) corresponding to f(x) = 13.8.

x	30	35	40	45	50
$f(x)$	15.9	14.9	14.2	13.4	12.5

P-6.15 if $y_0, y_{1,} y_2, \dots\dots\dots\ y_6$ be consecutive terms of a series. Show that

$$y_3 = 0.05\ (y_0 + y_0) - 0.3\ (y_1 + y_5) + 0.75\ (y_2 + y_4).$$

Use this relation to obtain y_3, if

$$y_0 = 72795, y_1 = 71561, y_2 = 70854, y_4 = 67909, y_5 = 66666, y_6 = 65000.$$

P-6.16 For the data of example 3, use Lagrange's interpolation formula to show that the estimate of (0.62) is 1.858927.

6.5 Linear Interpolation

Linear interpolation is used for interpolating the value of a function f(x) and x = x*, where $x_0 < x^* < x_1$ by using only two points $\left(x_0, f(x_0)\right)$ *and* $\left(x_1, f(x_1)\right)$ from the table which tabulates f(x) for various values of x. We apply this in tables of logarithm, exponential, $n^2, n^3, (n)!, n^{1/2}$, trigonometrical functions, such as sin x, cos x , tan x, sin h x, tan h x; Normal probability tables and other statistical tables. For example suppose we want $\log_{10}$ (64.6). We consult logarithmic table. We compute $\log_{10}$ (64.6) by using $\log_{10}$ x against x=64 and x=64 since (64, 65) includes 64.6. We may use either Lagrange's Linear Interpolation polynomial P_L(x).

$$P_L(x) = \frac{(x - x_0)}{(x_0 - x_1)} f(x_0) + \frac{(x - x_1)}{(x_1 - x_0)} f(x_1) \dots\dots\dots\dots\dots\dots. (6.24)$$

Use Linear Newtons divided difference interpolating polynomial $P_N(x)$.

$$P_N(x) = f(x_1) \frac{f(x_1) - f(x_0)}{x_1 - x_0} x - x_0$$

$$= f(x_0) + f[x_0, x_1](x - x_0) \dots\dots\dots\dots\dots\dots. (6.25)$$

6.5.1 Error in Linear Interpolation

From Eqn (6.17) for n = 1 we obtain

$$f(x) = P(x) + \left(\frac{1}{2}\right)(x - x_0)(x - x_1)f''(\xi)\ where\ \min(x_0, x_1, x) < \xi < max(x_0, x_1, x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(6.26)$$

Which gives the truncation error in linear interpolation.

$$E_1 = f(x) = P(x) = \left(\frac{1}{2}\right)(x - x_0)(x - x_1)f''(\xi)\ \ldots\ldots\ldots\ldots\ldots\ldots..(6.27)$$

If we can determine a bound for f' (x) in (x_0, x_1). i.e.

$$|f'(\xi)\ | \leq M \qquad ,x\ (x_0, x_1)$$

Then

$$|E_1| = f(x) - P(x) = \left|\left(\frac{1}{2}\right)\ (x - x_0)(x - x_1)f^{'}(\xi)\right| \leq \left(\frac{1}{2}\right) \max_{x_0<x<x_1} |(x - x_0)(x - x_1)f^{'}(\xi)|$$

$$\leq \left(\frac{1}{2}\right) \max_{x_0<x<x_1} |(x - x_0)(x - x_1)|\, M \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.28)$$

Since the maximum of $|(x - x_0)(x - x_1)|$occurs at $x = \frac{x_0+x_1}{2}$

Therefore, the upper bound of the error E_1 is

$$|E_1| = |f(x) - P(x)| \leq \left(\frac{1}{8}\right)(x_1 - x_0)^2 M \ldots\ldots\ldots\ldots\ldots..(6.29)$$

In case upper bound of f'(x) in $|x_0, x_1|$is not know we can estimate at follows:

By definition,

$$\Delta^2 f(x_0) = \Delta f(x_0 + h) - \Delta f(x_0)\ \ where\ x_1 = x_0 + h$$

$$= [f(x_0 + 2h) - f(x_0 + h)] - [f(x_0 + h) - f(x_0)]$$

$$= hf^{'(\xi_1)} -\ hf^{'(\xi_2)} \qquad for\ (x_0 + h) < \xi_1 < (x_0 + 2h)$$

$$and x_0 < (\xi_2) < (x_0 + h)$$

$$= h\left[f^{'(\xi_1)} - f'(\xi_2)\right]$$

$$= h^2 f''(\xi) \qquad for\ x_0 \leq \xi \leq x_0 + h.$$

Therefore,

$$f''(\xi) = \frac{\Delta^2 f(x_0)}{h^2} \qquad for\ x_0 \le \xi \le x_0 + h.$$

Or

$$M = \frac{\Delta^2 f(x_0)}{h^2} \qquad for\ x_0 \le \xi \le x_0 + h \dots\dots\dots\dots..(6.30)$$

Hence on putting M form Eqn. (4.28) in Eqn. (4.29), We get for h $= x_1 - x_0$.

$$E_{1(\max)} = \left(\frac{1}{8}\right)\Delta^2 f(x_0) \dots\dots\dots\dots\dots\dots\dots..(6.31)$$

Example 6.4: Given sin (0.1)= 0.09983 and sin (0.2)=0.19867. obtain approximate value of sin (0.16) by Lagrange's interpolation. Further estimate a bound on the truncation error.

Solution: Here,

$$P_1(0.16) = \frac{(0.16 - 0.2)}{(0.1 - 0.2)}(0.09983) + \frac{(0.16 - 0.1)}{(0.2 - 0.1)}(0.19867)$$

$= (0.4)(0.09983) + (0.6)(0.19867)$

=0.039932+0.119202

= 0.159134

The truncation error is

$$E_1 = \left(\frac{1}{2}\right)(x - 0.1)(x - 0.2)(-\sin \xi)\ \ where\ 0.1 < \xi < 0.2$$

Since

$$\frac{d^2}{dx}[\sin(x)] = \frac{e}{dx}[\cos x] = (-\sin x)$$

Further,

$$Max|-\sin \xi| = |-\sin(0.2)| = \sin(0.2) = 0.19867\ for\ 0.1 < \xi < 0.2$$

Thus

$$|E_1| = \left|\frac{(0.16 - 0.1)(0.16 - 0.2)}{2}\right|(0.19867)$$

$$= (0.0012)(0.19867)$$

$$= (0.000238404)$$

$$= (0.00024)$$

Example 6.5: Determine the step size h to be used in the tabulation of f''(x) = (-cos x,) in the interval [0,3] so that the linear interpolation will be correct to four decimal places.

Solution:

F(x) = Cos x f'(x) = (-sin x) and f'' (x) = (-cos x)

and

$$\max_{x_0<x<x_1} |-\cos x| = 1 = M$$

Hence, we get from Eqn. (26).

$$(1)\left(\frac{h^2}{8}\right) \le (5)(10)^{-5}$$

Which gives

$$h \le 0.02$$

You may now try the following exercises:

P-6.17: The error function f(x) is defined by the integral

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-t^2)dt$$

Approximate f(0.08) by linear interpolation in the given table of corrected rounded values. Estimate the total error.

x	0.05	0.10	0.15	0.20
$f(x)$	0.05637	0.11246	0.16800	0.22270

P-6.18: The function f(x)= 1/x is tabulation at unit interval from 1 to 12500. Obtain the possible error in the linear interpolation of this function when x = 752.

P-6.19: The function y=e* is tabulated at an intervals of which 0.01 in the interval (0,1). Show that the maximum error on linear interpolation is 0.000034.

P-6.20: Given sin 40^0= 0.6428, sin 50^0= 0.7660, sin 55^0= 0.8192 and sin 60^0= 0.8660, estimate the error in calculating sin 54^0 by using

(a) Lagrange's formula
(b) Newton's formula

6.6 Summary

In this unit the Lagrange's form of interpolating polynomial is derived. It is whether the arguments $x_0, x_1, x_2, \ldots \ldots \ldots . \ldots \ldots x_n$ are equidistant or of unequal width. It is shown that the interpolating polynomial is unique. We have also derived an expression for the general error which can be used to determine the accuracy of our calculation. Lastly the linear interpolating is discussed. The results are noted below.

(1) Lagrange's Polynomial

$$P_n(x) = \sum_{k=0}^{n} f(x_k) L_k(x)$$

$Where$

$$L_k(x) = \frac{(x - x_0)(x - x_1) \ldots \ldots . (x - x_{k-1})(x - x_{k+1}) \ldots (x - x_n)}{(x_k - x_0)(x_k - x_k) \ldots . (x_k - x_{k-1})(x_k - x_{k+1}) \ldots \ldots . (x_k - x_n)}$$

$$for\ k = 0,1,2, \ldots \ldots n$$

(2) **Interpolation Error**

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{\{(n+1)!\}} \prod_{i=0}^{n} (x - x_1) \quad where\ x_0 < \xi < x_n$$

If $f^{(n+1)}(x)$ is not known but the upper bound $\left|f^{(n+1)}(x)\right| < M$ is given then the upper bound of the error is

$$|E_n(x)| \le \frac{M(x - x_0)(x - x_1) \ldots \ldots . (x - x_n)}{\{(n+1)!\}}$$

6.7 Solution/ Answers

P-6.1: it is given that x_0= 1, x_1= 3, x_2= 4 and x_3= 6 and f(x_0)= -3, f(x_1)=9, f(x_2)=30 and f(x_3)= 132. The Lagrange's interpolating polynomial is:

$$P_3(x) = x^3 - 3x^2 - 6$$

$$At\ x = 2, \qquad f(x) = P_3(2) = 0$$

$$At\ x = 5, \qquad f(x) = P_3(5) = 69$$

P-6.2: f(8) = P_3(8)= 13.9 and f(10)= P_3(10)= 14.6667

P-6.3: p.d.f: $f(x) = \left(\frac{1}{2}\right)(-25x^2 - 453.5x - 2052.3)$

$$Mode: \ At\ x = 9.07$$

$$\left(Since\ \ \frac{d^2f}{dx^2} = -\frac{25}{6} < 0\right)$$

$$F_{max}(x) = f(x)|_{x=9.07} = 0.3602$$

P-6.4: Using second form of Lgrange's formula we have

$$\frac{f(x)}{x(x-1)(x-2)(x-3)(x-4)} = \frac{1}{8x} - \frac{1}{x-1} + \frac{11}{4(x-2)} - \frac{3}{x-3} + \frac{9}{8(x-4)}$$

$$= \frac{x^2 + 2x + 3}{x(x-1)(x-2)(x-3)(x-4)}$$

So that f(x) $= x^2 + 2x + 3$

P-6.5: $P_2(x) = -x^2 + 30x + 648;$

$$f(1) = P_2(1) = 667\ and\ \ f(5) = P_2(5) = 773$$

P-6.6: The eqn. (4.21) gives the magnitude of the associated error for the third degree polynomial as

$$|E(x)| = \frac{(x-1.2)(x-1.3)(x-1.4)(x-1.6)^{(4)}(\xi)}{4!}$$

Here, f(x) $= e^x$,

Therefore, $f^{(4)}(x) = e^x$ when xξ [1,2,1.6]

And max $|f^{(4)}(x)| = e^{1.6}$

Hence , at x= 1.25

$$|E(1.25)| \leq |\frac{(0.05)(-0.05)(-0.15)(-0.35)(4.9530)}{24}$$

P-6.8: $\log_{10} 40|$ = P_3 (401)= 2.60314

P-6.9: 12.450125

P-6.10: P_3 (x)= $2x^3 - 5x + 4$ and f(5)= P_3 (5)= 229.

P-6.11: Largange's interpolating polynomial yields P_3 (5280)= 24.5478

P-6.12: Given $x_0 = 3, x_1 = 7, x_2 = 9, 4 = 10 \; and \; f(x_0) = 168, f(x_1) = 120, f(x_2) = 72, f(x_3) = 63$ we get $f[x_0, x_1, x_2] = -2, \; f[x_0, x_1, x_2, x_3] = 1.$

Newton's divided difference interpolating polynomial is given by

$$P_3(x) = x^3 - 21x^2 + 119x - 27 \; and$$

$$f(6) = P_3(6) - 21(6^2) + (119)(6) - 27 = 147$$

P-6.13 : f(2.5)= 57.2656

P-6.14: Let x= g(y), then using Lagrange's inverse interpolation formula.

$P_4(y) is \; given \; by$:

$g(13.8) = P_4(13.8)$

$$= \frac{(13.8 - 14.9)(13.8 - 14.2)(13.8 - 13.4)(13.8 - 12.5)}{(15.9 - 14.9)(15.9 - 14.2)(15.9 - 13.4)(15.9 - 12.5)} \times 30 + \cdots \ldots \ldots \ldots \ldots \ldots$$

$$+ \cdots \ldots \ldots \ldots \ldots . + \frac{(13.9 - 15.9)(13.8 - 14.9)(13.8 - 14.2)(13.8 - 13.4)}{(12.5 - 15.9)(12.5 - 14.9)(12.5 - 14.2)(12.5 - 13.4)} \times 50$$

$= 42.636732.$

P-6.15 Given x= 0,1,2,34,5,6 and y= $y_0, y_1, y_2, y_5, y_6, y_3$ is required for n=3 Lagrange's interpolating polynomial second form yields,

$$\frac{-y_3}{3.2.1.1.2.3} = \frac{-(y_1 - y_6)}{3.1.2.4.5.6} + \frac{-(y_1 - y_5)}{2.1.1.3.4.5} + \frac{-(y_2 - y_4)}{1.2.1.2.3.4}$$

Or

$$y_3 = \frac{(y_1 + y_6)}{20} - \frac{3(y_1 + y_5)}{10} + \frac{3(y_2 + y_4)}{4}$$

Hence for given data

$$y_3 = 0.05(72795 + 65000) - 0.3(71561 + 66666) + 0.75(70854 + 67909)$$

$$= 69493.9 \approx 69494$$

P-6.16: To show that f (0.62)= 1.858927

P-6.17: Using Newton's linear interpolation formula

$$f(x) = f(x_0) + \frac{f(x_1)f(x_0)}{(x_1 - x_0)}(x - x_0)$$

We have, for x= 0.08,

$$f(0.08) = f(0.05637) + \frac{(0.11246 - 0.05637)}{(0.10 - 0.05)}(0.08 - 0.05)$$

$$= 0.09002$$

The error is

$$|E_n(x)| \approx |\Delta^{n+1} f(x_0)](x - x_0)(x - x_1) \ldots \ldots (x - x_n)$$

For n= 1, we have

E (0.08) = (0.08-0.05) (0.08-0.10) (-0.110) = 7.0 $\times 10^{-5}$

P-6.18: Here f(X) = (1/x); h= 1; f'(x)=-(1/x^2); f'' (x) = -(2/x^3)

Therefore, f'' (752) = 2/ $(752)^3$= M

$$(E_1)_{max} = \left(\frac{1}{8}\right) \times 1^2 \times \left[\frac{2}{(752)^3}\right] = 0.00000000059$$

P- 6.19: Here , h= 0.01; f(x_0)= e^x so that, f(x) = e^x and f'' (x) = e^x for 0≤ x ≤11,

$$F_{max} \text{ "}(n) = e1.0 = 2.7183 = [\Delta^2 f(x_0)]/h^2$$

$$\Delta^2 f(x_0) = (0.01)^2, 2.7183 = 0.00027183$$

Hence

$$(E_1)_{max} = \frac{|\Delta^2 f(x_0)]}{8} = \frac{0.00027183}{8} = 0.000034$$

P-6.20: Here, $x_0 = 40, x_1 = 50, x_2 = 55, x_3 = 60, x = 54, and$

$$f(40) = 0.6428\ f[40{,}50] = 0.01232, f[40{,}50{,}55] = -(0000112), f[40{,}50{,}55{,}60] = -(0.0000008).$$

Hence, n+1=3 n=2

Thus, $E_3(54) = 8.0 \times 10^{-7} \left[\frac{\{(54-40)(54-50)(54-50)(54-60)\}}{6}\right] = 0.0000478.$

P-6.21: x= 53

P-6.22: Let y= f(x), given $y_0, y_1 = 12, y_2 = 19,\ x_0 = 1, x_1 = 3, x_2 = 4\ for\ y = f(x) = 7$

$$x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(9-12)}(4)$$

$$= 1.8571.$$

6.8 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

U.P. Rajarshi Tandon Open University, Prayagraj

SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 3 Central Differences

Unit – 7 : Central Difference Interpolation Formulae

Unit – 8 : Inverse Interpolation

Unit – 9 : Numerical Differentiation

Unit – 10 : Numerical Integration

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Block & Units Introduction

The ***Block - 3 – Central Differences,*** is the third block. This block consists of four units regarding, central differences, inverse interpolation, numerical differentiation and numerical integration.

Unit - 7 – Central Difference Interpolation Formulae; deals with the concept of central difference interpolation. Gauss and Bessels's formulae are derived and their applications are successes.

In ***Unit – 8 – Inverse Interpolation***; of the block the problem of inverse interpolation is discussed and various methods for its solution are suggested.

In ***Unit – 9 – Numerical Differentiation***; the concept of numerical differentiation has been defined. Various formulae to solve the problem of numerical differentiation are discussed.

Finally, in ***Unit – 10 - Numerical Integration*** is taken into consideration. Trapezoidal rule, Simpson's rule and Weddle's rule are derived. Euler Maclaurin's summation formula is also given in this unit.

At the end of block/unit the summary, self assessment questions and further readings are given.

Unit-7: Central Difference Interpolation Formulae

Structure

7.1 Introduction

We have obtained Newton's polynomials and Larange's polynomial for approximating the function f(x) if the arguments are equally or unequally spaced. If the number of given set of arguments and entries or the intervals of differencing are quite large then the computational work in Lagrange's polynomial becomes cumbersome and large. The Newton's polynomials are fundamental and the use leading term and leading differences irrespective of the fact whether the unknown value of entry lies in the neighborhood of the leadings terms of is at a distance apart. The leading differences always lie on the diagonal and sloping upward or downward from the leading term but the coefficients of such higher order difference diminish very slowly with the increase in the order of differences. Therefore these fundamental formulae do not converge more rapidly. In fact, we refer a formula which converges more rapidly with increase of the order of the differences. The rate of convergence should be high if the formula uses the differences in the neighborhood of the unknown entry. It is therefore, desired to shift the origin to a point where the entries above and below, the unknown entry may have a greater influence on the convergence (that is on the performance of the formula) Generally in case of equally spaced arguments the difference table is constructed with arguments in the ascending order of magnitude. If f(x*) is to be estimated for x= x*, then the Newtons's forward interpolating performs better if x* is at the beginning of the table while the Newton's backward interpolation formula is preferred if x* is nearer to the end of the table. But, none of these formulae satisfy the highly rapid convergence criteria if x* is nearer to central value of x's. This lead us to search the central difference formulae for interpolation near the middle of the tabulated sets, when the arguments are equally spaced.

7.2 Objectives

After the study of this unit you shall be able to:

- Construct a central difference table,
- Expand a central difference in terms of function values or entries.
- Obtain a proper interpolating polynomial P(x) of f(x) for a given data,
- Estimate the value of f(x), when x lies near the middle of the table

7.3 Central Difference formulae

Let the function $y = y_x = f(x)$ be given for (2n+1) equi-distant values of the arguments $x_0 \pm h$, $x_0 \pm 2h$, …… nh. Let $y = y_0$ denote the central ordinate corresponding to $x = x_0$. For convenience, we are representing $x = x_0 + rh$, by xr and corresponding $y = y_{x_0+nh}$ by y_r for r = 0, ± 1, ± 2, ± 3, ……. $\pm$ n. The difference table used in the computational work of the central difference formulae is given on the next page. It is the ordinary difference table with x_0 in the middle of the arguments column. For the central difference formulae, the origin x_0 is taken near x*, the point about which f(x) is to be estimated.

x	f(x)=y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
x_{-4}	y_{-4}						
		Δy_{-4}					
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$				

		Δy_{-3}		$\Delta^3 y_{-4}$			
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		$\Delta^2 y_0$		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$			
x_3	y_3		$\Delta^2 y_2$				
		Δy_3					
x_4	y_4						

7.3.1 Gauss Forward Formula

For (2n+1) equi-distant arguments

The general Newton's formula is

f(x) = f(x0) + $(x - x_0)\ f[x_0, x_1] + (x - x_0)(x - x_1)\ \ f[x_0, x_1, x_2] +$

$(x - x_0)(x - x_1)(x - x_2)(x - x_3)\ f[x_0, x_1, x_2, x_3] +$

$(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)\ \ f[x_0, x_1, x_2, x_3, x_4, x_5] +$

$$(x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5) f[x_0, x_1, x_2, x_3, x_4, x_5, x_6] + \cdots \quad ..(7.1)$$

Let us define

$x_0 = x_0,\quad x_1 = x_0 + h,\quad x_2 = x_0 - h,\quad x_3 = x_0 + 2h,\quad x_4 = x_0 - 2h,\quad x_5 = x_0 + 3h,\quad x_6 = x_0 - 3h, \ldots\ldots\ldots..$

$and\ u = \frac{x - x_0}{h}$

Y = f(x_0) $+ hu\ f\ [x_0 x_0 + h] + hu\ (hu - h)\ f\ [x_0 - h, x_0, x_0 + h] +$

$hu\ (hu - h)\ (hu + h)\ f\ [x_0 - h, x_0, x_0 + h, x_0 + 2h] +$

$hu\ (hu - h)\ (hu + h)(hu - 2h)\ \ f\ [x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h] +$

$hu\ (hu - h)\ (hu + h)(hu - 2h)(hu + 2h)\ \ f\ [x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h] +$

$$hu\ (hu + h)\ (hu - h)(hu + 2h)(hu - 2h)(hu - 3h)\ \ f\ [x_0 - 3h, x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h] + \cdots \ldots \ldots$$

Using the relations,

$$f[x_0, x_0] = \frac{\Delta y_0}{h}, \qquad f[x_0 - h, x_0 + x_0 + h] = \frac{\Delta^2 y_{-1}}{2h^2}$$

$$f[x_0 - h, x_0, x_0 + h, x_0 + 2h] = \frac{\Delta^3 y_{-1}}{(3!)h^3},$$

$$f\ [x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h] = \frac{\Delta^4 y_{-2}}{(4!)h^4}$$

$$f\ [x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h, x_0 + 3h] = \frac{\Delta^5 y_{-2}}{(5!)h^6} \qquad \ldots \ldots . . \ldots (7.2)$$

Then the above equation reduces to

$$y_u = y_0 + \mu \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!} \Delta^3 y_{-1}$$

$$+ \frac{(u+1)u(u-1)(u-2)}{4!} \Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!} \Delta^5 y_{-2}$$

$$+ \frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!} \Delta^6 y_{-3} + - - - \qquad \ldots \ldots . (7.3)$$

$$= y_0 + {}^{\mu}C_1 \Delta y_0 + {}^{\mu}C_2 \Delta^2 y_{-1} + {}^{\mu+1}C_3 \Delta^3 y_{-1} + {}^{\mu+1}C_4 \Delta^4 y_{-2} + {}^{\mu+2}C_5 \Delta^5 y_{-2}$$

$+{}^{\mu+2}C_6\Delta^6 y_{-3} \pm --- \mp {}^{\mu+n-1}C_{2n-1}\Delta^{2n-1}y_{-n+1} + {}^{\mu+n-1}C_{2n}\Delta^{2n}y_{-n}$

$$\dots\dots(7.4)$$

Equations (7.3) and (7.4) are different forms of Gauss Forward formula. This formula uses the odd order differences falling just below and the even order differences on the horizontal line at y = y_0 in the difference table.

7.3.2 Gauss backward formula

For (2n+1) equi-distant arguments

Let us define

$x_0 = x_0, x_1 = x_0 + h, x_2 = x_0 + h, x_3 = x_0 - 2h, x_4 = x_0 + 2h, \dots\dots etc.$

In the Newton's divided difference formula Equ. (7.1), we get

Y = f(x_0) + (x-x_0) $f[x_0, x_0 - h] + (x - x_0)(x - x_0 + h)f[x_0, x_0 - h, x_0 + h] +$

$(x - x_0)(x - x_0 + h)(x - x_0 - h)f[x_0, x_0 - h, x_0 + h, x_0 - 2h]$

$+(x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h)f[x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h]$ + $(x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h)(x - x_0 - 2h)f[x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h, x_0 - 3h] + (x - x_0)(x - x_0 + h)(x - x_0 - h)(x - x_0 + 2h)(x - x_0 - 2h)(x - x_0 + 3h)f[x_0, x_0 - h, x_0 + h, x_0 - 2h, x_0 + 2h, x_0 - 3h, x_0 + 3h] + \cdots \dots\dots$

$Letting \quad u = \frac{x - x_0}{h} \; and\ using\ the\ fact$

$$f[x_0 - h, x_0] = \frac{\Delta y_{-1}}{h}$$

$$f[x_0 - h, x_0, x_0 + h] = \frac{\Delta^2 y_{-1}}{(2!)h^2}$$

$$f[x_0 - 2h, x_0 - h, x_0, x_0 + h] = \frac{\Delta^3 y_{-2}}{(3!)h^3}$$

$$f[x_0 - 2h, x_0 - h, x_0, x_0 + h, x_0 + 2h] = \frac{\Delta^4 y_{-2}}{(4!)h^4}$$

$$f[y_0 - 3h, y_0 - 2h, y_0 - h, y_0, y_0 + h, y_0 + 2h] = \frac{\Delta^5 y_{-3}}{(5!)h^5}$$

$$f[y_0 - 3h, y_0 - 2h, y_0 - h, y_0, y_0 + h, y_0 + 2h, y_0 + 3h] = \frac{\Delta^6 y_{-3}}{(6!)h^6} \qquad \dots \dots (7.5)$$

and so on.

We get,

$$y_u = y_0 + u\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2}$$

$$+ \frac{(u+2)u(u+1)u(u-1)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-3}$$

$$+ \frac{(u+3)(u+2)u(u+1)(u-1)(u-2)}{6!}\Delta^6 y_{-3} + - - - \cdots \qquad \dots . (7.6)$$

$$= y_0 + {}^{\mu}C_1 \Delta y_{-1} + {}^{\mu+1}C_2 \Delta^2 y_{-1} + {}^{\mu+1}C_3 \Delta^3 y_{-2} + {}^{\mu+2}C_4 \Delta^4 y_{-2} + {}^{\mu+2}C_5 \Delta^5 y_{-3}$$

$$+ {}^{\mu+3}C_6 \Delta^6 y_{-3} \pm - - - + {}^{\mu+n-1}C_{2n-1} \Delta^{2n-1} y_{-n} + {}^{\mu+n}C_{2n} \Delta^{2n} y_{-n}$$

$$\dots . (7.7)$$

Equations (7.6) and (7.7) are different forms of Gauss's backward formula. It is odd order differences falling just above and the even order differences on the central line through x = x_0.

Thus the path of Gauss's forward and backward formula are shown by star (***) and dotted lines (…….) in the difference table.

Short Different Table

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
x_{-1}	y_{-1}						

		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0						$\Delta^6 y_{-3}$
		Δy_0	$\Delta^3 y_{-2}$		$\Delta^4 y_{-4}$	$\Delta^5 y_{-2}$	
x_1	y_1			$\Delta^3 y_{-1}$			

7.2.3 Stirling Formula

For (2n+1) equi-distant arguments

The mean of Gauss's forward and Gauss's backward formula given by Eqns (7.3) and (7.6) respectively, we get

$$y_u = y_0 + u\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{[\Delta^3 y_{-1} + \Delta^3 y_{-2}]}{2} +$$

$$+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u^2(u^2-1^2)(u^2-2^2)}{5!}\frac{[\Delta^5 y_{-3} + \Delta^5 y_{-3}]}{2}$$

$$+\frac{u^2(u^2-1^2)(u^2-2^2)}{6!}\Delta^6 y_{-3} + - - - - - - - -$$

$$\frac{u(u^2-1^2)(u^2-2^2)\ldots\ldots..\{u^2-(n-1)^2].}{(2n-1)!}\frac{[\Delta^{2n-1} y_{-n+1} + \Delta^{2n-1} y_{-n}]}{2} +$$

$$\frac{u^2(u^2-1^2)(u^2-2^2)\ldots\ldots..\{u^2-(n-1)^2][\Delta^{2n} y_{-n}]}{(2n)!}\ldots\ldots \qquad \ldots.(7.8)$$

This is Stirling's formula.

These quantities that occur in Striring's formula are shown in the following difference table.

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
x_{-1}	y_{-1}						
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	

x_1 y_1

7.3.4 Bessel's formula

For (2n+2) equidistant arguments

Let us change the original in Gauss's backward formula from x_0 to x_1 then u will be replaced by u-1 in the formula; this gives us

$$y_u = y_1 + (u-1).\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1}$$

$$+\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-1} + \frac{(u+1)u(u-1)(u-2)(u-3)}{5!}\Delta^5 y_{-1}$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!}\Delta^6 y_{-1} + ------ \quad --(7.9)$$

Averaging the above formula of Eqn. (7.9) and the Gauss's forward interpolation formula given by Eqn. (7.3), we obtain.

$$y_u = \frac{(y_1+y_2)}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}.\frac{[\Delta^2 y_0 + \Delta^2 y_{-1}]}{2}$$

$$+\frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!} + \frac{[\Delta^4 y_{-1} + \Delta^4 y_{-2}]}{2}$$

$$+\frac{(u-1/2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2}$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!}\frac{[\Delta^6 y_{-2} + \Delta^6 y_{-3}]}{2} + -- \quad --\ (7.10)$$

This is Bessel's formula.

__**Alternative forms of Bessel's Formula**__

II Form The Eqn. (7.10) may be written as

$$y_u = \frac{(y_0 + y_1)}{2} + \frac{({}^{\mu}C_1 + {}^{\mu-1}C_1)}{2}\Delta y_0 + {}^{\mu}C_2 \frac{[\Delta^2 y_0 + \Delta^2 y_{-1}]}{2} + \frac{({}^{\mu+1}C_3 + {}^{\mu}C_3)}{2}\Delta^3 y_{-1}$$

$$+ {}^{\mu+1}C_4 \frac{[\Delta^4 y_{-1} + \Delta^4 y_{-2}]}{2} + \frac{({}^{\mu+2}C_5 + {}^{\mu+1}C_5)}{2}\Delta^5 y_{-2} \;\; + {}^{\mu+2}C_6 \frac{[\Delta^6 y_{-2} + \Delta^6 y_{-3}]}{2}$$

$$+ - \;\; -$$

$$\dots\dots\dots\dots\dots\dots\dots\dots..(7.11)$$

In this formula, we observe that various terms are either alternative means of the coefficients of Eqn. (7.4) and (7.7) or the mean of the differences lying on the horizontal line between y_o and y_1.

II Form Let

$$u = \frac{x - x_0}{h} \qquad \dots\dots\dots\dots\dots..(7.12)$$

Now we define v = u - (1/2) ⇒ u = v + (1/2)

Then Bessel's formula becomes

$$y_u = y_{v+\frac{1}{2}} = \frac{(y_1 + y_0)}{2} + v\Delta y_0 + \frac{v^2\left(\frac{1}{2}\right)^2}{2!}\frac{[v^2 y_0 + v^2 y_{-1}]}{2}$$

$$+ \frac{v + [(v^2 - (1/2)^2)]}{3!}\Delta^3 y_{-1} + \frac{\left[v^2\left(\frac{1}{2}\right)^2\right]\left[\left(v^2 - \left(\frac{1}{2}\right)^2\right)\right]}{4!}\frac{[v^4 y_{-1} + v^4 y_{-2}]}{2}$$

$$+ \frac{v + \left[\left(v^2 - \left(\frac{1}{2}\right)^2\right)\right]\left[v^2\left(\frac{3}{2}\right)^2\right]}{5!}\Delta^5 y_{-2} + - - - - - -$$

$$\dots\dots..(7.13)$$

This is the most convenient from of Bessel's formula for practical use provided

u >1/2. It employs alternatively the means of the coefficients and the mean of differences lying on the horizontal line between y_0 and y_1.

The following table depicts the differences which occur in the Bessel's formula.

x	y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-2}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_{-1}$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
x_2	y_2						

7.3.5 Bessel's formula for haves

On putting u = ½ in Eqn. (7.10) we get cases of Bessel's formula called Bessel's formula for haves. It is uses to estimate the values of the function mid way between two given values, that is for u = ½ = 0.5

$$y_{\frac{1}{2}} = \frac{(y_1 + y_0)}{2} - \frac{1}{8}\frac{[v^2 y_0 + v^2 y_{-1}]}{2} + \frac{3}{128}\frac{[v^2 y_{-1} + v^2 y_{-2}]}{2}$$

$$-\frac{5}{1024}\frac{[v^6 y_{-2} + v^6 y_{-3}]}{2} + -----$$

$$+(-1)^n \frac{[1.35 ----(2n-1)]^2}{2^{2n}(2n)!}\frac{[v^{2n} y_{-1} + v^{2n} y_{-2}]}{2} \qquad \ldots\ldots\ldots\ldots\ldots (7.14)$$

7.6 Choice of Formula

Gauss's forward is employed to interpolate the value of the function if u is in the interval (0,1) (i.e., 0<u<1).

Gauss's backward is employed to interpolate the value of the function if u is in the interval (-1, 0) (i.e., -1<u<0).

Stirling's formula is used if -0.25<u<0.25 while Bessel's formula gives better results if $0.25 \le u \le 00.75 \Rightarrow -0.25 \le v \le 0.25$.

Our x_0 should be so chosen that u satisfies the above inequality. Further, the choice depends on the order of the highest difference that could be neglected considering its insignificant contributions. The expected contribution due to neglected highest order difference and the subsequent higher order differences should be less than half a unit in the last decimal place of the observations. In general, if the neglected difference is order then Stirling's formula is used, while, the Bessel's formula is used if it is of even order.

Let us consider some examples:

Example 7.1: The following table gives the values of y = f(x) = $\sqrt{x}$.

x	2.50	2.55	2.60	2.65	2.70
f(x)	1.5811	1.5969	1.6125	1.6279	1.6432

Obtain an estimate of f (2.62) using

(a) Gauss forward interpolation formula
(b) Stirling formula
(c) Bessel's formula

Solution:

Central Difference Table

x	u=(x-2.60)/0.5	f(x)=y	Δy	Δ^2y
2.50	-2	y_{-2}=1.5811		
			Δy_{-2}=0.0158	
2.55	-1	y_{-1}=1.5969		$\Delta^2 y_{-2}$=-0.0002
			Δy_{-1}=0.0156	
2.60	0	y_{-0}=1.6125		$\Delta^2 y_{-1}$=-0.0002
			Δy_0=0.0154	
2.65	1	y_{-1}=1.6279		$\Delta^2 y_0$=-0.0001
			Δy_1=0.0153	
2.70	2	y_2=1.6432		

For x= 2.62, u= (2.62-2.60)/0.05= 0.4

Here the second difference almost, constant, therefore we stop at this stage and do not compute higher order differences.

Part (a) Gauss Forward interpolation formula.

$y_{(0.4)}$ = 1.1625+ $^{04}C_1$ (0.0154)+ $^{0.4}C_2$ (-0.0002)

$$= 1.6125 + (0.4)(0.0154) + (0.4)(-06)(-0.0002)\left(\frac{1}{2}\right)$$

$$= 1.6125 + 0.0061 + 0.000024$$

$$= 1.61868$$

Part (b) Stirling Formula yields

$$y_{(0.4)} = 1.6125 + (0.4)\frac{[0.0154 + 0.0156]}{2} + \frac{(0.4)^2}{2!}(-0.0002)$$

$$= 1.6125 + 0.00620 - 0.000016$$

$$= 1.61868$$

Part (C) Bessel's Formula yields

$$y_{(0.4)} = \frac{1}{2}(1.6279 + 1.6125) + (0.4 - 0.5)(0.0154)$$

$$+\frac{1}{2}[(0.4)(0.4 - 1)]\left[\frac{1}{2}(-0.0001 - 0.0002)\right]$$

$$= 1.6202 - 0.00154 + 0.000018$$

= 1.61868

In the above cases the interpolated value of $\sqrt{(2.62)}$ is 1.61868 regardless of which formula is used.

Example 7.2: Apply (a) Stirling's formula and (b) Bessel's formula to obtain the value of Φ(1.2) from the following table which gives the value of

$$\Phi(x) = \int_0^x \{1/\sqrt{2\pi}\} \exp(-z^2)\, dz \quad for\ x = 0.0\ to\ 2.0$$

x	0	0.5	1.0	1.5	2.0
$\Phi(x)$	0.0000	0.1915	0.3413	0.4332	0.4772

Solution: For h= 0.5 and x_0 = 1.0 and x= 1.2, we have u= (1.2-1.0)/ (0.5)= 0.4

x	u	$\Phi(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	2	y_{-2}=0.000				
			Δy_{-2}=0.1915			
0.5	-1	y_{-1}=0.1915		$\Delta^2 y_{-2}$=-0.0417		
			Δy_{-1}=0.1498		$\Delta^3 y_{-2}$=-0.0162	
1.0	0	y_0=0.3413		$\Delta^2 y_{-1}$=-0.0579		$\Delta^4 y_{-2}$=-0.0262
			Δy_0=0.0919		$\Delta^3 y_{-1}$=-0.0100	
1.5	1	y_{-1}=0.4332		$\Delta^2 y_0$=-0.0479		
			Δy_1=0.0440			
2.0	2	y_2=0.4772				

Part (a): Stirling formula gives

$$y_{(0.4)} = 1.3413 + (0.4)\frac{[0.0919 + 0.1498]}{2} + \frac{(0.4)^2}{2!}(-0.0579)$$

$$+\frac{0.4[(0.4)^2 - (1)^2]}{3!}\frac{[0.0100 + 0.0162]}{2} + \frac{(0.4)^2[(0.4)^2 - (1)^2]}{4!}(0.0262)$$

$$= 0.3413 + 0.04834 - 0.004632 - 0.0007336 - 0.00014678$$

$$= 0.38413 = 0.3841\ (approx)$$

Here, there are 5 observations, 5 is odd number, therefore the fourth order difference $\Delta^4 y_{-2} =$ 0.0262 will be dropped.

Part (b): Bessel's Formula Gives

$$y_{(0.4)} = \frac{1}{2}(0.4332 + 0.3413) + \left(0.4 - \frac{1}{2}\right)(0.0919)$$

$$+\frac{1}{2}[(0.4)(0.4-1)]\left[\frac{1}{2}(-0.0479 - 0.0579)\right]$$

$$+\left(\frac{1}{6}\right)\left[\left(0.4 - \frac{1}{2}\right)(0.4)(0.4-1)0.0100\right]$$

=0.38725-0.00919+0.005976+0.00004

=0.38408=0.3841 (approx)

Example 7.3: Use (a) Gauss's backward (b) Stirling (c) Bessels formula to find the sales of the year 1968, given that

Year (x)	1941	1951	1961	1971	1981	1991	2001
Sales (in Lac)(y)	9	15	20	27	39	59	90

Solution: Here h=10. Let us take 1971 (nearer to year 1968) as the origin x_0 and

$$u = \frac{x - x_0}{h} = \frac{x - 1971}{10}\ So\ that\ at\ x = 1968, u = \frac{1968 - 1971}{10} = 0.3$$

The difference table is computed below:

x	y	f (x)=y	Δy	$\Delta^2 y$	$\Delta^3 y$
1941	-2	9			
			6		
1951	-2	15		-1	
			5		3
1961	-1	20			

				2	
			7		3
1971	0	27		5	
			12		3
1981	1	39			
			20	8	
1991	2	59			
			31	11	
2001	3	90			

Here difference of third order are constant, therefore we stop at $\Delta^3 y$ column. We infer that the polynomial is degree 3.

Part (a) Gauss Backward formula is

$$y_{(0.4)} = 27 + (-0.3)(7) + \frac{(-0.3+1)(-0.3)}{2!}(5) + \frac{(-0.3+1)(-0.3)(-0.3-1)}{3!}(3)$$

$$= 27 - 2.1 - 0.525 + 0.1365$$

$$= 24.5115 = 24.5\ (approx.)$$

Part (b) Stirling formula is

$$y_{(-0.3)} = 27 + (-0.3) + \frac{(12+7)}{2} + \frac{(-0.3)^2}{2!}(5) + \frac{(-0.3)[(-0.3)^2-(1)^2](3+3)}{3!}\frac{3+3}{2}$$

$$= 27 - 2.85 + 0.225 + 0.1365$$

$$= 24.5115 = 24.5\ (approx.)$$

The use of Stirling formula is justified when u falls in the interval -0.25≤u≤0.25. In this case, u= 0.3 is close to -0.25

Part (c): Bessel’s Formula performs good if

0.25≤u≤0.25 (equivalently -0.25≤u≤0.25).

u= -0.3 does not lie in the interval, therefore we have to change the origin from x= 1971.

Let us take the new origin at x= 1961, then u=$\frac{x-1961}{10}$ and x= 1968, u= (1968-1961)/10= 0.7 and v=u-1.2= 0.2

Using Bessel's formula given by Eqn. (1.13) and

$y_0 = 20, y_1 = 27, \Delta y_0 = 7, \Delta^2 y_0 = 5, \Delta^2 y_{-1} = 2, \Delta^2 y_{-1} = 3 \; and \; v = 0.2$

We get

$$y_{(0.4)} = \frac{1}{2}(27+20) + (0.2) + (7) + \left[(0.2)^2 - \left(\frac{1}{2}\right)^2\right]\left[\frac{(5+2)}{2}\right]$$

$$= \left[(0.2)[(0.2)^2 - \left(\frac{1}{2}\right)^2\right](3)\left(\frac{1}{6}\right)$$

$$= 23.5 + 1.4 - (0.105)(3.5) + (0.1)(0.21)$$

$$= 23.5 + 1.4 - 0.3675 - 0.021$$

$$= 24.5115 = 24.5 \; (approx)$$

Example 7.4: For a locality, the expectations of life e_x^o (in years) are as follows. Obtain the expectation of life (at the age of 22.5 years).

Age (in year) x	10	15	20	25	30	35
Expectation of life (in year) e_x^o	55.4	52.2	49.1	46.1	43.2	40.5

Solution: x = 22.5 is the mid point of x=20 and x=25. If we shift the origin to x_0 =20 and take width of the interval h= 5 years, then

$$u = \frac{x - x_0}{h} = \frac{x-20}{10} \; So \; that \; at \; x = 22.5, u = 0.5 = 1/2.$$

The difference table is shown below:

x	u	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	-2	55.4			
			-3.2		
15	-1	52.2		0.1	
			-3.1		0
20	0	49.1		0.1	
			-3		0
25	1	46.1		0.1	
			-2.9		0
30	2	43.2		0.1	
			-2.8		
35	3	40.5			

Bessel's formula for halves given by Eqn. (1.14) yields

$$y_{\frac{1}{2}} = \left(\frac{1}{2}\right)(y_1 + y_0) - \left(\frac{1}{8}\right)(\Delta^2 y_0 + \Delta^2 y_{-1})$$

$$= \left(\frac{1}{2}\right)(26.1 + 29.1) - \left(\frac{1}{8}\right)[0.1 + 0.1]$$

$$= 27.6(.2)0.125$$

$$= 27.5875 = 27.6\ (approx)$$

You may try the following problems:

P-1.1 Apply a central difference formula to obtain y_{34} given that y_{25}= 0.27.7, y_{30}= 0.3027, y_{40}= 0.3794

P-1.2 Given the following table of values

x	1.1	1.2	1.3	1.4	1.5
f (x)	1.3357	1.5095	1.6984	1.9043	1.2193

Obtain an estimate of f(1.34) using Stirling's formula.

P-1.3 The following table gives the number of deaths in four successive 10 year age groups. Estimate the number of deaths between age groups 45-50 and 50-55 years

Age group (in years)	25-35	35-45	45-55	55-65
Deaths	13229	18139	24225	31496

P-1.4 Apply

(a) Gauss's forward formula

(b) Stirling's formula

To find a polynomial of degree four or less that takes the value of the function f(x)

x	1	2	3	4	5
f (x)	1	-1	1	-1	1

P-1.5 Apply Bessel's formula to obtain a polynomial of degree three or less which takes the values of the function f(y).

x	4	6	8	10
f (x)	1	3	8	20

P-1.6 Apply Gauss's formula to find the value of u_9 if u_0= 14, u_4= 25, u_8= 32, u_{12}= 35, u_{16}= 40

P-1.7 Use

(a) Gauss's forward formula

(b) Stirling's formula

To get an estimate of $u_{12.2}$ from the following table where u=$10^5(log_{10}^{26})$

x	10	11	12	13	14
u	23967	28060	31788	35290	38368

P-1.8 Use

(a) Gauss's forward formula

(b) Stirling's formula

To obtain the value of $25^0$40'30"

x	$25^0 40'0''$	$25^0 40'20''$	$25^0 40'40''$	$25^0 41'0''$	$25^0 41'20''$
Sin θ	0.4331347	0.43322218	0.4330956	0.43339695	0.443348433

P-1.9 Given

θ	0^0	5^0	1^0	15^0	20^0	25^0	30^0
$Tan\ \theta$	0.00	0.0875	0.1763	0.2679	0.3640	0.4663	0.5774

Use Stirling formula to show that tan 16^0= 0.28676.

P-1.10 Use Gauss's Interpolation formula to obtain u_{44} with the help of the following data. U_{30}= 3678, u_{35}= 2995, u_{40}= 2400, u_{45}= 1876, u_{50}= 1416.

P-1.11 Use Stirling formula to evaluate the value of y for x= 30 from the formula table.

x	21	25	29	33	37
y=f(x)	18.4708	17.8140	17.1070	16.3432	15.5134

7.4 δ and μ operators

7.4.1 The operator δ is called the central difference operator.

$$\delta \equiv E^{\frac{1}{2}} - E^{\frac{1}{2}} \dots\dots\dots\dots\dots\dots (7.15)$$

$$\Rightarrow \delta \equiv E^{-\frac{1}{2}}(E-1)$$

$$\Rightarrow \delta \equiv E^{-1/2}\,\Delta \qquad where\ E = 1 + \Delta \qquad \dots\dots\dots (7.16)$$

Eqn. (7.16) establishes a relation between $\delta\ and \nabla$.

$$\text{Again } \nabla \quad \equiv \quad E^{-\frac{1}{2}}(1 - E^{-1})$$

$$\Rightarrow \delta \qquad \equiv E^{1/2}\,\nabla \qquad \dots\dots\dots\dots\dots\dots\dots (7.17)$$

Eqn. (7.17) establishes a relation between $\delta\ and \nabla$.

Further the first order central difference δ f_r or f_r for f(x) at x = x_r

$$\delta f_r = f\left(x_r + \frac{h}{2}\right) - f\left(x_r - \frac{h}{2}\right) = f_{r+\left(\frac{1}{2}\right)} - f_{\left(r-\left(\frac{1}{2}\right)\right)} \ldots \ldots \ldots . (7.18)$$

If we operate Eqn. (1.18) with δ we get second order difference as $\delta^2 f_r$

$$\delta^2 f_r = \delta(\delta f_r)$$

$$= \delta[f(x_r + h/2) - f(x_r + h/2)]$$

$$= \delta f\, f\left(x_r + \frac{h}{2}\right) - \delta f\left(x_r + \frac{h}{2}\right)$$

$$= [f(x_r + h) f(x_r)] - [f(x_r) - f(x_r - h)]$$

$$= f(x_r + h) - 2f(x_r) + f(x_r + h)$$

$$f_{r+1} - 2f_r + f_{r-1} \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . . (7.19)$$

Similarly

$$\delta^3 f_r = f_{r+3/2} - 3f_{r+(1/2)} + 3f_{r-\left(\frac{1}{2}\right)} - f_{r-\left(\frac{3}{2}\right)} \qquad \ldots \ldots \ldots \ldots \ldots (7.20)$$

$$\delta^4 f_r = f_{r+2} - 4f_{r+1} + 6f_r - 4f_{r-1} + f_{r-2} \qquad \ldots \ldots \ldots \ldots \ldots \ldots (7.21)$$

In fact

$$\delta^{n-1} f_{r+\frac{1}{2}} - \delta^{n-1} f_{r-\frac{1}{2}} = \delta^n f_r = \left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{-\frac{1}{2}}\right)^n f_r \; for\; r = 1, 2, 3 - - - -$$

It shows that the even order differences $\delta^{2m}\, fr$ at a tabular value x_r are expressed in terms of tabular values of function f(x), while the odd order differences $\delta^{2n+1\, fr}$ at tabular value x_r are in terms of non-tabular values of f(x). The coefficients of are the same as those of the binomial expansion of $(1\text{-}x)^t$, t= 1,2,3,………...

7.3.2 The mean or average operator μ is defined as

$$\mu = \frac{1}{2}\left(\mathrm{E}^{\frac{1}{2}} - \mathrm{E}^{\frac{1}{2}}\right) \qquad \ldots \ldots \ldots \ldots \ldots . . \ldots \ldots (7.22)$$

The first order mean operator μf_r of the function f(x) at the argument $x = x_r$ is given as.

$$\mu f_r \equiv \frac{1}{2}\left(E^{\frac{1}{2}} - E^{\frac{1}{2}}\right) f(x_r)$$

$$= \frac{1}{2}\left[f\left(x_r + \frac{h}{2}\right) + f\left(x_r - \frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[f_{r+1/2} + f_{r-1/2}\right] \qquad \ldots\ldots\ldots\ldots..(7.23)$$

$$\mu^2 f_r = \mu(\mu f_r)$$

$$= \mu\left[\frac{1}{2}\left(f_{r+\frac{1}{2}} + f_{r-\frac{1}{2}}\right)\right]$$

$$= \frac{1}{2} f\left(\mu f_{r+\frac{1}{2}}\right) + \frac{1}{2}\left(\mu f_{r-\frac{1}{2}}\right)$$

$$= \frac{1}{2}\left[\frac{1}{2}(f_{f+1} + f_r)\right] + \frac{1}{2}\left[\frac{1}{2}(f_{r-1} + f_r)\right]$$

$$= \frac{1}{4}(f_{r+1} + 2f_r + f_{r-1}) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7.24)$$

and so on.

From Equations (7.15) and (7.22).

$$E^{\frac{1}{2}} \equiv \mu + \frac{1}{2}(\delta) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(7.25)$$

and

$$E^{-\frac{1}{2}} \equiv \mu - \frac{1}{2}(\delta) \qquad \ldots\ldots\ldots\ldots\ldots\ldots..(7.26)$$

The Stirling's central formula in terms of μ and δ operators may be written as

$$y_u = y_0 + u\left[\frac{\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}}{2}\right] + \frac{u^2}{2}\delta^2 y_0 + \frac{u(u^2 - 1^2)}{3!}\frac{\left[\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right]}{2}$$

$$+\frac{u^2(u^2-1^2)}{4!}\delta^4 y_0+\frac{u(u^2-1^2)(u^2-2^2)}{5!}\frac{\left[\delta^5 y_{\frac{1}{2}}+\delta^5 y_{-\frac{1}{2}}\right]}{2}$$

$$+\frac{u^2(u^2-1^2)}{6!}\delta^6 y_0+-------------$$

$$\frac{u(u^2-1^2)(u^2-2^2)\ldots.[u^2-(n-1)^2]}{(2n-1)!}\frac{\left[\delta^{2n-1} y_{\frac{1}{2}}+\delta^{2n-1} y_{-\frac{1}{2}}\right]}{2}$$

$$+\frac{u(u^2-1^2)(u^2-2^2)\ldots.[u^2-(n-1)^2]}{(2n!)}[\delta^{2n}y_0]\quad ----- \qquad \ldots\ldots\ldots\ldots(7.27)$$

OR

$$y_u=y_0+u(\mu\delta y_0)+\frac{u^2}{2!}(\delta^2 y_0)+\frac{u(u^2-1^2)}{3!}(u\delta^3 y_0)$$

$$+\frac{u^2(u^2-1^2)}{4!}(\delta^4 y_0)+\frac{u(u^2-1^2)(u^2-2^2)}{5!}(u\delta^5 y_0)$$

$$+\frac{u^2(u^2-1^2)}{6!}(\delta^6 y_0)+-------------$$

$$\frac{u(u^2-1^2)(u^2-2^2)\ldots.[u^2-(n-1)^2]}{(2n-1)!}(u\delta^{2n-1}y_0)$$

$$+\frac{u(u^2-1^2)(u^2-2^2)\ldots.[u^2-(n-1)^2]}{(2n!)}[\delta^{2n}y_0]----- \qquad \ldots\ldots\ldots\ldots..(7.28)$$

7.4 Summary

In this unit interpolation formulae, to interpolate the value of the function near the middle of the set of equi-distant arguments, have been derived.

The interpolation formulae derived in this unit with, u = (x - x_0)/h, are mentioned below:

(1) Gauss's forward interpolation formula with (2n + 1) equi-distant arguments:

Two line left

(2) Gauss's backward interpolation formula with (2n+1) equi-distant arguments:

Two line left

(3) Stirling's Central difference formula with (2n+1) equi-distant arguments:

$$y_u = y_0 + u\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{[\Delta^3 y_{-1} + \Delta^3 y_{-2}]}{2} +$$

$$+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u^2(u^2-1^2)(u^2-2^2)}{5!}\frac{[\Delta^5 y_{-3} + \Delta^5 y_{-3}]}{2}$$

$$+\frac{u^2(u^2-1^2)(u^2-2^2)}{6!}\Delta^6 y_{-3} + - - - - - - - -$$

$$\frac{u(u^2-1^2)(u^2-2^2)\ldots\ldots..\{u^2-(n-1)^2].}{(2n-1)!}\frac{[\Delta^{2n-1}y_{-n+1} + \Delta^{2n-1}y_{-n}]}{2} +$$

$$\frac{u^2(u^2-1^2)(u^2-2^2)\ldots\ldots..\{u^2-(n-1)^2]}{(2n)!}[\Delta^{2n}y_{-n}]$$

(4) Bessel's central difference formula for (2n+2) arguments

$$y_u = \frac{(y_1+y_2)}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}.\frac{[\Delta^2 y_0 + \Delta^2 y_{-1}]}{2}$$

$$+\frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!} + \frac{[\Delta^4 y_{-1} + \Delta^4 y_{-2}]}{2}$$

$$+\frac{(u-1/2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2}$$

$$+\frac{(u+2)(u+1)u(u-1)(u-2)(u-3)}{6!}\frac{[\Delta^6 y_{-2} + \Delta^6 y_{-3}]}{2}$$

Which by taking v=u-1/2, becomes

$$y_u = y_{v+\frac{1}{2}} = \frac{(y_1+y_0)}{2} + v\Delta y_0$$

$$+\frac{v^2(1/2)^2}{2!}\frac{[v^2y_0+v^2y_{-1}]}{2}+\frac{v+[(v^2-(1/2)^2)]}{3!}\Delta^3y_{-1}$$

$$+\frac{\left[v^2\left(\frac{1}{2}\right)^2\right]\left[\left(v^2-\left(\frac{1}{2}\right)^2\right)\right]}{4!}\frac{[v^4y_{-1}+v^4y_{-2}]}{2}$$

$$+\frac{v+[(v^2-(1/2)^2)][v^2(3/2)^2]}{5!}\Delta^5y_{-2}+--------$$

(5) Bessel's formula for haves for u=1/2 = 0.5

$$y_{\frac{1}{2}}=\frac{(y_1+y_0)}{2}-\frac{1}{8}\frac{[v^2y_0+v^2y_{-1}]}{2}+\frac{3}{128}\frac{[v^2y_{-1}+v^2y_{-2}]}{2}$$

$$-\frac{5}{1024}\frac{[v^6y_{-2}+v^6y_{-3}]}{2}+\ -----$$

$$+(-1)^n\frac{[1.35----(2n-1)]^2}{2^{2n}(2n)!}\frac{[v^{2n}y_{-1}+v^{2n}y_{-2}]}{2}$$

The criteria regarding the choice between Stirling's and Bessel's formulae are as follows

(1) If -0.25 ≤ u ≤0.25 then Bessel's formula is to be preferred and for 0.25 ≤ u ≤ 0.75 Stirling's formulas is suggested. The choice of x_0 should be so made that is satisfies the above condition.

(2) If the given set of observations are an odd number say 2n+1, then Stirling's formula is recompleted, whereas for Bessel's formula, the number of observations should be an even number, say (2n+2).

(6) The central difference operator and δ mean operator μ are defined as below:

$$\delta\equiv E^{\frac{1}{2}}-E^{\frac{1}{2}}\qquad\text{and}\qquad\mu=\frac{1}{2}\left(E^{\frac{1}{2}}-E^{\frac{1}{2}}\right)$$

The above formula may be written in terms of these operators to use central differences table.

7.5 Solutions/Answers

P-1.1 Taking x_0= 35, h=5, u= (34-35)/5=-0.2 we have.

$Y_{(34)}$= $y_{-0.2}$= 0.331844 = 0.3318(approx.)

P-1.2 Choose x_0= 1.3 so that u= (x-x_o)/h (1.34-1.30)/(0.10)=-0.4 .

Central Difference Table

x	u	f	δf	$\delta^2 f$	$\delta^3 f$	$\delta^4 f$
1.1	-2	1.3357				
			0.1738			
1.2	-1	1.5095		0.0151		
			0.1889		0.0019	
1.3	0	1.6984		0.0170		0.0002
			0.2059		0.0021	
1.4	1	1.9043		0.0191		
			0.2250			
1.5	2	1.293				

Now, f_0= 1.6984, $\delta^2 f_0$ = 0.0170, $\delta^2 f_0$ = 0.0002.

$$u\delta^2 f_0 = \frac{1}{2}\left(\delta^2 f_{\left(\frac{1}{2}\right)} - \delta^2 f_{(-1/2)}\right) = \frac{1}{2}(0.2059 + 0.1889) = 0.1974$$

$$u\delta^3 f_0 = \frac{1}{2}\left(\delta^3 f_{\left(\frac{1}{2}\right)} - \delta^3 f_{(-1/2)}\right) = \frac{1}{2}(0.002 + 0.0019) = 0.0020$$

Using Stirling formula given by Eqn. (1.25) we get

$f_{(1.34)}$ = $y_{0.4}$ = 1.6984+(0.4)(0.1974)+[(0.16)/2(0.170)]

+[(0.4)(-0.84)/6](0.0020)+[(0.16)-(-0.84)/24](0.0002)

= 17786 (approx)

P-1.3 First prepare a cumulative frequency table and then take xo=45, h=10, x=50, u= (x-x_o)/h = (50-45)/10 = 0.5, use Bessel's formula for haves hereafter.

Number of deaths between 45 and 50 = 42646 – 31368 = 11278

And the number of deaths between 50 and 55 = 24225-11278 = 12947

P-1.4 $u_x = (1/8)(2x^4-8x^2+3)$

P-1.5 For u= (x-6)/2, the polynomial is

$P_{(u)}= y_{(u)}= (1/6)[+30(u-1/2)+5u(u-1)+(2/3)(u-1/2)u(u-1)]$

Which is terms of x is

P(x) = (1/6) (33+15(x-7)+(15/4)(x-6) (x-8)+(1/2) (x-6) (x-7)(x-8)

P-1.6 U_9= 33.

P-1.7 $U_{12.2}$= 0.30495

P- 1.8 (a) 0.43322218

(b) 0.43326587

P-1.9 U_{44}= 1975

P-1.10 f(30) = 16.9217.

7.6 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit-8: Inverse Interpolation

Structure

8.1 Introduction

We considered the problem of direct interpolation or simply interpolation so far in which we intend to find the value of y=f(x) for a particular value of x lying between any two tabulated values of argument. But we may also face the problem of evaluating the unknown value of x for a known value of y, this technique is termed as inverse interpolation. That is *"**the process of determining the unknown value of the argument corresponding to given value of entry, with the help of a set of observations is known as inverse interpolation"***.

2.2 Objectives

After going through this unit you will learn

- What is inverse interpolation
- Problem solving by using different methods

2.3 Inverse Interpolation by Lagrange's Interpolation Formula

Given the (n+1) values of arguments x_0, x_1, x_2,……x_3 with their respective entries y_0, y_1, y_2,……y_3 the Lagrange's interpolation formula for direct interpolation is

$$y = \frac{(x-x_1)(x-x_2)\dots.(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots.(x_0-x_n)}(y_0) + \frac{(x-x_1)(x-x_2)\dots.(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots.(x_n-x_{n-1})}(y_1)$$

$$+\cdots.+\frac{(x-x_0)(x-x_1)\dots.(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\dots.(x_n-x_{n-1})}(y_n).$$

As in the problem of in the problem of inverse interpolation, we have to find out the value of x for given y, the above formula can be rewritten as follows:

$$x = \frac{(y-y_1)(y-y_2)\dots.(y-y_n)}{(y_0-y_1)(y_0-y_2)\dots.(y_0-y_n)}(x_0) + \frac{(y-y_1)(y-y_2)\dots.(y-y_n)}{(y_1-y_0)(y_1-y_2)\dots.(y_n-y_{n-1})}(x_1)$$

$$+\cdots.+\frac{(y-y_0)(y-y_1)\dots.(y-y_{n-1})}{(y_n-y_1)(y_n-y_2)\dots.(y_n-y_{n-1})}(x_n). \qquad \dots\dots\dots\dots \quad (8.1)$$

We illustrate the use of this formula in the following example.

Example 8.1: Using the following data find the value of x corresponding to y = f(x) = 3.0.

x	1.0	1.2	1.5	1.8	2.0
y	4.00	3.408	2.625	2.110	2.0

Solution: Using (8.1), we have

$$x = \frac{(y-3.408)(y-2.625)(y-2.112)\ (y-2.0)}{(4.0-3.408)(4.0-2.625)(4.0-2.112)(4.0-2.0)}(1.0)$$

$$+\ \frac{(y-4.0)(y-2.625)(y-2.112)\ (y-2.0)}{(3.408-4.0)(3.408-2.625)(3.408-2.112)(3.408-2.0)}(1.2)$$

$$+\frac{(y-4.0)(y-3.408)(y-2.112)\ (y-2.0)}{(2.625-4.0)(2.625-3.408)(2.625-2.112)(2.625-2.0)}(1.5)$$

$$+\frac{(y-4.0)(y-3.408)(y-2.625)\ (y-2.0)}{(2.112-4.0)(2.112-3.408)(2.112-2.625)(2.112-2.0)}(1.8)$$

$$+\frac{(y-4.0)(y-3.408)(y-2.112)\ (y-2.0)}{(2.0-4.0)(2.0-3.408)(2.0-2.625)(2.0-2.112)}(2.0)$$

Taking y=3, we get x= 1.4221.

8.4 Method of Successive Approximation

Assuming the function y=f(x) a polynomial of the degree n, we may use any of the interpolation formulae consider so far to find the value of x corresponding to the given value of y.

Consider the Newton's forward formula when (n+1) equidistant values of arguments x_0, x_1, x_2,……x_n with their respective entries y_0, y_1, y_2,……y_n are given .

Where u= $(x-x_0)/h$ and h is the interval of differencing. Since y is given, we have to solve the equation (which will be terms if u) for a real root. Thus we rearrange the equation by transposing its terms as follows

$$u\Delta y_0 = y - y_0 - \frac{u(u-1)}{2!}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 - \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 - \cdots \ldots ..$$

Dividing throughout by Δy_0 we get

$$u = \frac{y - y_0}{\Delta y_0} - \frac{u(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)(u-3)}{4!}\frac{\Delta^4 y_0}{\Delta y_0} + \cdots \ldots \qquad \ldots\ldots\ldots..(8.2)$$

For the first approximation for u we neglect all difference higher than the first degree and take

$$u^{(1)} = \frac{y - y_0}{\Delta y_0}$$

Where the symbol $u^{(1)}$denotes the first approximate value of u. The second approximation, say $u^{(2)}$, can be obtained by putting u=$u^{(1)}$ in the right hand side of (8.2). we then have

$$u^{(2)} = \frac{y - y_0}{\Delta y_0} - \frac{u^{(1)}(u^{(1)}-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u^{(1)}\left(u^{(1)}-1\right)(u^{(1)}-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} - \frac{u^{(1)}\left(u^{(1)}-1\right)\left(u^{(1)}-2\right)(u^{(1)}-3)}{4!}\frac{\Delta^4 y_0}{\Delta y_0} + \cdots \ldots\ldots\ldots\ldots..$$

Similarly, putting u=$u^{(2)}, u^{(3)}$……in (8.2) the process can be continued till two successive approximated values are nearly equal. Finally the value of x will be obtained from u using the relation x = x_n + uh. We illustrate the process in the following example.

Example 8.2: The following values of y=3050 by successive approximation method.

x	10	15	20
y	1804	2698	3614

Solution: We first prepare the difference table in usual manner.

x	y	Δy	Δy_0
10	1804		
		894	
15	2698		22
		916	
20	3614		

We observe from the difference table that second order differences are constant. Form we have (8.2),

$$u = \frac{y - y_0}{\Delta y_0} - \frac{\mathrm{u(u-1)}}{2!}\frac{\Delta^2 \mathrm{y}_0}{\Delta y_0}$$

Thus, we have the first approximation to u as follows

$$u^{(1)} = \frac{y - y_0}{\Delta y_0} = \frac{3050 - 1804}{854} = 1.45902$$

Putting u = 1.39 in (8.3) we get the second approximate value. Here,

$$u^{(2)} = \frac{y - y_0}{\Delta y_0} - \frac{\mathrm{u}^{(1)}(\mathrm{u}^{(1)} - 1)}{2!}\frac{\Delta^2 \mathrm{y}_0}{\Delta y_0}$$

That is,

$$u^{(2)} = 1.4502 - \frac{1.4502(1.4502-1)}{2} \times \frac{22}{894} = 1.44078$$

Now the required value is

X = $x_0 + uh$

= 10+5×1.44078 (since we have taken origin at 1.0 and h=5) = 17.2539.

Example 8.3: For the given values in table find the value of x corresponding to y = 2.285, using Bessel's formula:

Solution:

u	x	y	Δy^*	$\Delta^2 y^*$	$\Delta^3 y^*$	$\Delta^4 y^*$	$\Delta^5 y^*$
-2	0.736	2.2832974					
			8049				
-1	0.737	2.2841023		13			
			8062		-1		
0	0.738	2.2849085		12		3	
			8074		2		-6
1	0.739	2.2857179		14		-3	
			8088		-1		
2	0.740	2.2865247		13			
			8101				
3	0.741	2.2873348					

***** All the values of these columns are to be multiplied by 10^{-7}. Moreover the values in last two columns are too small can be ignored in calculation.

Here the Bessel's formula is

$$y = \frac{y_0 + y_1}{2} + u\Delta y_0 + \frac{(u_{-1/4}^2)\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{u(u^2 - 1/4)}{3!}\Delta^3 y_{-1}$$

After substituting the value of y and its difference and inverting the terms, we get u=-0.386732-($u^2-0.25$) (0.000805053)u(u^2-0.25)(0.000041285)   (8.4)

Here the first approximation is

$u^{(1)} = -0.386732$

For the second approximation we put the value of first approximation in (2.4) and get

$u^{(2)} = -0.386647993$

Similarly

$u^{(3)} = -0.386654283$

$u^{(4)} = -0.386652697$

Hence the value of x is

x=0.738+0.001 (-0.386652697+0.5)

= 0.7388867

Example 8.4: Obtain a real root of the equation x^3 - 6x – 11 = 0 by the inverse interpolation method which lies between 3.0 and 4.0.

Solution: We first prepare the following table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
3.0	-2.0				
		4.568			
3.2	2.568		0.768		
		5.336		0.48	
3.4	7.904		0.816		0
		6.152		0.48	
3.6	14.056		0864		0
		7.016		.048	
3.8	21.072		0.912		

		7.928	
4.0	29.0		

We have inverse interpolation formula (derived from Newton's forward formula) for as follows:

$$u = \frac{y - y_0}{\Delta y_0} - \frac{u(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} \dots\dots\dots\dots\dots (8.5)$$

Taking y = 0 (as there will be a root corresponding to y = 0), the first approximation is

$$u^{(1)} = \frac{y - y_0}{\Delta y_0} = 0.4378$$

Now we substitute the value of $u^{(1)}$ in (8.5) get

$$u^{(3)} = -0.4418$$

$$u^{(4)} = 0.4418$$

Hence the root of the equation is

U = 3.0+.02(0.4418) = 3.008836.

8.5 Method of Reversion of Series

The most suitable method of solving the problem of inverse interpolation is by reversion of series. Since most of the interpolation formulae developed so far can be interpreted easily in form of power series, the process is simple and fast. Moreover, these series are convergent and hence can be reverted to get the value of variable/ argument. Here we consider the power series.

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \dots \dots .. a_n x^n \qquad (8.6)$$

Then as a first approximation we consider the series up to first degree only i.e. let

$y = a_0 + a_1 x$ Then we have x = (y-a_0)/a_r

Now suppose that the inverse function y = f(x) can be expressed as a power series in (y-a_0)/a_1 in the form.

$$x = c_1\left(\frac{y-a_0}{a_1}\right) + c_2\left(\frac{y-a_0}{a_1}\right)^2 + \cdots \ldots + c_n\left(\frac{y-a_0}{a_1}\right)^n \qquad (8.7)$$

Again from (8.6), we have

$$\left(\frac{y-a_0}{a_1}\right) = x + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^3 + \cdots \ldots \ldots ..$$

Putting this value of from $\frac{y-a_0}{a_1}$ (8.8) and (8.7), we get

$$x = c_1\left(x + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^3\right) + c_2\left(x + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^3\right)^2$$

$$+c_3\left(x + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^3\right)^3 + c_4\left(x + \frac{a_2}{a_1}x^2 + \frac{a_3}{a_1}x^3 + \frac{a_4}{a_1}x^3\right)^4 + \cdots$$

$$= c_1 x + \left(\frac{c_1 a_2}{a_1} + c_2\right)x^2 + \left(\frac{a_3}{a_1}c_1 + 2c_2\frac{a_2}{a_1} + c_3\right)x^3 + \cdots \ldots.$$

Equating coefficients of like powers of x on both sides we get

$$c_1 = 1, \frac{c_1 a_2}{a_1} + c_2 = 0 \Rightarrow c_2 = -\frac{a_2}{a_1}, \qquad etc.$$

Thus we get

$$\begin{cases} c_1 = 1 \\ c_2 = -\dfrac{a_2}{a_1} \\ c_3 = -\dfrac{a_3}{a_1} + 2\left(\dfrac{a_2}{a_1}\right)^2 \\ c_4 = -\frac{a_4}{a_1} + 5\left(\frac{a_2 a_3}{a_1^2}\right) - 5\left(\frac{a_2}{a_1}\right)^3, \; etc \end{cases} \qquad (8.9)$$

and the value of x can be obtained by substituting the value of x in (8.8).

Thus in this method the values $a_0, a_{1,} a_2 \ldots . . \ldots ..$of will be obtain from interpolation formula by writing the formula in terms of power series then the values of $c_1, c_2 \ldots ..$will be obtained with the help of the values of a's using relation (8.9). As an illustration we shall now write Newton's and Stirling's formulae in the form of power series and then obtain the values of $a_0, a_{1,}\ a_2$ …… … ..for the same.

(i) Newton's formula:

$$y = y_0 + \mu\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+\frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= y_0 + \left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4}\right)u + \left(\frac{\Delta^2 y_0}{2} - \frac{\Delta^3 y_0}{3} + \frac{11\Delta^4 y_0}{4}\right)u^2 + \left(\frac{\Delta^3 y_0}{6} + \frac{\Delta^4 y_0}{4}\right)u^3 + \frac{\Delta^4 y_0}{4}u^4$$

$$a_0 = y_0$$

$$\left.\begin{aligned} a_1 &= \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} \\ a_2 &= \frac{\Delta^2 y_0}{2} - \frac{\Delta^3 y_0}{3} + \frac{11\Delta^4 y_0}{4} \\ a_3 &= \frac{\Delta^3 y_0}{6} + \frac{\Delta^4 y_0}{4} \\ a_4 &= \frac{\Delta^4 y_0}{4} \end{aligned}\right\} \quad (8.10)$$

(ii) Stirling's Formula:

$$y_u = y_0 + u\frac{\Delta y_{-1} - \Delta y_0}{2!} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{u(u^2-1)}{4!}\Delta^4 y_{-2} + \cdots$$

$$= y_0 + \left(\frac{\Delta y_{-1} - \Delta y_0}{2!} - \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2}\right)u + \left(\frac{\Delta^2 y_{-1}}{2} - \frac{\Delta^4 y_{-2}}{24}\right)u^2 + \left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12}\right)u^3$$

$$+ \frac{\Delta^4 y_{-2}}{24}u^4 + \cdots ....$$

Here,

$$\left\{\begin{aligned} a_1 &= y_0 \\ a_1 &= \frac{\Delta y_{-1} - \Delta y_0}{2!} - \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12} \\ a_2 &= \frac{\Delta^2 y_{-1}}{2} - \frac{\Delta^4 y_{-2}}{24} \\ a_3 &= \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12} \\ a_4 &= \frac{\Delta^4 y_{-2}}{24} \end{aligned}\right. \qquad (8.11)$$

Example 8.5: Find the value x corresponding to y= 15.0 in the following table, using the method of reversion of series.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30	15.9				
		-1.0			
35	14.9		0.2		
		-0.8		-0.4	
40	14.1		-0.2		1.0
		-1.2		0.6	
45	13.3		0.4		
		-0.8			
50	12.5				

To apply the method we use Newton's forward formula for which the value of a's are given in (8.10) which, using the above difference table, provide

$$a_1 = y_0 = 15.9$$

$$a_1 = \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} = (-1) - \frac{0.2}{2} - \frac{0.4}{3} - \frac{1}{24} = -1.48333$$

$$a_2 = \frac{\Delta^2 y_0}{2} - \frac{\Delta^3 y_0}{3} + \frac{11\Delta^4 y_0}{4} = \frac{0.2}{2} + \frac{0.4}{3} + \frac{11}{4} = 0.75833$$

$$a_3 = \frac{\Delta^3 y_0}{6} + \frac{\Delta^4 y_0}{4} = \frac{-0.4}{6} - \frac{1}{4} = -0.31667$$

$$a_4 = \frac{\Delta^4 y_0}{4} = \frac{1}{24} = 0.04167$$

Using these values of a's we get the values of c's as follows

$$c_1 = 1$$
$$c_2 = -\frac{a_2}{a_1} = \frac{-0.75833}{-1.48333} = 0.51123$$

Similarly,

$$c_3 = 0.309236157 \ and \ \ c_4 = -0.15855$$

Since

$$x = c_1 w + c_2 w^2 + c_3 w^3 + c_4 w^4,$$

$where$

$$w = (y - a_0)/a_1 = 0.60674$$

We have,

$$x = 1 \times 0.60674 + 0.51123 \times (0.60674)^2 + 0.309236157 \times (0.60674)^3 - 0.15855 \times (0.60674)^4$$

$$\Rightarrow x = 0.866301959$$

$The\ required\ value\ of\ x\ is$

X = 30 + 5(0.842523) = 34.212615.

8.6 Summary

There are several problems arise in mathematics in which we intend to find the value of x corresponding to a given value of f(x) when the functional from of f(x) is not known and we are only given the set of tabulated values of x and f(x). Secondly, if the explicit form of f(x) is given which is in form of complicated equation and we have to solve this equation. All the problems of this kind can be solved by the technique of inverse interpolation. Any of the methods described so far can be chosen according to the given situation.

8.7 Exercise

P - 1: Apply Lagrange's formula to find the value of age for which annuity value f(x) is 13.6.

x	30	35	40	45	50
f(x)	15.9	14.9	14.1	13.3	12.5

P - 2: Find the value of x corresponding to y = 14.0 for the values given in the following table:

x	0	5	10	15
y	16.35	14.88	13.5	12.46

P - 3: Find the root of the equation $x^3 - 2x - 5 = 0$, which lies between 2 and 3, correct to three decimal places by any suitable method of inverse interpolation.

P - 4: Find the root of the equation $x^3 + 15x + 4 = 0$ near to 0.3 using inverse interpolation with Bessel's formula.

Answer: P - 1: 40.1, **P - 2:** 8.34, **P - 3:** 2.0945, **P - 4:** 0.26795

8.8 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Analysis, McMillan Publishing Company, New York: M.J. Marom
6. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit-9: Numerical Differentiation

Structure

9.1 Introduction

In the previous part of this course we considered the process of interpolation in which we either establish an approximate function for the given tabular values or find out the value of the function at any given argument. A similar kind of problem is to find the numerical value of the derivative at any particular value of argument for a given set of observations which can be solved using the process of numerical differentiation.

Thus "*Numerical differentiation is a process of evaluating the derivative of a function at some particular value of the argument when the values of the function corresponding to the given value of argument are known*". The problem of numerical differentiation is solved by first fitting up a polynomial to the given set of values of the function and then differentiating it as many times as desired. The fitting of polynomial can be done by using any of the interpolation formula.

9.2 Objectives

After the study of this unit you shall be able to:

- Evaluate the derivative of a function at any value of independent variable for a given set of tabular values when the explicit for of the function is not known.

- Derive approximate expressions for the derivatives of functions based on the given tabular values.

9.3 Numerical Differential for Equal Intervals

As formulae for the derivatives in numerical differentiation can be derived using interpolation formula, we illustrate here the process for Newton's forward formula and the Stirling formula.

(i) Newton's forward formula

Suppose we are given (n+1) values of variables x_0, x_1, x_2,……x_n with their respective entries y_0, y_1, y_2,……y_n the Newton's forward interpolation formula is .

Where u= $(x-x_0)/h$ and h is the interval of differencing. Since y is given, we have to solve the equation (which will be terms if u) for a real root. Thus we rearrange the equation by transposing its terms as follows

$$y_u = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots ..$$

……(9.1)

Where $u = (x - x_0) / h$.

Now we get the first derivative of y in (9.1) with respect to x. Since $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

We have from (9.1).

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0 + \frac{4u^3 - 18u^2 + 22u - 6}{4!}\Delta^4 y_0 \ldots .\right]$$

…………..(9.2)

This formula will get be used to get the first derivative. Now differentiating (9.2) again, we get the formula for the second derivative as follows

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta y_0 + (u-1)\Delta y_0 + \frac{12u^2 - 36u + 22}{24}\Delta^4 y_0 + \cdots \ldots .\right] \qquad (9.3)$$

The differences of higher order can also be found similarly.

(ii) Stirling formula:

$$y_u = y_0 + u\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\frac{[\Delta^3 y_{-1} + \Delta^3 y_{-2}]}{2} +$$

$$+\frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \frac{u^2(u^2-1^2)(u^2-2^2)}{5!}\frac{[\Delta^5 y_{-3} + \Delta^5 y_{-3}]}{2}$$

$$+\frac{u^2(u^2-1^2)(u^2-2^2)}{6!}\Delta^6 y_{-3} + - - - - - - - -$$

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2!} + u\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{4u^3-2u}{4!}\Delta^4 y_{-2}\right.$$

$$\left. + \frac{5u^4-15u^2+4}{5!}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2} + \frac{6u^5-20u^3+8u}{6!}\Delta^6 y_{-3}\right]$$

$$\dots\dots(9.4)$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_{-1} + u\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{12u^2-2}{4!}\Delta^4 y_{-2} + \frac{20u^3-30u+4}{5!}\frac{\Delta^5 y_{-3} + \Delta^5 y_{-2}}{2}\right.$$

$$\left. + \frac{30u^4-60u^2+8}{6!}\Delta^6 y_{-3}..\right]$$

$$\dots\dots\dots\dots\dots(9.5)$$

The derivatives for another formula can be obtained similarly.

Example 9.1: Find the first and second derivatives of the function y = f(x), tabulated below at x = 1.1 & 1.8.

X =	1.0	1.2	1.4	1.6	1.8	2.0
f(x)	2.00	3.128	4.544	6.296	8.432	11.000

Solution: Here first all we prepare a difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	2.0000				
		1.128			

1.2	3.128		0.288		
		1.416		0.048	
1.4	4.544		0.366		0
		1.752		0.048	
1.6	6.296		0.384		0
		2.136		0.048	
1.8	8.432		0.432		
		2.568			
2.0	11.0000				

After observing the difference table we use formula (3.2) up to third order term thus

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{3!}\Delta^3 y_0\right]$$

Here u = (x - x_0) / h = (1.1-1.0)/0.2 = 1/2, thus

$$\left.\frac{dy}{dx}\right|_{x=1.1} = \frac{1}{2}\left[1.1280 + \frac{2.\frac{1}{2} - 2}{2}(0.2880) + \frac{3.\frac{1}{2} - 6.\frac{1}{2} + 2}{6}(0.0480)\right]$$

$$= \frac{1}{0.2}\left[1.1280 + 0 - \frac{1}{24}(0.0480)\right] = 5.65$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\Delta^2 y_0 + (u-1)\Delta^3 y_0]$$

$$= \frac{1}{0.4}[(0.2880) + (0.5-1)(0.0480)] = 7.8$$

To find the derivative at x=1.8, we have the Newton's backward formula

$$\mathrm{y_u} = \mathrm{y_0} + \mathrm{u}\nabla\mathrm{y_n} + \frac{\mathrm{u(u+1)}}{2!}\nabla^2\mathrm{y_n} + \frac{\mathrm{u(u+1)(u+2)}}{3!}\nabla^3\mathrm{y_n},$$

$$where\ u = \frac{(x - x_n)}{h}$$

Thus,

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2u+1)}{2!}\nabla^2 y_n + \frac{3u^2-6u+2}{3!}\nabla^3 y_n\right] \; and$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\nabla^2 y_0 + (u-1)\nabla^3 y_0]$$

Here u=(1.8-2.0)/0.2= -1, thus

$$\left.\frac{dy}{dx}\right|_{x=1.8} = \frac{1}{0.2}\left[2.568 + \frac{(-2+1)}{2}(0.432) + \frac{3-6+2}{6}(0.0480)\right] = 11.34$$

and

$$\left.\frac{d^2y}{dx^2}\right|_{x=1.8} = \frac{0.432}{0.04} = 10.8$$

Example 9.2: Find the first and second derivatives of the function tabulated below at the point x = 0.61

X:	0.4	0.5	0.6	0.7	0.8
Y:	2.583649	2.797442	3.044237	3.327505	3.651081

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	2.583649				
		0.21380			
0.5	2.797442		0.03299		
		0.24679		0.00349	
0.6	2.797442		0.3648		0.00034
		0.28327		0.00383	
0.7	3.327505		0.4031		
		0.32358			
0.8	3.651081				

We use Stirling formula for this problem. Ignoring the differences fifth and higher order we have from (9.4),

$$\frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_{-1} + \Delta y_0}{2!} + u\Delta^2 y_{-1} + \frac{3u^2-1}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{4u^3-2u}{4!}\Delta^4 y_{-2}\right]$$

Here x = 0.61, u = 0.1, h = 0.1. Substituting these values in the above formula for the first and second derivatives we get

$$\frac{dy}{dx} = \frac{1}{0.1}\left[\frac{\Delta y_{-1} + \Delta y_0}{2!} + (0.1)\Delta^2 y_{-1} + \frac{3u^2 - 1}{3!}\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{2} + \frac{4u^3 - 2u}{4!}\Delta^4 y_{-2}\right]$$

$$= \frac{1}{0.1}\left[\frac{0.24679 + 0.28327}{2} + 0.1(0.03648) + \frac{0.03 - 1}{12}(0.00732) + \frac{0.004 - 2}{24}(0.0034)\right]$$

$$= 2.642227$$

Similarly,

$$\frac{d^2x}{dy^2} = 4.06983$$

Remark: The function tabulated above is

$$y = 2e^x - x$$

Hence

$$\frac{dy}{dx} = 2e^x - 1 \quad and \quad \frac{d^2x}{dy^2} = 2e^x$$

Putting x=0.6 in these, we get

$$\frac{dy}{dx} = 2.644238 \quad and \quad \frac{d^2x}{dy^2} = 3.644238$$

as the correct values for the first and second derivatives. The values obtained by numerical differentiation are therefore approximately near to the values obtained by the numerical differentiation.

9.4 Numerical Differential for Unequal Intervals

We consider this case with the help of following example:

Example 9.3: Find first, second and third derivative at x = 3.2 in the following table:

x:	3.0	3.3	3.5	3.8	4.0
f(x):	35.0	90.466	135.813	218.258	284.000

Solution: Here the values are not at equal interval. We use divided difference table. Here Δf(x) stands for the first divided difference of f(x).

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
3.0	35.0				
		184.887			
3.3	90.466		83.695		
		226.734		15.595	
3.5	135.81		96.171		0.989
		274.82		16.584	
3.8	218.258		107.78		
		328.71			
4.0	284.000				

The Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)f(x_0, x_1) + (x - x_0)(x - x_1)f(x_{0,}x_1, x_2)$$
$$+ (x - x_0)(x - x_1)(x - x_2)f(x_{0,}x_1, x_2, x_3)$$

Differentiating w.r.t. x, we get

$$f^{'}(x) = f(x_0, x_1) + (2x - (x + x_1))f(x_{0,}x_1, x_2)$$

$$+[3x^2 - 2x(x_0 + x_1 + x_2) + (x_0x_1 + x_1x_2 + x_2x_3)]f(x_{0,}x_1, x_2, x_3).$$

Similarly the second and third derivatives are obtained as follows

$$f^{''}(x) = +2f(x_0, x_1, x_2) + [6x - 2(x_0 + x_1 +, x_2]f(x_{0,}x_1, x_2)$$

$$f^{'''}(x) = 6f(x_{0,}x_1, x_2, x_3)$$

Substituting the values in (9.5), we get

$$f^{'}(x)|_{x=3.2} = 184.887 + [6.4 - 6.3](83.695) + [3(10.24) - 6.4(9.8) + 3195](15.595)$$

$$+[4(32.768) - 3(10.24)(13.6) + 6.4(69.19) - 156.06](0.9892)$$

$= 191.978193$

Similarly

$f''(x) = 161.152$

$f'''(x) = 93.570$

9.5 Approximate Expressions for the Derivatives of a Function

We have

$$\delta y_x = y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}$$

$$u y_x = \frac{1}{2}\left(y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}\right)$$

Thus,

$$u\delta y_x = u\left(y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}\right)$$

$$= \frac{1}{2}(y_{x+1} + y_x - y_x - y_{x-1})$$

$$= \frac{1}{2}(y_{x+1} - y_{x-1})$$

$$= \frac{1}{2}(e^{hD} - e^{-hD})y_x \qquad [Since \quad y_{x+1} = Ey_x = e^{hD}y_x]$$

$$= \sinh\ (hD)$$

$$= hD + \frac{1}{3!}(hD)^2 + \ \dots\dots$$

$$= hD\ (neglecting\ higher\ order\ derivatives) \qquad (9.7)$$

Thus from (9.6) and (9.7), we obtain

$$hDy_x = \frac{1}{2}(y_{x+1} - y_{x-1})$$

$$Dy_x = \frac{1}{2h}(y_{x+1} - y_{x-1}) \qquad (9.8)$$

Now,

$$\delta^2 y_x = \delta\delta y_x$$

$$= \delta\left(y_{x+\frac{1}{2}} - y_{x-\frac{1}{2}}\right)$$

$$= (y_{x+1} + y_x - y_x - y_{x-1})$$

$$= y_{x+1} + 2y_x - y_{x-1}$$

$$= (E - 2 + E^{-1})y_x$$

$$= (e^{hD} + e^{-hD} - 2)y_x \qquad [y_{x+1} = Ey_x = e^{hD}y_x]$$

$$= \left[\left(1 + hD + \frac{1}{2!}(hD)^2 + \cdots\right) + \left(1 - hD + 1 + hD + \frac{1}{2!}(hD)^2 + \cdots\right) - 2\right]$$

$$= 2\left(\frac{1}{2}(hD)^2 + \cdots\right)y_x$$

$$= h^2D^2y_x \ (neglecting\ higher\ oredr\ derivatives)$$

Thus

$$h^2D^2y_x = \frac{1}{2}(y_{x+1} - 2y_x + y_{x-1})$$

Or

$$D^2y_x = \frac{1}{h^2}(y_{x+1} - 2y_x + y_{x-1})$$

Again,

$$h^3D^3y_x = hD(h^2D^2)y_x$$

$$= hD\delta^2 y_x \qquad [from\ (9.9)]$$

$$= u\delta(y_{x+1} - 2y_x + y_{x-1}) \qquad [from\ (9.7)]$$

$$= \frac{1}{2}(\Delta + \nabla)(y_{x+1} - 2y_x + y_{x-1})$$

$$= [(y_{x+2} - 2y_x + y_x) - (y_{x+1} - 2y_x + y_{x-1}) + (y_{x+1} - 2y_x + y_{x-1}) - (y_x - 2y_{x-1} + y_{x-2})]$$

$$= \frac{1}{2}[y_{x+2} - 2y_{x-1} + 2y_{x-1} - y_{x-2}]$$

Thus

$$D^3 y_x = \frac{1}{2h^3}(y_{x+2} - 2y_{x-1} + 2y_{x-1} - y_{x-2})$$

Now,

$$h^4 D^4 y_x = (h^2 D^2)(h^2 D^2) y_x$$

$$= \delta^2 \delta^2 y_x$$

$$= \delta^2 (y_{x+1} - 2y_x + y_{x-1})$$

$$= (\delta^2 y_{x+1} - 2\delta^2 y_x + \delta^2 y_{x-1})$$

$$= [(y_{x+2} - 2y_{x+1} + y) - 2(y_{x+1} - 2y_x + y_{x-1}) + (y_x - 2y_{x-1} + y_{x-2})]$$

$$= y_{x+2} - 4y_{x+1} + 6y_x - 4y_{x+1} + y_{x-2}$$

Thus

$$D^4 y_x = \frac{1}{h^4}(y_{x+2} - 4y_{x+1} + 6y_x - 4y_{x+1} + y_{x-2})$$

Thus we get the approximate expressions for the first four derivatives as follows:

$$\left\{ \begin{array}{l} y_x^{i} = \dfrac{1}{2h}(y_{x+1} - y_{x-1}) \\ y_x^{ii} = \dfrac{1}{h^2}(y_{x+1} - 2y_x + y_{x-1}) \qquad \ldots \ldots (9.10) \end{array} \right.$$

$$y_x^{iii} = \frac{1}{2h^3}(y_{x+2} - 2y_{x+1} + 2y_{x-1} - y_{x-2})$$

$$y_x^{iv} = \frac{1}{h^4}(y_{x+2} - 4y_{x+1} + 6y_{x-1} - 4y_{x-1} + y_{x-2})$$

Example 9.4: Assuming Bessel's interpolation formula obtain the following result

$$\frac{d}{dx}y_x = \Delta y_{x-\frac{1}{2}} - \frac{1}{24}\Delta^3 y_{x-\frac{3}{2}} + \cdots ..$$

Solution: The Bessel's formula is

$$y_x = \frac{y_0 + y_1}{2} + \left(x - \frac{1}{2}\right)\Delta y_0 + \frac{x(x-1)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{\left(x - \frac{1}{2}\right)x(x-1)}{3!}\Delta^3 y_{-1} + \cdots$$

$$y_{x+\frac{1}{2}} = \frac{y_0 + y_1}{2} + x\Delta y_0 + \frac{\left(x + \frac{1}{2}\right) + \left(x - \frac{1}{2}\right)}{2!}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \frac{x\left(x + \frac{1}{2}\right) + \left(x - \frac{1}{2}\right)}{3!}\Delta^3 y_{-1} + \cdots \ldots$$

$$\frac{d}{dx}y_{\frac{1}{2}} = \Delta y_0 + x\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} + \left(\frac{x^2}{2} - \frac{1}{24}\right)\Delta^3 y_{-1} + \cdots. \qquad (9.11)$$

Taking x = 0 in (9.11), we obtain

$$\frac{d}{dx}y_{\frac{1}{2}} = \Delta y_0 - \frac{1}{24}\Delta^3 y_{-1} + \ \ldots ..$$

Shifting origin to x-1/2, we get

$$\frac{d}{dx}y_x = \Delta y_{x-\frac{1}{2}} - \frac{1}{24}\Delta^3 y_{-1} + \ \ldots ..$$

Example 9.5: Prove that

$$y' = \frac{1}{h}\left(\delta y - \frac{1}{24}\delta^3 y + \frac{3}{640}\delta^5 y - \cdots\right)$$

and

$$y'' = \frac{1}{h^2}\left(\delta^2 y - \frac{1}{12}\delta^4 y + \frac{3}{90}\delta^6 y - \cdots\right)$$

Solution: We have

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= e^{\frac{hD}{2}} - e^{-\frac{hD}{2}}$$

$$= 2Sinh(hD/2)$$

Thus

$$\frac{hD}{2} = Sinh^{-1}(\delta/2) \qquad (9.12)$$

By Taylor's series expansion, we have

$$Sinh^{-1}x = x - \frac{x^3}{6} + \frac{3}{40}x^5 - \ \ldots\ldots..$$

Using expansion in (9.12), we get

$$\frac{hD}{2} = \frac{\delta}{2} - \frac{\delta^3}{6.8} + \frac{3}{640}\delta^5 - \ \ldots\ldots$$

Or,

$$D = \frac{1}{h}\left[\delta - \frac{1}{24}\delta^3 + \frac{3}{64}\delta^5 - \ \ldots\ldots\right] \qquad (9.13)$$

Or,

Squaring (9.13), we get

$$D^2 = \frac{1}{h^2}\left[\delta^2 - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 - \ \ldots\ldots\right]$$

Or,

$$y'' = D^2 y = \frac{1}{h^2}\left[\delta^2 y - \frac{1}{12}\delta^4 + \frac{1}{90}\delta^6 y - \ \dots\dots\right]$$

Example 9.6: Prove that

$$y'_x = \frac{1}{h}(y_{x+h} - y_{x-h}) - \frac{1}{2h}(y_{x+2h} - y_{x-2h}) + \frac{1}{3h}(y_{x+3h} - y_{x-3h}) - \cdots ..$$

Solution:

$$RHS = \frac{1}{h}(y_{x+h} - y_{x-h}) - \frac{1}{2h}(y_{x+2h} - y_{x-2h}) + \frac{1}{3h}(y_{x+3h} - y_{x-3h}) - \cdots.$$

$$= \frac{1}{h}(E - E^{-1})y_x - \frac{1}{2h}(E^2 - E^{-2})y_x + \frac{1}{3h}(E^3 - E^{-3})y_x$$

$$= \frac{1}{h}\left[\left\{E - \frac{1}{2}E^2 + \frac{1}{3}E^3 + \right\} - \left\{E^{-1} - \frac{1}{2}E^{-2} + \frac{1}{3}E^{-3} - \cdots\right\}\right]y_x$$

$$= \frac{1}{h}[\log(1 + E) - \log(1 - E^{-1})]y_x$$

$$= \frac{1}{h}log\left\{\frac{1 + E}{1 + E^{-1}}\right\}y_x$$

$$= \frac{1}{h}(log\ E)y_x$$

$$= \frac{1}{h}(loge^{hD})y_x$$

$$= \frac{1}{h}(hD)y_x$$

$$= Dy_x = y'_x = LHS$$

9.6 Summary

The derivative of a function by numerical differentiation can be found using any of the interpolation suitable to the problem. For the selection of appropriate formula, we follow the various cases of interpolation problem. That is, to choose any of the Newton's Sterling's or Bessel's formula for the derivative values. Moreover, if we intend to find the values of derivative of function at a point near the beginning or end of given set of values, we use Newton's forward or backward formula accordingly. If the derivative is to be found at a middle point any of the central difference formula can be used. Newton's divided difference and Lagrange's formulae are appropriate for the cases of unequal intervals.

9.7 Exercise

P - 1: Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ and at x= 1 from the following table by using appropriate interpolation formula.

X	1	2	3	4	5	6
Y	198669	295520	389418	479425	564642	644217

P - 2: Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ and at x= 9 from the following table by using appropriate interpolation formula.

X	1	2	4	8	10
Y	0	1	5	21	27

P - 3: Calculate f(7.50) from the values using Bessel's formula.

X	7.47	7.48	7.49	7.50	7.51	7.52	7.53
Y	0.193	0.195	0.198	0.201	0.203	0.206	0.208

P - 4: From the following table, find first and second derivative of f(x) at x=10.

X	3	5	11	27	34
Y	-13	23	899	17315	35606

[**Hint:** use Newton's divided difference formula]

Answers: P - 1: 98008, - 1986; **P - 2**: 6.0, 0.6667; **P - 3**: 17.60; **P - 4**: 233, 54.0

9.8 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

Unit-10: Numerical Integration (Quadrature)

Structure

10.1 Introduction

Numerical integration is the process of computing the value of a definite integral from a set of known numerical values of the integrands. When applied to the integration of a function of a single variable, the process is called numerical quadrature.

10.2 Objectives

After going through this limit, you will learn:

- The concept of numerical integratic.
- How to solve problems of numerical integration.
- How to sum a series using Eular-Macularian formula.

The problem of numerical integration is solved by replacing the integrand by an interpolating and then integrating this polynomial between the desired limits. Suppose some numerical values $f(a_0), f(a_1), \dots . f(a_n)\ of\ f(x) for\ x = a_0, a_l, \dots . a_{ll}$ respectively are given and we are required to find the integral.

$$I = \int_{a_0}^{a_n} f(x)dx$$

Here we shall consider some simple approximate methods of finding the value of a definite integral from a given set of numerical values of the integrand. This process is also known as numerical quaderature when the integrand is a function of a single variable.

Here the integrand is replace by a suitable interpolation formula, usually one involving differences, and then by term between the desired limits. We can get different quaderature formulae as they are called by terms up to different orders of difference. We shall obtain below some quaderature formulae by integrating Newton's forward formula.

In Newton's formula, u = (x-a0)/h and dx = hdu; and if the limits of integration for x are a_0 and a_n the limits in terms of u will be 0 and n. Hence

$$\int_{a_0}^{a_n} f(x)dx = h\int_{0}^{n} \left[f(a_0) + u\Delta f(a_0) + \binom{u}{2}\Delta^2 f(a_0) + \binom{u}{3}\Delta^3 f(a_0) + \cdots.\right] du$$

$$= h\left[nf(a_0) + \frac{n^2}{2}\Delta f(a_0) + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 f(a_0)}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 f(a_0)}{2!} + ..\right]$$

…………(10.1)

This formula is called general quaderature formula.

10.3 Trapezoidal Rule

Here was assume that integrand is such that it can be well represented by straight line in any interval of width h. That means f(x) can be represented by first-degree polynomial or, equivalently, can be regarded as a constant. Accordingly, putting in (10.1) n= 1 and neglecting difference of all orders higher than the first we get

$$\int_{a_0}^{a_1} f(x)dx = h\left\{f(a_0) + \frac{\Delta f(a_0)}{2}\right\} = \frac{h}{2}[f(a_0) + f(a_1)].$$

Similarly, we have for the other intervals,

$$\int_{a_1}^{a_2} f(x)dx = \frac{h}{2}[f(a_1) + f(a_2)], \dots \dots \dots \dots \dots \dots \dots . etc.$$

Adding all these expressions, we get

$$\int_{a_0}^{a_n} f(x)dx = \int_{a_0}^{a_1} f(x)dx + \int_{a_1}^{a_2} f(x)dx + \cdots . + \int_{a_{n-1}}^{a_n} f(x)dx$$

$$= \frac{h}{2}[f(a_0) + f(a_1)] + \frac{h}{2}[f(a_1) + f(a_2)] \dots . \frac{h}{2}[f(a_{n-1}) + f(a_n)]$$

$$= \frac{h}{2}[f(a_0) + 2f(a_1) + 2f(a_2) + \cdots . +2f(a_{n-1}) + f(a_n)].$$

$$= \frac{h}{2}[\{f(a_0) + f(a_n)\} + 2\{f(a_1) + f(a_2) + \cdots . +f(a_{n-1})\}] \qquad (10.2)$$

This is known as a Trapezoidal rule. It is useful where h is small, as for any small segment, a straight line can approximate a smooth curve.

10.4 Simpson's One –Third Rule

Taking n = 2 in the general quadrate formula (10.1) and neglecting the difference of third and higher order we get for the interval $[a_0, a_2]$,

$$\int_{a_0}^{a_2} ydx = \frac{h}{3}[f(a_0) + 4f(a_1) + f(a_2)].$$

For the next interval $[a_2, a_4]$ we have

$$\int_{a_2}^{a_4} ydx = \frac{h}{3}[f(a_2) + 4f(a_3) + f(a_4)].$$

Similarly, for the third interval $[a_4, a_6]$

$$\int_{a_4}^{a_6} ydx = \frac{h}{3}[f(a_4) + 4f(a_5) + f(a_6)] \; and \; so \; on \dots \dots$$

Finally, we have

$$\int_{a_{n-2}}^{a_n} ydx = \frac{h}{3}[f(a_{n-2}) + 4f(a_{n-1}) + f(a_n)], (where \; n \; is \; even)$$

Adding all such expression, we obtain

$$\int_{a_0}^{a_n} ydx = \frac{h}{3}[f(a_0) + 4f(a_1) + f(a_2) + f(a_2) + 4f(a_3) + f(a_4) + f(a_4) + 4f(a_5) + f(a_6)$$

$$+ \cdots .. + f(a_{n-2}) + 4f(a_{n-1}) + f(a_n)]$$

$$= \frac{h}{3}[f(a_0) + f(a_n) + 4[f(a_1) + f(a_3) + \cdots + f(a_{n-1})]$$

$$+ 2\big[\big(f(a_2) + f(a_4) + \cdots + f(a_{n-2})\big)\big] \qquad (10.3)$$

This is Simpson's 1/3 rule. It is simple, accurate and very useful. Here, we have assumed that interval is divided into an even number of intervals and geometrically it means that we have replaced the graph of the given function by n/2 acres of second degree polynomial.

10.5 Simpson's 3/8 Rule

Put n = 3 a in (10.1) and ignore all difference above third, we get

$$\int_{a_0}^{a_3} f(x)dx = h \left[3f(a_0) + \frac{9}{4}\{f(a_1) - f(a_0)\} + \frac{5}{4}\{f(a_2) - 2f(a_1) + f(a_0)\}\right.$$

$$\left. + \frac{3}{8}\{f(a_3) - 3f(a_2) + 3f(a_1) - f(a_0)\}\right]$$

$$= \frac{3h}{2}[f(a_0) + 3f(a_1) + 3f(a_2) - f(a_3)]$$

Similarly,

$$\int_{a_0}^{a_3} f(x)dx = \frac{3h}{2}[f(a_3) + 3f(a_4) + 3f(a_5) - f(a_6)] \ \textit{and so on} \dots ..$$

$$\int_{a_6}^{a_0} f(x)dx = \frac{3h}{2}[f(a_3) + 3f(a_4) + 3f(a_5) + f(a_6)]$$

Adding all these integrals, we have

$$\int_{a_6}^{a_0} f(x)dx = \frac{3h}{2}[\{f(a_0) - f(a_n)\} + 3\{f(a_1) + f(a_2) - f(a_4) + \cdots . + f(a_{n-1})\}$$

$$+ 2\{f(a_3) + f(a_6) + \cdots . + f(a_{n-1})\}] \qquad \dots \dots \dots \dots \dots \dots (10.4)$$

10.6 Weddle's Rule

Here we replace the integrand by a sixth degree polynomial. Accordingly we put n = 6 in (10.1). After simplification we get

$$\int_{a_0}^{a_6} f(x)dx = h\left[6f(a_0) + 18\Delta f(a_0) + 27\Delta^2 f(a_0) 24\Delta^3 f(a_0) + \frac{123}{10}\Delta^4 f(a_0) + \frac{33}{10}\Delta^5 f(a_0)\right.$$

$$\left. + \frac{41}{140}\Delta^6 f(a_0)\right]$$

Where replace the coefficient of $\Delta^6 f(a_0)$ which is 41/140 by 3/10 since the error due to this change is negligible. Thus we have after solving

For the next interval, we have

$$\int_{a_0}^{a_6} f(x)dx = \frac{3h}{10}[f(a_0) + 5f(a_1) + f(a_2) 6f(a_3) + f(a_4) + 5f(a_5) f(a_6)]$$

For the next interval, we have

$$\int_{a_0}^{a_{12}} f(x)dx = \frac{3h}{10}[f(a_7) + 5f(a_7) + f(a_8) 6f(a_9) + f(a_{10}) + 5f(a_{11}) f(a_{12})]$$

and so on…….

adding all these integrals we have

$$\int_{a_6}^{a_n} f(x)dx = \int_{a_6}^{a_6} f(x)dx + \int_{a_6}^{a_{12}} f(x)dx + \cdots \ldots + \int_{a_{n-6}}^{a_n} f(x)dx$$

$$= \frac{3h}{10}[f(a_0) + 5f(a_1) + f(a_2)6f(a_3) + f(a_4) + 5f(a_5)f(a_6)] \qquad (10.5)$$

This is Weddle's rule. This rule is very accurate. In usefulness it is second only to Simpson's 1/3 rule. Similarly, by replacing f(x) other higher degree polynomials.

10.7 Euler- Maclaurin Formula

This formula is based on the expansion of operators.

Suppose

$\Delta f(x) = f(x)$

$That\ is\ F(a_1) - F(a_0) = f(a_0)$

$Similarly\ F(a_2) - F(a_1) = f(a_1)$

$and\ so\ on \ldots..$

$F(a_n) - F(a_{n-1}) = f(a_{n-1})$

Adding all the above expression we get

Now,

$$\sum_{r=0}^{n-1} f(a + rh) = F(a_n) - F(a_0) = F(a + nh) - F(a).$$

$When\ an = a + nh$

Now,

$F(x) = \Delta^{-1} f(x) = [e^{hD} - 1]^{-1} f(x),$

$$= \left[hD + \frac{h^2D^2}{2} + \frac{h^3D^3}{6} + \frac{h^4D^4}{24} + \ \dots\dots\right]^{-1} f(x)$$

$$= (hD)^{-1}\left[1 + \left(\frac{hD}{2} + \frac{h^2D^2}{6} + \frac{h^3D^3}{24} + \cdots\dots\right)\right]^{-1} f(x)$$

$$= (hD)^{-1}\left[1 - \left(\frac{hD}{2} + \frac{h^2D^2}{12} - \frac{h^3D^3}{720} + \cdots\dots\right)\right]^{-1} f(x)$$

$[using\ the\ expansion\ of\ (1+x)^{-1}]$

$$= \frac{1}{h}\left[D^{-1}f(x) - \frac{h^2}{2}f'(x) - \frac{h^3}{720}f'''(x)\dots\right]$$

$as \quad Df(x) = f(x). so D^{-1}f(x) = \int f(x)dx.$

Thus,

$$\sum_{r=0}^{n-1} f(a = rh) = F(a+nh) - F(a)$$

$$= \frac{1}{h}\int_a^{a+nh} f(x)dx - \frac{1}{2}[f(a+nh) - f(a)] + \frac{h}{12}[f'(a+nh) - f'(a)]$$
$$- \frac{h^3}{720}[f'''(a+nh) - f'''(a)] + \cdots\dots$$

$$\frac{1}{h}\int_a^{a+nh} f(x)dx = \left[\frac{1}{2}f(a) + f(a+h) + \cdots + f\left(a + \overline{n-1h}\right) + \frac{1}{2}f(a+nh)\right]$$

$$- \frac{h}{12}[f'(a+nh) - f'(a)] + \frac{h^3}{720}[f'''(a+nh) - f'''(a)] \dots\dots\dots\dots\dots\dots\dots (10.6)$$

This is known as Euler-Maclaurin formula. In the form (10.6) it is useful in finding the sum of a series. Using the Bernoullian numbers.

$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \ldots\ldots\ldots\ldots\ldots..(4.6) can\ be\ expressed\ as$

$$\frac{1}{h}\int_{a}^{a+nh} f(x)dx = \left[\frac{1}{2}f(a) + f(a+h) + \cdots + f\left(a + \overline{n-1h}\right) + \frac{1}{2}f(a+nh)\right]$$

$$-\frac{B_1 h}{2!}[f'(a+nh) - f'(a)] + \frac{B_2 h^3}{4!}[f'''(a+nh) - f'''(a)] - \cdots \qquad \ldots\ldots...(10.7)$$

10.8 Examples and Exercises

Example 1 Evaluate the integral $\int_0^1 \frac{1}{1+x^2}dx$ by using Simpson's 1/3 rule. Hence obtain the approximate value of .

Solution: We have

$$h = \frac{1-0}{6} = \frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	1
y =f(x)	1	.97297	.9	.8	.69230	.59016	.5

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_2 + y_5 + \cdots. + y_{n-1}) + 2(y_2 + y_4 + \cdots. + y_{n-2})]$$

$$= \frac{1}{6\times 3}[(1+5) + 4(.97297 + .8 + .59016) + 2(.9 + .69230)]$$

$$= \frac{14.13712}{18} = .78539.$$

Exact value of

$$\int_0^1 \frac{1}{1+x^2}dx = [tan^{-1}x]_0^1 = \frac{\pi}{4} = 0.785142$$

$$Hence, \pi = 4 \times 0.7857142 = 3.14156$$

Example 2 Evaluate the integral $\int_0^1 \frac{1}{1+x^2} dx$ by using Simpson's 3/8 rule and obtain the approximate value of π .

Solution: We have

$$h = \frac{1-0}{6} = \frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	1
y = f(x)	1	.97297	.9	.8	.69230	.59016	.5

$$\int_0^1 \frac{1}{1+x^2} dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \cdots . + y_{n-2} + y_{n-1})$$

$$+ 2(y_3 + y_6 + \cdots . + y_{n-3})]$$

$$= \frac{3}{8} . \frac{1}{6}[(1+5) + 3(.97297 + .9 + .69230 + .59016) + 2(.8)]$$

$$= \frac{12.56575}{16} = .785411$$

Exact value of

$$2\int_0^1 \frac{1}{1+x^2} dx = [tan^{-1}x]_0^1 = \frac{\pi}{4}$$

$$Hence, \pi = 4 \times 0.785411 = 3.141644.$$

Example 3 Compute the value $\int_0^{1.5} \frac{x^3}{e^x - 1} dx$ by taking seven ordinates and using Simpson's 1/3 rule.

Solution: For seven ordinates we have to divide the range [0, 1.5] in to six equal sub interval each length,

$$h = \frac{1.5-0}{6} = 0.25. \ Thus$$

x	0	0.25	0.50	0.75	1.00	1.25	1.50

y =$\frac{x^3}{e^x-1}$	0	0.0550	0.1926	0.3736	0.5819	0.7842	0.9693

According to Simpson's 1/3 rule.

$$\int_0^{1.5} y\,dx = \frac{1}{3}h[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + 2y_4) + (y_4 + 4y_5 + y_6)]$$

$$\int_0^{1.5} y\,dx = \frac{1}{3}h[(y_0 + y_6) + 2(y_2 + y_4) + 4(y_1 + y_5)]$$

$$= \frac{0.25}{3}[(0 + 0.9693) + 2(0.1926 + 05819) + 4(0.0550 + 0.3776 + 0.7842)]$$

$$= \frac{0.25}{3} = [0.9693 + 2 \times 0.7745 + 4 \times 1.2168]$$

$$= \frac{0.25}{3}[0.9693 + 1.5490 + 4.8672]$$

$$= \frac{0.25}{3}(7.3855) = \frac{1}{12}(7.3855) = 0.6155$$

Example 4 Evaluate by Simpson's 3/8 rule dividing the range of integration into the six equal parts. Also find the error of approximation.

Solution: Dividing the range [4, 5.2] into six equal parts, we have

$$h = \frac{5.2 - 4}{6} = 0.2.\ and$$

x	4	4.2	4.4	4.6	4.8	50.	5.2
f(x) = x^2	16	17.64	19.36	21.16	23.4	25.0	27.4
	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	$f(x_5)$	$f(x_6)$

Applying Simpson's 3/8 rule, we have

$$\int_{x_0}^{x_6} f(x)\,dx = \frac{3}{8}h[f(x_0) + 3\{f(x_1) + f(x_2)\} + f(x_3) + f(x_3) + 3\{f(x_4) + f(x_5) + f(x_6)\}]$$

$$= \frac{3}{8} h[f(x_0) + f(x_6) + 3\{f(x_1) + f(x_2) + f(x_4) + f(x_6)\} + 2(x_3)]$$

$$\Rightarrow \int_4^{5.2} x^2 dx = \frac{3 \times 0.2}{8} [(16 + 27.04) + 3(17.64 + 19.36 + 23.4 + 25.00)2 \times 21.16]$$

$$= \frac{0.3}{4} [43.04 + 3 \times 85.04 + 42.32]$$

$$= \frac{0.3}{4} [43.04 + 255.12 + 42.32]$$

$$= \frac{0.3}{4} (340.48) = \frac{102.144}{4} = 25.536$$

$$Actul\ value, \quad \int_4^{5.2} x^2 dx = \left[\frac{x^3}{3}\right]_4^{5.2}$$

$$= \frac{1}{3} [(5.2)^3 - 4^3] = \frac{1}{3} [140.608 - 64]$$

$$= \frac{1}{3} (76.608) = 25.536$$

Example 5 Evaluate $\int_0^{\pi/2} \sqrt{cos\theta} d\theta$ by dividing the interval into 6 points.

Solution: We use Simpson's 1/3 rule to evaluate the integral.

$$h = \frac{\frac{\pi}{2} - 0}{6} = \frac{\pi}{12}$$

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$\sqrt{cos\theta} d\theta$	1	.98281	.93060	.84089	.70710	.50874	0

$$\int_0^{\pi/2} \sqrt{cos\theta} d\theta = \frac{1}{3} \cdot \frac{\pi}{12} [(1 + 0) + 4(.98281 + .84089 + .50874) + 2(.93060 + .70710)]$$

$$= \frac{\pi}{36} \times 13.60516 = 1.1873.$$

Example 6 Calculate the value of the define integral

$$\int_{1}^{2} \frac{dx}{x}$$

Correct to fine places of decimals, using the trapezoidal, Simpson's one-third, Simpon's three-eights and Weddle's rules, and also obtain the errors of approximation.

The divided the interval (1, 2) into six equal parts each of width h = 1/5. The value of the function y = 1/x are next tabulated for each of the seven boundaries:

x	1	$\frac{7}{6}$	$\frac{8}{6}$	$\frac{9}{6}$	$\frac{10}{6}$	$\frac{11}{6}$	2
$\frac{1}{x}$	1.000000	0.857143	0.750000	0.666667	0.600000	0.545455	0.500000

(a) By trapezoidal rule, the integral is evaluated as

$$I_T = \frac{1}{12}[1.500000 + 2 \times 3.419265]$$

$$= 0.69488, correct\ to\ five\ decimal\ places$$

(b) $\quad Simpson's\ \ one - third\ rule\ gives$

$$I_{1/3} = \frac{1}{18}[1.500000 + 4 \times 2.069265 + 2 \times 1.350000]$$

$$= 0.69317, correct\ to\ five\ decimal\ places$$

(c) $\quad Simpson's\ \ one - eight\ rule\ gives$

$$I_{3/8} = \frac{1}{16}[1.500000 + 3 \times 2.752598 + 2 \times 666667]$$

$$= 0.69320, correct\ to\ five\ decimal\ places$$

(d) $\quad By\ Weddle's\ rule\ we\ have\ for\ the\ integral\ the\ value$

$$I_w = \frac{1}{20}[2.85 + 4 \times 1.402598 + 6 \times 666667]$$

$$= 0.69315, correct\ to\ five\ decimal\ places$$

$The\ value\ of\ the\ integral\ is$

$$I = \int_1^2 \frac{dx}{x} = 0.69315.$$

Hence the absolute errors are

$$E_T = |I - I_T| = 0.00173,$$

$$E_{\frac{1}{3}} = \left|I - I_{\frac{1}{3}}\right| = 0.00002,$$

$$E_{\frac{3}{8}} = \left|I - I_{\frac{3}{8}}\right| = 0.00005,$$

$$and\ E_w = |I - I_w| = 0.00000,$$

Example 7 Find the sum $1^3+2^3+\ldots\ldots n^3$.

We use the Euler- Maclaurin formula in the form (10.6). In the present case we take f(x) = x^3, a = 1 and h = 1 and n = n. Thus

$$1^3 + 2^3 + \cdots . n^3 = \int_1^{1+n} x^3 dx - \frac{1}{2}[(n+1)^3 - 1] + \frac{3}{12}[(n+1)^2 - 1]$$

$$= \frac{[(n+1)^4 - 1]}{4} - \frac{[(n+1)^3 - 1]}{2} + \frac{[(n+1)^2 - 1]}{4} = \left[\frac{n(n+1)}{2}\right]^2$$

Example 8 Evaluating $\int_1^2 \frac{dx}{1+x^2}$ six points of decimal by using Weddle's rule

Solution: Let us take

x	0	1	2	3	4	5	6
$f(x) = \frac{1}{1+x^2}$	1/1+0	1/1+1	1/1+4	1/1+9	1/1+16	1/1+25	1/1+36

That is f(x): 1.000000 0.500000 0.200000 0.200000 0.058824 0.038462 0.027027

By Weddle's rule, we have

$$\int_0^6 \frac{1}{1+x^2}dx = \frac{3h}{10}[\{f(0)+f(6)\}+5\{f(1)+f(5)\}+\{f(2)+f(4)+6f(3)]$$

$$= \frac{3\times 1}{10}[\{1.000000+0.27027\}+5(0.500000+0.038462)+(0.200000+0.058824)+(6 \times 0.100000)]$$

$$= \frac{3}{10}[1.027027+2.692310+0.258824+0.600000]$$

$$= \frac{3}{10}[4.578161] = \frac{1}{10}(13734483)$$

$$= 1.3734483$$

Example 9 Find by Weddle's rule the value of the expression $\int_{4.0}^{5.2} log_e\, xdx$

Solution: Range = 5.2-4=1.2

Divide their range into six equal parts each of length

$h = \frac{5.2-4}{6} = 0.2$ Let

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$log_e\, x$:	1.3863	1.4351	1.4816	1.5260	1.5686	1.6094	1.6486

By Weddle's rule, we have

$$\int_{4.0}^{5.2} log_e\, xdx = \frac{3h}{10}[\{f(0)+f(6)\}+5\{f(1)+f(5)\}+\{f(2)+f(4)+6f(3)]$$

$$= \frac{3\times 0.2}{10}[(1.3863+1.6486)+5(1.4351+1.6094)+(1.4846+1.5686)+6\times 1.5260]$$

$$= \frac{0.6}{10}[3.0349+5\times 3.0445+3.0502+9.1560]$$

$$= 0.06[4.0349+15.2225+3.0502+9.1560]$$

$= 0.06[30.4636] = 1.827816$

Example 10: Compute the value of $\int_0^{15} \frac{x^3}{e^x - 1} dx$ by Weddle's rule.

Solution: Divide the range [0,1.5] into 6 equal parts each of length

$h = \frac{1.5}{6} = 0.25 \quad Let$

x	0	0.25	0.50	0.75	1.0	1.25	1.50
$f(x)\frac{x^3}{e^x - 1}$	0	0.0550	0.1923	0.3776	0.5819	0.7842	0.9693

By Weddle's Rule

$$\int_0^{1.5} \frac{x^3}{e^x - 1} dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3h}{10}[y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_5) + 6y_3]$$

$$= \frac{3 \times 0.25}{10}[0 + 0.1926 + 0.5819 + 0.9693 + 5(0.0550 + 0.7842) + 6 \times 0.3774]$$

$$= 0.075[1.7438 + 5 \times 0.8392 + 2.2656]$$

$$= 0.075[1.7438 + 4.1960 + 2.2656]$$

$$= 0.075[8.2054]$$

$$= 0.6154050 = 0.6154.$$

Exercises

P-1 Use Trapezoidal rule to find an approximate value of $\int_{-3}^{3} x^4 dx$ by taking 7 equidistant ordinates.

P-2 Evaluate $\int_0^1 \frac{1}{1+x1} dx$ by Simpson's 1/3 rule by dividing the range into 8 equal parts and obtain the value.

P-3 Evaluate the following using Simpson's 1/3 rule with h= ¼

$$\int_0^1 \frac{x^2}{1+x^3} dx$$

hence obtain the value of $\log_e 2$.

P-4 By Simpson's one third rule show that 1.62 is an approximate value of $\int_1^5 \frac{1}{x} dx$ when interval of differencing, h=1.

P-5 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using (i) Simpon's 1/3 rule and (ii) Simpons's 3/8 rule and find the error in all cases.

P-6 Evaluate $\int_0^1 \frac{1}{x} dx$ by Simpon's one third rule and prove that $\log_e 7$= 1.959 approximately.

P-7 Calculate an approximate value of $\int_1^{\pi/2} \sin x \, dx$ by sing 11 ordinates using

(i) The trapezoidal rule
(ii) Simpson's third rule.

P-8 Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Weddle's rule.

P-9 Compute the value of $\int_1^{\pi/2} [1 - 0.162 Sin^2\theta] d\theta$ by taking

$$\theta = 0^0, 15^0, 30^0, 45^0, 60^0, 75^0, 90^0.$$

P-10: Compute the value of the definite integral $\int_{0.2}^{1.4} (\sin x + log_e x + e^x) d\theta$ by Weddle's rule.

P-11 Fill in the blanks:

1. Numerical integration is the process of computing the value of …………………integral from a set of numerical values of the integrand.

2. When the process of numerical integration is applied to the integration of a function of single value, the process is called numerical…………………

3. To apply Simpson's 1/3 ruel, we need at least ……………functional values of f(x) corresponding to …………………values of x.

P-12 Choose the correct alternative:

1. Suppose only two functional values of f(x) are given for two values of the argument x_0 and x_1+h. then trapezoidal rule is-

$$(a) \int_{x_0}^{x_1} f(x)dx = \frac{1}{2}h[f(x_0) + f(x_1)]$$

$$(b) \int_{x_0}^{x_1} f(x)dx = h[f(x_0) + f(x_1)]$$

$$(c) \int_{x_0}^{x_1} f(x)dx = 2h[f(x_0) - f(x_1)]$$

$$(d) \int_{x_0}^{x_1} f(x)dx = \frac{1}{2}h[f(x_0) - f(x_1)]$$

P-13 For the equidistant values of the argument x_0, x_1, x_2, x_3, x_4 we cannot apply-

(a) Trapezoidal Rule
(b) Simpson's 1/3 rule
(c) Simpson's 3/8 rule
(d) Bothe a and b

P-14 $\int_{x_0}^{x_4} ydx \ is\ equal\ to$

$$(a)\ \frac{h}{3}[y_0 + y_4 + 4(y_1 + y_3) + 2y_2]$$

$$(b)\ \frac{h}{3}[y_0 + h(y_1 + y_2 + y_3 + y_4)]$$

$(c)\ \frac{h}{3}[y_0 + 2(y_1 + y_2) + 2y_2(y_3 + y_4)]$

$(d)\ \frac{h}{3}[y_0 + y_4 + 2(y_1 + y_3) + 4y_2)]$

Answers:

P-1 115 **P-2** 0.315 **P-3** 0.2310843

P-5 (i) 1.3661734 error: 0.39474279 (ii) 1.3670808, error: 0.038566849

P-6 1.9587 **P-7** (a) 0.9979 (b) 1.0001 **P-8** 1.3734474

P-9 1.444142 **P-10** 4.05098

P-12 (a) **P-13** (c) **P-14** (a)

10.9 Summary

Numerical integration is a powerful tool to evaluate any integral numerically. In this process we approximate first the integrand by means of any interpolation formula and then integrate it over a given range. We have seen that trapezoidal, Simpson one third, Simpson three eight and Weddle rule are special cases of general quadrature formula. Euler Maclauring formula can also be used as summation formula as well as integral formula.

10.10 Further Readings

1. Finite Difference & Numerical Analysis, S. Chand & Company, New Delhi: H.C. Saxena
2. Numerical Mathematical Analysis, John Hopkins Press, Baltimore New York; James B. Scarborough
3. Introductory Method of Numerical Analysis, Prentice Hall of India Pvt. Ltd.: S.S. Sastry
4. Introduction to Numerical Analysis, Tata McGraw Hill Publishing Company, New Delhi: S.T. Hildebrand
5. Numerical Method for Scientific & Engineering Computation, New Age International Publishers, New Delhi: M.K. Jain, S.R.K. Iyengar & R.K. Jain

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SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 4 Solutions of Differential Equations

Unit – 11: Numerical Solution of Ordinary Equations - I

Unit – 12: Numerical Solution of Ordinary Equations - II

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Block & Units Introduction

The ***Block - 4 – Solution of Differential Equations,*** is the fourth block. There are number of differential equations which do not possess closed form of finite solutions. Even if they possess closed form solutions, we do not know the method of getting it. In such situations, depending upon the need of the hour, we go in for numerical solutions of differential equations. In researches, especially after the advent of computer, the numerical solutions of the differential equations have become easy for manipulations. These methods yield solutions either as power series in x form which the values of y can be found by direct substitution, or as a set of values of x and y. The methods discussed in this block Picards and Taylor series belong to the former class of solutions whereas those of Euler, Rung-Kutta belong to the latter class. However Rung-Kutta forth order method is iterative and gives solution to desired accuracy.

Unit – 11 – Numerical Solution of Ordinary Differential Equations - I; this unit deals with the first order Picard's Iteration Method, Euler's Method and Runga-Kutta Methods. The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now available which reduce numerical work considerably.

Unit –12– Numerical Solution of Ordinary Differential Equations - II; this unit deals with the second order Picard's Iteration Method, Euler's Method and Runga-Kutta Methods as well as simultaneous methods.

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The end of block/unit the summary, self-assessment questions and further readings are given.

Unit-11: Numerical Solution of Differential Equations - I

Structure

11.1 Introduction

This unit deals with the first order Picard's Iteration Method, Euler's Method and Runga-Kutta Methods. The methods of solution so far presented are applicable to a limited class of differential equations. Frequently differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods. These methods are of even greater importance when we realize that computing machines are now available which reduce numerical work considerably.

11.2 Objective

A number of numerical methods are available for the solution of first order differential equations of the form:

$\frac{dy}{dx} = f(x, y)$, given $y(x_0) = y_0$

These methods yield solutions either as power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y. The methods of Picards and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class.

11.3 Picard's Method of Successive Approximations

Consider first order differential equation

$\frac{dy}{dx} = f(x,y)$ ………. (1)

with the initial condition

$$y = y_0 \text{at} x = x_0$$

Integrating (1) with respect to x between x_0 and x, we have

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

or $$y = y_0 + \int_{x_0}^{x} f(x,y)\,dx \qquad \text{……… (2)}$$

Now, we solve (2) by the method of successive approximation to find out the solution of (1). The first approximate solution (approximation) y_1 of y is given by

$$y_1 = y_0 + \int_{x_0}^{x} f(x,\, y_0)\,dx$$

Similarly, the second approximation y_2 is given by

$$y_2 = y_0 + \int_{x_0}^{x} f(x,\, y_1)\,dx$$

Similarly, the nth approximation y_n is given by

$y_n = y_0 + \int_{x_0}^{x} f(x,\, y_{n-1})\,dx$ ……….(3)

With $y(x_0) = y_0$.

Hence, this method gives a sequence of approximation $y_1,\ y_2, \dots \dots \dots\ y_n$ and it can be proved $f(x,y)$ is bounded in some regions containing the point $(x_0,\ y_0)$ and if $f(x,y)$ satisfies the Lipchitz condition, namely

$$|f(x,y) - f(x,\bar{y})| \leq k\,|y - \bar{y}|$$

Where k is a constant and $y_1, y_2, \ldots \ldots \ldots$ converges to the solution (2).

Example 1: Use Picard's method to obtain y for $x = 0.2$. Given

$$\frac{dy}{dx} = x - y$$

with the initial condition $y = 1$ at $x = 0$

Solution: Here, $f(x,y) = x - y$, $x_0 = 0$, $y_0 = 1$

We have first approximation

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 1 + \int_0^x (x-1)\,dx$$

$$= 1 - x + \frac{x^2}{2}$$

Second approximation,

$$y_2 = y_0 + \int_0^x f(x, y_1)\,dx$$

$$= 1 + \int_0^x (x - y_1)\,dx$$

$$= 1 + \int_0^x (x - 1 + x - \frac{x^2}{2})\,dx$$

$$= 1 - x + x^2 - \frac{x^3}{6}$$

Third approximation,

$$y_3 = y_0 + \int_0^x f(x, y_2)\,dx$$

$$= 1 + \int_0^x (x - y_2)\,dx$$

$$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{6})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{6} + \frac{x^4}{24}$$

Fourth approximation,

$$y_4 = y_0 + \int_0^x f(x,\ y_3)\, dx$$

$$= 1 + \int_0^x (x - y_3)\, dx$$

$$= 1 + \int_0^x (x - 1 - x - x^2 + \frac{x^3}{6} - \frac{x^4}{24})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120}$$

Fifth approximation,

$$y_5 = y_0 + \int_0^x f(x,\ y_4)\, dx$$

$$= 1 + \int_0^x (x - y_4)\, dx$$

$$= 1 + \int_0^x (x - 1 + x - x^2 + \frac{x^3}{3} - \frac{x^4}{12} + \frac{x^5}{120})\, dx$$

$$= 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}$$

When $x = 0.2$, we get

$$y_1 = .82,\ y_2 = .83867,\ y_3 = .83740,$$

$$y_4 = .83746,\ y_5 = .83746$$

Thus $\boldsymbol{y = 0.837}$ **when** $\boldsymbol{x = 0.2}$ **Ans**

Example 2: Find the solution of

$\frac{dy}{dx} = 1 + xy$, $y(0) = 1$

Which passes through (0,1) in the interval (0,0.5) such that the value of y is correct to three decimal places (use the whole interval as one interval only) Take $h = 0.1$.

Solution: The given initial value problem is

$$\frac{dy}{dx} = f(x, y) = 1 + xy \; ; \quad y(0) = 1$$

i.e. $y = y_0 = 1$ at $x = x_0 = 0$,

Here,

$$y_1 = 1 + x + \frac{x^2}{2}$$

$$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{15} + \frac{x^6}{48}$$

$$y_4 = y_3 + \frac{x^7}{105} + \frac{x^8}{384}$$

When $x = 0$, $y = 1.000$ at

$\therefore x = 0.1$, $y_1 = 1.105, y_2 = 1.223 = y_3$ (correct up to 3 decimals)

$\therefore y = 1.223$

$x = 0.3$, $y = 1.355, y_2 = 1.355 = y_3$ (correct up to 3 decimals)

$x = 0.4$, $y = 1.505$

$$x = 0.5, \; y = 1.677, y_4 = y_3 = 1.677$$

Thus,

x	0	0.1	0.2	0.3	0.4	0.5
y	1.000	1.105	1.223	1.355	1.505	1.677

We have numerically solved the given differential equation for

$x = 0.1, 0.2, 0.3, 0.4,$ and 0.5.

Example 3: Use Picard's method to obtain y for $x = 0.1$. Given

$$\frac{dy}{dx} = 3x + y^2$$

with the initial condition $y = 1$ at $x = 0$

Solution

Here, $f(x, y) = 3x + y^2$, $x_0 = 0$, $y_0 = 1$

We have First approximation

$$y_1 = y_0 + \int_0^x f(x, y_0)\,dx$$

$$= 1 + \int_0^x (3x + 1)\,dx$$

$$= 1 + x + \frac{3}{2}x^2$$

Second approximation

$$y_2 = 1 + x + \frac{5}{2}x^2 + \frac{4}{2}x^3 + \frac{3}{4}x^4 + \frac{9}{20}x^5$$

Third approximation

$$y_3 = 1 + x + \frac{5}{2}x^2 + 2x^3 + \frac{23}{12}x^4 + \frac{25}{12}x^5 + \frac{68}{45}x^6 + \frac{1157}{1260}x^7$$
$$+ \frac{17}{32}x^8 + \frac{47}{240}x^9 + \frac{27}{400}x^{10} + \frac{81}{4400}x^{11}$$

When $x = 0.1$, we have

$$y_1 = 1.115,\ y_2 = 1.1264,\ y_3 = 1.12721$$

Thus $y = 1.127$ when $x = 0.1$ Ans

Check Your Progress

1. Obtain y when $x = 0.1,\ x = 0.2$

   Given that $\frac{dy}{dx} = x + y\ ,\ \ y(0) = 1$, check the result with exact value.

11.4 Euler's Method

The oldest and simplest method was derived by Euler. In this method, we determine the change

Δy in y corresponding to small increment in the argument . Consider the differential equation.

$$\frac{dy}{dx} = f(x,y) \qquad \ldots\ldots\ldots.\ (1)$$

with the initial condition

$$y(x_0) = y_0$$

Integrating (1) with respect to x between x_0 and x_1, we have

$$\int_{y_0}^{y_1} dy = \int_{x_0}^{x_1} f(x,y)\,dx$$

$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx$ ------(2)

Now, replacing $f(x,y)$ by the approximation $(x_0,\ y_0)$, we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x_0,\ y_0)\,dx$$

$$= y_0 + f(x_0,\ y_0)(x_1 - x_0)$$

$$y_1 = y_0 + hf(x_0,\ y_0)(\because x_1 - x_0 = \Delta x = h)$$

This is the formula for first approximation y_1 is given by

$$y_2 = y_1 + hf(x_1,\ y_1)$$

In general,

$$y_{n+1} = y_n + hf(x_n,\ y_n)$$

Euler's Modified Method

Instead of approximating $f(x, y)$ by $f(x_0,\ y_0)$ in equation (2). Let the integral is appointed by Trapezoidal rule to obtain.

$$y_1 = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1,\ y_1)]$$

We obtain the iteration formula

$$y_1{}^{n+1} = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1, y_1{}^{n})]n = 0,1,2 \dots \dots$$

Where, $y_1{}^{n}$ is the nth approximation to y_1 .

The above iteration formula can be started by $y_1{}^{(1)}$ from Euler's method.

$$y_1{}^{(0)} = y_0 + h(x_0,\ y_0)$$

Example 1: Use Euler's method, compute y(0.5) for differential equation

$$\frac{dy}{dx} = y^2 - x^2$$

with the initial condition $y = 1$ when $x = 0$

Solution: Let $h = \frac{0.5}{5} = 0.1$

$x_0 = 0,\quad y_0 = 1\ ,\ f(x, y) = y^2 - x^2$

Using Euler's method we have

$$y_{n+1} = y_n + h(x_n,\ y_n)$$

But considering $n = 0,1,2, \dots \dots \dots$ in succession, we get

$$y_1 = y_0 + hf(x_0,\ y_0)$$

$$= 1 + 0.1(1^2 - 0) = 1.10000$$

$$y_2 = y_1 + hf(x_1,\ y_1)$$

$$= 1.10000 + 0.1[(1.10000)^2 - (0.1)^2] = 1.22000$$

$$y_3 = y_2 + hf(x_2,\ y_2)$$

$$= 1.22000 + 0.1[(1.22)^2 - (0.2)^2] = 1.36484$$

$$y_4 = y_3 + hf(x_3,\ y_3)$$

$$= 1.36484 + 0.1[(1.36484)^2 - (0.3)^2] = 1.54212$$

$$y_5 = y_4 + hf(x_4,\ y_4)$$

$$= 1.54212 + 0.1[(1.54212)^2 - (0.4)^2] = 1.76393$$

Hence, the value of $\boldsymbol{y}$ **at** $\boldsymbol{x = 0.5}$ **is 1.76393**. Ans

Example 2: Find the solution of differential equation

$$\frac{dy}{dx} = xy \qquad \text{with } y(1) = 5$$

in the interval [1, 1.5] using $h = 0.1$.

Solution: As per given we have

$$x_1 = 1, \quad y_0 = 5\ , \ f(x, y) = xy$$

Using Euler's method we have

$$y_{n+1} = y_n + h(x_n,\ y_n)$$

But considering $n = 0,1,2, \dots \dots \dots$ in succession, we get

For $n = 0 y_1 = y_0 + 0.1f(x_0,\ y_0)$

$$= 5 + 0.1(1 \times 5) = 5.5$$

For $n = 1 y_2 = y_1 + 0.1f(x_1,\ y_1)$

$$= 5.5 + 0.1(1.1 \times 5.5) = 6.105$$

For $n = 2 y_3 = y_2 + 0.1f(x_2,\ y_2)$

$$= 6.105 + 0.1(1.2 \times 6.105) = 6.838$$

For $n = 3 y_4 = y_3 + 0.1f(x_3,\ y_3)$

$$= 6.838 + 0.1(1.3 \times 6.838) = 7.727$$

For $n = 4 y_5 = y_4 + 0.1f(x_4,\ y_4)$

$$= 7.727 + 0.1(1.4 \times 7.727) = 8.809$$

Hence, the value of $\boldsymbol{y(1.5)}$**at is8.809**. Ans

Example 3: Given

$$\frac{dy}{dx} = x + y \text{ with initial condition } y(0) = 1.$$

Find $y(0.05)$ and $y(0.1)$ correct to 6 decimal places.

Solution:

Using Euler's method, we have

$$y_1{}^{(0)} = y_1 = y_0 + h(x_0,\ y_0)$$
$$= 1 + 0.05(0 + 1) = 1.05$$

We improve y_1 by using Euler's modified method

$$y_1{}^{(1)} = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1, y_1{}^0)]$$
$$= 1 + \frac{0.05}{2}[(0 + 1) + (0.05 + 1.05)]$$
$$= 1.0525$$
$$y_1{}^{(2)} = 1 + \frac{0.05}{2}[(0 + 1) + (0.05 + 1.0525)]$$
$$= 1.0525625$$
$$y_1{}^{(3)} = 1 + \frac{0.05}{2}[(0 + 1) + (0.05 + 1.0525625)]$$
$$= 1.052564$$
$$y_1{}^{(4)} = 1 + \frac{0.05}{2}[(0 + 1) + (0.05 + 1.0525625)]$$
$$= 1.0525641$$

Since, $y_1{}^{(3)} = y_1{}^{(4)} = 1.0525641$ correct to 6 decimal places. Hence we take $y_1 = 1.052564$

i.e., we have $y(0.05) = 1.052564$

Again, using Euler's method, we obtain

$$y_2{}^{(0)} = y_2 = y_1 + h[f(x_1,\ y_1)]$$
$$= 1.052564 + 0.05(1.052564 + 0.05)$$
$$= 1.1076922$$

We improve y_2 by using Euler's modified method

$$y_2{}^{(1)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1076922 + 0.1)]$$
$$= 1.1120511$$

$$y_2^{(2)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1120511 + 0.1)]$$
$$= 1.1104294$$

$$y_2^{(3)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1104294 + 0.1)]$$
$$= 1.1103888$$

$$y_2^{(4)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1103888 + 0.1)]$$
$$= 1.1103878$$

$$y_2^{(5)} = 1.052564 + \frac{0.05}{2}[(1.052564 + 0.05) + (1.1103878 + 0.1)]$$
$$= 1.1103878$$

Since, $y_2^{(4)} = y_2^{(5)} = 1.1103878$ correct to 7 decimal places. Hence we take $y_2 = 1.1103878$

Therefore, we have $\boldsymbol{y(0.1) = 1.1103878}$, correct to **6 decimal places. Ans**

Example 4: Find$y(2.2)$ using Euler's method for

$$\frac{dy}{dx} = -xy^2, \text{ where } \quad y(2) = 1. \quad (\text{ Take } h = 0.1 \text{ })$$

Solution:

By Euler's method, we have

$$y_1^{(0)} = y_1 = y_0 + h(x_0,\ y_0)$$
$$= 1 + 0.1(-2)(-1)^2 = 0.08$$

The value of y_1 by using Euler's modified method

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0,\ y_0) + f(x_1, y_1^{0})]$$
$$= 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8)^2]$$
$$= 0.8328$$

Similarly

$$y_1^{(2)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8328)^2]$$
$$= 0.8272$$
$$y_1^{(3)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8272)^2]$$
$$= 0.8281$$
$$y_1^{(4)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8281)^2]$$
$$= 0.8280$$

$$y_1^{(5)} = 1 + \frac{0.1}{2}[(-2)(1)^2 + (-2.1)(0.8280)^2]$$
$$= 0.8280$$

Since $y_1^{(4)} = y_1^{(5)} = 0.8280$. Hence, we take $y_1 = 0.828$ at $x_1 = 2.1$

Now, if y_2 is the value of y at $x = 2.2$. Then, we apply Euler's method to compute $y(2.2)$, i.e., we obtain

$$y_2^{(0)} = y_2 = y_1 + h(x_1,\, y_1)$$
$$= 0.828 + 0.1(-2.1)(0.828)^2 = 0.68402$$

Now, using Euler's modified formula, we obtain

$$y_2^{(1)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.68402)^2]$$
$$= 0.70454$$
$$y_2^{(2)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70454)^2]$$
$$= 0.70141$$
$$y_2^{(3)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70141)^2]$$
$$= 0.70189$$
$$y_2^{(4)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70189)^2]$$
$$= 0.70182$$
$$y_2^{(5)} = 0.828 + \frac{0.1}{2}[(-2.1)(0.828)^2 + (-2.2)(0.70182)^2]$$
$$= 0.70183$$

Since ${y_2}^{(4)} = {y_2}^{(5)} = 0.7018$, correct to 4 decimal places.

Hence, we take $y(2.2) = 0.7018$ **Ans.**

Check Your Progress

1. Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$, with $y_0 = 1$ find y for $x = 0.1$ in four steps by Euler's method. [Ans $y(0.1) = 1.0932$]

11.5 Runge-Kutta Method

The method is very simple. It is named after two German mathematicians Carl Runge (1856-1927) and Wilhelm Kutta (1867-1944). These methods are well-known as Runge-Kutta Method. They aredistinguished by their orders in the sense that they agree with Taylor's series solution upto term of h^r where $r$is a positive integer is the order of the method.

It was developed to avoid the computation of higher order derivations which the Taylors'method may involve. In the place of these derivatives extra values of the given function $f(x, y)$are used.

(i) First order Runge-Kutta method

Consider the differential equation

$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$ -------- (1)

By Euler's method, we know that

$y_1 = y_0 + hf(x_0, y_0) = y_0 + hy'_0$ --------- (2)

Expanding by Taylor's series, we get

$y_1 = y(x_0 + h) = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 +$ ---------(3)

It follows that Euler's method agrees with Taylor's series solution upto the terms in h. HenceEuler's method is the first order Runge-Kutta method.

(ii) Second order Runge-Kutta method

Consider the differential equation

$\frac{dy}{dx} = f(x, y) ; \quad y(x_0) = y_0$

Let h be the interval between equidistant values of x. Then the second order Runge-Kutta method, the first increment in y is computed from the formulae

$$k_1 = hf(x_0,\ y_0)$$

$$k_2 = hf(x_0 + h,\ y_0 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

Then, $x_1 = x_0 + h$

$$y_1 = y_0 + \Delta y = y_0 + \frac{1}{2}(k_1 + k_2)$$

Similarly, the increment in y for the second interval is computed by the formulae,

$$k_1 = hf(x_1,\ y_1)$$

$$k_2 = hf(x_1 + h,\ y_1 + k_1)$$

$$\Delta y = \frac{1}{2}(k_1 + k_2)$$

and similarly for other intervals.

.

(iii) Third order Runge-Kutta Method

The method agrees with Taylor's series upto the term h^3. The formula is as follows:

$$y_1 = y_0 + \tfrac{1}{6}(k_1 + 4k_2 + k_3);$$

$$x_1 = x_0 + h$$

Where, $k_1 = hf(x_0,\ y_0)$

$$k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h,\ y_0 + 2k_2 - k_1)$$

Similarly for other intervals.

(iv) Fourth order Runge-Kutta method

This method coincides with the Taylor's series solution upto terms of h^4 .

Consider the differential equation

$\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_0) = y_0$.

Let$h$ be the interval between equidistant vales of x. Then the first increment in y is computed from the formulae.

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3);$$

$$x_1 = x_0 + h$$

Where, $k_1 = hf(x_0,\ y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h,\ y_0 + 2k_2 - k_1)$$

Similarly for other intervals.

$$k_1 = hf(x_0,\ y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h\ , y_0 + k_3)$$

Then

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

and $x_1 = x_0 + h$

Similarly, the increment in y for the second interval is computed by

$$k_1 = hf(x_1,\ y_1)$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_1 + h\ , y_1 + k_3)$$

Then, $y_2 = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

and $x_2 = x_1 + h$

and similarly for the next intervals.

Example 1: Apply Runge-Kutta Method to solve

$$\frac{dy}{dx} = xy^{1/3}\ ,\ y(1) = 1 \text{ to obtain } y(1.1).$$

Solution : Here, $x_0 = 1$, $y_0 = 1$ and $h = 0.1$. Then, we can find

$$k_1 = hf(x_0,\ y_0)$$

$$= 0.1(1)(1)^{\frac{1}{3}} = 0.1$$

$$k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.1}{2}\right)^{\frac{1}{3}} = 0.10672$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1\left(1 + \frac{0.1}{2}\right)\left(1 + \frac{0.10672}{2}\right)^{\frac{1}{3}} = 0.10684$$

$$k_4 = hf(x_0 + h\ , y_0 + k_3)$$

Then $$= 0.1(1 + 0.1)(1 + 0.10684)^{\frac{1}{3}}$$

$$= 0.11378 = 0.11378$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 2 \times 0.10672 + 2 \times 0.10684 + 0.11378)$$

$= 1 + 0.10682$ = 1.10682 **Ans**

Example 2: Solve the equation $y' = (x + y)$ with $y_0 = 1$ by Runge-Kutta rule from $x = 0$ to $x = 0.4$ with $h = 0.1$.

Solution : Here, $f(x, y) = x + y$, $h = 0.1$, given $y_0 = 1$ when $x_0 = 0$

We have,

$$k_1 = hf(x_0,\ y_0)$$

$$= 0.1(0 + 1) = 0.1$$

$$k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right)$$

$$= 0.1(0.05 + 1.05) = 0.11$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1(0.05 + 1.055) = 0.1105$$

$$k_4 = hf(x_0 + h\,, y_0 + k_3)$$

Then $= 0.1(0.1 + 1.1105)$

$= 0.12105$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105)$$

$$= 1.11034$$

Similarly for finding $y_2 = y(x = 0.2)$, we get

$$k_1 = hf(x_1,\ y_1)$$

$= (0.1)[\,(0.1)\ + 1.11034]$

$= 0.121034$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$= (0.1)[0.15\ + 1.11034\ + 0.660517]$

$= 0.13208$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$= (0.1)[0.15\ + 1.11034\ + 0.06604]$

$= 0.13208$

$$k_4 = hf(x_1 + h\,, y_1 + k_3)$$

$= (0.1)[0.20\ + 1.11034\ + 0.13263]$

$= 0.14263$

$\therefore y_2 = y(x = 0.2) = y_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1.11034 + \frac{1}{6}[0.121034 + 2(0.13208 + 0.13263 + 0.14429)]$$

$$= 1.2428$$

Similarly for finding $y_3 = y(x = 0.3)$, we get

$$k_1 = hf(x_2,\ y_2)$$

$= (0.1)[\,(0.2)\ + 1.2428]$

$= 0.14428$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right)$$

$= (0.1)[0.25 \ + 1.2428 \ + 0.07214]$

$= 0.15649$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right)$$

$= (0.1)[0.25 \ + 1.2428 \ + 0.07824]$

$$= 0.15710$$

$$k_4 = hf(x_2 + h\, , y_2 + k_3)$$

$= (0.1)[0.30 \ + 1.2428 \ + 0.15710]$

$$= 0.16999$$

$\therefore y_3 = y(x = 0.3) = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= 1.3997$$

Similarly for finding $y_4 = y(x = 0.4)$, we

$k_1 = (0.1)[\ (0.3) \ + 1.3997]$

$= 0.16997$

$$k_2 = (0.1)[0.35 \ + 1.3997 \ + 0.08949]$$

$$= 0.18347$$

$$k_3 = (0.1)[0.35 \ + 1.3997 \ + 0.9170]$$

$$= 0.18414$$

$$k_4 = (0.1)[0.4 \ + 1.3997 \ + 0.18414]$$

$$= 0.19838$$

$\therefore y_4 = 1.3997 + \frac{1}{6}[0.16997 + 2(0.18347 + 0.18414 + 0.19838)]$

$$y_4 = 1.5836$$ **Ans**

Example 3: Given $\frac{dy}{dx} = y - x$ with $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ correct to 4 decimal places.

Solution: We have $x_0 = 0$, $y_0 = 2$, $h = 0.1$

Then, we get

$$k_1 = hf(x_0,\ y_0)$$

$$= 0.1(2 - 0) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1[2 + \frac{0.2}{2} - (0 + \frac{0.1}{2})] = 0.205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1[2 + \frac{0.205}{2} - (0 + \frac{0.1}{2})] = 0.20525$$

$$k_4 = hf(x_0 + h\,, y_0 + k_3)$$

Then $$= 0.1[2 + 0.20525 - (0 + 0.1)]$$

$$= 0.210525$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2 + 0.2051708 = 2.2051708$$

$\Rightarrow$ $y(0.1) = 2.2052$ Correct to 4 decimal places

For $y(0.2)$, we have $x_0 = 0.1$, $y_0 = 2.2052$, we get

$$k_1 = hf(x_0,\ y_0)$$

$$= 0.1(2.2052 - 0.1) = 0.21052$$

$$k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right)$$

$$= 0.1[2.2052 + \frac{0.21052}{2} - (0 + \frac{0.1}{2})] = 0.216046$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1[2.2052 + \frac{0.216046}{2} - (0.1 + \frac{0.1}{2})] = 0.2163223$$

$$k_4 = hf(x_0 + h\,, y_0 + k_3)$$

Then $$= 0.1[2.2052 + 0.2163223 - (0.1 + 0.1)]$$

$$= 0.22215223$$

$$y(0.2) = 2.2052 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 2.2052 + 0.2162348$$

$\Rightarrow$ $y(0.2) = 2.4214$ **Ans**

Example 4: Given$\frac{dy}{dx} = -2xy^2$ with $y(0) = 1$, and $h = 0.1$ on the interval [0.1] using Runge-Kutta fourth order method.

Solution: As per given, we have $x_0 = 0$, $y_0 = 1$, $h = 0.2$

$$k_1 = hf(x_0,\ y_0)$$

$$= -2(0.2)(0)(1)^2 = 0$$

$$k_2 = hf\left(x_0 + \tfrac{h}{2}, y_0 + \tfrac{k_1}{2}\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(1)^2 = -0.4$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= -2(0.2)\left(\frac{0.2}{2}\right)(0.98)^2 = -0.38416$$

$$k_4 = hf(x_0 + h\,, y_0 + k_3)$$

$$= -2(0.2)(0.2)(0.961584)^2 = -0.0739715$$

Hence $$y_1 = y_0 + \tfrac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 1 + \frac{1}{6}[0 - 0.0 - 0.076832 \; - 0.0739715]$$

$$= 0.9615328$$

Now,we have $x_1 = 0.2$, $y_1 = 0.9615328$, $h = 0.2$, we get

$$k_1 = hf(x_1,\, y_1)$$

$$= -2(0.2)(0.2)(0.9615328)^2 = -0.0739636$$

$$k_2 = hf\left(x_1 + \tfrac{h}{2}, y_1 + \tfrac{k_1}{2}\right)$$

$$= -2(0.2)(0.3)(0.924551)^2 = 0.1025754$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= -2(0.2)(0.3)(0.9102451)^2 = 0.0994255$$

$$k_4 = hf(x_1 + h\,, y_1 + k_3)$$

$$= -2(0.2)(0.4)(0.8621073)^2 = -0.1189166$$

Hence $y(0.4) = y_1 + \tfrac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$= 0.9615328 + \frac{1}{6}[-0.0739636 - 0.2051508 - 0.1988510 \; - 0.1189166]$$

$$= 0.8620525$$

Similarly, we can obtain

$$y(0.6) = 0.7352784$$

$$y(0.8) = 0.6097519$$

$$y(1.0) = 0.500073$$ **Ans**

Check Your Progress

1. Apply Runge-Kutta method to find the solution of the differential equation $\frac{dy}{dx} = 3x + \frac{1}{2}y$ with $y(0) = 1$ and $x = 0.1$.

11.6 Summary

Many problems in science and engineering can be reduce to the problem of solving differential equations satisfying certain given conditions. These methods are of even greater importance when we realize that computing machines are now available which reduce numerical work considerably.A number of numerical methods are available for the solution of first order differential equations. These methods yield solutions either as power series in x from which the values of y can be found by direct substitution, or as a set of values of x and y . The methods of Picards and Taylor series belong to the former class of solutions whereas those of Euler, Runge-Kutta belong to the latter class. However Runge-Kutta forth order method is iterative and gives solution to accuracy.

11.7 Exercise

1. Apply Picard's method to find the third approximation of the solution

   $\frac{dy}{dx} = x + y^2$ with the condition $y(0) = 1$.

   [Ans $= 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \dots\dots$]

2. Using Picard'smethod , obtain the solution

   $$\frac{dy}{dx} = x(1 + x^3 y); \quad y(0) = 3$$

   Compute the value of $y(0.1)$ and $y(0.2)$.

   [Ans. $3.005, \ 3.020$]

3. Solve the following initial value problem by Picard method

   $\frac{dy}{dx} = xe^y \quad y(0) = 0$, compute $y(0.1)$

   [Ans. 0.0050125]

4. Use Picard method to approximate y when $x = 0.2$, given that

   $\frac{dy}{dx} = x - y$ with $y(0) = 1$.

   [Ans. 0.0837]

5. Given $\frac{dy}{dx} = 1 + y^2$ where $y = 0$, when $x = 0$, find $y(0.2), y(0.4)$and $y(0.6)$ using Runge-Kutta formula of order four.

[**Ans** $y(0.2) = 0.2027, y(0.4) = 0.4228$ and $y(0.6) = 0.6841$]

6. Use classical Runge-Kutta method of fourth order to find the numerical solution at $x = 1.4$ for $\frac{dy}{dx} = y^2 + x^2, y(1) = 0$. Assume step size $h = 0.2$

[**Ans** $y(1.2) = 0.246326$ and $y(1.4) = 0.622751489$]

7. Solve the differential equation $\frac{dy}{dx} = \frac{2x-1}{x^2}y + 1$, where x_0= 1 ,y_0= 2 $h = 0.2$. Obtain $y(1.2)$ and $y(1.4)$ using Runge-Kutta method.

[**Ans** 2.658913 and 3.432851]

Unit – 12: Numerical Solution of Ordinary Differential Equation - II

Structure

12.1 Introduction

In this unit we shall discuss the methods for finding the numerical solution of Ordinary Differential Equations having numerical coefficients with given initial conditions to any desired degree of accuracy. The solution is obtained step by step through a series of equal intervals in the independent variable. In the previous two units, you have seen how a complicated or tabulated function call be replaced by an approximating polynomial so that the fundamental operations of calculus, differentiation and integration can be performed more easily. We shall solve a differential equation, that is, we shall find function which satisfies a culmination of the independent variable, dependent variable and its derivatives. In physics, engineering, chemistry other disciplines it has become necessary to build mathematical model to represent complicated processes. Differential equations are one of the most important mathematical tools used modelling problem in the engineering and physical sciences. As it is always possible to obtain the analytical solution of differential equations recourse must necessarily be to numerical methods for solving differential equations. In this unit, we shall Picard's, Euler's method and the Runge's Kutta Fourth Order Method. Taylor series method to obtain numerical solution of ordinary differential equations (ODEs).

12.2 Objectives

After studying this unit, you should be able to

• Obtain and find the Picard's Method of differential equations.

• Obtain and find the Euler's Method of solution of ordinary differential equations.

• Obtainand find the Runge's Kutta Fourth Order Method.

• Obtain and find the predictor-Corrector Method.

12.3 Picard's Method:

Consider the first order differential equation i.e.

$$\frac{dy}{dx} = f(x, y) \qquad \dots \text{(i)}$$

It is required to find that particular solution of (i) which assumes the value y_0 when $x = x_0$. Integrating (i) between limits, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx$$

or, $[y]_{y_0}^{y} = \int_{x_0}^{x} f(x, y)\, dx$

$$\Rightarrow \quad y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad \dots \text{(ii)}$$

This is an integral equation equivalent to (i), for it contains the unknown *y* under the integral sign.

As a first approximation *y*, to the solution, we put $y = y_0$ in $f(x, y)$ and integrate (ii) giving

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For a second approximation y_2, we put $y = y_1$ in $f(x, y)$ and integrate (ii), giving

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Similarly, a third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

Continuing the process, we obtain $y_4, y_5, \dots, y_n$ where

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

NOTE :-

Picard's method is of considerable theoretical value, but can be applied only to a limited class of equations in which the successive integrations can be performed easily, The method can be extended to simultaneous equations and equations of higher order.

Example.1: Employ Picard's method to obtain, correct to four places of decimal, solution of the differential equation.

$\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$, given that

$y = 0$ when $x = 0$.

Ans. $\left[\because\ y_2 = y_3\right]$

Solution: We have

$$y = 0 + \int_0^x (x^2 + y^2)\, dx$$

First Approximation: Put $y = 0$ in $x^2 + y^2$, giving

$$y_1 = \int_0^x x^2 dx = \left(\frac{x^3}{3}\right)_0^x = \frac{x^3}{3}$$

at $x = 0.4, y_1 = 0.02133$

Second Approximation: Put $y = \frac{x^3}{3}$ in $x^2 + y^2$ giving

$$y_2 = \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx = \left(\frac{x^3}{3} + \frac{x^7}{63}\right)_0^x = \frac{x^3}{3} + \frac{x^7}{63}$$

at $x = 0.4, y_2 = 0.02136$

Third Approximation: Put $y = \frac{x^3}{3} + \frac{x^7}{63}$ in $x^2 + y^2$ giving

$$y_3 = \int_0^x \left(x^2 + \frac{x^6}{9} + \frac{x^{14}}{3969} + \frac{2x^{10}}{189}\right) dx$$

$$= \frac{x^3}{3} + \frac{x^7}{63} + \frac{x^{15}}{59535} + \frac{2x^{11}}{2079}$$

at $x = 0.4, y_3 = 0.02136$

Thus the solution of the differential equation at $x = 0.4$ is $y = 0.0214$

Example.2: Obtain Picard's second approximate solution of the initial value problem

$$y' = \frac{x^2}{y^2+1}, y(0) = 0$$

Solution: We have

$$y = 0 + \int_0^x \frac{x^2}{y^2+1} dx$$

First Approximation: We put $y = 0$ in $x^2(y^2+1)$ giving

$$y_1 = 0 + \int_0^x x^2\, dx = \left(\frac{x^3}{3}\right)_0^x = \frac{x^3}{3}$$

Second Approximation: We put $y = \frac{x^3}{3}$ in $\frac{x^2}{(y^2+1)}$ giving

$$y_2 = 0 + \int_0^x \frac{x^2}{\frac{x^6}{9}+1} dx = 9\int_0^x \frac{x^2}{x^6+9} dx$$

Let $x^3 = u \Rightarrow 3x^2\, dx = du$

$$\therefore \quad y_2 = \frac{9}{3}\int_0^{x^3} \frac{du}{u^2+3^2} = 3 \cdot \frac{1}{3}\left[\tan^{-1}\frac{u}{3}\right]_0^{x^3}$$

$$= \tan^{-1}\left(\frac{x^3}{3}\right) - \tan^{-1} 0$$

$$= \tan^{-1}\left(\frac{x^3}{3}\right)$$

$$= \frac{x^3}{3} - \frac{1}{3}\left(\frac{x^3}{3}\right)^3 + \frac{1}{5}\left(\frac{x^3}{3}\right)^5 -$$

$$= \frac{x^3}{3} - \frac{x^9}{81} + \frac{x^{15}}{1215} - \ldots$$

Example.3: Using Picard's process to approximate y when $x = 0.1$ and 0.2

$\frac{dy}{dx} = y + x,$ given that $y = 1$ when $x = 0.$

Check your answer by finding the exact particular solution.

Solution: We have

$$y = 1 + \int_0^x (y + x)\, dx$$

First Approximation: We put $y = 1$ in $y + x$, giving

Solution: $$y_1 = 1 + \int_0^x (1 + x)\, dx = 1 + x + \frac{x^2}{2}$$

Second Approximation: We put $y = 1 + x + \frac{x^2}{2}$ in $y + x$, giving

$$y_2 = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} + x\right) dx = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

Third Approximation: We put $y = 1 + x + x^2 + \frac{x^3}{6}$ in $y + x$, giving

$$y_3 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{6} + x\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Fourth Approximation: We put $y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$ in $y + x$,

$$y_4 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24} + x\right) dx = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$$

Fifth Approximation: We put $y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120}$ in $y + x$, giving

$$y_5 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} + x\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{120} + \frac{x^6}{720} \qquad \ldots \text{(i)}$$

Now, given equation is

$$\frac{dy}{dx} - y = x \text{ so that } I.F. = e^{-\int dx} = e^{-x}$$

Its solution is

$$y \cdot e^{-x} = \int x e^{-x}\, x + C \Rightarrow y e^{-x} = -x e^{-x} - e^{-x} + C$$

$$\Rightarrow \quad y = -x - 1 + Ce^x$$

Since $y = 1$ when $x = 0$

$\therefore \quad 1 = C - 1 \;\Rightarrow C = 2$

$\therefore \quad y = 2e^x - x - 1$

$$= 2\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \ldots\right] - x - 1$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60} + \frac{x^6}{360} + \ldots\infty \qquad \ldots \text{(ii)}$$

From (i) and (ii), it is clear that, exact particular solution is upto the term $i\ x^5$.

$\therefore \quad y = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \frac{x^5}{60}$ is the required solution.

Example.4: Find the value of y for $x = 0.1$ by Picard's method, given that

$$\frac{dy}{dx} = \frac{y - x}{y + x}, y(0) = 1$$

Solution: We have

$$y = 1 + \int_0^x \frac{y - x}{y + x} dx$$

First Approximation: We put $y = 1$ in $\frac{y - x}{y + x}$ giving

$$y = 1 + \int_0^x \frac{1 - x}{1 + x} dx = 1 + \int_0^x \left(-1 + \frac{2}{1 + x}\right) dx$$

$$= 1 + \left[-x + 2\log(1 + x)\right]_0^x = 1 - x + 2\log(1 + x)$$

Second Approximation: Put $y = 1 - x + 2\log(1 + x)$ in $\frac{y - x}{y + x}$, giving

$$y_2 = 1 + \int_0^x \left[\frac{1 - x + 2\log(1 + x) - x}{1 - x + 2\log(1 + x) + x}\right] dx$$

$= 1 + \int_0^x \left[1 - \frac{2x}{1 + 2\log(1 + x)}\right] dx$ which is very difficult to integrate.

Hence we use the first approximation and taking $x = 0.1$, we obtain

$$y = 1 - (0.1) + 2\log(1 + 0.1)$$

$$= 0.9 + 2\log 1 \cdot 1$$

$$= 0.9828$$

Example.5: Find an approximate value of y when $x = 0.1$, if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$, using Picard's method.

Solution: We have $y = 1 + \int_0^x (x - y^2)\, dx$

First Approximation: Put $y=1$ in $x-y^2$, giving

$$y_1 = 1 + \int_0^x (x-1)\,dx = 1 + \frac{x^2}{2} - x$$

at $x = 0.1; y_1 = 0.905$

Second Approximation: Put $y = 1 + \frac{x^2}{2} - x$ in $x - y^2$, giving

$$y_2 = 1 + \int_0^x \left(x - 1 - \frac{x^4}{4} - x^2 - x^2 + x^3 + x \right) dx$$

$$= 1 + \int_0^x \left(-1 + 2x - 2x^2 + x^3 - \frac{x^4}{4} \right) dx$$

$$= 1 - x + x^2 - 2\frac{x^3}{3} + \frac{x^4}{4} - \frac{x^5}{5}$$

at $x = 0.1; y_2 = 0.9094$

12.4 Taylor's Series Method

Consider the first order equation $\frac{dy}{dx} = f(x, y)$ … (i)

Differentiating (i), we have with the initial condition

$$\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{\partial f}{\partial x}$$

i.e. $y'' = f_x + f_y f'$ … (ii)

Differentiating this successively, we can get y^{III}, y^{IV} etc. Putting $x = x_0$ and $y = y_0$, the values of can be obtained, hence the Taylor's series

$$y = y_0 + (x - x_0)(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \frac{(x-x_0)^3}{3!}(y''')_0 + \dots \quad \text{… (iii)}$$

gives the values of *y* for every value of *x* for which (3) converges.

On finding the value y_1 for $x = x_1$ from (3), y', y'' etc. can be evaluated at $x = x_1$ by means of (1), (2) etc. Then *y* can be expanded about $x = x_1$. In this way, the solution can be extended beyond the range of convergence of series (3).

Example.1: Find the Taylor's series method, the values of *y* at $x = 0.1$ and $x = 0.2$ to five places of decimals from $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$.

Solution: We have, $(y)_0 = 1$

and $y' = x^2 y - 1$ i.e $(y')_0 = -1$

$\therefore$ Differentiating successively and substituting, we get

$$y^{II} = x^2 y' + 2xy \text{ i.e } (y'')_0 = 0$$

$$y^{III} = x^2 y'' + 2xy' + 2xy' + 2y \quad \text{i.e} \quad (y''')_0 = 2(y)_0 = 2$$

$$y^{IV} = x^2 y''' + 2xy'' + 4xy'' + 4y' + 2y' \quad \text{i.e} \quad (y''')_0 = 6(y')_0 = -6$$

and so on

Putting these values in the Taylor's series, we have

$$y = (y)_0 + x(y^I)_0 + \frac{x^2}{2!}(y^{II})_0 + \frac{x^3}{3!}(y^{III})_0 + \frac{x^4}{4!}(y^{IV})_0 + \ldots$$

$$= 1 + x(-1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \ldots$$

$$= 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$$

$\therefore$ $y(0.1) = 0.90031$ and $y(0.2) = 0.80264$

Example.2: Using Taylor's series method, compute $y(0.2)$ to three places of decimal from $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.

Solution: Here $(y_0) = 0$

We have

$$y^I = 1 - 2xy \text{ i.e } (y^I)_0 = 1 \qquad [\because (y)_0 = 0]$$

$$y^{II} = -2xy^I - 2y \quad \text{i.e } (y^{II})_0 = -2(y)_0 = 0$$

$$y^{III} = -2xy^{II} - 2y^I - 2y^I \text{ i.e } (y^{III})_0 = -4(y^I)_0 = -4$$

$$y^{IV} = -2xy^{III} - 2y^{II} - 4y^{II} \text{ i.e } (y^{IV})_0 = -6(y^{II})_0 = 0$$

Subs. these values in Taylor's series, we get

$$\therefore \quad y = (y)_0 + x(y^{II})_0 + \frac{x^2}{2!} y^{II}(0) + \frac{x^3}{3!} y^{III}(0) + \ldots$$

$$= 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-4) + 0 + \dots$$

$$= x - \frac{2}{3}x^3 + \frac{32}{5!}x^5 \dots$$

$\therefore \quad y(0.2) = 0.195$

Example.3: Evaluate $y(0.1)$ correct to six places of decimals by Taylor's series method if $y(x)$ satisfies $y' = xy + 1, y(0) = 1$

Solution: Here $(y)_0 = 1$ means $x_0 = 0$ and $y_0 = 1$

We have

$y^I = xy + 1 \quad$ i.e $\quad (y^I)_0 = 1$

$y^{II} = xy^I + y \quad$ i.e $\quad (y^{II})_0 = (y)_0 = 1$

$y^{III} = xy^{II} + y^I + y^I \quad$ i.e $\quad (y^{III})_0 = 2(y^I)_0 = 2(1) = 2$

$y^{IV} = xy^{III} + y^{II} + 2y^{II} \quad$ i.e $\quad (y^{IV})_0 = 3(y^{II})_0 = 3(1) = 3$

$\therefore$ Subs. these values in Taylor's series i.e

$$y = y_0 + (x - x_0)(y^I)_0 + \frac{(x-x_0)^2}{2!}(y^{II})_0 + \frac{(x-x_0)^3}{3!}(y^{III})_0 + \dots$$

$$= y_0 + x(y^I)_0 + \frac{x^2}{2!}(y^{II})_0 + \frac{x^3}{3!}(y^{III})_0 + \dots \qquad [\because x_0 = 0]$$

$$= 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(3) + \dots$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \dots$$

$\therefore \quad y(0.1) = 1.105345$

Example.4: Solve $y^I = y^2 + x, y(0) = 1$ using Taylor's series method and compute $y(0.1) = 1$ and $y(0.2)$.

Solution: Here $(y)_0 = 1$

We have

$y^I = y^2 + x$ i.e $(y^I)_0 = (y)_0^2 + 0 = 1$

$y^{II} = 2yy^{I} + 1$ i.e $(y^{II})_0 = 2(y)_0(y^{I})_0 + 1 = 3$

$y^{III} = 2yy^{II} + 2(y^{I})^2$ i.e $(y^{III})_0 = 2(y)_0(y^{II})_0 + 2(y^{I})_0^2 = 2(1)(3) + 2 = 8$

$y^{IV} = 2yy^{III} + 2y^{I}y^{II} + 4y^{I}y^{II}$

$(y^{IV})_0 = 2(y)_0(y^{III})_0 + 6(y^{I})_0(y^{II})_0$

$= 2(1)(8) + 6(1)(3)$

$= 34$

Subs. These values in Taylor's series, we have

$$y = (y)_0 + x(y^{I})_0 + \frac{x^2}{2!}(y^{II})_0 + \frac{x^3}{3!}(y^{II})_0 + \ldots$$

$$= 1 + x(1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(8)_0 + \frac{x^4}{4!}(34) + \ldots$$

$$= 1 + x + \frac{3x^2}{2} + \frac{4}{3}x^3 + \frac{17}{12}x^4 + \ldots$$

$\therefore$ $y(0.1) = 1.11647$

$y(0.2) = 1.27293$

Example.5: Find an approximate value of y when $x = 0.1$, if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$, using Taylor's series upto 4 decimal places.

Solution: Here $(y)_0 = 1$ and we have

$y^{I} = x - y^2$ i.e $(y^{I})_0 = -(y)_0^2 = -1$

$\therefore$ $y^{II} = 1 - 2yy^{I}$ i.e $(y^{II})_0 = 1 - 2(y)_0(y^{I})_0 = 1 - 2(1)(-1) = 3$

$y^{III} = -2yy^{II} - 2(y^{I})^2$ i.e $(y^{III})_0 = -2(y)_0(y^{II})_0 - 2(y^{I})_0^2$

$= -2(1)(3) - 2(-1)^2 = -6 - 2 = -8$

and so on.

Subs. these values in Taylor's series, we get

$$y = (y)_0 + x(y^{I})_0 + \frac{x^2}{2!}(y^{II})_0 + \frac{x^3}{3!}(y^{III})_0 + \ldots$$

$$= 1 + x(-1) + \frac{x^2}{2!}(3) + \frac{x^3}{3!}(-8) + \ldots$$

$$= 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \ldots$$

$\therefore \quad y(0.1) = 0.91367$ or 0.9137

12.5 Euler's Method:

Consider the ordinary differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots \text{(i)}$$

With the initial condition $y = y_0$ at $x = x_0$.

Suppose, we have to find the value y_m of *y* corresponding to the value x_m of *x*, For this we divide the interval $x_m - x_0$ into $'m'$ equal parts each of width *h* so that $x_m = x_0 + mh.$ Let $x_1, x_2, \ldots, x_{m-1}$ be the intermediate points and corresponding values of *y* be denoted by $y_1, y_2, \ldots, y_{m-1}$.

In this method we assume the property that in small interval a curve is approximately a straight line. Therefore, at the point (x_0, y_0), the curve is approximated by the tangent at the point (x_0, y_0). We know, that the equation of tangent line at the point (x_0, y_0) is given by

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} \cdot (x - x_0)$$

$\Rightarrow \quad y - y_0 = f(x_0, y_0) \cdot (x - x_0) \qquad$ [using (i)]

$\Rightarrow \quad y = y_0 + (x - x_0)\, f(x_0, y_0) \qquad \ldots$ (ii)

$\therefore$ The value y_1 of *y* at $x = x_1$, from (ii)

$$y_1 = y_0 + (x_1 - x_0)\, f(x_0, y_0)$$

$\Rightarrow \quad y_1 = y_0 + h\, f(x_0, y_0) \qquad \ldots$ (iii)

$[\because\ x_1 - x_0 = h]$

With the same arguments, the approximate value y_2 of *y* at $x = x_2$ is given by

$$y_2 = y_1 + h\, f(x_1, y_1)$$

In general, i.e., the approximate value y_m of *y* at $x = x_m = x_0 + mh$ is given by

$$y_m = y_{m-1} + h\, f(x_{m-1}, y_{m-1})$$

Example.1: Use Euler's method to find $y(0.4)$ from the differential equation. $\frac{dy}{dx} = xy, y(0) = 1.$ Take for each step $h = 0.1.$

Solution: Given $\frac{dy}{dx} = xy \Rightarrow f(x, y) = xy$

Here,

$x_0 = 0, y_0 = 1$ and $h = 0.1$

$\because$ $h = 0.1,$ Therefore, interval 0 to 0.4 is broken into four steps.

First approximation of *y* at $x_1 = 0.1$:

$$y_1 = y_0 + hf(x_0, y_0) = 1 + 0.1\, f(0,1) = 1$$

Second approximation : of *y* at $x_2 = 0.2$:

$$y_2 = y_1 + hf(x_1, y_1) = 1 + 0.1\, f(0.1,1) \qquad [\because x_1 = x_0 + h]$$

$$= 1 + 0.1(0.1) = 1.01$$

Third approximation of *y* at $x_3 = 0.3$:

$$y_3 = y_2 + hf(x_2, y_2) = 1.01 + 0.1\, f(0.2,1.01) \qquad [\because x_2 = x_0 + 2h]$$

$$= 1.01 + 0.1(0.202) = 1.0302$$

Fourth approximation of *y* at $x_4 = 0.4$:

$$y_4 = y_3 + hf(x_3, y_3)$$

$$= 1.0302 + 0.1\, f(0.3,1.0302) \qquad [\because x_3 = x_0 + 3h]$$

$$= 1.0302 + 0.030906$$

$$= 1.061106$$

Example.2: Solve by Euler's method the following differential equation for $x = 0.1$ correct to four decimal places in five steps

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

with the initial condition $y(0) = 1.$

Solution: Given, $\frac{dy}{dx} = \frac{y - x}{y + x} \Rightarrow f(x, y) = \frac{y - x}{y + x}$ and $x_0 = 0, y_0 = 1.$

The range 0 to 0.1 is divided into five steps.

$\therefore \quad h = \dfrac{0.1-0}{5} = 0.02$ $\qquad\qquad \left[\because\ h = \dfrac{x-x_0}{h}\right]$

First approximation:

$$y_1 = y_0 + h\,f(x_0, y_0) = 1 + 0.02\,f(0,1)$$

$$= 1 + 0.02\left[\frac{1-0}{1+0}\right] = 1.02$$

Second approximation:

$$y_2 = y_1 + h\,f(x_1, y_1) = 1.02 + 0.02\,f(0.02, 1.02)$$

$$= 1.02 + 0.02\left[\frac{1.02-0.02}{1.02+0.02}\right] = 1.02 + 0.02\left(\frac{1}{1.04}\right)$$

$$= 1.02 + 0.0192 = 1.0392 \qquad [\because\ x_1 = x_0 + h]$$

Third approximation:

$$y_3 = y_2 + h\,f(x_2, y_2) = 1.0392 + 0.02\,f(0.04, 1.0392)$$

$$= 1.0392 + 0.02\left[\frac{1.0392-0.04}{1.0392+0.04}\right] = 1.0392 + 0.02\left(\frac{0.9992}{1.0792}\right)$$

$$= 1.0392 + 0.0185 = 1.0577 \qquad [\because\ x_2 = x_0 + 2h]$$

Fourth approximation:

$$y_4 = y_3 + h\,f(x_3, y_3) = 1.0577 + 0.02\,f(0.06, 1.0577)$$

$$= 1.0577 + 0.02\left(\frac{1.0577-0.06}{1.0577+0.06}\right) = 1.0577 + 0.02\left[\frac{0.9977}{1.1177}\right]$$

$$= 1.0577 + 0.0178 = 1.0755$$

Fifth approximation:

$$y_5 = y_4 + h\,f(x_4, y_4) = 1.0755 + 0.02\,f(0.08, 1.0755)$$

$$= 1.0755 + 0.02\left(\frac{1.0755-0.08}{1.0755+0.08}\right) = 1.0755 + 0.02\left[\frac{0.9955}{1.1555}\right]$$

$$= 1.0755 + 0.0172 = 1.0927$$

Hence,

$y = 1.0927$ at $x = 0.1$

Example.3: Apply Euler's method solve for y at $x = 0.6$ from $\dfrac{dy}{dx} = 1 - 2xy, y(0) = 0$ take $h = 0.2$.

Solution: Here,

$f(x,y) = 1 - 2xy, x_0 = 0$ and $y_0 = 0, h = 0.2$

We know that

$$m = \frac{x - x_0}{h} = \frac{0.6 - 0}{0.2} = 3$$

$\therefore$ interval 0 to 0.6 is broken into 3 steps.

First approximation:

$$y_1 = y_0 + h\, f(x_0, y_0) = 0 + 0.2\, f(0,0)$$

$$= 0.2 + [1 - 2(0)(0)] = 0.2$$

Second approximation:

$$y_2 = y_0 + h\, f(x_1, y_1) = 0.2 + 0.2\, f(0.2, 0.2)$$

$$= 0.2 + 0.2[1 - 2(0.2)(0.2)]$$

$$= 0.2 + 0.2[1 - 0.08] = 0.2 + 0.2(0.92)$$

$$= 0.384$$

Third approximation:

$$y_0 = y_2 + h\, f(x_2, y_2)$$

$$= 0.384 + 0.2\, f(0.4, 0.384)$$

$$= 0.384 + 0.2[1 - 2(0.4)(0.384)]$$

$$= 0.384 + 0.2[1 - 0.3072]$$

$$= 0.384 + 0.13856$$

$$= 0.52256$$

Thus, $y(0.6) = 0.52256$

12.6 Modified Euler's Method:

Consider the differential equation

$$\frac{dy}{dx} = f(x,y)$$

with the initial condition $y = y_0$ at $x = x_0$.

Now we introduce a new notation $y_1^{(1)}$ defined by

$$y_1^{(1)} = y_0 + h\, f(x_0, y_0).$$

Here, $y_1^{(1)}$ is nothing but the value y_1 of *y* for $x = x_1$ obtained by Euler's method.

In other words $y_1^{(1)}$ is the first approximation of *y* at $x = x_1$.

Then by Modified Euler's method, we have

$$y_{m+1} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

Example.1: Use modified Euler's method to compute *y* for $x = 0.05$, Given that $\frac{dy}{dx} = x + y$ with initial conditions $x_0 = 0, y_0 = 1$ result correct upto three decimal places.

Solution: Here $\frac{dy}{dx} = f(x, y)$, where $f(x, y) = x + y$ with initial conditions $x_0 = 0, y_0 = 1$.

Take $h = 0.05 - 0 = 0.05$

Let y_1 be the value of *y* at $x = x_1 = x_0 + h = 0 + 0.5 = 0.05$ and let $y_1^{(1)}$ be the first approximation of y_1.

From Euler's method, we have

$$y_1^{(1)} = y_0 + h\, f(x_0, y_0) = 1 + 0.05\, f(0,1) = 1 + 0.05(1)$$

$$= 1.05$$

Now, we shall improve this value by Modified Euler's method

$$y_{m+1} = y_m + \frac{h}{2}\left[f(x_0, y_0) + f(x_{m+1}, y_{m+1}^{1})\right] \quad \ldots \text{(i)}$$

For the second approximation of y_1, we put $m = 0, x_1 = x_0 + h$ i.e $x_1 = 0 + 0.05 = 0.05$ in (i) we get

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.05}{2}\left[f(0,1) + f(0.05, 1.05)\right]$$

$$= 1 + 0.025\left[0 + 1 + 0.05 + 1.05\right] = 1.0525$$

Third approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(0,1) + f(0.05, 1.0525)\right] = 1.0526$$

Fourth approximation to y_1 :

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + \frac{0.05}{2}\left[f(0,1) + f(0.05, 1.0526)\right] = 1.0526$$

Clearly, $y_1^{(3)} = y_1^{(4)}$, Thus $y = 1.052$ at $x = 0.05$.

Example.2: Using modified Euler's method, obtain a solution of the equation $\frac{dy}{dx} = 1 - y = f(x, y)$ with $y(0) = 0$ in the range $0 \le x \le 0.3$ by taking $h = 0.1$.

Solution: Here $\frac{dy}{dx} = 1 - y \Rightarrow f(x, y) = 1 - y$ with $y(0) = 0$ i.e $x_0 = 0; y_0 = 0, h = 0.1$

Let y_1 be the value of *y* at $x = x_1 = x_0 + h = 0 + 0.1 = 0.1$

Let $y_1^{(1)}$ be the first approximation of y_1.

From Euler's method, we have

$$y_1^{(1)} = y_0 + h\, f(x_0, y_0) = 0 + 0.1(1 - 0) = 0.1$$

By Euler's modified method:

$$y_{m+1} = y_m + \frac{h}{2}\left[f(x_m, y_m) + f(x_{m+1}, y_{m+1}^{(1)}) \right]$$

For 2nd approximate of y_1, we put $m = 0; x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y_1^{(2)} = y_0 + \frac{0.1}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)}) \right]$$

$$= 0 + 0.05\left[1 - y_0 + 1 - y_1^{(1)}\right]$$

$$= 0.05[2 - 0 - 0.1] = 0.095$$

Third approximation to y_1 :

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)}) \right]$$

$$= y_0 + \frac{h}{2}[f(0,0) + f(0.1, 0.095)]$$

$$= 0 + \frac{0.1}{2}[1 - 0 + 1 - 0.095] = 0.09525$$

Fourth approximation to y_1 :

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)}) \right]$$

$$= 0 + \frac{0.1}{2}[f(0,0) + f(0.1, 0.09525)]$$

$$= 0.05[1 - 0 + 1 - 0.09525] = 0.09523$$

Clearly, $y_1^{(3)} = y_1^{(4)}$, Thus $y = 0.0952$ at $x = 0.1$

Now we shall compute the value y_2 of *y* at $x = 0.2$.

First approximation to y_2 :

$$y_2^{(1)} = y_1 + h\, f(x_1, y_1) = 0.0952 + 0.1\, f(0.1, 0.0952)$$

$$= 0.0952 + 0.1[1 - 0.0952] = 0.18568$$

Second approximation to y_2 :

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)}) \right]$$

$$= 0.0952 + \frac{0.1}{2}\left[f(0.1, 0.0952) + f(0.2, 0.18568) \right]$$

$$= 0.0952 + 0.05\left[1 - 0.0952 + 1 - 0.18568\right] = 0.18116$$

Third approximation to y_2 :

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)}) \right]$$

$$= 0.0952 + 0.05\left[f(0.1, 0.0952) + f(0.2, 0.18116) \right]$$

$$= 0.0952 + 0.05\left[1 - 0.0952 + 1 - 0.18116\right] = 0.18138$$

Fourth approximation to y_2 :

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)}) \right]$$

$$= 0.0952 + 0.05\left[f(0.1, 0.0952) + f(0.2, 0.18138) \right]$$

$$= 0.0952 + 0.05\left[1 - 0.0952 + 1 - 0.18138\right] = 0.18137$$

Clearly, $y_2^{(3)} = y_2^{(4)}$, Thus $y = 0.1813$ at $x = 0.2$

Now we shall compute the value y_3 of y at $x = 0.3$.

From Euler's method, we have

$$y_3^{(1)} = y_2 + h\, f(x_2, y_2) = 0.1813 + 0.1\, f(0.2, 0.1813)$$

$$= 0.1813 + 0.1(1 - 0.1813) = 0.26317$$

Now by Modified Euler's method, we have

$$y_{m+1} = y_m + \frac{h}{2}\left[f(x_m, y_m) + f(x_{m+1}, y_{m+1}^{(1)}) \right]$$

For $m = 1$

$$y_3^{(2)} = y_2 + \frac{0.1}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)}) \right]$$

$$= 0.1813 + 0.05\left[f(0.2, 0.1813) + f(0.3, 0.26317) \right]$$

$= 0.1813 + 0.05[1 - 0.1813 + 1 - 0.26317] = 0.25908$

Third approximation to y_3 :

$$y_3^{(3)} = 0.1813 + 0.05\left[2 - 0.1813 - 0.25908\right] = 0.25928$$

Fourth approximation to y_3 :

$$y_3^{(4)} = 0.1813 + 0.05\left[2 - 0.1813 - 0.25928\right] = 0.25927$$

Clearly, $y_3^{(3)} = y_3^{(4)}$, Thus $y = 0.25927$ at $x = 0.3$

Example.3: Using modified Euler's method solve for y at $x = 0.1$ an 0.2 from $\frac{dy}{dx} = x^2 + y^2, y(0) = 1.$

Solution: Here $\frac{dy}{dx} = x^2 + y^2 \Rightarrow f(x, y) = x^2 + y^2$ with $x_0 = 0, y_0 = 1.$

Let $x_1 = 0.1$ and $x_2 = 0.2,$ Let y_1 and y_2 be the values of y at $x = x_1$ and $x = x_2$ respectively.

$$y_1^{(1)} = y_0 + h[f(x_0, y_0)] \qquad \text{[Here } h = 0.1 - 0 = 0.1]$$

$$= y_0 + 0.1 f(0,1)$$

$$= y_0 + 0.1[0 + 1^2] = 1.1$$

Now by modified Euler's method, the second approximation to y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$= 1 + \frac{0.1}{2}[f(0,1) + f(0.1, 1.1)]$$

$$= 1 + 0.05\left[1 + (0.1)^2 + (1.1)^2\right] = 1.111$$

3rd Approximation :

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$

$$= 1 + 0.05\left[f(0,1) + f(0.1, 1.111)\right]$$

$$= 1 + 0.05\left[1 + (0.1)^2 + (1.111)^2\right] = 1.1122$$

4th Approximation :

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$

$$= 1 + 0.05\left[f(0,1) + f(0.1, 1.1122)\right]$$

$$=1+0.05\left[1+(0.1)^2+(1.1122)^2\right]=1.1123$$

5th Approximation :

$$y_1^{(5)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(4)})\right]$$

$$=1+0.05\left[f(0,1)+f(0.1,1.1123)\right]$$

$$=1+0.05\left[1+(0.1)^2+(1.1123)^2\right]$$

$$=1.1123$$

Since $y_1^{(4)}=y_1^{(5)}$ and so the value of y at $x=0.1$ is 1.1123

To compute y_2 : We have $x_1=0.1, y_1=1.1123, x_2=0.2$

I Approximation:

$$y_2^{(1)}=y_1+h\,f(x_1,y_1)=1.1123+0.1\,f(0.1,1.1123)$$

$$=1.1123+0.1\left[(0.1)^2+(1.1123)^2\right]=1.2370$$

By modified Euler's method, the second approximation to y_2 is given by

$$y_2^{(2)}=y_1+\frac{h}{2}\left[f(x_1,y_1)+f(x_2,y_2^{(1)})\right]$$

$$=1.1123+\frac{0.1}{2}\left[f(0.1,1.1123)+f(0.2,1.2370)\right]$$

$$=1.1123+0.05\left[(0.1)^2+(1.1123)^2+(0.2)^2+(1.2370)^2\right]=1.2532$$

$$y_2^{(3)}=y_1+\frac{h}{2}\left[(x_1,y_1)+f(x_2,y_2^{(2)})\right]$$

$$=1.1123+0.05\left[(0.1)^2+(1.1123)^2+(0.2)^2+(1.2552)^2\right]=1.2552$$

$$y_2^{(4)}=1.1123+0.05\left[(0.1)^2+(1.1123)^2+(0.2)^2+(1.2552)^2\right]=1.2554$$

Since $y_2^{(3)}=y_2^{(4)}$, we can say that the value of y at $x=0.2$ is 1.255

12.7 Runge's Method:

Consider the differential equation

$$\frac{dy}{dx}=f(x,y),\ y(x_0)=y_0 \qquad \ldots \text{(i)}$$

Working Rule to solve equation (i) by Runge's method:

Calculate successively

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f\left[x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right]$$

$$k' = h\, f[x_0 + h, y_0 + k_1]$$ and

$$k_3 = h\, f[x_0 + h, y_0 + k']$$

Finally compute,

$$k = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Which gives a sufficiently accurate value of k and also of

$$y = y_0 + k$$

Example.1: Apply Runge's method to find an approximate value of y when $x = 0.2$ given that

$\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution: Here $f(x, y) = x + y$

and $x_0 = 0, y_0 = 1, h = 0.2$

$\therefore \quad f(x_0, y_0) = f(0,1) = 0 + 1 = 1$

$\therefore \quad k_1 = h\, f(x_0, y_0) = 0.2(1) = 0.2$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\, f\left[0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right]$$

$$= 0.2\, f(0.1, 1.1) = 0.2[0.1 + 1.1] = 0.2(1.2) = 0.24$$

$$k' = h\, f(x_0 + h, y_0 + k_1) = 0.2\, f[0 + 0.2, 1 + 0.2] = 0.2\, f(0.2, 1.2)$$

$$= 0.2[0.2 + 1.2] = 0.2(1.4) = 0.28$$

$$k_3 = h\, f(x_0 + h, y_0 + k') = 0.2\, f[0 + 0.2, 1 + 0.28]$$

$$= 0.2\, f(0.2, 1.28) = 0.2[0.2 + 1.28] = 0.2(1.48)$$

$$= 0.296$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}[0.2 + 4(0.24) + 6.296]$$

$$= 0.2426$$

Hence the required approximate value of

$$y = y_0 + k$$

$= 1 + 0.2426$

$= 1.2426$

Example.2: Use Runge's method to approximate y when $x = 1.1$, given that $y = 1.2$ when $x = 1$ and $\frac{dy}{dx} = 3x + y^2.$

Solution: Here we have $x_0 = 1, y_0 = 1.2, h = 0.1$

$\therefore \quad f(x_0, y_0) = f(1, 1.2) = 3(1) + (1.2)^2 = 4.44$

$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.1 \times 4.44 = 0.444$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,f\left(1 + \frac{0.1}{2}, 1.2 + \frac{0.444}{2}\right)$$

$= 0.1\,f(1.05, 1.422) = 0.5172$

$k' = h\,f(x_0 + h, y_0 + k_1) = h\,f(1 + 0.1, 1.2 + 0.444)$

$= 0.1\,f(1.1, 1.644) = 0.6003$

$\therefore \quad k_3 = h\,f(x_0 + h, y_0 + k') = 0.1\,f(1 + 0.1, 1.2 + 0.6003)$

$= 0.1\,f(1.1, 1.8003) = 0.6541$

$$\therefore \quad k = \frac{1}{6}(k_1 + 4k_2 + k_3) = \frac{1}{6}[0.444 + 4(0.5172) + 0.6541]$$

$= 0.5278$

$\therefore \quad y = y_0 + k = 1.2 + 0.5278$

$= 1.7278$

12.8 Runge-Kutta Method

Fourth Order R-K Method:

This method is most commonly used and is often referred to as Runge-Kutta method only.

Working Rule: For finding the increment *k* of *y* corresponding to an increment *h* of *x* by Runge-Kutta Method from

$$\frac{dy}{dx} = f(x, y);\ y(x_0) = y_0$$

is as follows:

Calculate successively

$k_1 = h\,f(x_0, y_0)$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Finally compute: -

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Which gives the required approximate value as

$$y = y_0 + k$$

Note: -*k* is the weighted mean of k_1, k_2, k_3 and k_4.

Obs: - One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

Example.1: Apply Runge-Kutta fourth order method to find an approximate value of *y* when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Solution: Here $f(x, y) = x + y$ and $x_0 = 0, y_0 = 1, h = 0.2$.

$$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.2\,f(0,1) = 0.2$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\,f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2\,f(0.1, 1.1) = 0.2400$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\,f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$$= 0.2\,f(0.1, 1.12) = 0.244$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = 0.2\,f(0 + 0.2, 1 + 0.244)$$

$$= 0.2\,f(0.2, 1.244) = 0.2888$$

$$\therefore \quad k_1 = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.2 + 0.48 + 0.488 + 0.2888)$$

$$= \frac{1}{6}(1.4568) = 0.2428$$

Hence the required approximate value of *y* is

$$y = y_0 + k = 1 + 0.2428$$

$= 1.2428$

Example.2: Using Runge-Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with $y(0) = 1$ and $x = 0.2, 0.4.$

Solution: Here $f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}, x_0 = 0, y_0 = 1, h = 0.2$

$$k_1 = h\, f(x_0, y_0) = 0.2\, f(0,1) = 0.2$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2\, f(0.1, 1.1) = 0.19672$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\, f\left(0 + \frac{0.2}{2}, 1 + \frac{0.19672}{2}\right)$$

$$= 0.2\, f(0.1, 1.09836) = 0.1967$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = 0.2\, f(0 + 0.2, 1 + 0.1967)$$

$$= 0.2\, f(0.2, 1.1967) = 0.1891$$

$$\therefore \quad k = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}\left(0.2 + 2(0.19672) + 2(0.1967) + 0.1891\right)$$

$$= 0.19599$$

$$\therefore \quad y(0.2) = y_0 + k = 1 + 0.19599$$

$$= 1.196$$

To find $y(0.4) = x_1 = x_0 + h = 0 + 0.2$

Here, $x_1 = 0.2, y_1 = 1.196, h = 0.2$

$$k_1 = h\, f(x_1, y_1) = 0.2\, f(0.2, 1.196) = 0.1891$$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.2\, f\left(0.2 + \frac{0.2}{2}, 1.196 + \frac{0.1891}{2}\right)$$

$$= 0.2\, f(0.3, 1.2906) = 0.1795$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.2\, f\left(0.2 + \frac{0.2}{2}, 1.196 + \frac{0.1795}{2}\right)$$

$$= 0.2\, f(0.3, 1.2858) = 0.1793$$

$$k_4 = h\, f(x_1 + h, y_1 + k_3) = 0.2\, f(0.2 + 0.2, 1.196 + 0.1793)$$

$$= 0.2\, f(0.4, 1.3753) = 0.1688$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1891 + 2(0.1795) + 2(0.1793) + 0.1688) = 0.1792$$

Hence,

$$y(0.4) = y_1 + k$$

$$= 1.196 + 0.1792$$

$$= 1.3752$$

Example.3: Apply Runge-Kutta method to find approximate value of y for $x = 0.2$ in steps of 0.1, if $\frac{dy}{dx} = 0.1,$ if $\frac{dy}{dx} = x + y^2$, given that $y = 1$ where $x = 0$.

Solution: Given $f(x, y) = x + y^2$

Here, $h = 0.1, x_0 = 0, y_0 = 1$

$$k_1 = h\, f(x_0, y_0) = 0.1\, f(0,1) = 0.1$$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = h\, f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1\, f(0.05, 1.05)$$

$$= 0.1152$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = h\, f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = h\, f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1152}{2}\right)$$

$$= 0.1\, f(0.05 + 1.0576) = 0.1168$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = h\, f\left(\frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1\, f(0 + 0.1, 1 + 0.1168)$$

$$= 0.1\, f(0.1 + 1.1168) = 0.1347$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}[0.1 + 0.2304 + 0.2336 + 0.1347] = 0.1165$$

giving, $y(0.1) = y_0 + k = 1 + 0.1165 = 1.1165$

To find $y(0.2)$:

We know that

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = 11.65, h = 0.1$$

$\therefore \quad k_1 = h\,f(x_1, y_1) = 0.1\,f(0.1, 1.1165) = 0.1347$

$$k_2 = h\,f\left(x_1 + \frac{h}{2}, k_1 + \frac{k_1}{2}\right) = 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1347}{2}\right)$$

$$= 0.1\,f(0.15, 1.1838) = 0.1551$$

$$k_3 = h\,f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1\,f\left(0.1 + \frac{0.1}{2}, 1.1165 + \frac{0.1551}{2}\right)$$

$$= 0.1\,f(0.15, 1.194) = 0.1576$$

$$k_4 = h\,f(x_1 + h, y_1 + k_3) = 0.1\,f(0.1 + 0.1, 1.1165 + 0.1576)$$

$$= 0.1\,f(0.2, 1.1576) = 0.1823$$

$$k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] = \frac{1}{6}[0.1347 + 0.3102 + 0.3152 + 0.1823]$$

$$= 0.1571$$

Hence $y(0.2) = y_1 + k = 1.1165 + 0.1571 = 1.2736$

Example.4: Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\frac{dy}{dx} = x^2 + y^2, y(0) = 1$ taking $h = 0.1.$

Solution: We have, $\frac{dy}{dx} = \frac{x^2 + y^2}{10}$ so $f(x, y) = \frac{x^2 + y^2}{10}$

Here, $x_0 = 0, y_0 = 1$ and $h = 0.1$

$\therefore \quad k_1 = h\,f(x_0, y_0) = 0.1\,f(0,1) = 0.01$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.1\,f\left(\frac{0.1}{2}, 1 + \frac{0.01}{2}\right) = 0.1\,f(0.05, 1.005)$$

$$= 0.0101$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.1\,f\left(0 + \frac{0.1}{2}, 1 + \frac{0.0101}{2}\right)$$

$$= 0.01\,f(0.05, 1.005) = 0.0101$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3) = h\,f(0 + 0.1, 1 + 0.0101)$$

$$= 0.01\,f(0.1, 1.0101) = 0.0103$$

$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$$= \frac{1}{6}[0.01 + 0.0202 + 0.0202 + 0.0103] = 0.0101$$

$\therefore \quad y(0.1) = y_0 + k = 1 + 0.0101 = 1.0101$

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1, y_1 = 1.0101, h = 0.1$

$\therefore \quad k_1 = h\, f(x_1, y_1) = 0.1\, f(0.1, 1.0101) = 0.203$

$$k_2 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.203}{2}\right)$$

$$= 0.1\, f(0.15, 1.1116) = 0.1256$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = 0.1\, f\left(0.1 + \frac{0.1}{2}, 1.0101 + \frac{0.1256}{2}\right)$$

$$= 0.1\, f(0.15, 1.0729) = 0.0117$$

$$k_4 = h\, f(x_1 + h, y_1 + k_3) = 0.1\, f(0.1 + 0.1, 1.0101 + 0.0117)$$

$$= 0.1\, f(0.2, 1.0218) = 0.0108$$

$$\therefore \quad k = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.203 + 2(0.1236) + 2(0.0117) + 0.0108] = 0.0814$$

$$\therefore \quad y(0.2) = y_1 + k = 1.0101 + 0.814 = 1.0915$$

Now, $x_2 = x_1 + h = 0.2 + 0.1 = 0.3$

12.9 Predictor-Corrector Methods

In the predictor-corrector methods, however, four prior values are required for finding the value of *y* at x_{i+1}. A predictor formula is used to predict the value of *y* at x_{i+1} and then a corrector formula is applied to improve this value. We now discuss two methods, namely Milne's method and Adams-Bashforth method.

Method I : Milne's Method

Consider the differential equation of first order $\frac{dy}{dx} = f(x, y)$ … (i)

With the initial condition $y = y_0$ when $x = x_0$.

In order to solve the differential equation (i) by Milne's method, we first obtain the approximate value of y_{n+1} by predictor formula and then we improve this value by corrector formula. These formulas are as follows :

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

is Milne's predictor formula and

$$y_{n+1}^{(1)} = y_{n-1} + \frac{h}{3}\left(y_{n-1}^{'} - y_{n}^{'} + 2y_{n+1}^{'}\right)$$

is Milne's corrector formula

If we put $n = 3$ the Milne's predictor formula becomes

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1^{'} - y_2^{'} + 2y_3^{'}\right) \text{ and}$$

Milne's corrector formula becomes

$$y_4 = y_2 + \frac{h}{2}\left(y_2^{'} + 4y_3^{'} + y_4^{'}\right)$$

Where h is the interval of differencing.

Example.1: Use Milne's predictor corrector method and find $y(4.4)$ given that $5xy' + y^2 - 2 = 0$ and $y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143.$

Solution: Here, $y' = \frac{2-y^2}{5x} = f(x, y)$ and $x_0 = 4, y_0 = 1$ and $h = 0.1$ so that $x_1 = x_0 + h$

$\Rightarrow$ $x_1 = 4.1, x_2 = x_0 + 2h = 4.2, x_3 = x_0 + 3h = 4.3$ and $x_4 = x_0 + 4h$ or $x_3 + h = 4.4$

Given that, $y_0 = y(x_0) = y(4) = 1$

$$y_1 = y(x_1) = y(4.1) = 1.0049$$

$$y_2 = y(x_2) = y(4.2) = 1.0097$$

$$y_3 = y(x_3) = y(4.3) = 1.0143$$

$$y_4 = y(x_4) = y(4.4) = ?$$

Hence,

$$y_1^{'} = f(x_1, y_1) = \frac{2-y_1^2}{5x_1} = \frac{2-(1.0049)^2}{5(4.1)} = 0.0483$$

$$y_2^{'} = f(x_2, y_2) = \frac{2-y_2^2}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)} = 0.0466$$

$$y_3^{'} = f(x_3, y_3) = \frac{2-y_3^2}{5x_3} = \frac{2-(1.0143)^2}{5(4.3)} = 0.0451$$

Now using Milne predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1^{'} - y_2^{'} + 2y_3^{'}\right)$$

$$= 1 + \frac{4(0.1)}{3}\left[2(0.0483) - (0.0466) + 2(0.0451)\right]$$

$=1.0186$

Hence, $y_4' = f(x_4, y_4) = \frac{2-y_4^2}{5x_4} = \frac{2-(1.0186)^2}{5(4.4)} = 0.0437$

Using Milne's corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$=1.0097 + \frac{0.1}{3}\left[0.0466 + 4(0.0451) + 0.0437\right]$$

$$=1.0187$$

Hence $y_4 = y(x_4) = y(.4) = 1.0187$

Example.2: Given $2\frac{dy}{dx} = (1+x^2)y^2$ and $y(0)=1, y(0.1)=1.06, y(0.2)=1.12, y(0.3)=1.21,$ evaluate $y(0.4)$ by Milne's predictor-corrector method.

Solution: Here, $2\frac{dy}{dx} = (1+x^2)y^2 \Rightarrow y' = \frac{1}{2}(1+x^2)y^2 = f(x,y)$

$x_0 = 0, y_0 = 1$ and $h = 0.1$ so that $x_1 = x_0 + h = 0.1$

$x_2 = x_0 + 2h = 0.2, x_3 = x_0 + 3h = 0.3, x_4 = x_0 + 4h = 0.4.$

Given that

$$y_0 = y(x_0) = y(0) = 1; y_1 = y(x_1) = y(0.1) = 1.06$$

$$y_2 = y(x_2) = y(0.2) = 1.12; y_3 = y(x_3) = y(0.3) = 1.21$$

$$\therefore \quad y_1' = \frac{1}{2}(1+x_1^2)y_1^2 = \frac{1}{2}[1+(0.1)^2](1)^2 = 0.56742$$

$$y_2' = \frac{1}{2}(1+x_2^2)y_2^2 = \frac{1}{2}[1+(0.2)^2](1.12)^2 = 0.65229$$

$$y_3' = \frac{1}{2}(1+x_3^2)y_3^2 = \frac{1}{2}[1+(0.3)^2](1.21)^2 = 0.79793$$

Now using Milne's predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$=1 + \frac{4(0.1)}{3}\left[2(0.56742) - (0.65229) + 2(0.79793)\right]$$

$$=1.27712$$

Hence, $y_4' = f(x_4, y_4) = \frac{1}{2}(1+x_4^2)y_4^2$

$$= \frac{1}{2}\left[1+(0.4)^2\right](1.27712)^2 = 0.946$$

Using Milne's corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + y_3' + y_4'\right)$$

$$= 1.12 + \frac{0.1}{3}\left[0.65229 + 4(0.79793) + 0.946\right]$$

$$= 1.2797$$

Hence, $y_4 = y(x_4) = y(0.4) = 1.2797$

Example.3: Given that

$\frac{dy}{dx} = x^2(1+y), y(1) = 1, y(1.1) = 1.233, y(1.2) = 1.548, y(1.3) = 1.979.$ Evaluate $y(1.4)$ by Milne's predictor-corrector method.

Solution: Here, $y' = x^2(1+y) = f(x,y)$ and $x_0 = 1,$ $y_0 = 1$ and $h = 0.1$ so that $x_1 = x_0 + h = 1.1;\ x_2 = x_0 + 2h = 1.2;\ x_3 = x_0 + 3h = 1.3;\ x_4 = x_0 + 4h = 1.4;$

Given that : $y = y(x_0) = y(1) = 1$

$$y_1 = y(x_1) = y(1.1) = 1.233$$

$$y_2 = y(x_2) = y(1.2) = 1.548$$

$$y_3 = y(x_3) = y(1.3) = 1.979$$

Hence, $y_1' = x_1^2(1+y_1) = (1.1)^2(1+1.233) = 2.70193$

$$y_2' = f(x_2, y_2) = x_2^2(1+y_2) = (1.2)^2(1+1.548) = 3.66912$$

$$y_3' = f(x_3, y_3) = x_3^2(1+y_3) = (1.3)^2(1+1.979) = 5.03451$$

Now using Milne's predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' + y_2' + 2y_3'\right)$$

$$y_4 = 1 + \frac{4(0.1)}{3}\left[2(2.70193) - 3.66912 + 2(5.03451)\right]$$

$$= 2.57383$$

Hence, $y_4' = f(x_4, y_4) = x_4^2(1+y_4) = (1.4)^2(1+2.57383)$

$$= 7.0047$$

Using Milne's corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 1.548 + \frac{0.1}{3}\left[3.66912 + 4(5.03451) + 7.0047\right]$$

$= 2.575$

Hence, $y_4 = y(x_4) = y(1.4) = 2.575$

Example.4: Use Milne's predictor-corrector method to obtain the solution of the equation :

$\frac{dy}{dx} = x - y^2$ at $x = 0.8$ given that $y(0) = 0, y(0.2) = 0.0200,$ $y(0.4) = 0.0795, y(0.6) = 0.1762.$

Solution: Here $y' = x - y^2 = f(x, y)$ and $x_0 = 0, y_0 = 0$ and $h = 0.2$ so that $x_1 = x_0 + h = 0.2,$ $x_2 = x_0 + 2h = 0.4,$ $x_3 = x_0 + 3h = 0.6,$ $x_4 = x_0 + 4h = 0.8.$

Given that $y_0 = y(x_0) = y(0) = 0$

$$y_1 = y(x_1) = y(0.2) = 0.0200$$

$$y_2 = y(x_2) = y(0.4) = 0.0795$$

$$y_3 = y(x_3) = y(0.6) = 0.1762.$$ Hence

$$y_1' = f(x_1, y_1) = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$$

$$y_2' = f(x_2, y_2) = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$$

$$y_3' = f(x_3, y_3) = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5689$$

Now using Milne's predictor formula, we get

$$y_4 = y_0 + \frac{4h}{3}\left(2y_1' - y_2' + 2y_3'\right)$$

$$= 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5689)\right]$$

$$= 0.30488$$

Hence $y_4' = f(x_4, y_4) = x_4 - y_4^2 = 0.8 - (0.30488)^2$

$$= 0.70705$$

Using Milne's corrector formula, we get

$$y_4 = y_2 + \frac{h}{3}\left(y_2' + 4y_3' + y_4'\right)$$

$$= 0.0795 + \frac{0.2}{3}\left[0.3937 + 4(0.5689) + 0.70705\right]$$

$$= 0.30459 = 0.3046$$

Hence, $y_4 = y(x_4) = y(0.8) = 0.3046$

Method II : Adams-Bashforth Method

Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots \text{(i)}$$

With the initial condition $y = y_0$ when $x = x_0$.

$$y_1 = y_0 + \frac{h}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right)$$

is known as Adams-Bashforth predictor formula.

$$y_1 = y_0 + \frac{h}{24}\left(9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}\right)$$

is known as Adams-Bashforth corrector formula.

Example.1: Given $\frac{dy}{dx} = x - y^2$ and $y(0) = 0, y(0.2) = 0.02,$ $y(0.4) = 0.0795, y(0.6) = 0.1762;$ evaluate $y(0.8)$ by Adams-Bashforth method.

Solution: Given $\frac{dy}{dx} = x - y^2 = f(x, y)$ 'say' consecutive starting values for Adams-Bashforth method with $h = 0.2$ are given by $x_{-3} = 0, y_{-3} = 0, f_{-3} = x_{-3} - y_{-3}^2 = 0 - 0 = 0$

$$x_{-2} = 0.2, y_{-2} = 0.02, f_{-2} = x_{-2} - y_{-2}^2 = 0.2 - (0.02)^2 = 0.1996$$

$$x_{-1} = 0.4, y_{-1} = 0.0795, f_{-1} = x_{-1} - y_{-1}^2 = 0.4 - (0.0795)^2 = 0.39368$$

$$x_0 = 0.6, y_0 = 0.1762, f_0 = x_0 - y_0^2 = 0.6 - (0.1762)^2 = 0.56895$$

By Adams-Bashforth predictor formula

$$y_1 = y_0 + \frac{h}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right)$$

The value of y_1 at $x = 0.8$ is given by

$$y_1 = 0.1762 + \frac{0.2}{24}\left[55(0.56895 - 59(0.39368) + 37(0.1996) - 9(0)\right]$$

$$= 0.30495$$

$$\therefore \quad f_1 = x_1 - y_1^2 = 0.8 - (0.30495)^2 = 0.707$$

By Adams-Bashforth corrector formula, we have

$$y_1 = y_0 + \frac{h}{24}\left(9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}\right)$$

$$= 0.1762 + \frac{0.2}{24}\left[9(0.707) + 19(0.56895) - 5(0.39368) + 0.1996\right]$$

$$= 0.30457$$

$$\therefore \quad y(0.8) = 0.30457$$

Example.2: Given $\frac{dy}{dx} = x^2 - y, y(0) = 1$ and the starting values $y(0.1) = 0.90516, y(0.2) = 0.82127,\ y(0.3) = 0.74918$; evaluate $y(0.4)$ using Adams Bashforth method.

Solution: Here $\frac{dy}{dx} = x^2 - y = f(x, y)$ 'say'.

Consecutive starting values for the Adams-Bashforth method for $h = 0.1$ are :

x	y	$f(x,y) = x^2 - y$
$x = 0$	$y_{-3} = 1$	$f_{-3} = -1.0000$
$x = 0.1$	$y_{-2} = 0.90516$	$f_{-2} = -0.89516$
$x = 0.2$	$y_{-1} = 0.82127$	$f_{-1} = -0.78127$
$x = 0.3$	$y_0 = 0.74918$	$f_0 = -0.65918$

Using Adams Bashforth predictor formula

$$y_1 = y_0 + \frac{h}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right)$$

$$= 0.74918 + \frac{0.1}{24}\left[55(-0.65918) - 59(-0.78127) + 37(-0.89516) - 9(-1)\right]$$

$$= 0.68968$$

$$\therefore \quad f_1 = (0.4)^2 - 0.68968 = -0.52968$$

Using Adams Bashforth corrector formula

$$y_1 = y_0 + \frac{h}{24}\left(9 f_1 + 19 f_0 - 5 f_{-1} + f_{-2}\right)$$

$$= 0.74918 + \frac{0.1}{24}\left[9(-0.52968) + 19(-0.65918) - 5(-0.78127) + (-0.89516)\right]$$

$$= 0.68968$$

Hence, $y(0.4) = 0.68968$

Example.3: Use Adams Bashforth method to find $y(0.4)$ given that $\frac{dy}{dx} = \frac{1}{2}xy$ and $y(0) = 1,\ y(0.1) = 1.01, y(0.2) = 1.022, y(0.3) = 1.023.$

Solution: Given $\frac{dy}{dx} = \frac{1}{2}xy = f(x, y)$ 'say'.

Consecutive starting values for the Adams Bashforth method for $h = 0.1$ are :

x	y	$f(x,y) = \frac{1}{2}xy$

$x=0$	$y_{-3}=1$	$f_{-3}=0$
$x=0.1$	$y_{-2}=1.01$	$f_{-2}=-0.0505$
$x=0.2$	$y_{-1}=1.022$	$f_{-1}=-0.1022$
$x=0.3$	$y_0=1.023$	$f_0=-0.15345$

Using Adams Bashforth predictor formula

$$y_1 = y_0 + \frac{h}{24}\left(55 f_0 - 59 f_{-1} + 37 f_{-2} - 9 f_{-3}\right)$$

$$= 1.023 + \frac{0.1}{24}\left[55(-0.15345) - 59(-0.1022) + 37(-0.0505) - 9(0)\right]$$

$$= 1.0408$$

$$\therefore \quad f_1 = \frac{1}{2}x_1 y_1 = \frac{1}{2}(0.4)(1.0408) = 0.20816$$

Now, using Adams Bashforth corrector formula

$$y_1 = y_0 + \frac{h}{24}\left(9 f_1 + 19 f_0 - 5 f^{-1} + f_{-2}\right)$$

$$= 1.023 + \frac{0.1}{24}\left(9(0.20816 + 19(-0.15345) - 5(-0.1022) + (-0.0505)\right)$$

$$= 1.041$$

Hence, $y(0.4) = 1.041$

12.10 Summary

In this unit, we have covered the following we shall work with three different methods of solving differential equations by numerical approximation. In choosing among them there is a trade-off between simplicity and efficiency. Picard's and Euler's method is relatively simple to understand and to program, for example, but almost hopelessly inefficient. The third and final method we shall use, the order fourth method of Runge-Kutta, is very efficient, but rather difficult to understand and even to program. The improved Euler's method lies somewhere in between these two on both grounds. In each method one step goes from x_n to $x_{n+1} = x_{n+h.}$ The methods differ in how the step from y_n to y_{n+1} is performed. More efficient single steps come about at the cost of higher complexity for one step. But for a given desired accuracy, the overall savings in time is good.

12.11 Exercise

1. Use Picard's Iteration method for second order differential equation $\frac{d^2x}{dx^2} = f(x, y)\ wehere\ y = y_{0,} \frac{dy}{dx} = z_0, when\ x = x_0$.

2. Apply Picard's method to solve $\frac{dy}{dx} = 3e^x + 2y$ upto third approximation, given that $y = 0, when\ x = 0$.

3. Apply Picard's method to solve $\frac{dy}{dx} = 2 - \frac{y}{x}$ up to third approximation, given that $y = 2, when\ x = 1$.

4. Use Milne's method to solve $\frac{dy}{dx} = x + y$ with the initial conditions $x_0 = 0, y_0 = 1\ from\ x = 0.20\ to\ x = 0.30$.

5. Use Euler's method with $h = 0.1$ to find the solution of the differential equation $\frac{dy}{dx} = x + y$ with the initial condition $x_0 = 0, y_0 = 1$.

6. Obtain by Euler's modified method to five consecutive starting values for the numerical solution of $\frac{dy}{dx} = \log_{10}\left(\frac{x}{y}\right) with\ x_0 = 20\ and\ y_0 = 5$

7. Use Euler's method with $h = 0.1$ to find the solution of the differential equation $\frac{dy}{dx} = x^2 + y^2$ with the initial condition $x_0 = 0, y_0 = 1$.

8. Use Euler's modified method to determine $y(0.02)$ in two steps from $y' = x^2 + y, y(0) = 1$.

9. Obtain $y\ for\ x = 0.1$ from differential equation $\frac{dy}{dx} = x^2 + y$, given $y = -1\ when\ x = 0$ by Using the fourth order Runge-Kutta method.

10. Using the fourth order Runge-Kutta method; solve $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}, y^{(1)} = 1.0$, to find $y\ at\ x = 1.1$.

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SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 5 Computer

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SBSSTAT – 04 **Numerical Methods & Basic Computer Knowledge**
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Block & Units Introduction

The ***Block - 5 – Computer,*** is the fifth block. This block deals with theory of computer and consists of three units.

Unit – 13 – Introduction to Computer; presents a brief introduction to computers including their historical bolution, generation and classification.

Unit – 14 – Hardware; gives a brief account of hardware in CPU, I/O Devices, Blick diagram and memory organization.

Unit – 15 – System Software; deals with system software, MS-Dos, Files names, Creating, Editing and printing of files, other file management commands etc.

At the end of block/unit the summary, self assessment questions and further readings are given.

Unit-13: Introduction to Computers

Structure

13.1 Introduction

We all are familiar with calculations in our day to day life. We apply mathematical operations like addition, subtraction, multiplication, etc. and many other formulae for calculations. Simpler calculations take less time but complex calculations take much longer time. Moreover, sometimes we are not able to do all the calculations manually. Another factor that arises in these tedious calculations is the accuracy of the results. As the necessity is the mother of invention, man in ancient time intended to develop a machine which can perform this type of arithmetic calculations faster and with full accuracy. This gave birth to a device or machine called 'computer'.

Although the original objective for inventing the computer was to develop a fast calculating machine the computer we see today is quite different from the one made in the beginning. The number of applications of computer has increased; the speed and accuracy of calculation have also increased. Most of the work done by computer today is non- numerical nature. For example, reservation of tickets in airlines and railways, payment of telephone and electricity bills business data processing medical diagnosis, weather forecasting etc. are various areas of applications of computers. More accurately a computer may be defined as a device that operates upon the information or data. The data thus comes in various shapes and sizes depending upon the type of application. So due to this capability of data processing people have started calling it a 'Data processor'.

13.2 Objectives

After the study of this unit you will be in a position to

- Define a computer and identify its characteristics
- Know the origin and evolution of computer
- Identify capability of computer in terms of speed and accuracy
- Appreciate the evolution of computer through five generations
- Know the classification of computers based on their performance.

13.3 What is a Computer?

Computer is an electronic device. As mentioned in the introduction it can do arithmetic calculation faster and with good accuracy. But it does much more than that. It can be compared to magic box, which serves different purpose to different people. In general, *computer is a machine capable of solving problems and manipulating data. It accepts data, processes the data by doing some mathematical and logical operations and gives us the desired output.*

Therefore we may define computer as a device that transforms data. Data can be anything like marks obtained by a student in various subjects, income, saving, investments, etc., of a state. Thus a computer

i) Accept data,
ii) Store data,
iii) Process data as desired,
iv) Retrieve the stored data as and when required, and
v) Print the result in a desired format.

13.4 Characteristics of Computer

The major characteristics of computer are as follows:

Speed: In comparison to man, computer works very fast. It takes only a few seconds for calculations that we take hours to complete. It can perform millions of instructions and even more per second. For example, the weather forecasting that we see every day on TV is the result of compilation and analysis of huge amount of data on temperature, humidity, pressure, etc. of various places through computers.

Accuracy: If we try to do calculations very fast the chances of error increase. Another important aspect is to maintain the accuracy of the results. The degree of accuracy of computer is very high and every calculation is performed with the same accuracy. For example if we want to have an accuracy of 12 decimal places in our result, it will be difficult manually but for a computer it will be a simple task. The accuracy level is determined on the basis of design of computer. The errors in computer are due to human and inaccurate data.

Diligence: It is really interesting that computer is free from tiredness, lack of concentration, fatigue, etc. It can work for hours or days without creating any error. If billions of calculations are to be performed a computer will perform every calculation with the same accuracy.

Versatility: It means the capacity to perform completely different types of work. As we already discussed in introduction we may use our computer for various kinds of task.

Storage: The computer has an in-built memory where it can store a large amount of data. We can also store data in secondary storage devices such as floppies which can be kept outside the computer and can be carried to other computer. Computer has the power of storing any amount of information or data. Any information can be stored and recalled as long as we require it, for any number of days or years.

No IQ: Computer is just a machine and it works according to given instructions. It cannot do any work without instruction from the user. it performs the instructions at tremendous speed and with good accuracy. Thus, the user has to decide what he wants to do and in what good accuracy. In other words, computer cannot take its own decision as we can.

13.5 Historical Evaluation of Computer

History of computer could be traced back to the effort of man in search of a device to do fast calculations with good accuracy. We will briefly discuss some of the path breaking inventions in the field of computing devices.

Calculating Machines: It took over generations for early man to build mechanical devices for counting large numbers. The first calculating device called ABACUS was developed by the Egyptian and Chinese people. The word ABACUS means calculating board. It has a number of horizontal bars each having ten beads. Horizontal bars represent units, tens, hundreds, etc.

Nepier's Bones: This is a mechanical device built by English mathematical John Napier for the purpose of multiplication in 1617 AD.

Slide Rule: An English mathematician Edmund Gunter developed the slide rule.l This machine could perform operations like addition, subtraction, multiplication, and division. It was widely used in Europe in 16^{th} century.

Pascal's Adding and Subtracting Machine: Basic Pascal developed a machine that could add and subtract. The machine consisted of wheels gears and cylinders.

Leibniz's Multiplication and Division Machine: The German mathematician Gottfried Leibniz built a mechanical device around 1673 that could do multiplication and division.

Babbage's Analytical Engine: Charles Babbage is called the father of the computer. In 1823 he built a mechanical machine to do complex mathematical calculations. It was called difference engine. After that he developed a general purpose calculating machine called analytical engine.

Mechanical and Electrical Calculator: In the beginning of 19^{th} century, the mechanical calculator was developed to perform all sorts of mathematical calculation and it was widely used up to 1960s. Later the rotating part of it was replaced by electric motor. After that it was called the electrical calculator.

Modern Electronic Calculator: The electronic calculator used in 1960s was run with electron tubes which was quite bulky. Later it was replaced with transistors and as a result the size of calculators became too small.

The modern electronic calculator can compute all kinds' of mathematical computations and mathematical functions. It can also be used to store data permanently. Some calculators also have in built programs to perform some complicated calculations.

13.6 Computer Generations

We discussed in previous section that the evolution of computer started from 16^{th} century and resulted in the form that we see today. The present day computer however has also undergone rapid change during the last five decades. This period during which the evolution of computer took place can be divided into five different phase known as Generations of Computers. Each generations of computer is characterized by major technological development that fundamentally changed the way computers operate, resulting in increasingly smaller, cheaper and more powerful and efficient and reliable devices. Here we give a review of different generation and the developments that led to the current devices.

First Generation Computers [1940-1956]

The first computers used vacuum tubes for circuitry and magnetic drums for memory. These computers were large in size and writing programs on them was difficult. They were very expensive to operate and in addition to using a great deal of electricity, generated a lot of heat, which was often the cause of malfunctions. First generation computers relied on machine language to perform operations, and they could only solve one problem at a time. Input was based on punched cards and paper tape, and output was display on printouts. Some of the computers of this generation were:

ENIAC (Electronic Numerical Integrator and Calculator): It was the first electronic computer built in 1946 at the University of Pennsylvania, USA, by John Eckert and John Mauchly. The ENIC was 30 x 50 feet long, weighed 30 tons, contained 18,000 vacuum tubes, 70,000 registers, 10,000 capacitors and required 150,000 watts of electricity. The ENIC's field of application included weather prediction, atomic-energy calculations, cosmic-ray studies, thermal ignition, random-number studied, wind-tunnel design, and other scientific uses.

EDV AC: (Electronic Discrete Variable Automatic Computer): It was developed in 1950. The concept of storing data and instructions inside the computer was introduced here. This allowed much faster operations due to the rapid access to both data and instructions.

EDSAC (Electronic Delay Storage Automatic Computer): It was developed by M.V. Wilkes at Cambridge University in 1949.

UNIVAC-I: Eckert and Mauchly produced it in 1951 by Universal Accounting Computer setup.

Following were the major drawbacks of First generation computers.

1. The operating speed was quite low.
2. Power consumption was very high.
3. It required large space for installation and Air conditioning room.
4. The programming capability was quite low and
5. It was non portable.

Second Generation Computers [1956-1963]

Around 1955, a device called Transistor replaces the bulky vacuum tubes in the first generation computers. The transistor was far superior to the vacuum tube, allowing computers to become smaller, faster, cheaper, more energy-efficient and more reliable than the first generation computers. Thus the size of the computer reduced considerably.

Second Generation computers moved from binary machine language to symbolic or assembly languages, which allowed programmers to specify instructions in words. High level programming languages were also being developed at this time, such as early versions of COBOL and FORTRAN. These were also the first computers that stored their instructions in their memory, which change from a magnetic drum to a magnetic core technology. It is in the second generation that the concept of Central Processing Unit (CPU), memory, programming language and input and output units were developed. Some of the computers of the Second Generation were

1. IBM 1620: Its size was smaller as compared to First Generation computers and mostly used for scientific purposes.

2. IBM 1401: Its size was small or medium and used for business applications.
3. CDC 3600: It was used for scientific purposes.

The major advantages of the computer of this generation were:

1. Smaller in size than the first generation computers
2. Less heat generated.
3. Faster than the first generation computers
4. Less prone to hardware failures, etc.

Drawbacks:

1. Air conditioned rooms were required.
2. Frequent maintenance required
3. Commercial production was difficult and costly.

Third Generation [1964-1971]

The development of the integrated circuits (ICs) was the hallmark of the third generation of computers. These ICs are popularly known as Chips. A single IC has many transistors, registers and capacitors built on a thin slice of silicon. Thus the size of the computer reduced considerably. Instead of punched cards and printouts, users interacted with third generation computers through keyboards and monitors and interfaced with an operating system, which allowed. The device to run many different applications at one time a central program that monitored the memory.

Some of the computers developed during this period were IBM-360, ICL-1900, IBM-370 and V AX-750. Higher level language such as BASIC (Basic All purpose Symbolic Instruction Code) was developed during this period.

Computers of this generation were small in size, low in cost, large enough with memory space and processing speed was also very high. The major advantages of the computers of this generation were:

1. Smaller in size than the computers of previous generations.
2. Computational timings were further reduced in comparison generation computers.

3. Easily portable
4. Commercial production was also easier and cheaper

There were no major drawbacks of the computers of these generations except that sophisticated technology was required for the manufacturing of IC chips.

Fourth Generation [1971-Present]

The microprocessor brought the fourth generation of computers, as thousands of integrated circuits were built onto a single silicon chip. Due to the development of microprocessor it is possible to place computer's *central processing unit (CPU)* on a single chip of silicon. These computers are called microcomputers. Thus the computer which was occupying a very large room in earlier days can now be placed on a table. The personal computer (PC) that you see today is a Fourth Generation Computer.

In 1981, IBM introduced its first computer for the home users, and in 1984, Apple introduced the Macintosh. Microprocessors also moved out of the realm of desktop computers and into many areas of life as more and more everyday products began to use microprocessors.

As these small computers became more powerful they could be linked together to form networks, which eventually led to the development of the Internet.

The major advantages of the computers of this generation were:

1. Very reliable
2. Heat generation is negligible
3. Faster than the computers of the previous generations.
4. Easily portable
5. Cheapest among all generations.

Fifth Generation [Present and Beyond]

Fifth generation computing devices, based on artificial intelligence, which means to allow the computers to take their own decisions are still under the development phase, though there are some applications, such as voice recognition, that are being used today. The speed is expected to be extremely high in fifth generation computers. Apart this it can perform *parallel* *processing.* The use of parallel processing and superconductors is helping to make artificial intelligence a reality. Quantum computation and molecular and nanotechnology will radically change the face of computers in: years to come. The goal of fifth generation computing is to develop gives that respond to natural language input and are capable of learning and self organization.

13.7 Classification of Computers

The varieties of computers that we see today can be divided into different categories depending upon the size, efficiency, memory and number of users. Broadly they can be divided it to the following categories.

1. **Microcomputer:** This computer is at the lowest end of the computer range in terms of speed and storage capacity. Its CPU is a microprocessor.
   The most common application of personal computers (PC) is in this category. The first microcomputers were built of 8-bit microprocessor chips. An improvement of 8-bit chip is 16-bit and 32-bit chips. Examples of microcomputer are IBM PC,-AT, etc.
2. **Mini Computer:** This computer is superior to the microcomputer. It is used in multi-user system in which several users can work at the same time. It has large storage capacity and operates at a higher speed. This type of computer is generally used for processing large amount of data in various organizations including science and technology laboratories. They are also used as server in Local Area Networks (LAN).
3. **Mainframe:** The computers of this type operate at tremendous speed, have very large storage capacity and can handle the work load of many users at a time. These are generally 32-bit microprocessors and generally used in centralized databases. They are also used as controlling nodes in Wide Area Networks (WAN). Example of mainframes are DEC, ICL and IBM 3000 series.
4. **Supercomputer:** These are the fastest and the most expensive machines used to fulfill the need of any organization which has a very heavy load of data processing. They have high processing speed compared to other computers. They have also multiprocessing technique. The areas of applications of these computers are whether forecasting,

biomedical research, remote sensing, aircraft design. Examples of supercomputers are CRA Y YMP, CRA Y2, NEC SX-3, CRA Y XMP and PARAM from India.

13.8 Summary

In this lesson we have discussed about the major characteristics of computer. The speed, accuracy, memory and versatility are some of the features associated with a computer. But the computer that we see today has not developed over night. It has taken centuries of human effort to see the computer in its present form. There are five generations of computers. Over these generations the physical size of computer has decreased, but on the other hand the processing speed of computer has improved tremendously. We also discussed about the varieties of computers available today.

13.9 Exercises

1. What is a computer? Why is it known as data processor? Write the important characteristics of a computer.
2. Distinguish between Microcomputer and Mainframe computer.
3. Explain various types of computers.
4. Explain in brief the various generation in computer technology.
5. Write a short note on Fifth Generations of computer. What makes it different from Fourth Generation computer?
6. What is the first mathematical device built and when was it built ?
7. Who is called the father of Computer Technology?
8. In how many generations the evolution of computer is divided?
9. Why did the size of computer get reduce in third generation computer?
10. Write short note on the following

(a) Versatility (b) Storage (c) Slide Rule (d) Babbage's Analytical Engine

13.10 Further Readings

1. Rajaraman, V.; Fundamentals of Computers 5th ed., Prentic-Hall of India, New Delhi, (2007).

2. Hennessy, J.L. and Patterson, D. A.; Computer Oranization and Design: The Hardware/Software Interfaces, Morgan Kauffman Publishers, San Mateo, CA, (1994) .
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Unit-14: Hardware

Structure

14.1 Introduction

COMPUTER: A computer is an electronic data processing device which can read and write compute and compare, store and process large volume of data with high speed, accuracy and reliability. It stores the instructions given to it and then executes them at a terrific speed automatically without manual inte4rvention. It works on stored program concept. Once the data and the instruction set are fed into its memory, it reads the instructions and executes them to produce results. A computer thus consists of a machine called 'Hardware' which works with the help of a set of instructions used software.

HARDWARE: Hardware refers to the physical components of a data processing system. Input, storage, processing and control devices are hardware. Hardware professionals deal with the manufacturing and maintenance of computers.

SOFTWARE: Software refers to a set of computer programs and procedures for the effective operation of a data processing system. Without software, hardware is of no use and cannot produce any result.

The main components of a computer are:

- Central Processing Unit

- Input Devices
- Output Devices

14.2 Objectives

After going through this unit you will be able to understand:

- Main components of computer
- Functions of different components of computers

14.3 Central Processing Unit (CPU)

It is the brain of the computer. Its basic functions are to perform calculation and various logical functions. This unit will process the data, which is delivered by the input unit. The CPU has three components:

(i) The Control Unit
(ii) The Arithmetic and Logic Unit
(iii) The Memory Unit

(i) Control Unit: It consists of electronic circuits. It controls the overall operation of the computer system. It is considered as the heart of the computer system. It controls all the other units, directs them to operate in a proper way and coordinates various operations perform structure the input device to transfer the data and instructions to the main memory and then to the Arithmetic and Logical Unit (ALU). Then, it sends the processed results from ALU to the memory unit for storage and transfers it to the visual display. The control unit co-ordinates the various parts of the computer system – the Arithmetic and Logic Unit, the memory unit and the peripheral units. Besides, it controls the flow of data into, from and within the main storage as per the program instructions to perform its control operations effectively and quickly. The control unit has four basic components. These are:

(a) Instruction Register
(b) Decoder
(c) Address Register
(d) Instruction Counter

The instruction register receives one by one the instructions to be executed in the required sequence. Then the operation code of the instructions is transferred to decoder Arithmetic and Logic Unit to perform the operation. The address register enables the data in the location specified in the instruction to be transferred to a specified accumulator for the arithmetic and logic unit.

(ii) Arithmetic and Logic Unit

It consists of electronic circuits. It works at tremendous speed and executes millions of instructions per second (MIPS). This unit performs two kinds of operations, the arithmetic processing and logical processing. In arithmetic processing it performs all mathematical operations such as addition, subtraction, multiplication and division. In logical processing, it performs the relation and logical operation operations such as comparing larger or smaller values, true or false statements, etc.

The Arithmetic and Logic Unit of the CPU includes several types of sub-units and special purpose circuitry such as registers, counters, adders etc.

(a) Register: Registers are areas of high speed storage circuitry used as "work area" for the temporary storage of instructions and data during the operation of the control and Arithmetic and Logic Unit. The number, function and capacity of the registers and other submits in a CPU depend on the internal architecture of each particular computer. Besides, general purpose registers, there may be other registers named according to their functions.

(i) Storage Register: It temporarily holds data or instructions taken from or being sent to primary storage.

(ii) Address Register: It may hold the address of the storage location of data, or the address of an input/output device or a control function.

(iii) Instruction Register: It contains the instructions being executed by the

(iv) Accumulator: It is a register which accumulates the result of arithmetic or logic operations.

(v) The multiplier-quotient register: It holds either a multiplier or a quotient.

(vi) Floating point register: It is used for floating point arithmetic operations.

(b) Counter: The counter is closely related to the register. It is a device whose contents can be increased or decreased by a specific amount. The instruction counter is also called the instruction address register and contains the storage location address of the computer instruction being executed. The index register is a counter specifically set aside for modifying the portion of an instruction that indicates the address of data to be manipulated. This result in an operation known as "indexing" in whom the CPU automatically repeats the performance of the same instructions until of the data covered by instruction is processed.

(c) Adders: Adders are sub-units that perform the arithmetic operations of the arithmetic and logic unit receive data from two or more sources, perform the specific arithmetic operation desired and then convey the result to a receiving register such as the accumulator.

(iii) Memory Unit

It is also called as main memory or primary memory. It consists of very fast memories like magnetic memory or semiconductor memory. In this unit, the data and instructions are stored in the form of words, bytes and bits and are transferred to ALU during processing. Similarly, the processed results are stored again for further calculations or sent the output unit. The main storage consists of several static and dynamic memory cell. The computer need not remember all the data all the time. Some of the data or instructions can be kept elsewhere for retrieval at a later stage. Only the data is being processed and the instructions to process it are stored internally whereas the data and instruction that are required later are stored externally.

The internal storage is commonly called as the primary storage or the main memory. This is usually limited. The external storage or the secondary storage or auxiliary storage is the unlimited storage.

The memory stores binary instructions and data for the micro "Processor" and provides them to the microprocessor on request. Sometimes results are also stored on the memory.

14.4 Memory Organization

Primary Storage/ Internal Storage

The main memory, also called random access memory (RAM) is the work area of the computer. It stores program instructions or part of data for immediate needs.

(a) Magnetic Core Memory

In the past magnetic core memory was used as internal memory. It was a non volatile memory i.e., its contents were not lost if the power supply was interrupted. However, the necessity to store after reading was technological disadvantage of core storage.

(b) Semiconductor Memory

These days, internal memory consists of extremely small bit storage circuits (flip-flops) etched on a silicon chip. All the electronic elements to store a bit are placed in such a small area of the chip that a single chip can store millions of bits. The individual chips are arranged in groups to form a memory module.

Types of Semi-conductor Memory

(i) **Random Access Memory (RAM):** Any information can be read from and written into a RAM. It is a read/write memory. It is a volatile memory i.e. its contents are lost if the power supply is interrupted or turned off. The main memory of the computer is RAM.

(ii) **Read Only Memory (ROM):** Rom is permanently programmed with information during manufacture, by implementing the appropriate pattern of two state values. It cannot be changed subsequently by a normal write operation. It is thus completely non-volatile. It is mainly used to hold those programs which are required permanently.

(iii) **Programmable Read only Memory (PROM):** This can be programmed to record information using a special electronic equipments known as a PROM programmer. However, it cannot be changed subsequently.

(iv) **Erasable Programmable Read Only Memory (EPROM):** EPROM is a PROM which can be reversed by exposing it to an ultraviolet light source. The device can be re-erased and re-programmed again and again.

(v) **Cache Memory:** It is a small capacity high speed memory used to make processing faster. The main memory can process information very fast, but it takes much longer to transfer data to and from the input/output devices. The cache memory compensates for this mismatch in operating speeds. It holds those parts of data and the active program which are most frequently used. Thus the performance rate of the CPU improves. However, cache memory is very expensive as compared to the main memory, so its size is normally much smaller than the size of the main memory.

13.5 Input-Output Devices

Data is entered into the computer system by means of an input device. The keyboard is one of the most common input devices. The data is recorded on a material called the media. Similarly, the computer system needs an input device to communicate the processed information to the user. Common types of output devices are monitor, printer, etc. But there are certain devices which can serve as input as well as output devices like tape drive, floppy drive, disk drive etc.

Keyboard

Computer keyboard is an electromechanical component designed to create special standardized electronic codes when a key is pressed. The codes are transmitted along the cable that connects the keyboard to the computer system unit or terminal, where the incoming code is analyzed and converted into the appropriate computer usable code.

Keyboards come in a variety of sizes and shapes having a number of features in common:

1. Standard Type Writer keys
2. Function keys
3. Special Purpose keys
4. Cursor Movement keys
5. Numeric keys

Mouse

The mouse is an input device that is much in use nowadays in graphics as well as working with a GUI (Graphic User Interface). About the size of an audio cassette, it slides on rubber ball and has two or more buttons on the top. When a mouse is slid across a flat surface, the screen cursor also moves in the direction of the movement of the mouse. With a click of the button, the system can be notified of the selected position.

MICR

Magnetic ink character recognition devices were developed to assist the banking industry. It is used in the processing of cheques. The most commonly used character set by MICR devices is known as E13B font that consist of numerals 0-9 and 4 special characters. These help are in sorting of cheques/drafts based on the code printed on the cheque using the E13B font. Processing is speeded up using MICR but the main limitation is that only 10 digits and 4 characters are used.

Scanners

These are basically input devices that are capable of recognizing marks or characters. They are used for direct entry of data into the computer. Scanners eliminate the duplication of human effort required to get data into the computer. Reduction in human intervention improves data accuracy. Since scanners are direct data entry devices so they demand high quality documents.

TYPES:

1. **OCR**

   These are scanner devices that are capable of detecting alphabetic and numeric character by comparing the shapes with internally stored patterns. These are expensive and are used only for large volume processing applications e.g. by credit card companies.
2. **OMR**

   These scanners are capable of recognizing a pre-specified type of mark made by a pencil. These are normally used for validation of input documents, evaluating answer-sheets in objective-type tests e.g. GRE, GMAT.

3. **Bar Code Reader**

   Data coded in the form of light and dark lines or bars are known as bar codes. Bar codes are used particularly by the retail trade for labeling goods. Bar code reader is a device used for reading bar coded data. Bar code reading is performed by a laser-bean scanner which is linked to a computer.

4. **Desk Scanning**

   During the scanning process, the scanner applies a light source to your image. The light is reflected back from the image into the scanner optics where the varying levels of lights are interpreted. Your image is then reconstructed digitally and displayed on your screen. Unlike a camera image, scanning software lets you alter the information from the image stored in the computer and used in any application.

Magnetic Tape

Tape is a very popular sequential access storage device, used widely all over the world. Data is stored as tiny spots on the magnetizable material that coats one side of a plastic tape. The coated side of the tape is divided into vertical columns (frames) and horizontal rows (tracks).

An 8-bit code is used with a 9-track tape. The ninth track is used for recording the parity bit. A parity bit or check bit is used to detect errors that may occur due to loss of a bit from a string of 8-bits during data input or output operations. If the basic code for a character requires an odd number of one bit, an additional one bit is added to the check bit location so that there will always be an even number of one bit. This is an example of even parity. Similarly in odd parity the check bit is used to always produce an odd number of one bit. That is check bit will be 1 if the total number of one bits for representing a particular character is even and it will be 0 otherwise.

Since a magnetic tape is a continuous-length medium, records stored on the tape may be separated by blank spaces called Inter-Record Gaps (IRGs). IRGs are normally 0.5 inches in length. Tape records can be of varying lengths. If a tape contains a large number of very short records and if each record is separate by an IRG, then a lot of tape is wasted. To avoid this inefficient situation, several records can be combined into a tape block. Each block contains a fixed number of records & blocking factor control number of record in data blocks. An IBG (inter block gap) separate two blocks. The inter block gap allows the tape drive to detect the end of the block & also to accelerate to the speed required for reading & to deaccelarate after reading if a number of files are stored on one tape then the tape is called a multifile reel and if file is stored on more than one tape reel the file is called multireel file.

The tape density is the number of frames per inch of tape. It is measured in BPI (bytes per inch). Common densities are 1600 BPI, 6250 BPI.

Every file on a tape starts with a header label and ends with a trailer label.

Header label: A header label is a block of data containing the identifying information about the files such as file name, date of creation, retention period etc.

Trailor label: A trailer label is a block of data written as the last record of a file. It contains the same information as the header label but in addition in contains a block count specifying the number of data blocks in the file.

Cartridge Tape

It is a plastic ribbon coated on one side an iron-oxide material that can be magnetized. It is 1/4" wide and varies from 140 to 450 feet in length. It is encased in a 3 W'/5 1;4" pocket size dust protected jacket. Used in minicomputers and personal computers. Used to take backup of hard disks. It takes approximately 5-20 minutes to copy the contents of a hard disk. Tape cartridges have a capacity ranging from 60 MB to 32 GB.

Floppy Diskette

The floppy diskette is a direct access storage device although its capacity is much less than a hard disk. The diskette is made out of a flexible plastic material. This base is coated with an iron-oxide recording material. Data is recorded as tiny magnetic spots. The surface is divided into tracks and sectors, same as in a hard disk. The number of tracks on a diskette depends on the recording density. The size of each sector is fixed (512 bytes). Data is stored on both sides of the diskette. There is one head per surface in the floppy drive for reading/writing data on the diskette. The circular plastic disk is enclosed within a smoothly lined protective square jacket to protect it from dust and scratches.

The standard sizes available nowadays are: 5.25 inch, 3.5 inch. There capacities are as follows:

1. 5.25 inch
   (a) DSDD 48TPI 360KB
   (b) DSHD 96TPI 1.2MB
2. 3.5 inch
   (a) 120 TPI 1,44MB
   (b) 240 TPI 2.88MB

Salient Points

1. The index hole marks the beginning of the first sector.
2. The outermost track is labeled 0
3. The write protect notch can be covered to disable writing.

Advantages

1. Low cost.
2. Convenient to transport.
3. Compatible between computers.

Disadvantages

1. Floppy diskettes are prone to frequent errors due to mishandling.

Magnetic Disk

Magnetic disks are the most popular INPUT/OUTPUT device for Direct Access Storage. These are metal plates coated on both sides with a thin film of magnetic material. A set of such magnetic material plates are fixed to a spindle one below the other to make up a disk pack. The disk pack is mounted on a disk drive (which consists of a motor to rotate the disk pack about its axis). The disk drive has a set of magnetic heads mounted on arms which move radically in and out. There is a very small gap between the head and the plate surface because if the head comes in contact with the surface it can destroy the data as well, it would lead to wearing out of the head. Data is stored on both surfaces in a number of invisible concentric circle which are called tracks, each track is further divided into sector which can store a fixed number of bytes.

The capacity of diskette in wide use today ranges 360KB to 1.44 MB. Microcomputer's hard disk capacity ranges from 10MB to 1GB.

Hard disks have the following characteristics.

1. They are rigid metal platters connected to a central spindle.
2. The entire disk unit (disks and read/write heads) is placed in a permanently sealed container.
3. Air that flow through the container is filtered to prevent contamination.
4. The disks are rotated at very high speed (usually around 3,600 RPM: floppy disks rotate at about 300 RPM). These disk drives can have four or more disk platters in a sealed unit. In most of these disk units (which are often called Winchester disk drives). The read/write heads never touch the surface of the disks. Instead, they are designed to float from 0.5 to 1.25 millionths of an inch from the disk surface.

Removable Disk Packs

In large computer systems, hard disks are sometimes contained in packs that are removable. Meaning that they can be removed from the computer and replace at will. Disk packs typically hold 6 to 12 platters that are usually 14 inch in diameter. In disk packs, all tracks with the same track number are lined up, one above the other. All tracks with heads so that both sides of the disk can be read. The read/write heads move together and so are always on the same cylinder at the same time. Data that needs more space than one track is continued on to the same track on other disk, so the read/write heads do not need to move. (Only one read/write head's active at one time, but they are very fast). When all the tracks in cylinder full the read/write heads move to another cylinder, the cylinder numbers are used by the computer operating system to determine data addresses.

The capacity of removable disk packs varies by manufacturer and ranges from 150 to 250 MB. The total storage capacity could be dramatically increased by having a dozen or so extra packs to be interchanged with the packs in the disk drive. These are obsolete now days.

Fixed Disks

The capacity of fixed units has been increased upto 16GB. The typical capacity of fixed disk drives for desktop is 4 GB where as typical server will have more than one disk drive of 8 GB each. The average access time of such drives is of the order of few milliseconds, and data transfer rate of typically few million bits per second.

Access Time

This is defined as the time taken to locate and then transfer data from the disk into internal storage. It includes there elements namely.

1. **Seek Time:** This is the amount of time needed to move a read write head to the desired track from its current position.
2. **Latency Time:** Also known as Rotational Delay time. This is the time require for the portion of the track to be read or written to come beneath the read-write head.
3. **Transfer Time:** The time taken to transfer data from disk to internal storage.

Optical Disk

Optical technology can overcome some of the limitations faced by users while using magnetic storage technology.

In optical disks, a laser beam is used to read/write data on to the disk. The data is stored in the form of microscopic dots. There are about 54000 spiral tracks on a disk. The track density is about 16000 TPI. Optical disks can store video, text, music and graphics.

An optical disk is protected from rough handling by a plastic layer. Data is written on the disk by blasting microscopic pits on the surface using a laser beam. These pits are darker than the background of the shiny disk. The data is read by passing a lesser powerful beam over the surface and the changes in reflectivity indicate a 1 or 0 bit.

Advantages

1. Not as fragile as floppy disks.
2. Longer lasting.

3. More storage.
4. No heads required for reading/writing.

The main limitation of optical storage is that data once written can't be erased.

NOTE: New technologies are being develop for erasable, rewritable optical media. **CD-ROM**

Compact disk, read-only memory optical disks have a very large storage density and the access time is relatively low, made mostly of aluminum and plastic. Only 1.2 mm thick, a CDROM is tough. Not only is a CD-ROM more resistant to damage, there is no chance that you will ever accidentally overwrite any information or infect it with a virus. A standard 12 cm diameter CD-ROM supports up to 680 MB data capacity. By value of its immense storage capacity CDROM currently provides an affordable medium through which reasonably sophisticated multimedia can be distributed. CD-ROM addressing is carried out using measurement of time and data blocks read. Minutes, Seconds and blocks provide information to locate an item of information.

A track beginning mid way through the CD-ROM for instance is addressed 29:29:37 (minutes: seconds: blocks)

A CD-ROM player reads information optically using laser light.

WORM

Write-once, read many system, are based writable optical storage devices. A laser recording device writes 1 bit by deforming a thin sensitive layer of material on the disk surface. Unmodified areas represent 0 bits. The deformed WORM disk can't be restored to its original condition, so writing is indelible.

Reading of stored data occurs when a lower power laser beam passes over the disk and detects differences in the reflections coming from 0 and I-bits.

Digital Versatile Disc

Digital versatile disk is in genre of optical disk with same overall dimension of CD but much higher capacity. These can store at least 7 times more data than D ROM. These drives support MPEG-2 standard for the compression of data. High compression of DV D films necessitates either a fast Pentium II processer or an MPEG Card for decoding. Dual layer DVD disks have 8.5GB capacity on single side, using both sides capacity comes up to17 GB. Large storage capacities on the DVD broaden its multimedia possibilities.

These discs have backward compatibility and can read CD-ROM, CD-RW and CD-audio storage media. One of the important factors is that you don’t need any special interface for a DVD drive.

Visual Display Terminal

This is the most popular I-a device used nowadays for interactive processing. A keyboard is used to enter data into a processor and a video display unit called monitor- is used to display the key data and to receive processed information and message from the computer. VDTs are classified as.

Dumb terminals: These are simple devices that immediately transmit each keyed data character to the processor.

Intelligent Tenninals: These combine VDT hardware with built in microprocessor, They can process small jobs without the need to interact with the main computer.

Cathode Ray Tube

The cathode-ray tube probably the most popular softcopy output device is used with terminals connected to large computer systems and as a monitor for microcomputer systems. This type of video display screen is used to allow the operator to view data entry and computer output. Monitors that display only letter, and special characters such as $, * and ? are called alphanumeric monitors (or alphanumeric terminals). They look like television screens and display 80 characters per line, with 24 lines visible at one time. Screens that can display both alphanumeric data and graphics are called graphic monitors (or graphic terminals).

The CRT’s screen display is made up of small picture elements, called pixels for short. The smaller pixels (the more points that can be illuminated on the screen) have better than image clarity, or resolution.

The terms resolution refers to the crispness of the image displayed on screen. Three factors used to measure resolution: lines of resolution (vertical & horizontal), raster scan rate & band width.

The term raster scan rate refers to how many times per second the image on the screen can be refreshed that is lit up again. Because the phosphors hit by the electron beam do not glow very long, the electron beam must continuously sweep across the screen will seem to flicker, which can be very hard on the eyes. The higher the raster scan rate, the better the image quality and, the less eyestrain.

Bandwidth, this terms refers to the rate at which data can be sent to the electron gun to control its movement, positioning and firing. The higher the bandwidth, the faster the electron gun can be directed to do its job.

Monochrome and Color Monitors: A monochrome monitor (a monitor capable of displaying only a single-color image) and an ROB color monitor (ROB stands for red, green, blue) differ in two principal ways, First they have different numbers of electron guns. A monochrome monitor has only one electron gun.

Printers

Printers are the primary output devices used to prepare permanent documents for human use. Printers are classified as

1. **Impact Printers**

   These operate like typewriter, pressing a typeface against paper and linked ribbon E.G. daisy wheel printer, dot-matrix printer.

(i) **Letter Quality Printer:** Letter quality printers also called character printers or serial printers because they print one character at a time, produce a very high quality print image (one that is very clear and precise) because the entire character is formed with a single impact.

(a) **DMP (Dot Matrix Printer):** These are serial printers, i.e., they print one a character at a time. Each character is printed as a pattern of dots. The print head comprises of a matrix of tiny needles typically a 9 row * 7 columns matrix, which hammers out characters in the form of patterns of tiny dots. These printers are faster than daisy wheel printers although their letter quality is a bit inferior. The additional advantage of DMPs is the they don't have a fixed character font set (as in line printers), so they can print different shapes, e.g., charts, graphs, diagrams etc.

(b) **Daisy Wheel Printer:** The Daisy wheel printers has a print "wheel" with a set of print character of the outside tips of the flat spokes. To print a specific character, the wheel is spun until the appropriate spoke, or petal, is lined up with the print hammer. The print hammer is then fired, and the print character is forced against the ribbon and paper with sufficient force to make a clear, crisp impression.

(ii) Line Printers

These are the high speed printers which cater to huge volumes of output requirements of large computer organizations. These are known as Line printers because they use impact methods to produce one line at a time printed output, e.g., chain printers, band printer, drum printer.

From 300 to 3000 lines per minute can be printed depending on the printer used.

These printers have several copies of each printable character on a drum, a bell, or a print chain, with a separate print hammer for each position across the width of the paper guide. As the drum, bell or print chain revolves; the hammers are activated as the appropriate character pass in front of them. The speed achieved by this type of printer ranges from 200 to 3,000 lines per minute (LPM).

Speed is the obvious advantage of this type of printer. The main disadvantages are noise and poor image quality.

2. Non-Impact Printers

These are thermal, electrostatic chemical and inkjet technologies.

(i) Thermal Printers: They use heat to produce an image on special paper. The print mechanism, rather like a dot-matrix print head, is designed to heat the surface of chemically treated paper so that a dot is produced based on the reaction of the chemically treated paper so that a dot is produced based on the reaction of the chemical to heat. No ribbon or ink is involved. For users who want the highest quality desktop color printing available, thermal printers are the answer. However, they are also expensive and they require special expensive paper.

(ii) **Ink Jet Printer:** The ink jet printer ejects a steady stream of ink drop towards the printed page. The drops are selectively discarded by electrostatic attraction to leave only those that are needed to form the desired symbol. Those that are not needed are captured in tiny gutter & filtered to remove impurities. They are then recalculated through the drop generating mechanism.

(iii) **Laser Printer Technology:** This is much less mechanical than impact printing (that is, no print heads move, no print hammers hit) resulting in much higher speeds and quieter operation. The process resembles the operation of a photocopy machine. A laser beam is directed across the surface of a light sensitive drum and fared as needed to record an image in the form of a pattern of tiny dots. The image is than transferred to the paper, a pager at a time, in the same fashion as a copy machine using a special toner.

The main advantages of laser printer are:

1. Very high speed.
2. Low noise level.
3. Low maintenance requirements.
4. Very high image quality.
5. Excellent graphics capabilities.
6. A variety of type sizes and styles.
7. On large high speed laser printers, form can be printed at the same time data is recorded in them.

PLOTTERS

These are line-drawing devices which move a pen under computer control in such a way that continuous lines and curves can be drawn. These are used for drawing maps, engineering drawing etc.

COM (Computer Output Microfilm)

COM Technology is used to record computer output information as microscopic filmed images. Thus, COM is basically an output device that records information on a roll of microfilm.

COM recording technology consists of a microfilm recorder that receives information. The recorder in turn projects the characters of output on to a CRT screen. A high speed camera, inbuilt into the system takes pictures of the displayed information. The COM recording process produces characters that are about 50 times smaller than those producted by conventional printers. A special device known as MICROFILM READER is used to view the information recorded. A COM system is ideal for use in applications where there is a large amount of information to be retained.

14.6 Exercises

Q.1. Fill in the blanks:

1. CPU consists of --------------, ----------------- and -----------------------------.
2. The three components of a computer are -----------, ----------- and---------.
3. --------- is the component of the CPU which performs arithmetic and logical operations on data.
4. Semiconductor memory is ----------------- than core memory is speed.
5. The main memory of a computer may be made of ----------- or ------------.

Q.2. State True/False:

1. The data processing capabilities of a computer can be increased by increasing the memory Size.
2. Memory consisting of magnetic core is non-volatile memory.
3. Core memory is cheaper than semi-conductor memory.
4. Whenever power is interrupted, data stored in semi conductor memory cannot be regained.

Q.3. Expand the following:

a) RAM b) PROM c) ROM d) ALU e) EPROM

Q.4. Answer the following questions:

1. What is a computer?

2. State the essential components of a compute and give function of each of them in brief.
3. Outside CPU, which unit supplements the main memory of a computer/
4. What is Auxiliary or Secondary Memory and why is it required?
5. What is different media and related storage devices?
6. What is the purpose of having several mass storage devices with a computer?
7. What is difference between intelligent and dumb terminals?
8. Differentiate among Line Printer, Dot Matrix printer, Serial printer, Laser Printer and Plotter?
9. List the advantage of Magnetic disk over Magnetic tape?
10. Explain all the type of Non-Impact types of printers.
11. Explain the following for recording of data on a tape reel:
    a) Header Record
    b) Trailer Record
    c) Inter Record Gap
    d) Data Block
    e) Blocking Factor

Q.5. Fill in the blanks:

1. A beam ---------------- light is used to remove and retrieve data on optical disks.
2. The number of records in a data block on tape is controlled by the ……………..
3. ……………….is one form of optical storage.
4. The surface of magnetic disk is divided into a number of …………….. Each ………… Is further divided into a number of………………………
5. Data can be stored on ………………….. sides of a magnetic disk.
6. …………….is a device that prints a whole line of characters at one time one time under computer control.
7. Average time needed to retrieve a data item from the storage unit is called is …………………….
8. The period of time elapsed between input of a query and receipt of a response at the terminal is called …………………….of CRT.

9. ……………refers to the crispness of the image displayed on screen.
10. ……………refers to how many times per second the image on the screen can be refreshed.

Q.6. State True/False:

1. Magnetic tape is a random access device.
2. The VDU is both an input and output device.
3. The floppy disk can be used only as an input media.
4. The Line Printer is both an input and output device.
5. The magnetic disk is both an input and output media.
6. A dot Matrix Printer is an input device.
7. A PC can have one or more hard disks.
8. Magnetic Disk can be used as random and sequential access storage media.
9. Mouse is an input and output device.
10. Ink Jet Printer is an impact type of printer.
11. A deformed WORM can be restored to its original condition.
12. Magnetic tape is a secondary storage media.

14.7 Summary

This unit covers description of various components of a computer. The various parts of computers categorized as central processing unit input and output devices. The central processing unit is most important part of computer where the mathematical and logical operations are done in addition to processing of the data received from input-unit. The CPU can further be categorized into (i) Control unit (ii) Arithmetic Logic Unit and (iii) Memory unit. The control unit consists of instruction register, decoder, address, register and instruction counter. The arithmetic and logic unit performs various operations as per instructions received by the computer.

14.8 Further Readings

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Unit-15: System Softwares

Structure

15.1 Introduction

In the previous unit you have learnt that a computer works with the help of hardware and software both. There are some software which contain general program design to control the operations of computer. These programs are called system software.

15.2 Objectives

After studying this unit you should be able to understand.

- System software
- File, editing and disk command
- Number theory

15.3 System Software

If consists of sets of general programs designed to control the operation of a computer system. Without system software, application packages cannot be executed. Examples are control programs, processing programs etc.

1. Control Programs

They take care of all the system activities and handle all the input, output, scheduling, interrupts etc. These consist of programs like operating systems, Job control programs, I/O management programs etc.

Operating System (O.S.)

Operating System is a set of programs written specially to manage all the resources and operations of a computer. Operating system manages automatically all the application programs, special programs needed in between the application programs by calling the whenever needed. It also takes care of hardware functioning. On the basis of functioning and facilities provided by them, Operating system can be classified as follows:

(i) Single User Operating System

These Operating Systems allow only one user to work on a computer at a time. **Example:** MS-DOS, CP/M.

(ii) Multi User Operating System

This Operating System allows more than user to work on the computer at the same time. These Operating Systems allocate memory in such a way that different users can work simultaneously without disturbing each other. It also allocates the processing time in such a way that every user gets a very quick response from the machine. These are also known as Time Sharing Operating Systems.

Example: UNIX, XENIX, VMS, Windows NT.

Various function of Operating System:

- Memory Management
- Processor Management
- Device Management
- File Management

(i) Memory Management Functions

Operating system manages the primary memory of the system. It allocates the memory on the request of a process, which is being run at that time. It also keeps a check at a particular time, how many bytes of memory are in used and which process is using it. It also keeps track of what part of it is free. In case of a multi-user system, it decides on the priority basis, that which user will have access to memory and when. How much of it is used depends on the requirements.

(ii) Processor Management Functions

Operating system also takes care of the processor. It allocates the processor to the user. In case of multi-user system, it allocates the processor time to different users as and when needed and in such a way that every user has minimum waiting time.

(iii) Device Management Functions

It keeps track of all the devices i.e. peripherals attached to the computer such as I/O devices etc. When needed, it allocates the devices in such a way that each can be efficiently used. It initiates the I/O operations and allocates them along with other devices to the user.

(iv) File Management Functions

Writing and retrieving the information on/from the secondary storage device is the function of an Operating System. It follows a complete methodology for maintaining, the files, so that different sets of information do not get mixed up and exactly the same set of information is supplied, which is required by the user.

2. Processing: Programs

Processing programs are the programs which take care of application programs running on the computer. These programs work under the supervision of control programs. They help the application program to do the actual data processing: These are basically of two types:

- **Language Translators** are system program that translate program written by the user in high level language to machine language. Examples are compliers and assemblers.
- **Service Programs or Utility Programs** are a set of programs which execute tasks frequently required in data processing. Those tasks are of a routine nature that all computer users require their machine to perform from time to time (storing, copying etc.). Examples are sort/merge programs to arrange unsequenced data into specified sequence, debugging tools to help the user to locate and correct logical errors in the program etc.

Some Processing Programmes

Linker

A program that links the separately translated modules to form an absolute load module to be run as a unit.

Loader

A program that loads the absolute load module into main memory.

Interpreter

A program that translates each instruction of high level language and also executes instructions before passing on to the next instruction.

15.4 File Commands

i) HELP

PURPOSE: This command displays information about the DOS commands. One can seek information in two ways.

FORMATE: a) HELP [command]

A screenful of descriptive information is displayed about the named command.

b) Command

ii) DATE

PURPOSE: To set the system date. The change date shall now be used for the date stamping of files. Format of date can be changed.

FORMATE: DATE [dd-mm-yy]

iii) TIME

PURPOSE: To set the system time. The changed time shall now be used for the time stamping of files.

FORMATE:	TIME [HH: MM: SS: cc]

iv) DIR

PURPOSE:	To list all or specified files of the connected area on the specified device.
FORMATE:	DIR [drive:] [pathname] [/P] [lW] [lA:x] [/B] [/L] [/O:z] [IS]
/P	To see the page-wise listing of directory. In this case if the output of the command is more than one page, it shall pause after each screen full. On pressing any key, generally the space bar, next screen is shown.
/W	To see the width-wise listing of directory. It displays only file name an extension. Each line contains five file names. The directory names are enclosed in square Brackets.
/A:a	Displays files having certain file attributes, where attribute(a) is one of the following:

h	hidden files	-d	files only (no directory names)
-h	non-hidden files	a	files that have been archived
s	system files	-a	files that are not archived
-s	non-system files	r	read only files
d	directory names only	-r	files that are not read only

v) SET DIRCMD

PURPOSE:	The DIR command parameters can be preset using this command. It can be entered directly from the DOS prompt.
FORMATE:	you can use any valid combination of DIR parameters and switches with the SET DIRCMD command, including the location and name of a file.

vi) ATTRIB

PURPOSE:	To set or show file attributes.
FORMATE:	ATTRIB [+R] [-R] [+A] [+S] [-S] [drive:] [path] <filename> [IS]
+	Sets an attribute
-	Clear an attribute
R	Read only file attribute
A	Archive file attribute
S	System file attribute
H	Hidden file attribute
/S	Process files in all subdirectories in the specified path.

vii) DOSKEY

PURPOSE:	This command stores all DOS commands typed from the DOS prompt into a memory buffer. Commands can then be recalled using the up and down arrow keys. By default only a single command is stored in the buffer. It is kept in the buffer as long as another command is not executed. With the DOSKEY command we can store more than one command in the buffer, which can subsequently be recalled.
FORMATE:	DOSKEY [/REINSTALL] [BUFSIZE=n] [/HISTORY] [/INSERT] [LOVER TRIKE]
/INSERT	Puts DOSKEY in the insert mode. This lets you insert text within a display command. To temporarily activate the overstrike mode the Ins key can be pressed.
/OVERSTRIKE	Puts DOSKEY in the insert mode. This lets you insert text within a display command. To temporarily activate the insert mode, Ins key can be pressed.

/RINSTALL: Clears the buffer.

/HISTORY Displays all commands presently in the DOSKEY buffer.

viii) CLS

PURPOSE: Clears the display screen and DOS prompt appears on the top left comer of the screen.

FORMATE: CLS

ix) TYPE

PURPOSE: Display the contents of specified file.

FORMATE: TYPE [drive:] [path] <filename>

x) COPY

PURPOSE: Copies one or more files to specified or files on specified disk.

FORMATE: Copy <source-file-spec> <target-file>[Iv]

/V Causes DOS to verify that the sectors written on the target diskette are recorded properly.

xi) MOVE

PURPOSE: Move one or more files to the location you specify. The MOVE command can also used to rename directories.

FORMATE: MOVE [/Y I I-Y] [drive:] [path] [filename] [drive] [path] filename […] destination

/Y Indicates that you want MOVE to replace exiting files(s) without prompting you for confirmation. By default, if you specify an existing file. MOVE will ask you if you want to overwrite the existing file.

/-Y	Indicates that you want MOVE to prompt you for confirmation when replacing an existing file

xii) REPLACE

PURPOSE:	Used to selectively replace files on the target disk with files having the same name on the source disk.
FORMATE:	REPLACE [DRIVE 1:] [PATH 1] FILENAME [DRIVE 2:] [PATH 2] [/A] [IP] [/R] [/W]
	REPLACE [DRIVE 1:] [PATH 1] FILENAME [DRIVE 2:] [PATH 2] [IP] [/R] [IS] [IU] [1W]
/A	Copies specified files that are not present on the target disk. Cannot use with /S or /U switches
/P	Prompts you as each file is encountered on the target drive.
/R	Also replaces read-only files on the target drive.
/U	Searches all directories on the target drive for filenames that match those on the source drive.
/W	Waits for you to insert a diskette before beginning.

xiii) RENAME

PURPOSE:	Changing the name of a file.
FORMATE:	RENAME [drive:] [path] <old name> <new name>

xiv) PRINT

PURPOSE:	Prints a queue(list) of data on the printer.
FORMATE:	PRINT [/D:device] [IQ:n] [IT] [drive: [path] [filename[….]][/C][IP]

/D:device — Specifies a print device. The valid values for parallel ports are LPT1, LPT2 and LPT3. Valid values for serial parts are COM1, COM2, COM3 and COM4. The values LPT1 and PRN refer to the same parallel port which is also the default.

/Q:n — By default 10 files are allowed in the print queue, otherwise the range is 4-32 /Q:n switch is used to specify the maximum number of files you want in the print queue. This should be used before giving any print command.

/T — Terminates print queue i.e. removes all files from print queue.

/C — Cancels printing of the preceding filename and subsequent filenames.

/P — Adds the preceding filename and subsequent filenames to the print queue.

xv) PRINT

PURPOSE: To delete specified files from specified diskette.

FORMATE: [drive:] [path] filename [IP]

/P — Prompts you before the deletion actually occurs.

xvi) MEN

PURPOSE: Displays the amount of used and free memory.

FORMATE: MEN

15.5 Editing Commands

1. How to Start Edit

To invoke full screen editor the following command is used.

EDIT

PURPOSE: Edit is the full screen editor program available with DOS which allows us to create, change and display program and text files. It can be used to:

- ❖ Create new files and save them on disk.
- ❖ Update existing files and save both the updated and original files.
- ❖ Delete, edit, insert and display lines in files.
- ❖ Search, delete or replace txt within one or more lines in a file.

FORMATE: EDIT [drive:] [path] filename]

The EDIT program, which is a convenient full screen editor was introduced as a standard feature with the release of DOS 5.0. EDIT does not operate without the presence of vertically and horizontally.

2. USING PULL DOWN MENUS

Once full screen editor is involved you press FI key to display help about the current operation.

The Edit program has four pull down menus which are assessed by pressing the AL T key. Once the AL T key is pressed, one can use the left or right arrow key to pull down file, Edit, Search and Options. To pull down the menu, either first highlight the required option by use of left right arrow keys and press the return key or just type the first letter of the menu name viz. F, E, S, or O. Use escape to come out of the menu operations.

i) **FILE:** This menu perform all operations required to open and save files or to exit the EDIT program. The various options are:

NEW To create a new document, clear the current documents. If any changes have been made after the last save, it shall prompt you for saving before clearing it.

OPEN Open a documents; it prompts for a file name and display a list of file names with extension TXT in the current directory. One may type a file name and press return or press tab to point into the filename by moving the highlight to it using an arrow key, and press return to open the file. Wild cards may be used to list specific filenames. One may even use tab key to choose a different directory or drive.

SAVE Save the current file using the existing file name.

SAVE AS Save the current file with a new file name.

PRINT Print either the complete document or a selected part of text. The part is selected by highlighting it using shift and arrow keys.

EXIT Exit the editing session. If the open file has been modified then a chance is given to save it before quitting.

ii) EDIT: This menu performs all operations like cut, copy, paste or delete, selected text. Text is selected using the shift and arrow keys.

CUT Removes selected text from the screen and puts it on clipboard. The shift-Del key combination.

COPY Places the selected text on the clipboard without cutting it from the screen. The short-cut is to press Ctrl-Ins key combination.

PASTE Insert text from the clipboard to the present cursor location. The shortcut is to press Shift-Ins key combination.

CLEAR Delete select text without putting in on the clipboard. The shortcut is to press Del key when the cursor is in the selection.

iii) SEARCH: This menu perform all operations like locating a specified text string within the current document, replacing it with another text string.

FIND Finds a specified string.

REPEAT Finds the next match in the current document.

LASTFIND The shortcut is to press Ctrl-L or F3 key.

CHANGE Finds a specified string and replaces it with another. One can verify the change before replacement and advance to the next occurrence of the search string.

iv) OPTIONS: It can be used to change. Display attributes. This menu performs all functions like setting up display colors, turning the scroll bar on or off, changing the tab stop setting the file path for the EDIT help file.

DISPLAY Picks foreground and background color from a list, turn the scroll bars on or off, and set tab stops.

HELP PATH Picks the file path in which the EDIT. HELP file is located.

v) HELP: It can be used to get help on usage of EDIT.

15.6 Disk Management Commands

1. External Commands

The command files are required every time an external command is to be used. To execute any external command, the corresponding file (with COM or EXE extension) is read, and the action is carried out on the target diskette/disk. The action may be as per the external storage media(s) used.

2. Commands

i) Format

PURPOSE: Initializes the disk on the designated drive to a recording format acceptable to DOS.

All new diskettes and fixed must be formatted before they can be used by DOS.

Formatting destroys any previously existing data.

FORMAT FORMAT [drive:] [/S] [/B] [/V][:label] [/1] [/4] [/U] [/Q] [F:size]

COMMANDCOM. After formatting the target diskette/disk.

/B Reserves area for system files which can be copied later.

/V[:label] Assigns label to the floppy. Label means assigning name to floppy for further reference. Label can be maximum of 11 characters. If not specified now, system asks for the label after formatting.

/1 Formats a diskette for single sided use.

/4 Formats a 5.25 inch 360 KB floppy disk in a high density (1.2MB) drive

/U Specifies unconditional format, which destroys all data on the target disk to prevent subsequent unformatting with the command.

/Q Quick format takes less time, this command removes the file allocation table and root directory. The disk is not scanned for bad areas.

/F:Size Sepcifies the size of floppy disk to format such as 160, 180, 320,360, 720, 1.2, 1.44, 2.88. The unit measurement of disk space is KB, but it is not to be mentioned in the command syntax

ii) UNDELETE

PURPOSE: Restore files that were previously deleted by using the 7 command.

FORMAT UNDELETE [drive: Hpath] [filenameHIDTIIDSI/DOS] UNDELETE [ILISTIII ALL I/PURGE [drive] I/ST A TUSI/LOAD I/UNLOAD JIS [drive] I/Tdrive[-entries]]

/LIST List the delete files that are available to be recovered, but does not recover any file.

/ALL Recovers deleted files without prompting for confirmation on each file.

/DOS Recovers only those file that are internally listed as deleted by MS-DOS, prompting for confirmation on each file.

/DT Recovers only those files listed in the deletion tracking file, prompting for confirmation on each file.

/DS Recovers only those files listed in the SENTRY directory, prompting for confirmation on each file.

/LOAD Loads the Undelete memory resident program into memory using information defined in the UNDELETE.INI file.

/LOAD Unloads the memory resident portion of the undelete program from memory, turning off capability to restore deleted files.

/PURGE [drive] Deletes the contents of the SENTRY directory. If no drive is specified, UNDELETE searches the current drive for the directory.

/STATUS Display the type of delete protection in effect for each drive.

/S[drive] Enables the Delete Sentry Level of protection and loads the memory resident portion of the UNDELETE program.

IT drive [-entries] Enables the Delete Tracker level of protection and loads the memory resident portion of the UNDELETE program.

Delete Sentry:

Delete Sentry provides the highest level of protection to ensure that you can recover deleted files. This level of protection creates a hidden directory named SENTRY. When you delete a file, undelete moves the file from its current location to the SENTRY directory without changing the record of the file's location in the file allocation table (F AT). If you undelete the file, MS-DOS moves the file back to its original location.

Delete Tracker:

Delete Tracker provides an intermediate level of protection. It uses a hidden file named PCTRACKER.DEL to record the location of delete file. When you delete a file, MS-DOS changes the F AT to indicate that the location of the file is now available for another file.

Standard:

The Standard level of protection is automatically available when you switch on your computer. Of the three levels of guarding against accidental file deletion, it provides the lowest level of protection. It also has the advantage of requiring neither memory nor disk space.

iii) RECOVER

PURPOSE If a file or directory cannot be read there may be one or more damaged sectors on the disk. To recover the parts of the file or directory that are not damaged, the recover command may be used.

FORMAT RECOVER [drive:][path] filename

You cannot retrieve the part of a file that is stored in a defective sector, but you can recover the rest of it by using the Recover command. MS-DOS reads the file one sector at a time. If any of the sectors are damaged, MS-DOS removes them from the file. Ms-DOS marks the bad sectors so that information cannot be stored there in the future. When the operation is complete, the recovered file is stored in the root directory of the disk it came from. It is name sequentially beginning with FILEOOOO.REC.

iv) UNDELETE

PURPOSE: This command restore the directories and files on a disk after it has been formatted or in other words, the UNFORMAT command is used to recover an unintentionally formatted disk. It can also restore the disk restructured by RECOVER command.

FORMAT UNFORMAT drive: [/J]

UNFORMAT drive: [IU][/L][/TEST] [/P]

UNFORMAT IPARTN [/L]

/P Sends listed information to the connected printer.

/TEST Shows how UNFORMAT would recreate the information but does not actually unformat the disk.

/L Lists all files and sub-directories. It shall not use the file create by MIRROR program.

v) DISKCOPY

PURPOSE a) Copies track by track, contents of one diskette on to another diskette.

b) Previous contents of target diskette get erased.

FORMAT DISKCOPY [drive 1[drive 2:][/1]

Copies only the first side of the disk.

iv) DISKCOMP

PURPOSE: Compares the contents of two diskettes track by track and issues a message if tracks are not equal. However the source or target cannot be a hard disk.

FORMAT DISKCOPY [drive 1[drive 2:][/l][N]

/I Compares only the first side of the disk.

/V Verifies that the information is copied correctly.

vii) CHKDSK

PURPOSE: This command scans the disk in the specified drive and checks it for errors.

FORMAT CHKDSK [drive:[[pathname] filename][/F][N]

/V Displays the name of each directory along with the full path specification as it checks the disk.

/F Fixed errors on the disk.

viii) SCANDISK (E)

PURPOSE: It starts Microsoft Scandisk, a disk analysis and repair tool that checks a drive for errors and corrects any problems that it finds.

FORMAT SCANDISK [drive:[drive:…..] I/ALL][CHECKONLY I/AUTOFIX] [/NOSAVE] [NO SUMMARY]

/ALL Checks and repairs all local drives.

/AUTOFIX Fixes damages without prompting you first.

/CHEKONLY Checks a drive for errors, but does not repair any damage. You cannot use this switch with /AUTOFIX.

/NO SAVE Directors Scan Disk to delete any lost cluster it finds.

/NO SUMMARY Prevents Scan Disk from displaying a full screen summary after checking each drive.

ix) DEFRAG

PURPOSE: Reorganizes the files on a disk to optimize disk performance.

FORMAT DEFRAG [drive:][/F][/S [:]order] [IB[/H]

DEFRAGE [drive:] [IV] [IB] [H]

/F Defragments files and ensures that the disk contains no empty spaces between files.

/U Defragments files and leaves empty spaces, if any between files.

/S Controls how the files are sorted in the directories. The possible orders are

N In alphabetic order by name

-N In alphabetic order by extension

E In alphabetic order by extension

-E In reverse alphabetic order by extension

D By date and time, earlier first

-D By date and time, latest first

S By size smallest first

-S By size largest first

/B Restarts the computer after files have been reorganized.

/H Moves hidden files

x) BACKUP

PURPOSE Back up one more files from one disk to another. This command can backup files on disks of different media (hard-disks and floppy-disks).

FORMAT BACKUP [drive 1:[path] [file name H drive 2:] [IS] [1M] [IA] [:size]]

[D:date[/T:time]][:drive;][path] logfile]]

Drive 1 is the disk drive you want to backup.

Drive 2 is the target drive to which the files are backed up.

/S Backs up contents of sub-directories as well

/M Backs up only those files that have changed since the last backup.

/A Adds the files to be backed up to those already on the backup disk. It does not erase old files on the backup disk. This switch will not be accepted if files exist that were backup using backup from version 3.20 or earlier.

/F:[size] Specifies the size of the disk to be formatted.

/D: date Backs up only those files that were last modified on or after the specified date.

/T: time Backs up only those files that were last modified on or after the specified time.

/L[:drive:HpathHlogfile]] Makes a backup log entry in the specified file. If you do not specify <filename>, backup places a file called BACKUP.LOG in root directory of the disk that contains the files being backed up.

xi) RESTORE

PURPOSE Restores files that were backed up using the BACKUP command

FORMAT RESTORE drive 1: drive 2: [path [filename]] [IS] [/P] [/B:date] [/A:date] [/E:time] [/L:time] [1M] [IN] [/D]

Drive 1 specifies the drive on which the backup files are stored.

Drive 2 is the target drive on which files are to be restored.

	Path and filename identifies the file(s) you want to restore.
/S	Restore sub-directories also.
/P	Prompts for permission to restore any file matching the file specification that are read only or that have changed since the last backup (if appropriate attributes are set)
/B:date	Restores only those files last modified on or before date.
/A:date	Restores only those files last modified on or after date.
/E: time	Restores only those files last modified at or earlier than time.
/L:time	Restores only those files last modified at or later than time.
/M	Restores only those files last modified since that last backup.
/N	Restores only those files that no longer exist on the target disk.
/D	Displays files on the backup disk that match specification.

xii) XCOPY

PURPOSE	Copies files and directories, including lower level directories if they exist.
FORMAT	XCOPY source [destination] [/YII-Y] [ID:date] [/P] [IS] [IE] [N] [/W]
	DEFRAG [drive:] [IV] [IB] [/H]
/D: date	Copies source files that were modified on or after the specified date.
/S	Copies directories and lower level directories unless they are empty.
/E	Copies any sub-directories even if they are empty. You must use the IS switch with this switch.
/V	Verifies each file while writing.
/P	Prompts before creating each destination file.

/Y	Replace existing file(s) without prompting for confirmation. This switch was not available before version 6.
/-Y	Prompts for confirmation before replacing an existing file.
/W	Displays the following message and waits for you response before starting to copy files:
	Press any key to begin copying file(s)

15.7 Number System

Any system for representing numeric values or quantities utilizes a number of digits, e.g. the decimal system utilizes ten digits 0 to 9. These digits may be arranged in groups, the contribution of each digit being made according to the value of the digit and the significance of its position in the group.

Decimal Notation: The system of writing numbers in which successive digit positions are represented by successive powers of 10.

Examples: 6235 means

1000s(10^3)	100s(10^2)	10s(10^1)	1s(10^0)	
6	2	3	5	
=1000×6	+100×2	+10×3	+1×5	
=6000	+200	+30	+5	=6235

Binary Notation: A positional notation system for representing numbers in which the radix or base in 2. In this system, numbers are represented by two digits, 0 and 1, and each digit position represents a power of 2.

Examples: 1010_2 is 10 in decimal system as shown below:

8s(2^3)	4s(2^2)	2s(2^1)	1s(2^0)	
1	0	1	0	
=8×1	+4×0	+2×1	0+1×0	
=8	+0	+2	+0	=10

Octal Notation: The number system using 8 as base or radix. This system uses the digits from 0 to 7 and each digit position represents a power of 8.

Example: 232_8 is 154 in decimal system as shown below:

$8s(8^2)$	$8s(8^1)$	$1s(8^0)$	
2	3	2	
$+64\times2$	$+8\times3$	$+1\times2$	
=128	+24	+2	=154

Hexadecimal Notation: A notation of numbers with 16 as base or radix. Ten decimal digits from 0 to 9 are used and in addition six more character a, b, c, d, e and f are used to represent ten, twelve, thirteen, fourteen and fifteen as single characters. Each digit position represents a power of 16.

Example: 21316 means:

$256s(16^2)$	$16s(16^1)$	$1s(16^0)$	
2	1	3	
$+256\times2$	$+16\times3$	$+1\times3$	
+512	+16	+3	$=531_{10}$

Converting from one Number System to Another

Any number value in one number system can be represented in any other number system. There are many methods that can be used to convert numbers from one base to another.

1. **Converting to Decimal from another Base**

   Following steps are used to convert to a base 10 value from any other number system:

   Step 1: Determine the positional value of each digit. (This depends on the position of the digit and the base of the number system)

   Step 2: Multiply the obtained column Values (in step 1) by the digits in the corresponding position.

   Step 3 Sum the products calculated in Step 2. The total obtained is equivalent value in decimal.

   i) Binary to Decimal

This conversion can be done by assigning the values to each position and then adding these values together.

Example: $(1101)_2$ is to be converted to its decimal equivalent

According to the steps given above:

Step 1 & 2:

Column Number (from right to left)	Column Value (step 1)	Digit Column Value (step 2)
1	$2^0=1$	$1\times1=1$
2	$2^1=2$	$0\times2=0$
3	$2^2=4$	$1\times4=4$
4	$2^3=8$	$1\times8=8$
		=13

Step 3: Sum of the products

So, $(1101)_2 = (13)_{10}$

$(1101)_2 = (13)_{10}$ can also be represented as:

2^3	2^2	2^1	2^0
1	1	0	1
8×1	$+4\times1$	$+2\times0$	$1\times1=1$
			$=13_{10}$

ii) Hexadecimal to Decimal

Example: $(IAC)_{16}$ is to be converted to its decimal equivalent.

Step 1 & 2:

Column Number (from right to left)	Column Value (step 1)	Digit Column Value (step 2)
1	$16^0=1C\times1$	$12\times1=12$
2	$16^1=16$	$1\times16=10\times16=160$

3	$16^2=256$	$1\times256=256$
		$=428$

Step 3: Sum of the products

So, $(IAC)_{16} = (428)_{10}$

Similarly, $(F5)_{16} = (245)_{10}$ can be represented as

$16^1 \times 15+16^0 \times 5$

$=16\times15+1\times5$

$=240+5$

$=(245)_{10}$

iii) Octal to Decimal

Example: $(4706)_8$ is to be converted to its decimal equivalent.

Step 1 & 2:

Column Number	Column Value	Digit Column Value
(from right to left)	(step 1)	(step 2)
1	$8^0=1$	$6\times1=1$
2	$8^1=8$	$0\times8=0$
3	$8^2=64$	$7\times64=448$
4	$8^3=512$	$4\times512=2048$

Step 3: Sum of the products = 2502

So, $(4706)_8 = (2502)_{10}$

Similarly, (356)8 = $(238)_{10}$ can be represented as

$8^2 \times 3+8^1 \times 5+8^0 \times 6$

$=64\times3+8\times5+1\times6$

$=192+40+6$

$=(238)_{10}$

1. Converting from Base 10 to a New Base

Following steps are used to convert a number from base 10 to a new base:

Step 1: Divide the decimal number to converted by the value of the new base.

Step 2: Record the remainder from step 1 as the rightmost of the new base number.

Step 3: Divide the quotient of the previous division by the of the new base.

Step 4: Record the remainder from step 3 as the next digit (to the left) of the new base no.

Repeat step 3 and 4, recording remainders from right to left, until the quotient become zero in step. The last remainder obtained will be the leftmost digit of the base number.

i) Decimal to Binary

Example: $(62)_{10} = (111110)_2$ can be represented as:

Divisor	Quotient	Remainder
2	62	
2	31	0
2	15	1
2	7	1
2	3	1
2	1	1
	0	1

ii) Decimal to Octal

Example: $(428)_{10}$ can be converted to its Octal number as follows:

Divisor	Quotient	Remainder
8	952	

8	119	0
8	14	7
8	1	6
8	0	1

Hence $(428)_{10}$= $(1670)_8$

iii) Decimal to Hexadecimal

Example: $(428)_{10}$ can be converted to its equivalent hexadecimal number.

Divisor	Quotient	Remainder in hexadecimal
16	428	
16	26	12=C
16	1	10=A
	0	1

15.8 Exercises

Q.1. Answer the following questions:

1. What are different types of software?
2. What is an operating system? Discuss briefly differently O.S.
3. What are the functions of an operating system?
4. What are the different kinds of operating system?
5. Explain the following terms:
   a) Linker
   b) Loader
   c) Interpreter

Q.2. Fill in the blanks:

1. MS-DOS is a ------(single/multi) user operating system.

2. WINDOWS-NT is a (single/multi) user operating system.

3……………. are the programs which take care of application programs running on computer.

16. --------------- are a set of programs which execute tasks frequencly required in data processing.

Q.3. Fill in the blanks:

1. An operating system is an application software.

2. System software and Soft ware package is one and the same thing.

3. UNIX is a single operating system.

4. MS-DOS operating system can be used with all IBM compatible personal computer.

Q. 4. Convert the following decimal numbers into binary numbers, octal numbers and hexadecimal numbers:

a) 54

b)123

c) 259

d) 101

e) 78

Q. 5. Convert the following hexadecimal numbers into octal and decimal numbers.

a) ABCD

b)F23

c) 49 Al

d) 1230

e) 7000

Q. 6. Convert the following binary numbers, into decimal numbers:

a) 101110111

b)110110101

15.9 Summary

This unit covers a detailed discussion about system software along with various commands which is to be given to perform a specific function. A brief description of various number systems is also provided you may be able to convert a number in a given number to another number system after reading this unit.

15.10 Further Readings

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SBSSTAT – 04
Numerical Methods and Basic Computer Knowledge

Block: 6 Basics of Computer Programming

Unit – 16: Algorithm and Flow Chart

Unit – 17: Programming Language

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Block & Units Introduction

The ***Block - 6 – Basics of Computer Programming,*** is the sixth block. This block includes two units regarding to basics of computer programming and programming languages.

In ***Unit – 16 – Algorithm and Flow Charts***; described the said topics and various example related to these techniques are worked out.

In ***Unit – 17 – Programming Language***; elements ideas related to various programming languages rearranges from machine language to object oriented programming are discussed.

At the end of block/unit the summary, self assessment questions and further readings are given.

Unit-16: Algorithm and Flow Chart

Structure

16.1 Introduction

A Computer is a machine which responds to a specific set of instructions in a well-defined manner. Unfortunately, computers do what we tell them to do and not necessarily what we want them to do. There must be no ambiguity in the instructions that we give to a program no possibility of alternative interpretations. The computers will always take some course of action so that the results we get are those we anticipated.

Most interesting problems appear to be complex from programming point of view. For some problems this complexity must be inherent in the problem itself. In many cases, however in can be due to other factor that may be within our control; for example, incomplete or unclear specification of the problem. In the development of the computer programs, complexity need not always be a problem if it is properly handled and controlled.

Computer programming can be a difficult task. It is difficult largely because it itself is a complex activity, combining many mental processes at a time. We can do a great deal however to make it easy. For instance, the task of programming can be made much more manageable by systematically breaking it up into a number of less complex subtasks. This may be referred to as the *divide and conquer approach*. The approach of subdividing a task has made considerable success in practice.

Given a task, we separate it into two important phases, namely the *problem-solving phase* and the *implementation phase.* In the problem-solving phase, weconcentrate on subdividing a task into a number of comparatively simpler subtasks. This is equivalent to say that we concentrate on designing an algorithm to solve the stated problem (see figure 1.1 below). Only after we are satisfied that we have formulated a suitable algorithm, we turn to the details of the implementation of this algorithm in some programming language. Thus, given an algorithm that is sufficiently precise the translation to a computer program becomes quite straightforward.

16.2 Objectives

After the study of this unit, you will be able to:

- Define a algorithm
- Set various steps in a program using algorithm
- Define a flowchart
- Draw flow chart of various problem

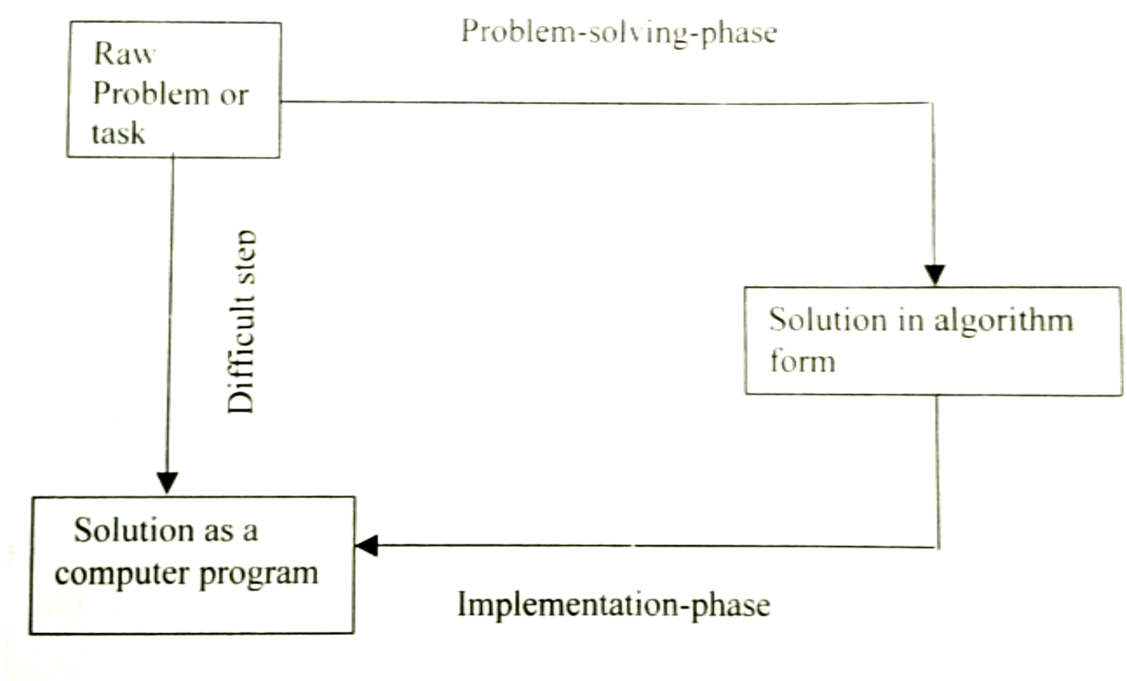


Figure 1.1: Problem solving and Implementation phases

16.3 Algorithm

An algorithm can be defined as an unambiguous, ordered sequences of steps that leads to the solution of a given problem. Any algorithm will have a start point and a termination point. Algorithms can be expressed in any language, from natural languages like English or French to programming languages.

Algorithm the term itself may be new the concept of an algorithm should be familiar. Directions given to a particular street constitute an algorithm for finding the street. A recipe is a very familiar form of algorithm. A blue print serves the same purpose in a construction project. At Christmas time many parents spend exasperating hours following algorithms for assembling children's new toys.

We shall require that our algorithm have several important properties. First the steps in an algorithm must be simple and unambiguous and must be followed in a carefully prescribed order. Second, we shall always insist that our algorithm be effective; that is they must always solve the problem in a finite number of steps. We could ill afford to pay the costs computing if this was not the case.

Examples of algorithm are omnipresent even in our everyday life. An interesting example is given below:

Example 1. Sorting mail:

A detailed algorithm for sorting mail is as follows.

Step 1: Get all mail from mailbox

Step 2: Put mails on table

While more mails to sort

Step 3: Get piece of mail from the table

Step 4: If piece is personal

Read it

Step 5: If piece is magazine

Put in the magazine rack

Step 6: Else if piece is bill

Pay it

Step 7: Stop

Typically, when an algorithm is associated with processing information, data are read from an input source or device, written to an output sink or device and/ or stored for further processing. Stored data are regarded as part of the internal state of the entity performing the algorithm. In practice the state is stored in a data structure, but an algorithm requires the internal data only for specific operation sets.

For any computational process, the algorithm must be rigorously defined and specified in the way it applies to all possible circumstances that could arise. That is any conditional steps must be systematically dealt with case-by-case; the criteria for each case must be clear and computable.

Because an algorithm is a precise list of steps the order of computation will almost always be critical to the functioning of the algorithm. Instructions are usually assumed to be listed explicitly, and are described as starting ‘from the top’ and going ‘down to the bottom’, an idea that is described more formally by flow of control.

Algorithms can be expressed in many kinds of notation including natural languages, pseudocode, flow charts, and programming languages. Natural language expressions of algorithms tend to be verbose and ambiguous, and are rarely used for complex or technical problems. Pseudocode and flowcharts are structured ways to express algorithms that avoid many of the ambiguities common in natural language statements, while remaining independent of a particular implementation language. Programming languages are primarily intended for expressing algorithms in a form that can be executed by a computer, but are often used as a way to define or document algorithms. Sometimes it is helpful in the description of an algorithm to supplement small flow charts with natural language and/ or arithmetic expressions written inside block diagrams to summarize what the flow charts are accomplishing. We shall provide below a few standard examples to help clarify the designing of an algorithm. The proposed algorithms are meant for illustration only, there is always a possibility of alternative designing.

Example 2: Algorithm for finding out the frequency of a definite integer in a sequence of integers:

Step 1: Let the total number of integers =0

Step 2: Let the frequency of desired integers =0

Step 3: Repeat steps 4,5,6 and 7 until the end of the sequence is reached.

Step 4: Read one integer from the sequence

Step 5: Add 1 to the total number of integers

Step 6: If the integer read is the desired integer, add 1 to the frequency of desired integer.

Step 7: Move to the next integer in the sequence. If no more integer is left in the sequence to go step 8 otherwise go back to step 4

Step 8: Write the frequency of desired integers.

Step 9: Stop.

Example 3: Algorithm to pick the largest of three numbers:

Step 1: Input the numbers X, Y and Z

Step 2: If X >Y, go to step 3

Otherwise go to step 5

Step 3: If X > Z, Write X as the largest number

Otherwise Write Z as the largest number

Step 4: Stop

Step 5: If Y > Z Write Y as the largest number

Otherwise Write Z as the largest number

Step 6: Stop

Example 4: Algorithm to find the largest number in an unsorted list of numbers:

The solution necessarily requires looking at every number in the list, but only once at each. From this follows a simple algorithm, which can be stated in a high-level description, say English prose, as:

Step 1: Assume that the first item is largest.

Step 2: Look at each of the remaining items in the list and if it is largest than the largest item so far, make a note of it.

Step 3: The last noted items is the largest in the list when the process is complete.

Step 4: Stop the process.

Written in prose but much closed to the high-level-language of a computer programme, the following is a more formal coding of the algorithm in pseudocode.

Step 1: Input the non-empty list of number L.

Step 2: Largest @ L_0

Step 3: for each item in the list L ≥1, do

If the item > largest, then

Largest @ the item

Step 4: Write largest

Step 5: Stop.

Example 5: Algorithm to count the number of non-zero observations in a list of n observations where n is any positive integer.

The strategy is to read a particular observation from the list. Check if it is non zero and increment a counter. The same procedure is repeated unless all the n observations are entertained. The complete algorithm is given below.

Step 1: initialize non zero observation counter ‘nzo’ to ‘zero’

Step 2: Repeat for the values of I from 1 to n.

Step 3: Input an observation, say O.

Step 4: if O_i = zero go to step 5

Otherwise nzo= nzo+1

Step 5: Go to Step 2 for next i unless i ≤ n.

Step 6: Write the counter nzo.

Step 7: Stop.

Example6: Algorithm to find the roots of a quadratic equation $ax^2+bx+c=0$ when discriminant is non negative. The roots are to be stored in R_1 and R_2:

Step 1: Input a, b, c

Step 2: Evaluate the discriminant D = b^2-4ac

Step 3: Check; if D < zero, go to step 4

Otherwise evaluate $R_1 = \{(-b-\sqrt{D})/2a\}, R_2 = \{(-b-\sqrt{D})/2a\}$ and go to step 6

Step 4: Writ4e a message “discriminant is negative”

Step 5: Go to Step 7

Step 6: Write R_1 and R_2

Step 7: Stop.

Different algorithm may complete the same task with a different set of instructions in less or more time, space or effort than others. For example, given two different recipes for making potato salad, one may have *peeled the potato* before *boil thepotato* while the other presents the steps I the reverse order, yet they both call for these steps to be repeated of all potatoes and end when the potato is ready to be eaten.

The analysis and study of algorithm is a discipline of computer science and is often practiced abstractly without the use of a specific programming language or implementation. In this sense, algorithm analysis resembles other mathematical disciplines in that if focuses on the underlying properties of the algorithm and not on the specifics of any particular implementation. Usually, pseudocode is used for analysis as it is the simplest and most general representation.

16.4 Flow Chart

Before writing a programme of any significant complexity, it is necessary to first specify it clearly. In the earliest days when programs were to be written it was not clear how they should be specified. As there was no sense at the time of software engineering as a proper discipline it was thought that specifying the execution sequence of a program was all that was required and flow charts were born.

Flow chart is a pictorial representation of an algorithm which is primarly drawn to formulate and understand the technicalities of the programme. A standard convention, consisting of various shapes is used to draw a flow chart. Each shape denotes a particular instruction. The step-by-step process is shown with lines, arrows, and boxes of different shapes demonstrating the flow of the process.

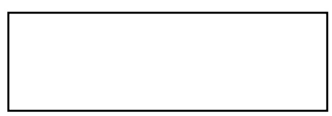
Flow charts are very useful in program development and provide excellent documentation with a good visual impact. The main advantage of drawing a flow chart is that is that one is not concerned with the nuances of programming language and the major concentration remains on the logic of the task to be executed. Moreover, a flow chart, being in pictorial form, helps to detect errors in logical sequence if any.

The commonly used symbols in a flow chart are given below.

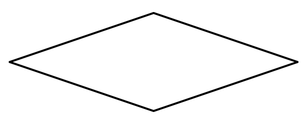
1. **Terminator:** An elongated oval flow (rectangle with rounded ends) chart shape indicating the start or end of the process. It is the first and the last symbol that is used in the program logic and usually contains the word “Start” and “End”.

2. **Input/ Output:** A box in the shape of aparallelogram denotes either an input (such as a Read), or an output (such as a Write).

3. **Processing:** The symbol used for processing is a rectangle and is used in a flowchart to represent arithmetic and data movement instructions.

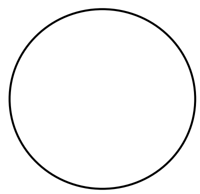


4. **Decision:**The symbol used for this purpose is a rhombus (a diamond shaped box), the point at which decision has to made and that allows branching. The condition for making the decision should be clearly mentioned in the dialogue box. A diamond usually has one arrow leading in, and two or more leading out, denoting different ways the control can proceed from that point. A diamond is used in cases of decisions statements like, “ If A is less than 10, proceed to add B to C; else multiply C and D,”.

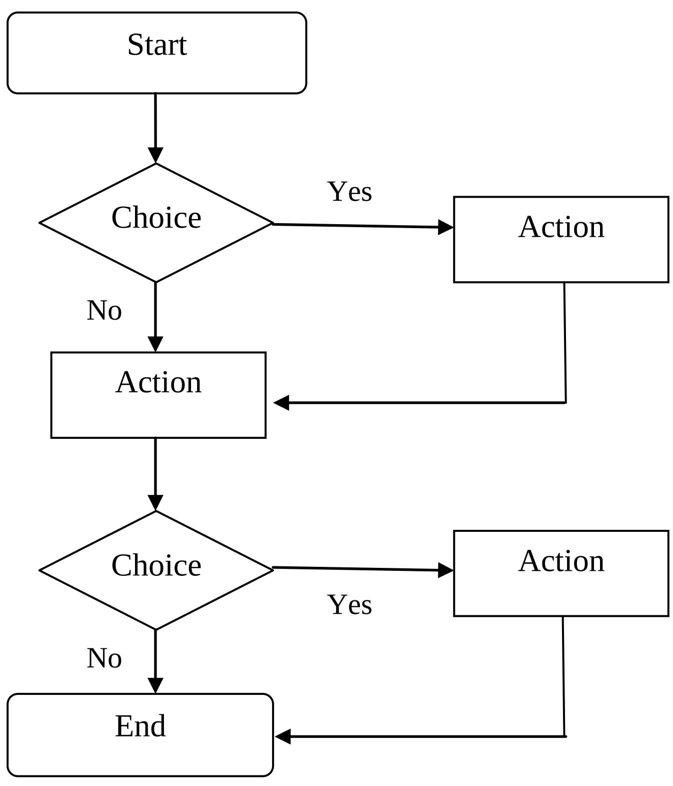


5. **Flow lines:** Flow lines, represented with arrows are used to indicate the flow of operation. Thus, the application of flow lines is to represent the exact sequence in which the instructions are to be executed.

6. **Connector:**Whenever the number and direction of flow lines become messy, it is useful utilize the connector symbol as a substitute for the flow lines. A connector is represented by a circleand sometimes a letter or a digit is placed within the circle to indicate the link.



A basic flowchart identifies the starting and ending points of a process, the sequence of actions in the process and the decision or branching points along the way, A basic flow chart looks like.



The benefits of flowchart are as follows:

1. **Communication:** Flow charts are better way of communicating the logic of a system to all concerned.
2. **Effective analysis:** With the help of flow chart a problem can be analysed in more effective way.
3. **Proper documentation:** Program flow chart serve as a good program documentation, which is needed for various purposes.
4. **Efficient coding:** The flow charts act as a guide or blueprint during program development phase.
5. **Proper debugging:** The flow chart helps in debugging process.
6. **Efficient program maintenance:** The maintenance of operating program becomes easy with the help of flow chart. It helps the programmer to put efforts more efficiently on that part.
7. There are certain limitations of flow charts as well, we have listed below some of the important limitations of flow charting. Also, since flow chart is merely a pictorial representation of an algorithm most of the benefits and limitations are applicable to algorithm as well except those which are specific to diagrams only.

1. Complex logic: Sometimes the program logic is quite complicated. In that case, flow chart becomes complex and clumsy.
2. Alternations and modifications: If alterations are required the flow chart may require re drawing completely.
3. Reproduction: As the flow chart symbols cannot be typed reproduction of flow chart becomes a problem.
4. The essentials of what is done can easily be lost in the technical details of how it is done.

Some examples on flow- chart are given below

Example 6: Flow chart to pick the largest of three numbers:

Algorithm to pick the largest of three numbers was attempted in example 3. The following figure shows the flow chart of the same example.

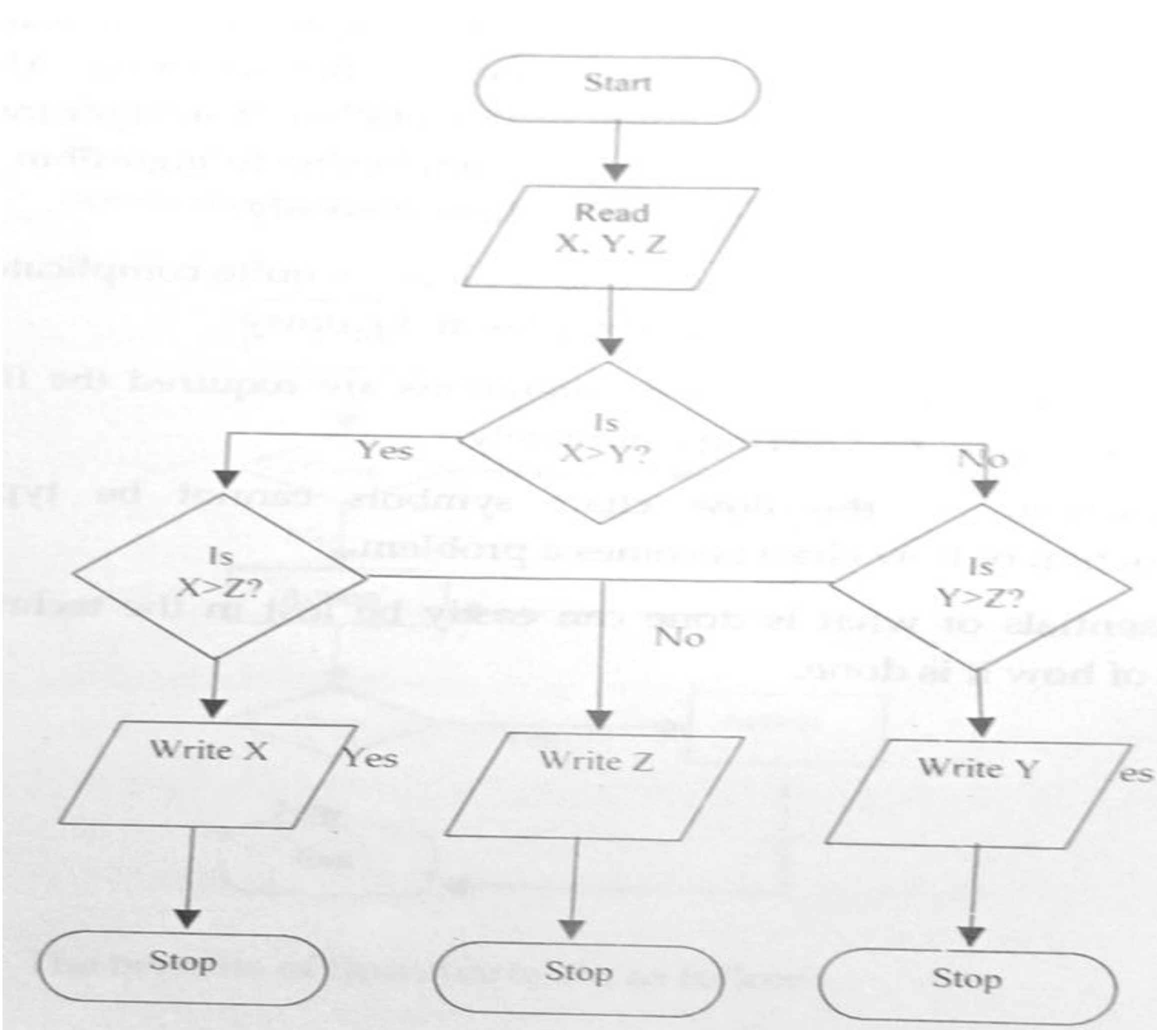


Figure 1.2: A flow chart to pick the largest of three numbers X, Y and Z.

Example 7: Flow chart to count number of non zero observation in a list of n observations:

The solution to this example in the form of algorithm was provided in example 5. We shall draw below a flow chart based on the various steps of the algorithm. In the illustration, we have used a new pictorial representation, a hexagon, in which the number of repetitions and the last of repetition are shown.

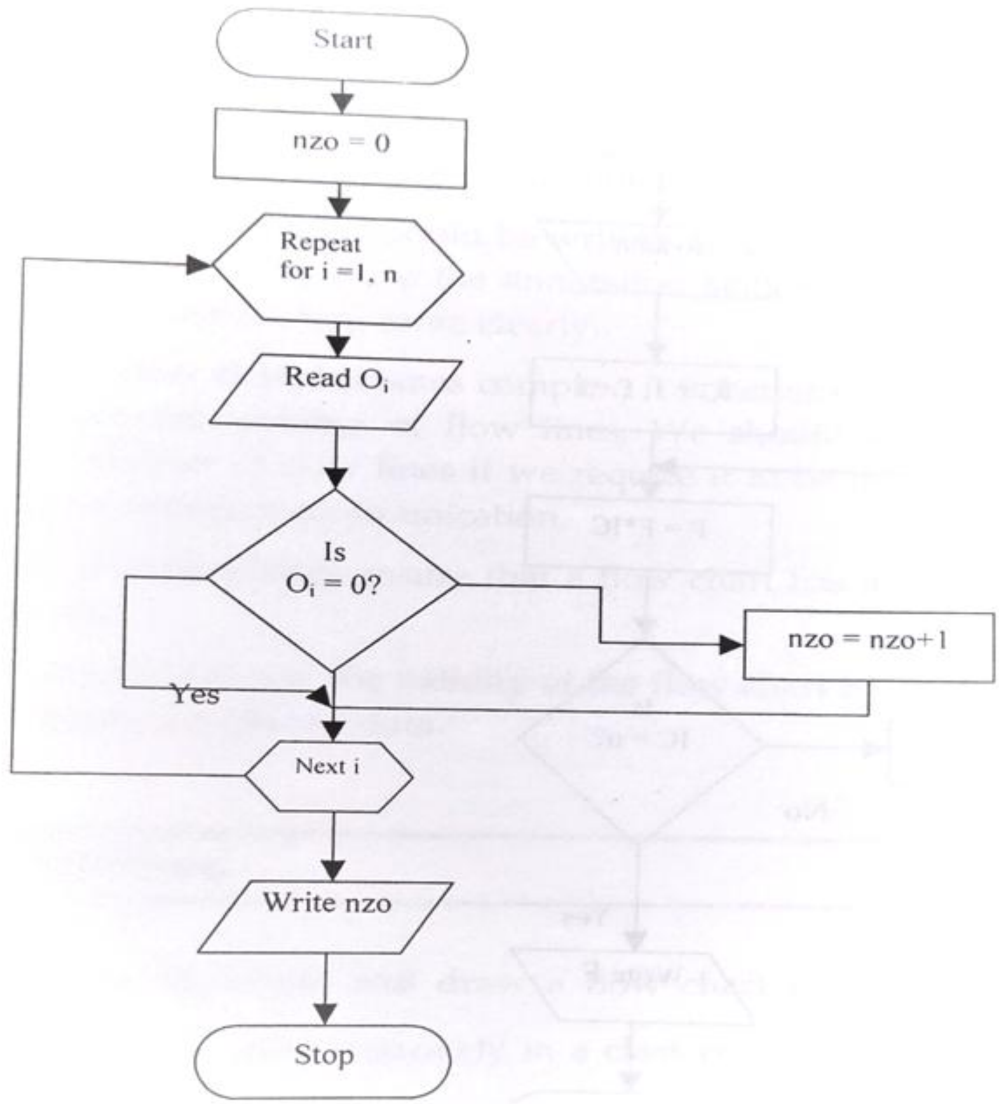


Figure 1.3: A flow chart to count number of non-zero observation in a list of n observations.

Example 8: Flow Chart to obtain factorial of positive integer n:

The factorial of n can be defined as the product of first n natural numbers. In order to draw a flow chart, we first require to input the number of observations, and then initialize and set two variables to unity. The first of these two variables will simply act as a counter whereas the second will be updated at each step and finally result in the required factorial. The following figure shows the flow chart for obtaining the factorial of a given number n. The algorithm of the problem has not been shown but it is presumed that it can be done routinely once the flow chart is understood.

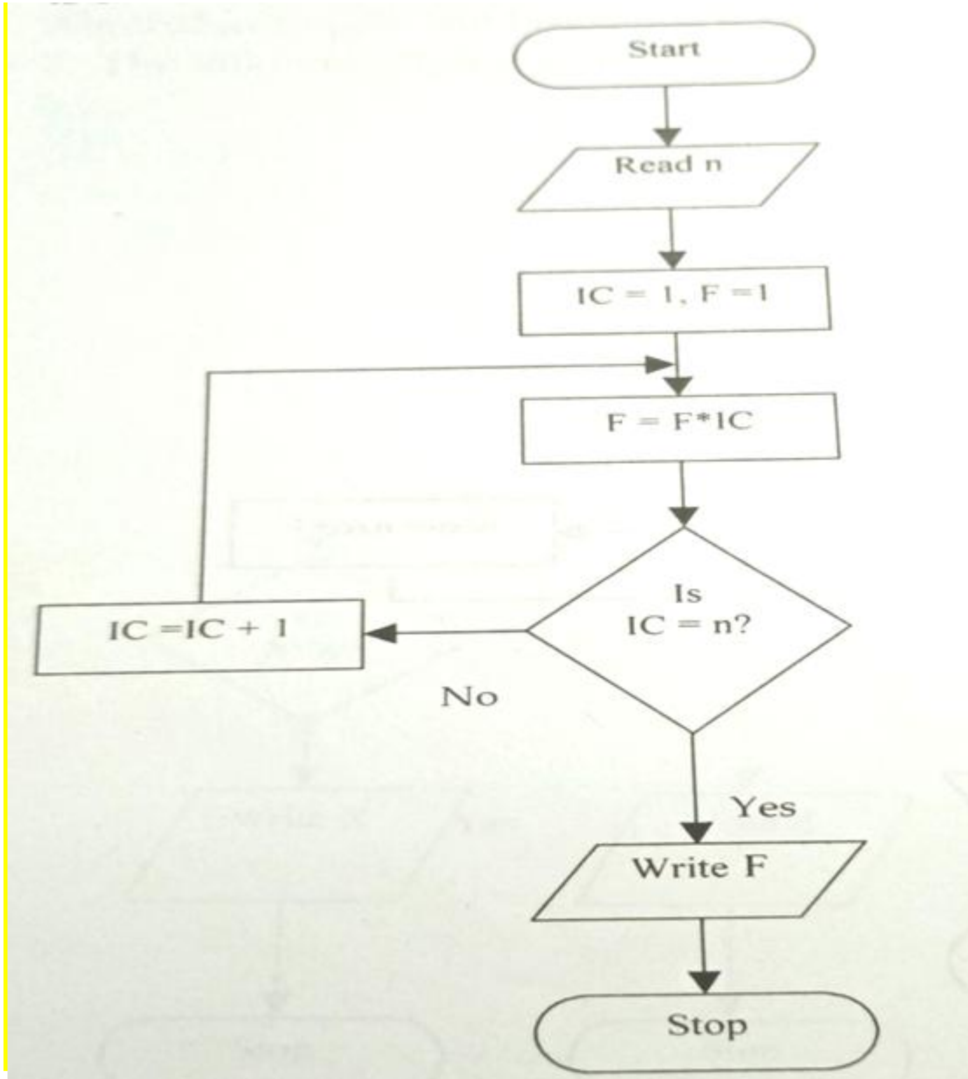


Figure 1.4: A flow chart to obtain factorial of a positive integer n.

Some Guidelines on Flow charting

a. While drawing a flow chart all necessary requirements should be listed out in logical order.

b. The flow chart should be clear, neat and easy to follow. There should not be any room for ambiguity in understanding the flow chart.

c. The usual direction of the flow of a procedure must be from left to right or from top to bottom.

d. Only one flow line should come out from a process symbol.

e. Only one flow line should enter a decision symbol but two or three flow lines, one for each possible answer, may leave the decision symbol.
f. Only one flow line is used in conjunction with a terminal symbol.
g. A brief description should be written within a standard symbol. If necessary, we can use the annotation symbol to describe data or computational steps more clearly.
h. If theflow chart becomes complex, it is better to use connectors to reduce the number of flow lines. We should always avoid the intersection of flow lines if we require it to be more effective and better means of communication.
i. We should always ensure that a flow chart has a logical start and finish.
j. It is useful to test the validity of the chart by passing through it with a simple test data.

16.5 Exercises

E-1. Write an algorithm and draw a flow chart to obtain the average height of boys and girls separately in a class of n students. The data for each student is available on separate record in the form of roll number, sex identification code, and height.

E-2. Write an algorithm and draw a flow chart to pick the largest of four real numbers.

E-3. Write an algorithm and draw a flow chart to convert Centigrade temperature to Fahrenheit.

E-4. Write an algorithm and draw a flow chart to evaluate $S_1 = \sum_{i=1}^{n} X_i$ and $S_2 = \sum_{i=1}^{n} X_1^2 - (s_1/n)^2$

E-5. A Fibonacci sequence is defined as 0,1,1,2,3,5,8,13,21,34,55, 89……… The first and second terms in the sequence are 0 and 1, respectively and the third and subsequent terms are found by adding the preceding two terms. Draw a flow chart to obtain all the numbers in Fibonacci sequence that are less than 200.

E-6. Draw a flowchart to arrange a given set of data in an ascending order.

E-7 For given a, b, c and d draw a flow chart to evaluate the function

$$f(x) = \begin{cases} ax^3 + bx + c.exp(ab^x), & if \quad x < d \\ ax^3 + bx + c, & otherwise. \end{cases}$$

16.6 Summary

Algorithm is a set of steps for solving a particular problem. To be an algorithm, a set of rules must be unambiguous and have a clear stopping point. Algorithm can be expressed in any language, from natural languages like English or French to programming languages like FORTRAN or C.

We use algorithms every day, for example, a recipe for baking a cake is an algorithm. Most programs with the exception of some artificial intelligence applications, consist of algorithms. Inventing elegant algorithms- algorithm that are simple and require the fewest steps possible – is one of the principal challenges in programming.

Flow chart is a visual explanation of an activity or process by means of pictorial representations. Each action is represented by a shape which leads on to the next action or actions, each shape attached to the next by a line to denote the flow of the activity.

16.7 Further Readings

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Unit-17: Programming Languages

Structure

17.1 Introduction

A Computer consists of two basic parts Hardware and Software. The process of software development is called programming. A computer can neither think nor make judgments on its own. It needs a program to tell it what to do. Programming, which is critical step in data processing, is a challenging and detailed process which begins with formulating the algorithms. Once an algorithm is obtained the next step for a solution using a computer would be to program the algorithm using mathematical and data processing techniques.

A **programming language** is an artificial language that can be used to control the behavior of a machine, particularly a computer. Programming languages like natural languages are defined by syntactic and semantic rules which describe their structure and meaning respectively. Many programming languages have some form of written specification of their syntax and semantics; some are defined only by an official implementation.

Programming languages are used to facilitate communication about the task of organizing and manipulating information, and to express algorithms precisely. Some authors restrict the term “programming languages” to those languages that can express all possible algorithms; sometimes the term “computer language” is used for more limited artificial languages.

Prominent purpose of programming languages is to provide instructions to a computer. As such programming languages differ from most other forms of human expression in that they require a greater degree of precision and completeness. When using a natural language to communicate, speakers can be ambiguous and make small errors, and still expect their intent to be understood. However, as mentioned earlier, computers do exactly what they are told to do, an cannot understand the code the programmer “intended” to write. The combination of the language definition and the program’s inputs must fully specify the external behavior that occurs when the program is executed.

Many languages have been designed from scratch, altered to meet new needs, combined with other languages and eventually fallen into disuse. Although there have been attempts to design one “universal” computer language that serves all purposes, all of them have failed to be accepted in this role. The needs for diverse computer languages arises from the diversity of contexts in which languages are used.

One can of courses use any language for writing a computer program according to the need. The language that any computer can actually understand and execute is its own native binary machine code (in the form of sequences of 1s and 0s). Programs written in any other language must be translated to the binary representation of the instructions before they can be executed by the computer. Computer programs can be broadly classified as follows.

17.2 Objectives

A computer consists of two basic parts Hardware and Software.

- What is programming language
- The difference between machine language, assembly language and high-level language.
- The concept of object-oriented programming
- About various programming language generations.

17.3 Machine Language

The machine language can be defined as a sequence of instructions written in the form of binary numbers consisting of 1s and 0s. Thus, in machine language the instructions are patterns of bits with different patterns corresponding to different commands to the machine. Since a machine language consists only of 1s and 0s, a computer can respond directly to it. The machine language is lowest possible level of language in which it is possible to write a computer programme. All other languages are said to be high level according to how closely they can be said to resemble machine code. Initially the machine language was referred to as a code but now-a-days the word code has a broader meaning and it refers to any program text.

Every Central Processing Unit (CPU) model has its own machine code, or instruction set. Successor or derivative processor designs may completely include all the instructions of a predecessor and may add additional instruction. Some nearly completely compatible processor designs may have slightly different effects after similar instructions. Occasionally a successor processor design will discontinue or alter the meaning of a predecessor's instruction code, making migration of machine code between the two processors more difficult. Even if the same model of processor is used two different systems may not run the same example of machine code if they differ in memory arrangement operating system or peripheral devices; the machine code has now embedded information about the configuration of the system.

A machine code instruction set may have all instructions of the same length, or may have variable-length instructions. How the patterns or organized depends largely on the specification of the machine code. Common to most is the division of one field (often known as the 'opcode') which specifies the exact operation (for example "add"). Other fields may give the type of the'operands', their location, or their value directly(operands contained in an instruction are called immediate). Some exotic instruction sets do not have an opcode field (such as Transport Triggered Architectures or the Forth virtual machine) and have operands(s) only. Other instruction sets lack any operand fields.

The main advantage of a machine language is that a program written in a low-level language can be extremely efficient, making optimum use of both computer memory and processing time since the computer directly starts executing it. However, writing as well as understanding a machine language is a tedious task since writing a low-level program takes a substantial amount of time as well as a clear understanding of the inner workings of the processor itself. This is the cause that a low-level programming language is typically used only for very small programs, or for segments of code that are highly critical and must run as efficiently as possible.

17.4 Assembly Language and Assembler

An assembly language is a low-level language for programming computers. It implements a symbolic representation of the numeric machine codes and other constants needed to program a particular CPU architecture. This representation is usually defined by the hardware manufacturer, and is based on abbreviations (called *mnemonics*) that help the programmer remember individual instructions, registers, etc. An assembly language is thus specific to a certain physical or virtual computer architecture (as opposed to most high level languages, which are portable).

A program written in assembly language consists of a series of *instructions* mnemonics that correspond to a stream of executable instructions that can be loaded into memory and executed. For example, an x86/IA-32 processor can execute the following binary instruction as expressed in machine language:

- Binary: 10110000 01100001 (Hexadecimal: 0xb061)

The equivalent assembly language representation is easier to remember (more mnemonic):

- Moval, #061h

This instruction means:

- Move the hexadecimal value 61 (97 decimal) into the processor register named "al".

The mnemonic "mov" is an *operation code or opcode*, and was chosen by the instruction set designer to abbreviate "move". A comma- separated list of arguments or parameters follows the opcode; this is a typical assembly language statement. (Actually, 10110000 is the actual opcode; "mov" is its corresponding assembly language *opcode mnemonic.* However, in practice many programmers drop the word *mnemonic* and technically incorrectly, call "mov"an *opcode*.

When they do this, they are referring to the underlying binary code which it represents. To put it another way, a mnemonic such as “mov” is not an opcode, but as it symbolizes an opcode, one might refer to “the opcode mov” for example, when one intends to refer to the binary opcode it symbolizes rather than to the symbol the mnemonic- itself.

Instructions in assembly language are generally very simple and typically consists of an *operation or opcode*plus zero or more operands. Most instructions refer to a single value, or pair of values. Generally, an opcode is a symbolic name for a single executable machine language instruction. Operands can be either immediate or the addresses of data elsewhere in storage.

Assembly languages were first developed in the 1950s, when they were referred to as second generation programming languages. They eliminated much of the error prone and time consuming first generating programming needed with the earliest computers, freeing the programmer from tedium such as remembering numeric codes and calculating addresses. They were once widely used for all sorts of programming. However, by the 1980s (1990s on small computers), their use had largely been supplanted by high level languages in the search for improved programming productively. Today assembly languages in the search for improved programming productivity. Today assembly language is used primarily for direct hardware manipulation or to address critical performance issues. Typical uses are device drivers, low-level embedded systems, and real time systems.

We illustrate assembly language by giving an example of adding two numbers and then storing the result in some memory location. The codes defined here are used for illustration only and they are not matched with actual mnemonics.

LDX, 10 : Load register X with 10

LDY, 23 : Load register Y with 23

ADD X, Y : $X \leftarrow X+Y$

LD (100), X : Save the result in the location 100

HALT : Halt process

Assembly language is a low-level language because the structure of the language reflects the instruction set (and architecture) of the CPU and is considered to be a second-generation language. LD, ADD, HALT, etc. used in the above program are called mnemonic codes. The use of mnemonics no doubt increases the readability of the program as compared to machine language. However, this readability is only for the benefits of the programmer and not for the computer. A computer cannot directly execute an assembly language since it is not in the form of binary codes. The assembly language program must be translated into machine code so that the computer can execute it. The program used for this purpose is called an *assembler*.

The assembler program recognizes the character strings that make up the symbolic names of the various machine operations and substitutes the required machine code for each instruction. At the same time, it also calculates the required address in memory for each symbolic name of a memory location, and substitutes those addresses for the names. The final result is a machine language program that can be directly executed by a computer at any time. Once the assembler has converted the assembly language into binary form or in machine language form, the assembler and the assembly language program are no longer needed. To help distinguish between the "before" and "after" versions of the program the original assembly language program is known as the *source code,* while the final machine language program is designated the *object code*. Fig 2.1 explains the above process explicitly.

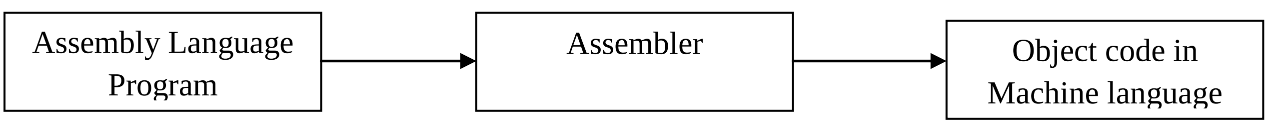

Fig. 2.1 Functioning of an assembler

Most assemblers also support *pseudo-operation,* which aredirectives obeyed by the assembler at assembly time instead of the CPU at run time. For example, pseudo-ops would be used to reserve storage areas and optionally set their initial contents. Often the names of *pseudo-ops* start with a dot to distinguish them from machine instructions. Some Assemblers also support *pseudo- instructions*, which generate two or more machine instructions.

An important task of assembler is to check for possible errors in the symbolic program. This is called *error diagnostics.* One such error may be an invalid machine code symbol if it is detected in the program. An assembler cannot translate a symbol if it is not taken from the list of available codes and, as such, it does not know its binary equivalent. In such a case, the assembler prints an error message to inform the programmer that his symbolic program has an error at a specific line of code. Another possible error may occur if the program has a faulty symbolic address. Other errors may also occur if the program has a faulty should detect all such error may also occur and practical assembler should detect all such errors and convey messages accordingly.

One of the greatest demerits of assembly language is that it is specific to particular machine architecture. Assembly language programs written for one processor will not work on a different processor if it is architecturally different. In other words, we can say that assembly language programs are not portable. Programmers still use assembly language when speed is an essential component or when they need to perform an operation that is not possible using a high-level language.

17.5 High-Level Language

The main disadvantages of machine and assembly languages were that the time and cost of creating these languages were quite high. This was the prime motivation for the development of high-level languages. A high-level programming language is more abstract to use and portable in comparison to low level languages.

The terms “high level language” does not imply that the language is always superior to low level programming languages. Rather high-level languages refer to the higher level of abstraction from machine language. Rather than dealing with registers, memory addresses and call stacks, high level languages deal with variables, arrays and complex arithmetic or Boolean expressions. In other words, we can say that thigh level languages make complex programming simpler, while low level languages tend to produce more efficient code. Portability is also a feature of high-level languages. Programs written in high level language will run on a variety of different systems and will produce the same results regardless of the platform. Besides high-level languages are more expressive and secure. Fortran 90, FORTRAN 77, ADA, Pascal C, JAVA etc. are examples of high-level languages. The following figure gives the hierarchy of programming languages.

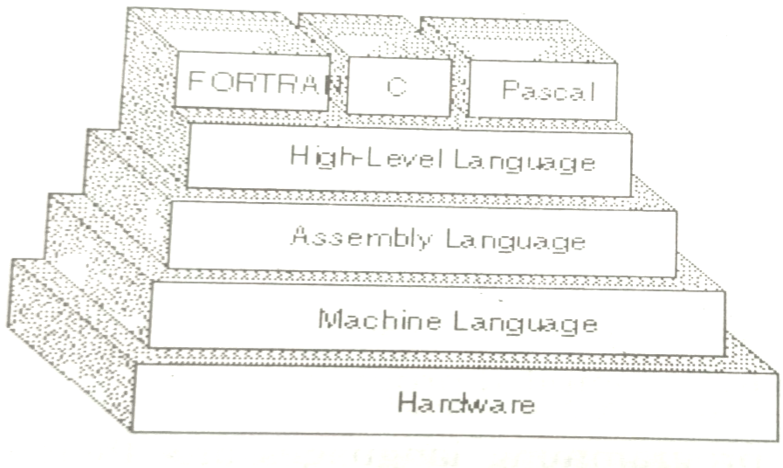

Figure 2.2: Hierarchy of programming languages

A high-level source program must be translated into the form the machine can understand. This is achieved by compiling (translating) a high-level language program with a special piece of software called a *complier*. A complier takes the source code as input and produces as output the machine language code of the machine on which it is to be executed. During the process of translation, compiler reads the source program statement wise and checks the syntax errors. If there is any error the computer generates the print out of the errors. This action is known as *debugging.*Typically, the complied machine code is less efficient than the code produced when using assembly language. This means that it runs a bit more slowly and uses a bit more memory than the equivalent assembled program. To offset this drawback however we also program, so it can be ready to go sooner than the assembly language program.

Interpreter is another type of translator which does the translation from high level language to low level language to low machine language. Interpreters are easy to write and they do not require large memory and spaces in the computers. Such a program resides in memory and directly executes the high-level program by taking one statement at a time and translating it into the machine instruction. This process directly executes the use's program has both advantages and disadvantages.

The primary advantage is that you can run the program to test its operation make a few changes, and run it again directly. There is no need to recompile because no new machine code is ever produced. This can enormously speed up the development and testing process. On the down side, this arrangement requires that both the interpreter and the user's program reside in memory at the same time. In addition, because the interpreter has to scan the user's program one line at a time and execute internal portions of itself in response; execution of an interpreted program is much slower than for a complied program.

Higher level programming languages are, therefore divided for convenience into complied languages and interpreted languages. However, there is rarely anything about a language that requires it to be exclusively complied or exclusively interpreted. The categorization usually reflects the most popular or widespread implementations of a language – for instance, BASIC is thought of as interpreted language, and C a complied one, despite the existence of BASIC compilers and C interpreters.

17.6 Object Oriented Programming

The leaps and bound growth of the computer sophistication in modern times as incited in users to demand for software that harness the capabilities of sophisticated hardware such as the Macintosh-II and the PS/2 family of computers. In addition to these users are no more satisfied with the applications that are not user friendly or that have numerous bugs that have to be worked around. A recent approach in this direction is *object-oriented programming language (OOPL)*. The genesis of this technology dates back to the early 1960s with the work of Nygaard and Dahl in the development of the first object-oriented language called *Simula67.* Research progressed through the 1970s with the development of *Smalltalk* at Xerox.

Object oriented programming (OOP) is a programming paradigm that uses "object" and their interactions to design applications and computer programs. Here, objects are the modular units that are made by combining the groups of operations and data. Every object has both state (data) and behavior (operations on data). In that they are not much different from ordinary physical objects. This resemblance to real things has given objects much of the power and appeal. They can not only model components of real systems, but equally as well fulfill assigned roles as components in software systems. OOPL lets us to combine objects into structured networks to form a complete program. One of the principal advantages of object-oriented programming techniques over procedural programming techniques is that they enable programmers to create modules that do not need to be changed when a new type of object is added. A programmer can simply create a new object-oriented program easier to modify. In an OOPL objects and object interactions are the basic elements of design. In addition to this it is based on several other techniques viz., Inheritance Modularity, Polymorphism and Encapsulation.

Critical to understanding of OOPL is the concept of inheritance. Objects are defined and then used to build a hierarchy of descendant objects. Each of the descendant objects has the inheritance to access to methods used by ancestor objects.

Modularity is the property that measures the extent to which the programs have been composed out of separate parts called modules. A modular approach to programming is gaining popularity in fields of artificial intelligence system integration, where a large-scale general system is composed of modules that each serve a specific purpose and communicate with each other to produce the system's overall behavior.

Polymorphism in object-oriented programming is the ability of objects belonging to different data types to respond to method calls of method of the same name, each one according to an appropriate type specific behavior. One method or an operator such as +, -, or *, can be abstractly applied in many different situations. Polymorphism is the use of one operator such as "+", to operator, for example may be used to perform integer addition, float addition, list concatenation or string concatenation.

Encapsulation conceals the functional details of a class from objects that send messages to it. Encapsulation is achieved by specifying which classes may use the members of an object. The result is that each object exposes to any class a certain interface – those members accessible to that class. The reason for encapsulation is to prevent clients of an interface from depending on those parts of the implementation that are likely to change in future thereby allowing those changes to be made more easily, that is without changes to clients.

Object oriented (OO) applications can be written in either conventional languages or OOPLs, but they are much easier to write in languages especially designed for OO programming. OO language experts divide OOPLs into two categories, hybrid languages and pure OO languages. Hybrid languages are based on some non-OO model that has been enhance with OO concepts. C++ (a superset of C), Ada95, and CLOS (an object-enhanced version of LISP) are hybrid languages. Pure OO languages are based entirely on OO principles; Smalltalk, Eiffel, Java and Simula are pure OO languages.

17.7 Programming Language Generations

In this section, we have categorized the languages according to various generations. Languages in some of the categories have not been one or other category of the previous descriptions.

1GL or first- generation language was (and still is) *machine language* or the level of instructions and data that the processor is actually given to work on (which in conventional computers is a string of 0s and 1s).

2GL or second-generation language is assembly language. A typical 2GL instruction looks like this:

ADD 12, 8

As assembler converts the assembly language statements into the machine language.

3GL or third generation language is a programming language designed to be easier for a human to understand, including things like named variables. A fragment might be

b= c+2*d

Fortran, ALGOL and COBOL are early examples of this sort of language. Most "modern" language such BASIC, C, C++, Java and including COBOL, Fortran, ALGOL and third generation languages. Most 3GLs support structured programming. A complier converts the statements of a specific programming language into machine language. A 3GL requires a considerable amount of programming knowledge.

4GL or fourth generation language is designed to be closer to natural language than a 3GL. Languages for accessing databases are often described as 4GLs. A 4GL statement might look like this:

EXTRACT ALL CUSTOMER WHERE “PREVIOUS PURCHASES” TOTAL MORE THAN Rs. 5000

5GL or fifth generation language is programming that uses a visual or graphical development interface to create source language that is usually complied with a 3GL or 4GL complier. Microsoft, Borland, IBM, and other companies make 5GL visual programming products for developing applications in Java, for example. Visual programming allows you to easily envision object-oriented programming class hierarchies and drag icons to assemble program components.

17.8 Exercises

1. What do you mean by programming language? Mention its uses.
2. Distinguish between Machine Language and programming language?
3. How does assembly language differ from programming language and machine language?
4. Describe high level language.
5. Explain the meaning of object-oriented programming?

17.9 Summary

A **programming language** is an artificial language that can be used to control the behavior of a machine, especially a computer.

In machine language the Instructions are patterns of bits with different patterns corresponding to different commands to the machine. It is specific to a particular processor and therefore, writing a machine language program requires detailed knowledge of the internal structure.

Assembly Language is a symbolic representation of the numeric machine codes and other constants needed to program a particular CPU architecture. This representation is usually defined by the hardware manufacture, and is based on abbreviations, called mnemonics. This is also machine dependent. The assembly language program is translated into machine code before it is executed by the computer. The program used for this purpose is called an assembler.

High level language refers to the higher level of abstraction from machine language. Rather than dealing with registers, memory addresses and call stacks, high level languages deal with variables, arrays and complex arithmetic or Boolean expressions. Theses languages are machine independent and procedure oriented. A program written in high level language is translated to the equivalent machine code by a translator program.

The translator program can be either **Interpreter** or **Complier**. The former translates the program one statement at a time whereas the latter translates the entire program to machine code before the execution.

Object- oriented programming (OOP) is a paradigm that uses “object” and their interactions to design applications and computer programs. Here objects are the modular units that are made by combining the groups of operations and data. OOP language lets us to combine objects into structured networks to form a complete program.

17.11 Further Readings

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