



Uttar Pradesh Rajarshi Tondon
open University

Bachelor of Science

UGMM -105

Mechanics - I (Statics and Dynamics)

MECHANICS –I (STATICS AND DYNAMICS)

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BLOCK

1

Statics

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Course Design Committee

Prof. Ashutosh Gupta, School of Computer and Information Science, UPRTOU, Prayagraj	Chairman
Prof. Sudhir Srivastav Dept. of Mathematics, DDU Gorakhpur University, Gorakhpur	Member
Prof. P. K. Singh Dept. of Mathematics, Allahabad University, Prayagraj	Member
Prof. Mona Khare Dept. of Mathematics, Allahabad University, Prayagraj	Member
Dr. A. K. Pandey Associate Professor, ECC Prayagraj	Member
Dr. Vikas Singh Academic Consultant, UPRTOU, Prayagraj	Member
Dr. S. S. Tripathi Academic Consultant, UPRTOU, Prayagraj	Member

Course Preparation Committee

Dr. Archana Shukla Assistant Professor (Dept. of Mathematics), Nehru gram bharti university, Prayagraj	Author (Unit – 1 -3, 7)
Ms. Samriddhi Saxena Assistant Professor (Dept. of Mathematics) Invertis University, Bareilly (U.P.)	Author (Unit – 4 - 6)
Dr. S. S. Tripathi Academic Consultant School of Science, UPRTOU, Prayagraj	Editor (Unit 1 - 7)
Prof. Ashutosh Gupta Director, School of Computer and Information Science, UPRTOU, Prayagraj	

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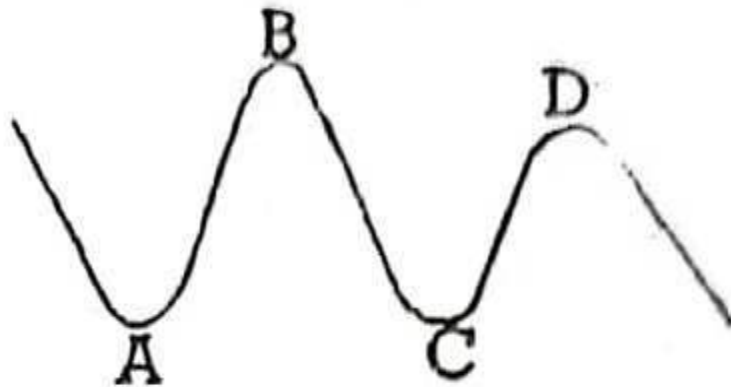
UNIT-1: Stable and Unstable equilibrium

Structure

- 1.1. Introduction**
- 1.2. Objectives**
- 1.3. Stable and unstable equilibrium (Definition and Examples)**
- 1.4. Stability**
- 1.5. Condition of stability (One degree of freedom)**
- 1.6. A heavy body on a fixed body**
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1.1 Introduction:-

Consider the motion of a body on a smooth curve in a vertical plane as shown in the figure, obviously the body can rest at points A, B, C and D. Which are points of maxima or minima of the curve. If the body be slightly displaced from its position of rest at A or C (i.e. the points of minima), it will tend to return to its original position of rest, while if displaced from its position of rest at B or D (i.e. the points of maxima). It will tend to move still further away from its original position of rest. In the first case the equilibrium of the body is said to be stable and in the second case it is said to be unstable



Take one more illustration. Consider the equilibrium of a rigid body fixed at one point say A. For the equilibrium of the body the centre of gravity G of the body must lie on the vertical line through the point of support A. There arise three cases.

Case 1- Let us consider the center of gravity G lies below the point of support A. In this case if the body be slightly displaced from its position of equilibrium its centre of gravity will be raised. If the body be then let free, the force of gravity will bring the body back to its original position of equilibrium. In this case the body is said to be in stable equilibrium.

Case 2- Next let us consider that the centre of gravity G lies above the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be lowered. If the body be then let free, the force of gravity will still further move away the body from its original position of equilibrium. In this case the body is said to be in unstable equilibrium.

Case 3- If the centre of gravity G is at the point of support A , the body will still be in equilibrium when displaced. In this case we say that the body is in a state of neutral equilibrium.

Remark- It can be seen that among all the possible positions of the body, In the case 1 the height of the centre of gravity of the body above some fixed plane is minimum and in the case2 it is maximum.

1.2 Objectives-

After reading this unit Students must be able to-

- ☐ Difference between stable and unstable condition
- ☐ How will be use stable and unstable condition
- ☐ Understand one degree of freedom
- ☐ Understand a heavy body on a fixed body

1.3. Stable and unstable equilibrium (Definitions and example.)

- (1) Stable equilibrium- A body is said to be in stable equilibrium If when slightly displaced from its position of equilibrium, the forces acting on the body tends to make it return toward its position of equilibrium.
- (2) Unstable equilibrium- The equilibrium of a body is said to be unstable If when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.
- (3) Neutral equilibrium- A body is said to be in neutral equilibrium If the forces acting on it are such that they keep the body in equilibrium in any slightly displaced position.

Some more example of stable, unstable and neutral equilibrium :-

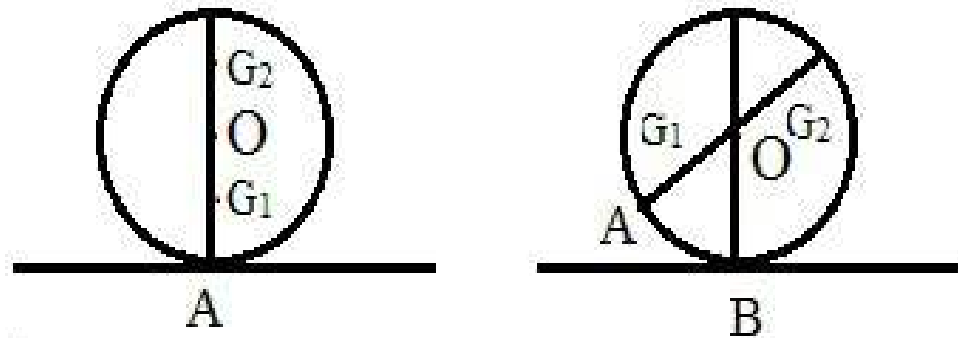
Consider the case of a heavy sphere resting on a horizontal plane. Suppose the centre of gravity of the sphere is not at its geometric centre O . It is obvious that for the equilibrium of the sphere its point of contact A with the plane, its geometric centre of gravity O and its center of gravity must be in the same in the same vertical line. One position of equilibrium is in which the. Center of gravity. G_1 is below the geometric centre O . In this case If the sphere be slightly displaced. It would tend to come back to its original position of

equilibrium. This is the position of stable equilibrium. The other position of equilibrium is in which the centre of gravity, G_2 is above the geometric centre O. In this case If the sphere be slightly displaced it would not come back to its original position of equilibrium but would go further away that position. This is the position of unstable equilibrium.

If however the centre of gravity of the sphere is at its geometric centre O the sphere will still be in equilibrium when displaced. In this case the equilibrium is neutral.

If a right circular cone rests on a horizontal plane with its base in contact with the plane and its axis vertical, its equilibrium is stable but If it rests with its vertex in contact with the plane and its axis vertical, it equilibrium is unstable. Again If it rests along a generator, it is in neutral equilibrium.

The equilibrium of a pendulum is stable when it is displaced from its vertical position of equilibrium, because it returns to the vertical position again. Anytop, heavy body or a stick placed vertically on a finger is an example of unstable equilibrium.



1.4. Stability:-

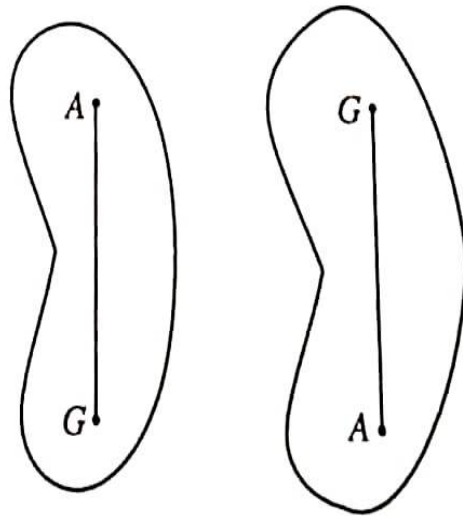
Consider an object hinged at a point O. There is equilibrium so long as its centre of gravity G lies in the vertical line from O because its weight then acts through O and there is no moment tending to turn the body about O. But the position nature of equilibrium will differs with the position of G. There are three cases.

Case 1- The G lies below the point of support. If the body is slightly displaced from the position of rest, there are two equal and opposite forces acting on the body on through from O and the other through from G. These form a couple which tends to restore the body to its original position. In this case the body is called to be in stable equilibrium.

Case 2- The center of gravity G lies above the point of support. In this case If the body is slightly displaced from its position of equilibrium, its centre of gravity would be lowered. If the body is then let free, the force of gravity will still further more away. The body from its original position of equilibrium, In this case the body is called to unstable equilibrium.

Case 3- If the centre of gravity G is at the point of support A , the body would still be in equilibrium when displaced. We can say that the body is in a state of natural equilibrium.

Remark:- It can be seen that among all the possible position of the body, In case 1 the height of the C.G, of the body above some fixed plane is minimum and in case 2 it is maximum.



Example. 1-

A body rests in equilibrium upon another fixed body, the portion of two bodies in contact. Being spheres of radius r and R respectively, and the straight line joining the centre of the spheres being vertical, If the first body be slightly displaced to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent sliding.

Solution-

Let O is the centre of the spherical surface of the lower body, and O_1 be the upper body. Let A_1G_1 is h .

Let the upper body be slightly displaced by rolling, so that O_2 is the new position of the center of the upper body. A_2 is the new contact point. G_2 be the

new position of centre of gravity and the new position of the point B_1 will be C hence CG_2 in h.

Draw B_2L vertically to meet O_2C in L, and O_2M vertically to meet a horizontal line through B_2 in M.

Let $\angle B_2OB_1 = \theta$ and $\angle B_2OC = \phi$ so that $\angle CO_2M = (\theta + \phi)$

since the upper body has rolled into its new position, the arc B_1B_2 = the arc CB_2 so that

$$R \cdot \theta = r \cdot \phi \dots\dots\dots (1)$$

Where R and r are respectively the radius of the lower and upper surfaced. The equilibrium is stable or unstable according to G_1 lies to the left, or right of the line B_2L

i.e. According as the distance of G_2 from O_2N is $>$ or $<$ B_2N

i.e. According as $(r - h)\sin(\theta + \phi)$ is $>$ or $<$ $r\sin\theta$.

i.e. According as $(r - h)\sin\left(\frac{R+r}{r}\theta\right)$ $>$ or $<$ $r\sin\theta$.

i.e. According as $(r - h)\left[\frac{R+r}{r}\theta - \frac{1}{\text{L}3}\left(\frac{R+r}{r}\right)^3\theta^3 + \dots\dots\dots\right]$ $>$ or $<$ $\left[\theta - \frac{\theta^3}{\text{L}3} + \dots\dots\dots\right]$ -----(2)

by substituting the expansion for the sine of angle in forms of the angle,

i.e. according as

$$(r - h)\left[\frac{R+r}{r} - \frac{1}{\text{L}3}\left(\frac{R+r}{r}\right)^2\theta^2 + \dots\dots\dots\right] > \text{or} < r\left[1 - \frac{\theta^3}{\text{L}3} + \dots\dots\dots\right]$$

i.e. when θ is made indefinitely small, according as

$$(r - h)\frac{R+r}{r} > \text{or} < r$$

i.e. according as $rR >$ or $h(R + r)$

i.e. according as $\frac{1}{h} >$ or $<$ $\frac{1}{R} + \frac{1}{r}$ -----(3)

in the case when $\frac{1}{h} = \frac{1}{R} + \frac{1}{r}$, i.e. when $h = \frac{Rr}{R+r}$,

we must return to equation (2) and the equilibrium is stable or unstable according as

$$\frac{r^2}{R+r} \left[\frac{R+r}{r} \theta - \frac{1}{\frac{1}{3}} \left(\frac{R+r}{r} \right)^3 \theta^3 + \dots \right] > \text{or} < r \left[\theta - \frac{\theta^3}{\frac{1}{3}} + \dots \right]$$

i.e. according as $-\frac{1}{6}(R+r)^2\theta^3 + \text{higher powers of } \theta > \text{or} < -\frac{1}{6}r^2\theta^3 + \dots$

i.e. according as $(R+r)^2 - sqs.$ etc of $\theta < \text{or} > r^2 - sqs.$ etc. of θ ,

i.e. according as $(R+r)^2 < \text{or} > r^2$ when θ is made indefinitely small.

Hence in this case equilibrium is unstable. The equilibrium is thus only stable when $\frac{1}{h} > \frac{1}{R} + \frac{1}{r}$ In all other cases it is unstable.

If the curvature of the lower surface is in the other direction as in the second figure then in this case the angle

$$CO_2N = \phi - \theta = \frac{R-r}{r} \theta$$

and the equilibrium is stable or unstable according as G_2 is to the left or right of B_2L .

i.e. according as $O_2G_2 \sin(\phi - \theta) < NB_2$

i.e. according as $(h-r) \sin \frac{R-r}{r} \theta < r \sin \theta$.

i.e. according as $(h-r) \left[\frac{R-r}{r} \theta - \frac{1}{\frac{1}{3}} \left(\frac{R-r}{r} \theta \right)^3 + \dots \right]$

$> \text{or} < \left[\theta - \frac{\theta^3}{\frac{1}{3}} + \dots \right] \text{-----} (4)$

i.e. according as $(h-r) \left[\frac{R-r}{r} - \frac{1}{6} \left(\frac{R-r}{r} \right)^3 \theta^2 + \dots \right] > \text{or} < r \left[1 - \frac{\theta^2}{6} + \dots \right]$

i.e. according as $(h-r) \frac{R-r}{r} > \text{or} < r$, if θ is indefinitely

i.e. according as $h > \text{or} < \frac{Rr}{R-r}$, i.e. accordingly as $\frac{1}{h} > \text{or} < \frac{1}{r} - \frac{1}{R}$

In the critical case when $h = \frac{Rr}{R-r}$ we must return to (4) and proceed to higher powers then becomes

$$\frac{r^2}{R-r} \left[\frac{R-r}{r} \theta - \frac{1}{3} \left(\frac{R-r}{r} \theta \right)^3 + \dots \right] > \text{or} < r \left[\theta - \frac{\theta^3}{3} + \dots \right]$$

$$\text{i.e. } -\frac{1}{6} \frac{(R-r)^2}{r^2} \theta^3 + \dots > \text{or} < 1 - \frac{\theta^3}{6} + \dots$$

$$\text{i.e. } \frac{(R-r)^2}{r^2} - \text{powers of } \theta > \text{or} < -\text{powers of } \theta$$

Hence when θ is indefinitely small, we see that the equilibrium θ is stable or unstable according as

$$(R-r)^2 > \text{or} < r^2$$

$$\text{i.e. according as } R > 2r \text{ \& } R < 2r$$

$$\text{In this case when } R = 2r \text{ and hence } h = \frac{Rr}{R-r} = 2r$$

$$\text{then } \phi = \frac{R}{r} \theta = 2\theta$$

$$\text{and } O_2 G_2 \sin(\phi - \theta) = (h - r) \sin(\phi - \theta) = r \sin \theta = NB_2 \text{ always.}$$

Hence, in this particular case, G_2 always coincides with L ; and the upper body would always rest from. Whatever angle it be rolled since its C.G is now always vertically above the point of contact

Cor.1. If the upper body have a plane face in contact with the lower body, as in the following figure r is infinite in value. Hence the equilibrium is stable if

$$\frac{1}{h} be > \frac{1}{R} \text{ i.e. if } h be < R$$

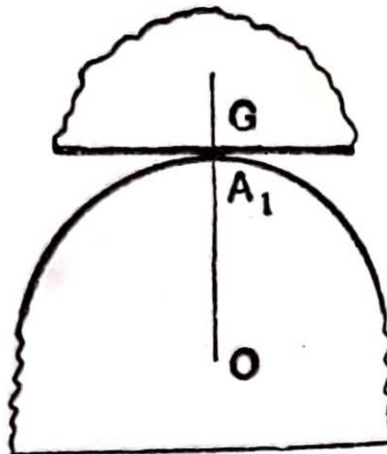


Fig.

Hence the equilibrium is stable, if the distance of the C of G of the upper body through its plane face be less than the radius of the lower body; otherwise the equilibrium is unstable.

Cor. 2. If the lower body be a plane so that R is infinite, the equilibrium is stable if

$$\frac{1}{h}be > \frac{1}{r} \text{ i.e. if } hbe < r$$

Hence if a body of spherical base be placed on a horizontal table, it is in stable equilibrium, If the distance of its centre of gravity through the point of contact be less than the radius of the spherical surface.

Check your progress

Question.1 A hemisphere rests in equilibrium on a sphere of equal radius; show that the equilibrium is unstable when the curved surface, rests on the sphere and stable when the flat surface of hemisphere rests on the sphere.

Question. 2 A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that however large a weight is attached to the highest of the sphere, the equilibrium is stable.

Question. 3 A heavy cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube is $\frac{\pi r}{2}$, show that the cube can rock through a right angle without falling.

Question.4 A rod SH of length 2c and whose centre of gravity G is at a distance d from its centre, has a string of length $2c \sec \alpha$, tied to its two ends. and the string is then slung over a small smooth peg p, find the position of equilibrium and show that the position which is not vertical is unstable.

Question.5 A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse, If the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.

1.5. Condition of stability- One degree of freedom-

A body is called to have one degree of freedom when it is so constrained that only one geometrical quantity is required to fix its position. For such body, the work done in bringing it from any standard position may be given as

$$W = f(\theta)$$

where θ is the quantity that defines the position of the body. Now If F is the resultant force on the body and δs the displacement of the body in the direction of F then work done

$$d \cdot w = Fds, \quad i.e. f'(\theta) = Fds$$

or
$$F = f'(\theta) \frac{d\theta}{ds} \dots\dots\dots(1)$$

In a position of equilibrium, $F=0$,

then
$$f'(\theta) \frac{d\theta}{ds} = 0$$

As there are a number of geometrical quantities, each of which could be used to fix the position of a body with one degree of freedom we choose θ to be that one which continually increase or decreases as the body moves in the same direction with this assumption, $\frac{d\theta}{ds} \neq 0$

then for equilibrium

$$f'(\theta) = 0 \dots\dots\dots(2)$$

Assume that a root of this equation is θ_0 . Then we have to test the equilibrium whether it is stable or unstable for $\theta = \theta_0$. Now If the force and the displacement have opposite signs, equilibrium will be stable, thus for stability, $\frac{dF}{ds}$ must be negative. from (1)

$$\left(\frac{dF}{ds}\right) \theta_0 = f''(\theta_0) \left(\frac{d\theta}{ds}\right)^2 \theta_0 + f'(\theta_0) \left(\frac{d^2\theta}{ds^2}\right) \theta_0$$

$$f''(\theta_0) = \left(\frac{d\theta}{ds}\right)^2 \theta_0$$

since $f'(\theta_0) = 0$. Hence for stability $f''(\theta_0)$ must be negative.

It follows, therefore that If $w = f(\theta)$, the conditions for stable equilibrium are

$$f'(\theta_0) = 0 \text{ and } f''(\theta_0) < 0$$

These are also the conditions for a maximum w for $\theta = \theta_0$ similarly it may be shown that If w is a minimum, the equilibrium is unstable.

If w is neither a maximum nor a minimum the equilibrium is either neutral or unstable. The equilibrium is neutral. If dF is absolutely zero. But If dF has the same sign as ds for displacement in one direction, and the opposite sign for displacement in the other direction, then after any small displacement, the body

certainly moves on the unstable side. after a short time and To equilibrium is really unstable.

When the forces are conservative the potential energy V of the body is given by

$$V = C - W$$

Where C is a constant. Thus in the case of a conservative system the equilibrium is stable or unstable according as the potential energy is a true minimum or maximum.

For example. If gravitational energy is the only form of potential energy the height of the C.G must be a minimum for stable equilibrium.

Example 1- Show that in the position of stable equilibrium, the potential energy of the body is minimum.

Solution- When the force of gravity is the only external force acting on a body, we arrive at the following rule for deciding whether a given equilibrium position is stable or unstable.

Suppose that the body is tilted from its position of equilibrium, the measure of tilt being given by θ . Express the height of the centre of gravity of the body in terms of the variable θ . If this height be given by $z = f(\theta)$, the position of equilibrium are given by.

$\frac{dz}{d\theta} = f'(\theta) = 0$ we then find the sign of $\frac{d^2z}{d\theta^2}$ for the roots of $f'(\theta) = 0$ If the sign is positive for a root the height of the C.G is minimum and the equilibrium is stable for that θ , If the sign is negative the C.G. is at a maximum height and the equilibrium is unstable.

In other cases it is convenient to find instead of the height Y of the centre of gravity above a fixed plane, the depth x of the centre of gravity below a fixed plane, In such cases, If the depth x is minimum, the equilibrium is unstable and If the depth x is maximum the equilibrium is stable.

1.6 A heavy body resting on a fixed body:-

We shall assume that

- (i) the upper body is free to roll, without, sliding on the lower body
- (ii) the portion of the two bodies in contact are spherical of radius r and R . respectively, and
- (iii) the straight line joining their centres is vertical initially.

Let O and O' be the centres of the spherical surface. The figure below is a section of the bodies through G , the centre of gravity of the upper one.

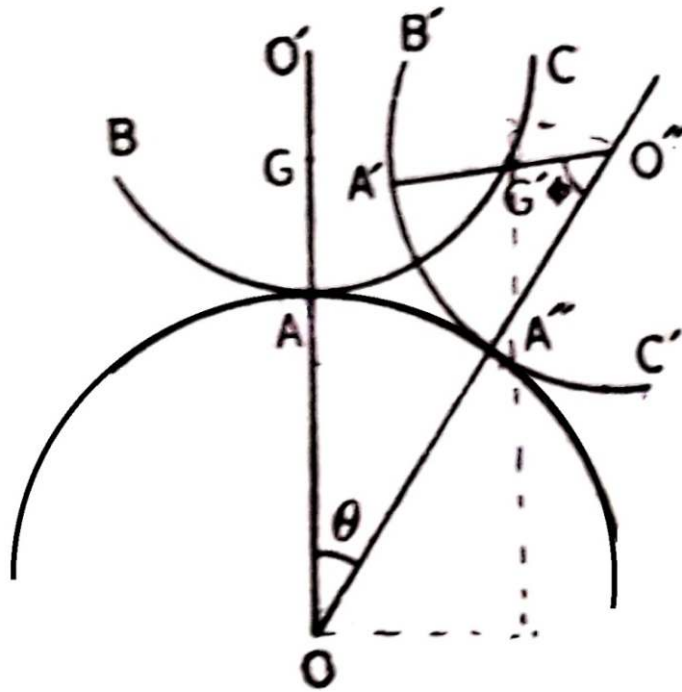


Fig. - 3

Let the upper body BAC be displaced into the position, $B'A'C'$, the new point of contact being A'' . Let C' be the new position of the centre of gravity of the upper body, which now lies on $O''A'$, the new position of $O'A$. Let

$$\angle AOA'' = \theta, \quad \angle A'O''A'' = \phi$$

Since we suppose that the upper body is displaced by rolling on the lower, we have

$$\text{arc } AA'' = \text{arc } A'A''. \text{ i.e. } R\theta = r\phi$$

If z be the height of G' above O , we have

$$\begin{aligned} z &= OO'' \cos \theta - O''G' \cos(\theta + \phi) \\ &= (R + r) \cos \theta - (r - h) \cos \left(1 + \frac{R}{r}\right) \theta \end{aligned}$$

where $AG = h$

Now the equilibrium, $\frac{dz}{d\theta} = 0$ hence

$$-(R+r)\sin\theta + (r-h)\left(\frac{r+R}{r}\right)\sin\left(\frac{r+R}{r}\theta\right) = 0$$

This is satisfied by $\theta = 0$

To find the sign of $\frac{d^2z}{d\theta^2}$ for $\theta = 0$

we have

$$\frac{d^2z}{d\theta^2} = -(R+r)\cos\theta + (r-h)\left(\frac{r+R}{r}\right)^2 \cos\left(\frac{r+R}{r}\theta\right),$$

Which for $\theta = 0$ reduces to

$$-(R+r) + (r-h)\left(\frac{r+R}{r}\right)^2, \text{ i.e. } \left(\frac{r+R}{r}\right) \left[-\frac{r^3}{r+R} + r \cdot h\right]$$

$$\text{i.e. } \left(\frac{r+R}{r}\right)^2 \left[\frac{rR}{r+R} - h\right]$$

The equilibrium is stable or unstable according as

$\frac{d^2z}{d\theta^2}$ is positive or negative,

$$\text{i.e. } \frac{rR}{r+R} > \text{ or } r < \text{, i.e. } \frac{1}{h} \text{ or } < \frac{1}{r} + \frac{1}{R}$$

Particular cases

Case 1- The lower surface is concave instead of being convex near A. In this case R is negative and the equilibrium is stable or unstable according as

$$\frac{1}{h} > \frac{1}{r} - \frac{1}{R}$$

Case 2- The lower surface near A is plane. In this case R is infinite and the equilibrium is stable or unstable according as

$$h < \text{ or } > r$$

Case 3- The upper body has a plane face in contact with the lower body. In this case r is infinite and the equilibrium is stable or unstable according as

$$h < \text{ or } > R$$

Case 4- The critical case when

$$\frac{1}{h} = \frac{1}{r} + \frac{1}{R}$$

The determination of the stability in this case is a question of some difficulty and we will not discuss it here.

1.7. Solved Example:-

Example 1-

A lamina in the form of an isosceles triangle, whose vertical angle is B , is placed on A sphere of radius r , so that its plane is vertical and one of its equal sides is in contact with the sphere.

Show that If the triangle be slightly displaced in its own plane the equilibrium is stable if $\sin \beta < \frac{3r}{A}$, where A is one of the equal sides.

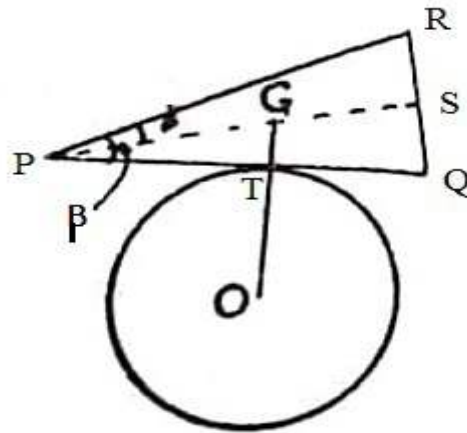


Fig. - 4

Solution-

Let PQR be the triangular lamina whose side, PQ, PR are equal, each of length A , and whose vertical angle QPR is β .

Let the lamina rest on the sphere whose centre is O and radius is r with point T of the side PQ in contact

Let PS be the median bisecting QR at right angles then the C.G. of the lamina is G , where $PG = \frac{2}{3}PS$

For equilibrium, G must be vertically above T . The height of G above T is

$$h = TG = PG \sin \frac{1}{2} \beta$$

$$\text{where } PG = \frac{2}{3}PS = \frac{2}{3}PQ \sin \frac{1}{2} \beta = \frac{2}{3}A \sin \frac{1}{2} \beta$$

Therefore $h = \frac{2}{3} A \cos \frac{1}{2} \beta \sin \frac{1}{2} \beta$

$$= \frac{1}{3} A \sin \beta$$

The equilibrium is stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

with the usual notation.

here r is infinite and $R=r$ Hence.

$$\frac{1}{h} > \frac{1}{r} \text{ i.e. } h < r$$

$$\text{i.e. } \frac{1}{3} A \sin \beta < r \text{ i.e. } \sin \beta < \frac{3r}{A}$$

Example 2-

A uniform rod resting with one end against a smooth vertical wall and with point in its length resting on a smooth peg. Find the position of equilibrium, and show that it is unstable.

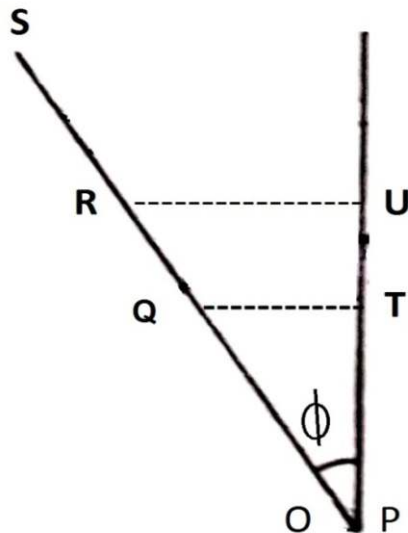


Fig.

Solution-

Let PQRS be the rod resting with its end P against the wall and point Q resting on a figure.

The C.G. of the rod is in middle point at R.

Let the rod make an angle θ with the help of wall and let RU, QT be the $\perp$ draw from R, Q on the wall.

Let us consider the length of rod is $2a$. i.e. $PR=a$ and the distance from the smooth peg from the wall is b i.e. $QT=b$

The height of R above Q is $Y=PU-PT$

Where $PU = PR\cos\theta = a\cos\theta$ and

$$AT = QT\cot\theta = b\cot\theta$$

Thus $Y = a\cos\theta - b\cot\theta$

For equilibrium,

$$\frac{dy}{d\theta} = -a\sin\theta + b\csc^2\theta = 0$$

Which gives $\sin^2\theta = \frac{b}{a}$ i.e. $\theta = \sin^{-1}\left(\frac{b}{a}\right)^{\frac{1}{3}}$

$$\text{Also } \frac{d^2y}{d\theta^2} = -a\cos\theta - \frac{2b\cos\theta}{\sin^2\theta} < 0$$

When θ has the above value, since $\sin\theta$ as well as $\cos\theta$, are positive.

Hence Y is a maximum in the position of equilibrium and the equilibrium is unstable.

Example-3

A thin hemispherical bowl of radius and weight W , rests in equilibrium on the highest point of a fixed sphere of radius b , which is rough enough to prevent any sliding inside the bowl is placed a small smooth sphere of weight w . Show that equilibrium is not stable unless, $w < \frac{a-b}{2b} W$.

Solution-

Let us consider initially the bowl, Centre O, rests on the sphere, centre C, contact with the point p.

When displaced, let us consider the contact point is Q, where $\text{arc } PQ = \text{arc } P'Q$ and $\angle PCQ = \theta$, $\angle P'OQ = \phi$

The small weight w , now we take the lowest position R in the bowl.

Since the radius of the sphere and the bowl are b and a respectively, we get

$$b\theta = \text{arc } PQ = \text{arc } p'p = a\emptyset$$

For stability of equilibrium (In the original position) the moment of W (In the displace position) about Q should have be greater than the moment of w, then we will get

$$W(OG \sin GOR - OQ \sin (QOR) > w(OQ \sin QOR)$$

$$\text{i.e. } W\left\{\frac{1}{2}asin(\theta + \emptyset) - asin\theta\right\} > w(bsin\theta),$$

$$\text{i.e. } W\left\{\frac{1}{2}\sin\left(1+\frac{b}{a}\right)\theta - \sin\theta\right\} > w\sin\theta,$$

or, Since θ is small, $W\left\{\frac{1}{2}\left(1 + \frac{a}{b}\right)\theta - \theta\right\} > W\theta$,

Which gives $W \frac{(a-b)}{2b} > w$, i.e. $w < W \frac{(a-b)}{2b}$

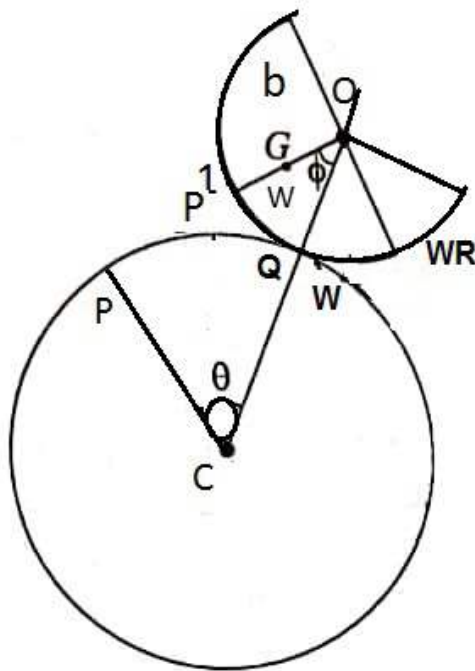


Fig. - 6

Solved examples

Example- A uniform rod of length L is attached by smooth rings at both ends to a parabolic wire, fixed with its vertical axis and vertex downwards, and of latus rectum $4a$, show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium, is given by $\cos^2 \theta = \frac{2a}{L}$

The condition of equilibrium of this position had been obtained by method of virtual work.

The height of centre of gravity of the rod above the vertex of the parabolic wire is

$$Y = \left(\frac{L^2}{4a} \right) \cos^2 \phi + a \tan^2 \phi$$

For equilibrium, $\frac{dy}{d\phi} = 0$ we get

$$-\left(\frac{L^2}{2a} \right) \cdot \cos \phi \sin \phi + 2a \tan \phi \sec^2 \phi = 0$$

$$-\left(\frac{L^2}{2a} \right) \cos \phi \sin \phi + 2a \tan \phi \sec^2 \phi = 0$$

Whichever either $\sin \phi = 0$ i.e. $\phi = 0$ (in horizontal position)

$$\cos^4 \phi = \frac{4a^2}{L^2}$$

$$\cos^2 \phi = \frac{2a}{L}$$

Example-4

A lamina in the form of a cycloid whose generating circle is a radius a , rests on the top of another cycloid whose generating circle is of radius b , their vertices being in contact and their axes vertical. If h be the height of C.G. of upper cycloid above O its vertex, Show that the equilibrium is stable only if $h < \frac{4ab}{a+b}$, and is unstable.

If $h \geq \frac{4ab}{a+b}$

Solution-

Cycloid $S = 4a \sin \Psi$ (upper)

$$\rho = \frac{ds}{d\Psi} = 4a \cos \Psi$$

A+ vertex A(Point of common contact)

$\Psi=0, \rho_1 = 4a$ at A

Similarly, for lower cycloid ρ_2 at A is $4b$

The equilibrium is stable. If

$$\frac{1}{h} > \frac{1}{4a} + \frac{1}{4b} \text{ or}$$

$$h < \frac{4ab}{a+b} \text{ and}$$

unstable If $h > \frac{4ab}{a+b}$

$$\text{If } h = \frac{4ab}{a+b}$$

$$\begin{aligned} \text{Now } \frac{d}{ds} \left(\frac{1}{\rho} \right) + \frac{d}{ds} \left(\frac{1}{\rho_2} \right) &= - \left(\frac{1}{\rho_1^2} \frac{d\rho_1}{ds} + \frac{1}{\rho_2^2} \frac{d\rho_2}{ds} \right) \\ &= \frac{1}{\rho_1^2} \cdot 4a \sin \Psi + \frac{1}{\rho_2^2} \cdot 4b \sin \Psi \\ &= 4 \left(\frac{a}{\rho_1^2} + \frac{b}{\rho_2^2} \right) \sin \Psi \\ &= 0, \text{ when } \Psi = 0 \end{aligned}$$

which gives no information.

Further, more

$$\begin{aligned} \frac{d^2}{ds^2} \left(\frac{1}{\rho_1} \right) + \frac{d^2}{ds^2} \left(\frac{1}{\rho_2} \right) &= \frac{d}{ds} \left(-\frac{1}{\rho_1^2} \frac{d\rho_1}{ds} \right) + \frac{d}{ds} \left(-\frac{1}{\rho_2^2} \frac{d\rho_2}{ds} \right) \\ &= \frac{d}{ds} \left(\frac{4a}{\rho_1^2} \sin \Psi \right) + \frac{d}{ds} \left(\frac{4b}{\rho_2^2} \sin \Psi \right) \\ &= -2 \frac{4a}{\rho_1^3} \frac{d\rho_1}{ds} \sin \Psi + \frac{4a}{\rho_1^2} \cos \Psi \frac{d\Psi}{ds} \\ &\quad + 4b \left(-\frac{2}{\rho_2^3} \frac{d\rho_2}{ds} \sin \Psi + \frac{1}{\rho_2^2} \cos \Psi \frac{d\Psi}{ds} \right) \\ &= \frac{8a \times 4a}{\rho_1^3} \sin^2 \Psi + \frac{4a}{\rho_1^2} \cos \Psi \frac{\sec \Psi}{4a} + 4a \left(\frac{8b}{\rho_2^3} \sin^2 \Psi + \frac{\cos \Psi}{\rho_2^2} \cdot \frac{1}{4b \cos \Psi} \right) \end{aligned}$$

$$= \frac{32a^2}{\rho_1^3} \sin^2 \Psi + \frac{1}{\rho_1^2} + \frac{32b^2}{\rho_2^3} \sin^2 \Psi + \frac{1}{\rho_2^2}$$

$$\text{So, } \frac{d^2}{ds^2} \left(\frac{1}{\rho_1} \right) + \frac{d^2}{ds^2} \left(\frac{1}{\rho_2} \right) + \frac{(\rho_1 + \rho_2)(\rho_2 + 2\rho_1)}{\rho_1^2 \rho_2^3}$$

$$= \frac{32a^2}{\rho_1^3} \sin^2 \Psi + \frac{1}{\rho_1^2} + \frac{32b^2}{\rho_2^3} \sin^2 \Psi + \frac{1}{\rho_2^2} + \frac{(\rho_1 + \rho_2)(\rho_2 + 2\rho_1)}{\rho_1^2 \rho_2^3}$$

$$= \frac{1}{(4a)^2} + \frac{1}{(4b)^2} + \frac{4(a+b)(b+2a)}{(4a)^2(4b)^3} \text{ when } \Psi = 0 > 0$$

showing that the equilibrium is unstable. Therefore the equilibrium is

$$\text{unstable when } h \geq \frac{4ab}{a+b}$$

$$\text{stable when } h < \frac{4ab}{a+b}.$$

1.8 Summary

Definition of stable and unstable is studied and stability is studied and related some example. Condition of stability (One degree of freedom) is described. A heavy body on a fixed body is described and examples to be done with the help of stable and unstable condition are also provided for better understanding.

1.9. Terminal Questions

Question. 1 A solid frustum of paraboloid of revolution of height and latus rectum $4a$ rests with its vertex on the vertex of paraboloid of revolution whose latus rectum is $4b$. Show that the equilibrium is stable if $h < \frac{3ab}{(a+b)}$

.....

Question. 2 A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect. If the inclinations of the plane to the horizontal are α and β ($\alpha > \beta$), show that the inclination of the beam to the horizontal in one of the equilibrium position is given by

$$\tan \theta = \frac{1}{2} (\cot \beta - \cot \alpha) \text{ and show that the beam is unstable in this position.}$$

.....

Question.3 A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg, find the position of equilibrium and show that it is unstable.

.....

Question.4 A sphere of weight W and radius a lies within a fixed spherical shell of radius b and a . Particle of weight w is fixed to the upper end of the vertical diameter. Prove that equilibrium is stable, If $\frac{W}{w} > \frac{b-2a}{a}$ and that If

$$\frac{W}{w} = \frac{b-2a}{a},$$

Then the equilibrium is essentially stable.

.....

Question.5 A heavy cube balances on the highest point of a sphere whose radius is r . If the sphere is rough enough to prevent sliding and If the side of the cube is $\frac{r\pi}{2}$, show that the cube can rock through a right angle without falling.

.....

Unit - 2 : Virtual work

Structure

- 2.1. Introduction**
- 2.2. Objectives**
- 2.3. Definition of work**
- 2.4. Virtual work and virtual displacement (definition)**
- 2.5. Principal of virtual work for a system of coplanar forces acting on a particle.**
- 2.6. Principal of virtual work for a system of coplanar forces acting at different points of a rigid body.**
- 2.7. The tensions of an inextensible string (non extensible).**
- 2.8. Process of solving the problem**
- 2.9. Summary**
- 2.10. Terminal Questions**

2.1. Introduction

The principal of virtual work states that in equilibrium of the virtual work of the forces applied to a system is zero. Newton's laws state that at equilibrium the applied forces are equal and opposite to the reaction, or constraint forces. This means the virtual work of the constraint forces must be zero as well.

2.2. Objective

After reading this unit students should be able to

- Understand the definition of work
- Understand virtual work and virtual displacement
- Understand principal of virtual work for a system of coplanar forces acting on a particles and different points of a rigid body
- The tension of an inextensible string (non extensible)

2.3. Definition of work

Work :-

A force is said to do work when its point of application displaces from one position to another position.

Suppose a force F action on a particle at O in the direction OB and the particle is displaced from one position O to another position A . Let OA makes an angle θ with OB , the direction of the forces.

work done by the force F =

$$\begin{aligned} F \times OB &= F \times OA \cos\theta \\ &= F \times \text{Projection of } OA \text{ on } OB \end{aligned}$$

Again work done by the force F

$$\begin{aligned} &= F \times OB = F \times OA \cos\theta \\ &= (F \cos\theta) \times OA \end{aligned}$$

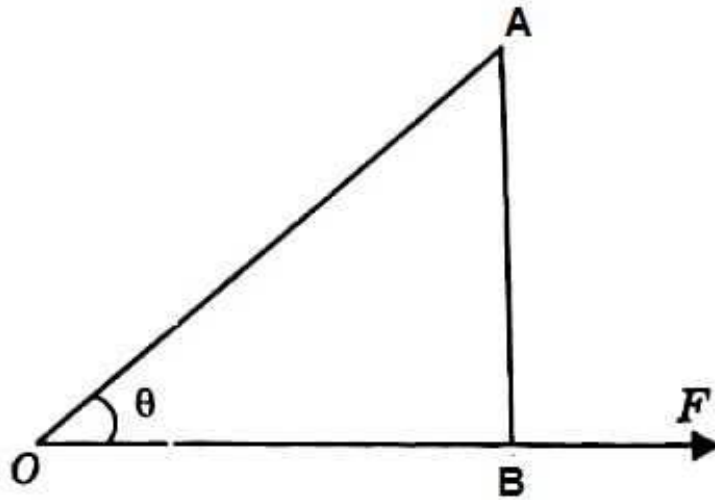


Fig. - 1

=Resolved part of the forces in the direction of actual displacement $\times$ actual displacement.

So, the product of the force and the orthogonal projection of the displacement on the line of action of the force is said to be work done by the force.

or

product of resolved part of the forces in the direction of actual displacement if it is in the direction opposite to the direction of the force. If the displacement is zero or it is in the direction.

perpendicular to the direction of the forces, then the work done is zero.

Theorem - 1

The work done by a force in displacing a particle from one position to another position is equal to the algebraic sum of works done by the resolved part of the forces.

proof ; let OX and OY are be two mutually perpendicular axes A force F acts at a particle placed at O this force displaces the point of application O to a point A . Let A be is in the XOY plane. OA makes an angle ϕ from the X axis. Force F makes an angle ϕ_2 from this axis. Let OB and OC are the components of the displacements OA in the direction OX and OY axes

respectively F_1 and F_2 be the components of the forces along OX and OY respectively.

Now the work done by the force F

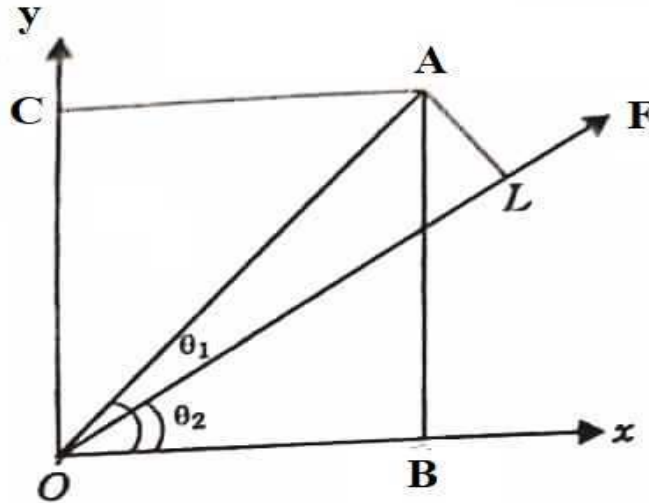


Fig. - 2

= Force $\times$ displacement in the direction of the force F

$$= F \times OL = F \times OA \cos(\theta_1 - \theta_2)$$

$$= F \times OA (\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)$$

$$= (OA \cos\theta_1)(F \cos\theta_2) + (OA \sin\theta_1)(F \sin\theta_2)$$

$$= OB \times F_1 + OC \times F_2$$

= $F_1 \times$ displacement in the direction of F_1 + $F_2 \times$ displacement in the direction of F_2

= Work done by the component F_1

+ Work done by the component F_2

= Algebraic sum of work done by the components F_1 and F_2

= Algebraic sum of the work done by the resolved parts of the forces F.

Example :- Theorem:

The algebraic sum of the works done by B number of coplanar force acting on a particle is equal to the work done by their resultant.

Proof: Let the forces $F_1, F_2, F_3, F_4, F_5, \dots$ acts on a particle at O.

These forces displace the point of application from O to B. forces $F_1, F_2, F_3, F_4, F_5, \dots$ make angles, $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \dots$ with OB respectively. Let F is the resultant of these forces, which makes an angle ϕ with OB.

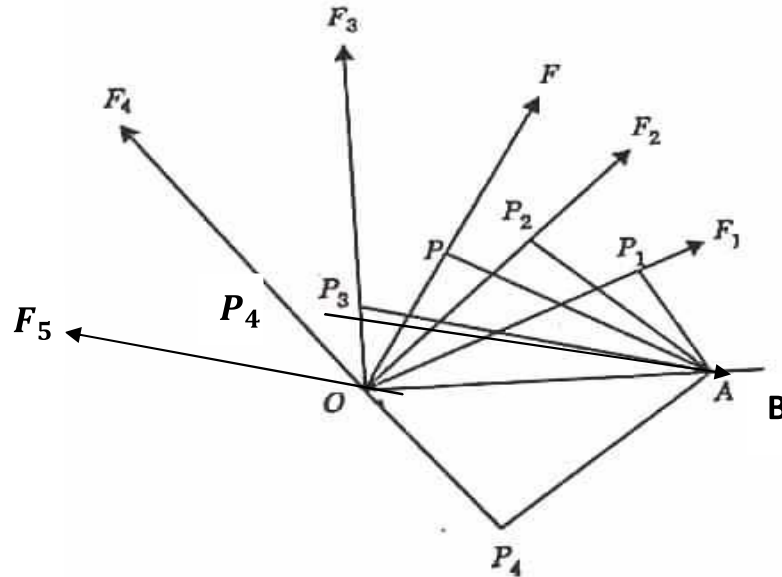


Fig. - 3

The algebraic sum of the work done by the forces $F_1, F_2, F_3, F_4, \dots$

= work done by force F_1 + work done by the force F_2 + work done by the force F_3 + work done by the force F_4 +

= $F_1 \times OP_1 + F_2 \times OP_2 + F_3 \times OP_3 + F_4 \times OP_4 + F_5 \times OP_5 + \dots$

= $F_1 \times OB \cos \phi_1 + F_2 \times OB \cos \phi_2 + F_3 \times OB \cos \phi_3 + F_4 \times OB \cos \phi_4 + F_5 \times OB \cos \phi_5 + \dots$

= $OA (F_1 \cos \phi_1 + F_2 \cos \phi_2 + F_3 \cos \phi_3 + F_4 \cos \phi_4 + F_5 \cos \phi_5 + \dots)$

OA $\times$ resolved part of the resultant along OB.

= $OB \times F \cos \phi = F \times (OB \cos \phi)$

= $F \times OP$ = work done by the resultant.

Check your progress

Q.(1) The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the separate forces.

Q.(2) A particle acted on by constant forces $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is displaced from the point $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ to the point $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Find the total work done by the forces.

Q.(3) The virtual work done by the reaction of any smooth surface with which the body is in contact is as follows:

- (a) Positive
- (b) Negative
- (c) Zero
- (d) None of these.

Q.(4) The virtual work done by the tension of an extensible string is:

- (a) Zero
- (b) Negative
- (c) Positive
- (d) None of these

2.4. Virtual work and virtual displacement (definition)

If a number of forces act on a body and displace it, these forces do some work actually. But if the forces are in equilibrium then they do not displace their points of application and so there is actually no work done by these forces. However if we imagine that the forces in equilibrium undergo some small displacement and find out the work done by the forces during that displacement then such a displacement and such a work is called virtual work.

2.5. Principal of virtual work for a system of coplanar forces acting on a particle.

Theorem

Statement:-

The necessary and sufficient condition that a particle acted upon by a number of coplanar forces be in equilibrium is that the sum of the virtual work done by the forces in any small virtual displacement consistent with the geometrical conditions of the system is zero.

Proof :- Suppose the point O be in equilibrium under the action of coplanar forces $P_1, P_2, P_3, P_4, P_5, \dots$. Suppose the component of the force P_1 along X-axis is X_1 and along Y-axis is Y_1 . Similarly, component of P_2 along X-axis be X_2 , along Y-axis be $Y_2, \dots$.

Necessary condition:-

Suppose the point O is displaced to a point O' by the action of these forces, co-ordinates of O' are (x_1, y_1) , therefore virtual work done the force P_1 .

$$= X_1 x_1 + Y_1 y_1$$

work done by force $P_2 = X_2 x_1 + Y_2 y_1$, so on algebraic sum of the virtual work done by all forces.

$$= (X_1 x_1 + Y_1 y_1) + (X_2 x_1 + Y_2 y_1) + \dots$$

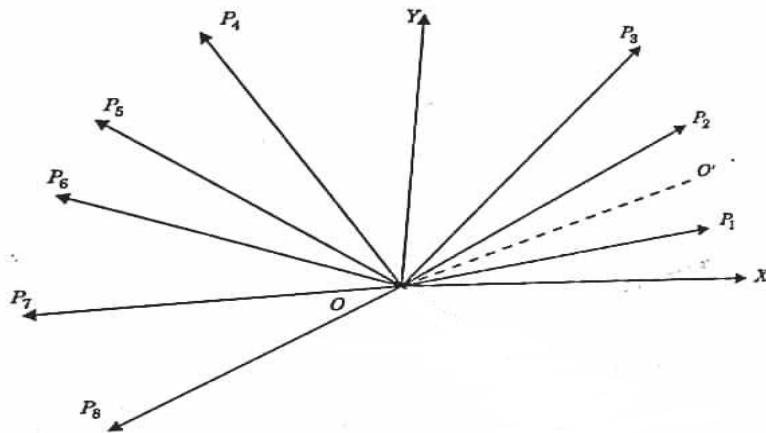


Fig. - 4

$$= X_1 x_1 + X_2 x_1 + \dots + Y_1 y_1 + Y_2 y_1 + \dots$$

$$= x_1 (X_1 + X_2 + \dots) + y_1 (Y_1 + Y_2 + \dots)$$

$$= x_1 \sum X_i + y_1 \sum Y_i$$

Hence the sum of virtual work done by all forces=0

Sufficient condition:

Suppose the virtual work done by all the forces are zero,

ie; $x_1 \sum X_1 + y_1 \sum Y_1 = 0$

case(i) Let $x_1 \neq 0$ and $y_1 = 0$

therefore $x_1 \sum X_1 = 0$. Since $x_1 \neq 0$, we have $\sum X_1 = 0$

case(ii) When $y_1 \neq 0$ and $x_1 = 0$

therefore $y_1 \sum Y_1 = 0$

since $y_1 \neq 0$, we have $\sum Y_1 = 0$

and x_1, y_1 are arbitrary, we must have $\sum X_1 = 0$ and $\sum Y_1 = 0$.

This shows that the particle a+o is in equilibrium.

2.6. Principle of virtual work for a system of coplanar forces. Acting at different points of a rigid body.

Theorem

Statement:

The necessary and sufficient condition that a rigid body acted upon by a number of coplanar forces are in equilibrium is that the algebraic sum of the virtual work done by the forces in any small virtual displacement consistent with the geometrical condition of the system is zero.

Proof: Suppose the number of coplanar forces $F_1, F_2, \dots$ acting on different points $P_1(x_1, y_1), P_2(x_2, y_2), \dots$

respectively of a rigid body.

Suppose the components of the force F_1 along X-axis, Y-axis are x_1, y_1 respectively.

suppose co-ordinates of $P_1, P_2, \dots$ with respect to O as the pole, X-axis as initial line, is $(r_1, Q_1), (r_2, Q_2), \dots$ respectively.

A small virtual displacement is given to the system, so that point $P_1(x_1, y_1)$ takes the new position $Q_1(x_1 + \delta x_1, y_1 + \delta y_1)$. This is done by rotating the

body through a small angle so, then moving it a distance a parallel to X-axis; and moving a distance b parallel to Y-axis.

therefore,

$$x_1 + \delta x_1 = OA + AB = OA + MN = r_1 \cos(\phi_1 + \delta\phi) + a$$

$$= r_1 (\cos\phi_1 \cos\delta\phi - \sin\phi_1 \sin\delta\phi) + a$$

$$= r_1 \cos\phi_1 - r \sin\phi_1 \delta\phi + a$$

(since $\cos\delta\phi \Rightarrow 1$, $\sin\delta\phi = \delta\phi$) as $\delta\phi$ is very small

$$= x_1 - y_1 \delta\phi + a$$

therefore $\delta x_1 = a - y_1 \delta\phi$

$\therefore$ displacement of P_1 along X-axis be .

$$a - y_1 \delta\phi$$

$$y_1 + \delta y_1 = BN + NQ = AM + NQ = r_1 \sin(\phi_1 + \delta\phi) + b$$

$$= r_1 (\sin\phi_1 \cos\delta\phi + \cos\phi_1 \sin\delta\phi) + b$$

$$= r_1 \sin\phi_1 + r_1 \cos\phi_1 \delta\phi + b = y_1 + x_1 \delta\phi + b$$

therefore, $\delta y_1 = x_1 \delta\phi + b$

$\therefore$ displacement of P_1 along Y-axis is $x_1 \delta\phi + b$.

work done by the force $F_1 = X_1 \delta x_1 + Y_1 \delta y_1$

$$= X_1 (a - y_1 \delta\phi) + Y_1 (b + x_1 \delta\phi)$$

$$= aX_1 + bY_1 + \delta\phi (x_1 Y_1 - y_1 X_1)$$

similarly we can find the work done by the other forces $F_2, F_3 \dots$ each time a, b and $\delta\phi$ remain the same for each force.

therefore total work done by all the forces $F_1, F_2, \dots$ are

$$a \sum X_1 + b \sum Y_1 + \delta\phi (x_1 Y_1 - y_1 X_1)$$

if the system is in equilibrium.

therefore, $\sum X_1 = 0$, $\sum Y_1 = 0$, $\sum (x_1 Y_1 - y_1 X_1) = 0$,

$\therefore$ Virtual work done by all forces are zero.

Hence the condition is necessary.

Let the work done by the forces are zero.

$$\text{therefore } a\sum X_1 + b\sum Y_1 + \delta\phi\sum(x_1Y_1 - y_1X_1)=0$$

The quantities a, b and $\delta\phi$ are arbitrary.

Let there is only displacement along X-axis.

$$\text{therefore, } a \neq 0, \quad b = 0, \quad \delta\phi = 0$$

$$\text{so that } a\sum X_1 = 0 \text{ or } \sum X_1 = 0$$

$$\text{if } a = 0, \quad b \neq 0, \quad \delta\phi = 0 \text{ so that } b\sum Y_1 = 0$$

$$\therefore \sum Y_1 = 0$$

$$\text{Let } a = 0, \quad b = 0, \quad \delta\phi \neq 0$$

$$\text{therefore } \delta\phi\sum(x_1Y_1 - y_1X_1)=0 \therefore \sum(x_1Y_1 - y_1X_1)=0$$

$$\text{therefore } \sum X_1 = 0; \sum Y_1 = 0; \sum(x_1Y_1 - y_1X_1)=0$$

Hence the system is in equilibrium. The condition is sufficient.

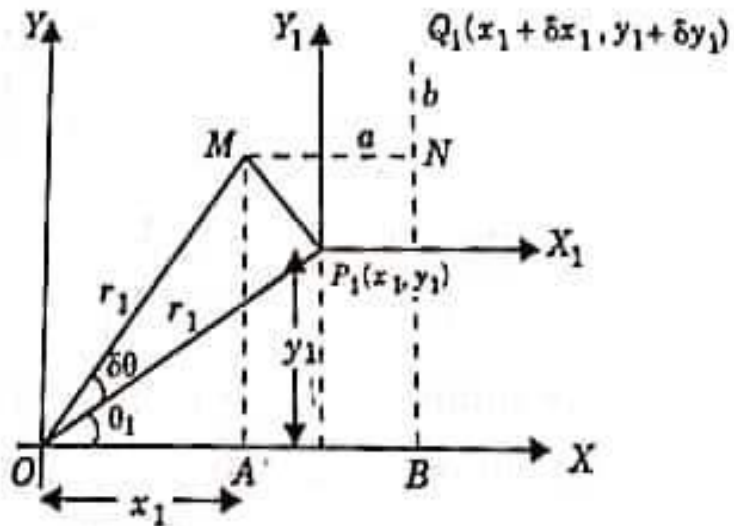


Fig. – 5

2.7. The tension of an inextensible string (non-extensible)

Suppose T is tension of the string AB .

This tension is replaced of two equal forces T ,

T acted inward in opposite.

direction, String AB be

displaced to new position $A'B'$.

Which makes an small angle θ

with the direction of AB .

Draw perpendicular $A'G$ from

the point A' an AB and draw.

a perpendicular $B'E$ from B' on AB

after producing it to point E

sum of the virtual work done by the tension T

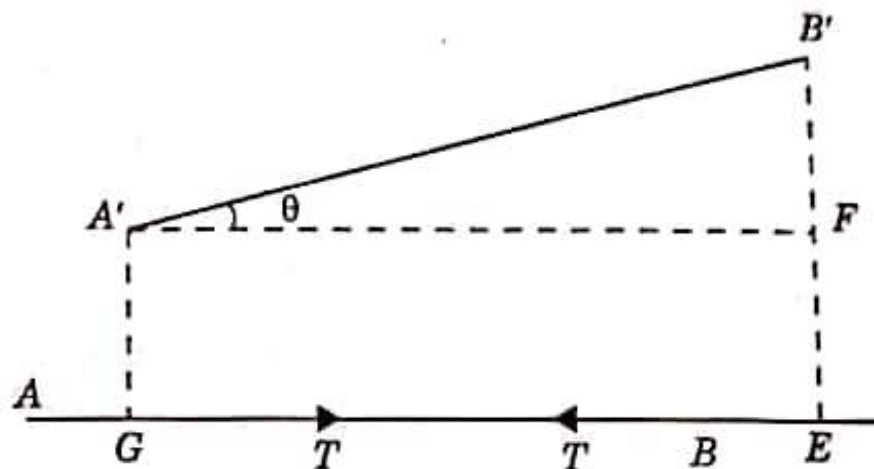


Fig. - 6

$$=T \cdot AG - T \cdot BE$$

$$=T \cdot (AG - GB) - T \cdot (GB + BE) = T \cdot AB - T \cdot GE$$

$$=T.AB - T.A'B' = T(AB - A'B \cos\theta)$$

$$=T.AB \left\{ 1 - \left(1 - \frac{\theta^2}{2!} \dots \dots \dots \right) \right\}$$

$$(\therefore AB = A'B')$$

=0, since θ is very small

therefore work done is =0

forces which can be omitted in writing the equation of virtual work for a body in equilibrium.

- i. Reaction of any smooth surface with which the body is on contact.
- ii. Tension of inextensible string or thrust is in a light rod.
- iii. Internal action and reaction between parts of a same body.
- iv. Reaction at a fixed point or a fixed axis about which the body is rotates.

2.8. Process of solving the problem

First of all draw the figure.

- i. If it is a string, replace tension T in two equal forces T and T acted inward in opposite direction.
If L be the length of string in equilibrium then the virtual work done by the tension T be $-T\delta L$.
- ii. If it is a rod then tension T of the rod be replaced in two equal forces. T and T acted outwards in opposite directions. If L be the length of the rod then the virtual work done by the thrus is $T\delta L$
- iii. Distances of the action of forces are measured from a fixed line or a fixed point. If distance measured is in the direction of the force, then the virtual work done by the force is taken to be positive. If it is in opposite direction then it is taken to be negative.
- iv. Followed this rule problem is solved.

Some typical examples:-

Q.1 Two equal uniform rods AB and AC each of length 2a are freely joined at A and rest on a smooth vertical circle of radius b, show that if 2θ be the inclination between them,

then $a \sin^3 \theta = b \cos \theta$

Solution:-

Let AB and AC are two rods resting on vertical circle of centre O. Because vertical circle is fixed we will be measure the distance from centre of the circle.

Let G_1 and G_2 are the centre of gravity of rod AB and AC respectively. Let W be the weight of each rod.

therefore the weigh $2W$ will be acting vertically downward.

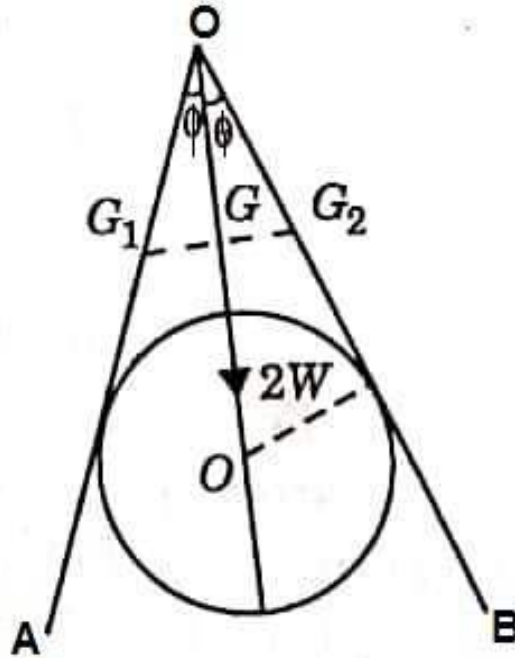


Fig. 7

from the point G, G be the middle point of GG_2

A small displacement be given to the system; consisted with the geometrical couditions so that ϕ be comes to $\phi + \delta\phi$

$$OG = OA - GA = b \operatorname{Cosec}\phi - a \cos\phi$$

By the principal of virtual work $2W \cdot \delta(OG) = 0$

$$\delta(OG) = 0, \text{ since } W \neq 0$$

$$\text{therefore } \delta(b \operatorname{Cosec}\phi - a \cos\phi) = 0$$

putting the value OG.

$$\text{or } -b \operatorname{Cosec} \phi \cot \phi \delta \phi + a \sin \phi \delta \phi = 0$$

$$(-b \operatorname{Cosec} \phi \cot \phi + a \sin \phi) \delta \phi$$

But $\delta \phi \neq 0$ therefore $(-b \operatorname{Cosec} \phi \cot \phi + a \sin \phi) = 0$

$$a \sin^3 \phi = b \cos \phi$$

Example : 2 A regular hexagon P,Q,R,S,T,U

consists of six equal rods which are each of weight W and are freely joined together. The hexagon rests in a vertical plane and PQ in contact with a horizontal table. If R and U be connected by a light string, prove that its tension is $W\sqrt{3}$

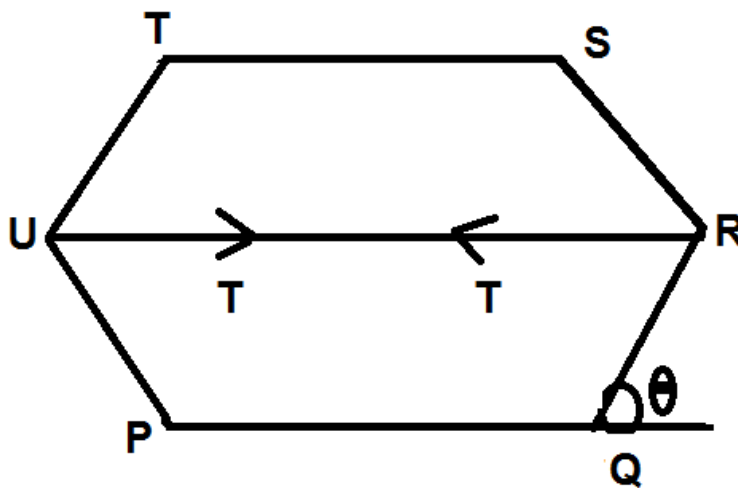


Fig. - 8

Solution:-

Let each rod be length $2a$. Replace the string UR in two equal forces T, T acted inwards in opposite direction. Let the rod QR makes an angle ϕ with the horizontal.

Therefore $UR = 2a + 4a \cos \phi$, $GL = 2a \sin \phi$

A small virtual displacement be given to the system. So that ϕ becomes $\phi + \delta \phi$ and length l of the string becomes $l + \delta l$

therefore equation of virtual work be

$$-T \cdot \delta l - 6W \cdot \delta(GL) = 0$$

$$\text{or } T\delta(2a+4a \cos\phi) + 6W\delta 2a \sin\phi = 0$$

$$12aW \cos\phi \delta\phi - 4a T \sin\phi \delta\phi = 0$$

$$(3W \cot\phi - T) \delta\phi = 0 \quad \text{since } \delta\phi \neq 0$$

therefore $T = 3W \cot\phi$. In equilibrium $\phi = 60^\circ$

$$\therefore T = 3W \cot 60^\circ = \frac{3W}{\sqrt{3}} \quad \therefore T = W\sqrt{3}$$

Example.3. Four rods of equal weight W form a rhombus $ABCD$, with smooth hinges at the joints. The frame is suspended by the point A and a weight W is attached to C . A stiffening rod of negligible weight joins the middle points of AB and AD , keeping these inclined at α to AC . Show that the thrust in this stiffening rod is.

$$(2W+4w)\tan\alpha$$

Solution:-

$ABCD$ is a framework formed of four equal rods each of weight w and let length $2a$. It is suspended by the point A and a weight W is attached to C . To keep the system in the form of a rhombus of light rod EF joins the middle points E and F of AB and AD respectively. The line AC must be vertical and BD is horizontal.

We have $\angle BAC = \angle DAC = \alpha$

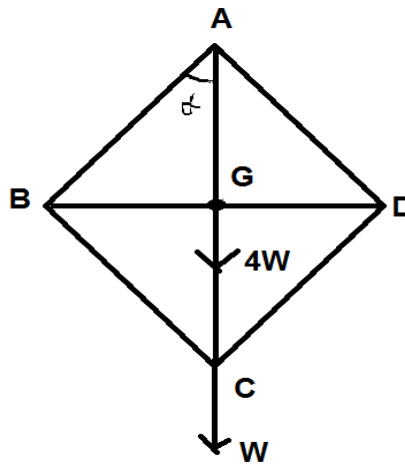


Fig. - 9

Let T be the thrust in the rod EF . The total weight $4w$ of all the four rod can be taken acting at the point of intersection O of the diagonals A and BD . Replace the rod EF by two equal and opposite forces T . see the figure.

The system a small symmetrical displacement about the vertical AC in which α changes to $\alpha + \delta\alpha$. The point A remains fixed and so the distances of the point of application of the weights $4w$ and W will be measured from A . The length of the rods AB , BC , CD and DA do not change. The length EF changes. The $\angle AOB$ remains 90° and the points O and C changes.

We have $EF = 2 \cdot AE \sin \alpha = 2a \sin \alpha$

$AO =$ depth of O below the fixed point A

$= AB \cos \alpha = 2a \cos \alpha$ and

$AC = 2AO = 4a \cos \alpha$

By the principal of virtual work, we have

$$T \cdot \delta(2a \sin \alpha) + 4w \delta(2a \cos \alpha) + W \delta(4a \cos \alpha) = 0$$

$$2aT \cos \alpha \delta \alpha - 8aw \sin \alpha \delta \alpha - 4aW \sin \alpha \delta \alpha = 0$$

$$\text{or } 2a(T \cos \alpha - 4w \sin \alpha - 2W \sin \alpha) \delta \alpha = 0$$

$$T \cos \alpha - 4w \sin \alpha - 2W \sin \alpha = 0$$

$$T \cos \alpha = (4w + 2W) \sin \alpha \quad \because \delta \alpha \neq 0$$

$$T = (2W + 4w) \tan \alpha.$$

Example.4:- Four uniform rods are freely jointed at their extremities and form a parallelogram $PQRS$, which is suspended by the point A and is kept in the shape of by string PR . Prove that the tension is half of the whole weight.

Solution: Let $PQRS$ be a parallelogram which is suspended from a point P .

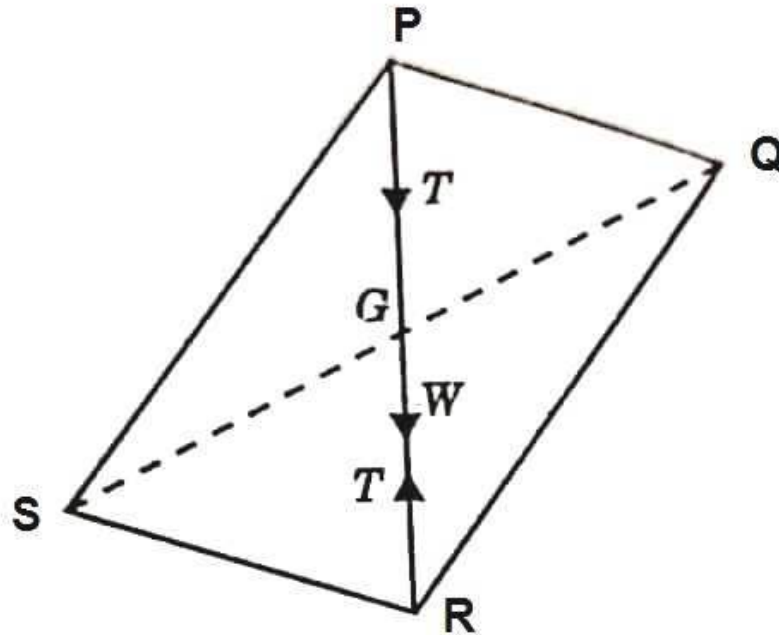


Fig. - 10

Point P and R are jointed by a string PR. Therefore total weight W of these four rods will be acting vertically downwards from the point G. Replacing tension of the string PR by two forces T, T acting inward in opposite directions, distances are measured from a fixed point P. Let $PG=x$

Therefore $PR=2x$, P virtual displacement is given to the system, so that x becomes $x+\delta x$

Principal of virtual work $W \cdot \delta(PG) - T \cdot \delta(PR) = 0$

$W \cdot \delta(x) - T \cdot \delta(2x) = 0$ on putting the value of PG and PR.

or $W\delta x - 2T\delta x = 0$

or $(W-2T) \delta x = 0$; $\delta x \neq 0 \therefore W-2T=0$

$T = \frac{W}{2}$ = half of the weight of the rods

Example 5:- The middle points of opposite sides of a jointed quadrilateral are connected by length l and l'. If T and T' be the tensions in these rods, prove that

$$\frac{T}{l} + \frac{T'}{l'} = 0$$

Solution :-

Let P, Q, R, S be the middle points of the rods AB, CD, DA, and BC respectively.

Let T and T' be the Tension in the rod PQ and RS respectively replacing the tension by two forces acting outwards in opposite direction.

A small virtual displacement is given to the system. Which changes angles but not the lengths of sides.

Therefore the equation of the virtual work be

$$T\delta(PQ) + T'\delta(RS) = 0 \dots\dots\dots(1)$$

In the ΔAOB

$$OA^2 + OB^2 = 2(OP^2 + AP^2)$$

$$\text{or } 2(OA^2 + OB^2) = PQ^2 + AB^2 \quad (2)$$

similarly

$$2(OB^2 + OC^2) = RS^2 + BC^2 \dots\dots\dots(3)$$

$$2(OC^2 + OD^2) = PQ^2 + CD^2 \dots\dots\dots(4)$$

$$2(OD^2 + OA^2) = RS^2 + DA^2 \dots\dots\dots(5)$$

Subtracting (3) from (2), we get

$$2(OA^2 - OC^2) = PQ^2 + AB^2 - RS^2 - BC^2 \dots\dots\dots(6)$$

Subtracting (5) from (4), we get

$$2(OC^2 - OA^2) = PQ^2 + CD^2 - RS^2 - DA^2 \dots\dots\dots(7)$$

Adding (6) and (7)

$$0 = 2(PQ^2 - RS^2) + AB^2 + CD^2 - BC^2 - DA^2$$

Taking differentials

$$2[2PQ \delta(PQ) - 2RS \delta(RS)] = 0$$

Since AB, BC, CA, DA are constant.

$$\text{therefore } \delta(PQ) = \frac{RS}{PQ} \delta(RS) \dots\dots\dots(8)$$

on putting the value of $\delta(PQ)$ from (8) in (1)

$$T \frac{RS}{PQ} \delta(RS) + T' \delta(RS) = 0 \text{ or } \left(\frac{T}{PQ} + \frac{T'}{RS} \right) \cdot \delta(RS) = 0$$

but $\delta RS \neq 0$ therefore $\frac{T}{PQ} + \frac{T'}{RS} = 0$ or

$$\frac{T}{l} + \frac{T'}{l'} = 0$$

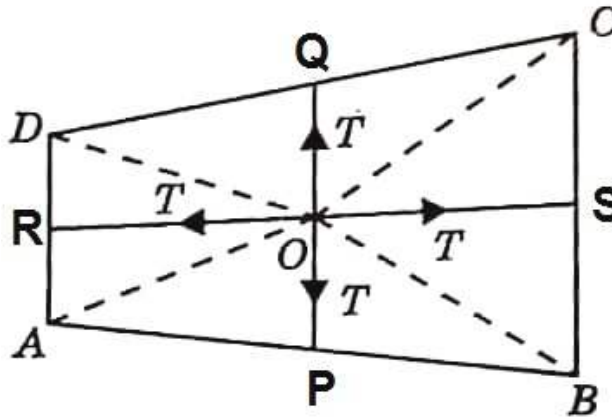


Fig. - 11

2.9. Summary

Definition of work, virtual work and virtual displacement are discussed. and theorem discussed based on virtual work. and principal of virtual work for a system of coplanar forces acting on a particles and at different points of a rigid body discussed the tension of an inextensible string (non-extensible) discussed. Process of solving the problems discussed examples to be done with the help of virtual work. are also provided for better understanding.

2.10. Terminal Question Self learning questions.

Terminal Question.

- (1) A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal and rests with its lower end on the quadrant of the curve which is further removed from the focus.

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- (2) Six equal heavy rods freely hinged at the ends from a regular hexagon. ABCDEF which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrust of these rods are $(5\sqrt{3}/2)W$ and $(\sqrt{3}/2)W$, where W is the weigh of each rod.

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- (3) Two uniforms rod AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is $\frac{W}{\tan B + \tan C}$ Where W is the total weight of the rods AB and AC.

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- (4) Five weightless rods of equal length are jointed together so as to from a rhombus ABCD with one diagonal BD. If a weight W be attached to C and the system be suspended from A, show that there is a thrust in BD equal to $W/\sqrt{3}$

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- (5) Six equal rods AB, BC, CD, DE, EF, and FA are each of weight W and are freely jointed at their extremities. So as so to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; Prove that its tension is 3W.

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Unit - 3 : Common Catenary

Structure

- 3.1. Introduction**
- 3.2. Objectives**
- 3.3. The Catenary (Definition)**
- 3.4. Intrinsic equation of the common Catenary**
- 3.5. Cartesian equation of common catenary**
- 3.6. Definitions. related to catenary**
- 3.7. Some important relation for the common catenary**
- 3.8. Approximations to the common catenary**
- 3.9. Sag of tightly stretched wires**
- 3.10. Summary**
- 3.11. Terminal Questions**

3.1. Introduction

In the present chapter we shall consider the equilibrium of perfectly flexible strings. All those strings which offer no resistance on bending at any point are called flexible strings. In such cases the resultant action across any section of the string consists of a single force. Whose line of action is along the tangent to the curve formed by the string. The normal section of the string is taken to be a small that is may be regarded as a curved line. A chain with short and perfectly smooth links approximates a flexible string.

3.2. Objective

After reading this unit students should be able to

- Define the Catenary.
- Understand the intrinsic equation if the common Catenary and cartesian equation of the common Catenary
- Understand the need and axis of the Catenary, vertex of the Catenary, parameter of the Catenary, directrix of the Catenary, span and sag.
- Understand some important relation for the common Catenary.
- Understand approximations to the common Catenary
- Understand sag of tightly stretched wires

3.3. The Catenary (Definition)

When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called a Catenary.

A uniform of common Catenary If the weight per unit length of the suspended flexible string or chain is constant, then the Catenary is called the uniform or common Catenary.

NOTE :- Here we will frequently use the word ‘Catenary’ for the common Catenary ie the word ‘Catenary’ will always mean the common Catenary in this chapter.

3.4. Intrinsic equation of the common Catnary

Suppose the uniform flexible string BCA hang in the form of a uniform. Catenary with C as its lowest point. Let P be any point on the portion. CB of the string and s the distance of P from C measured along the arc length of the string. If w is the weight per unit length of the string, then the weight of the portion CP will be ws and will act vertically downwards through the centre of

gravity of CP. The portion CP of the string is in equilibrium under the action of the following three, forces.

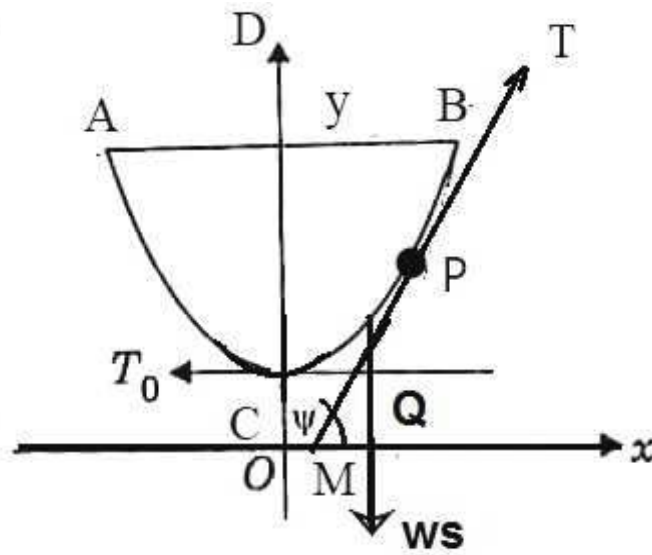


Fig. 1.21

1. The weight ws of the string CP acting vertically downwards through its centre of gravity.
2. The tension T_0 at the lowest point C acting along the tangent to the curve at C which is horizontal.
3. The tension T at P acting along the tangent to the curve at P inclined at an angle Ψ to the horizontal.

since the string CP is in equilibrium under the action of the three forces acting in the same vertical plane therefore the line of action of the weight ws must pass through the point Q which is the point of intersection of the lines of action of the tensions T_0 and T . Resolving the forces acting on CP horizontally and vertically, we get

$$T \cos \Psi = T_0 \dots \dots \dots (1)$$

$$T \sin \Psi = ws \dots \dots \dots (2)$$

dividing (2) by (1), we have

$$\tan \Psi = ws/T_0 \dots \dots \dots (3)$$

now let $T_0 = ws$, (4)

ie; Let the tension at the lowest point be equal to the weight of the length c of the string, then from (3), we get-

$$\tan \Psi = s/c \text{ or } s = c \tan \Psi$$

which is the intrinsic equation of the common Catenary.

Remark. 1- From the equation (1) it is clear that the horizontal component of the tension at every point of the Catenary is the same and is equal to T_0 , the tension at the lowest point.

Remark 2- From the equation (2) we conclude that the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and that point.

Remark 3- From the relation 4 it follow that the tension at the lowest point is equal to the weight of the string of length c .

Example:1- If T be the tension at any point p of a Catenary and T_0 that at the lowest point A, prove that;

$$T^2 - T_0^2 = W^2$$

W being the weight of the arc AP of the Catenary.

Solution: For a Catenary we know that

$$T \cos \Psi = T_0$$

and $T \sin \Psi = ws$

squaring and adding these equations, we get

$$T^2 = T_0^2 + w^2 s^2 \quad \text{ie} \quad T^2 - T_0^2 = W^2$$

where $W = ws$ =weight of the arc AP of the length s .

3.5. Cartesian equation of common catenary

To show that the Cartesian equation of common Catenary is

$$Y = c \cosh \frac{x}{c}$$

Proof: We know that $s = c \tan \Psi$ and $\frac{dy}{dx} = \tan \Psi$

therefore $s = c \frac{dy}{dx}$

differentiating with respect to x , we get

$$\frac{ds}{dx} = c \frac{d^2y}{dx^2} \quad \text{or} \quad \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = c \frac{d^2y}{dx^2}$$

on putting $\frac{dy}{dx} = p$, we get

$$\sqrt{1+p^2} = c \frac{dp}{dx} \text{ or } dx = c \frac{dp}{\sqrt{1+p^2}}$$

on integrating, we get

$$x = c \sinh^{-1}P + \alpha_1 \text{ where } \alpha_1 \text{ is a const.}$$

at the lowest point C of the Catenary; $x=0$

$$p = \frac{dy}{dx}=0. \text{ therefore } \alpha_1=0. \text{ hence.}$$

$$x = c \sinh^{-1}P, \text{ which gives } \frac{dy}{dx} = \sinh \frac{x}{c} \therefore P = \frac{dy}{dx}$$

$$\text{or } dy = \sinh \frac{x}{c} dx$$

on integrating, we get

$$y = c \cosh \frac{x}{c} + \alpha_2$$

at the lowest point C, $x=0$, $Y=C$.

therefore $\alpha_2 = 0$. hence.

$$y = c \cosh \frac{x}{c}$$

which is the required cartesian equation of the Catenary.

3.6. Definitions. related to catenary

1. Axis of the Catenary:-

Since $\cosh \frac{x}{c}$ is an even function of x , therefore the curve is symmetrical about the axis of Y which is along the vertical through the lowest point of the Catenary. This vertical line of symmetry is called the axis of the Catenary.

2. Vertex of the Catenary:-

The lowest point C of the common Catenary at which the tangent is horizontal is called the vertex of the Catenary.

3. Parameter of the Catenary:-

The quantity c occurring in the cartesian equation $y = c \cosh \frac{x}{c}$ of the Catenary is called parameter of the Catenary.

4. Directrix of the Catenary:-

The horizontal line at a depth c below the lowest point ie; the axis of x , is called the directrix of the Catenary.

5. Span and Sag:-

Suppose the string be suspended from the two points A and B in the same horizontal line. Then the distance AB is called the span of the Catenary and the depth DC (see the fig 1.21) of the lowest point below BA is called the Sag of the Catenary.

3.7. Some important relation for the common catenary

1. Relation between x and s :-

For a Catenary, we get

$$\begin{aligned} s &= c \tan \psi \left[\because \frac{dy}{dx} = \tan \psi \right] \\ &= c \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{s}{c} \dots\dots\dots (1) \end{aligned}$$

also

$$y = c \cosh \frac{x}{c}$$

differentiating, we get

$$\frac{dy}{dx} = \sinh \frac{x}{c} \dots\dots\dots (2)$$

from (1) and (2) we get

$$\frac{s}{c} = \sinh \frac{x}{c} \quad \text{or} \quad s = c \sinh \frac{x}{c} \dots\dots\dots (3)$$

Which is the relation between x and s .

2. relation between y and s :-

For a Catenary, we have

$$y = c \cosh \frac{x}{c}$$

$$\text{also } s = c \sinh \frac{x}{c} \quad \text{see relation (3)}$$

squaring and adding then we get

$$y^2 = s^2 + c^2 \left[\cosh^2 \frac{x}{c} - \sinh^2 \frac{x}{c} \right] = c^2$$

$$y^2 = c^2 + s^2 \dots\dots\dots(3)$$

which is the relation between Y and s.

3. Relation between y and Ψ :-

for the curve, we have

$$\begin{aligned} \frac{dy}{ds} &= \sin\Psi \\ \frac{dy}{d\Psi} &= \frac{dy}{ds} \cdot \frac{ds}{d\Psi} = \sin\Psi \cdot \frac{d}{d\Psi} (c \tan \Psi) \quad [\because s = c \tan \Psi] \\ &= c \sin\Psi \cdot \sec^2\Psi = c \sec\Psi \tan\Psi \\ \frac{ds}{d\Psi} &= c (\sec\Psi) \tan\Psi \end{aligned}$$

integrating, we get $y = c \sec\Psi + K$ where K is const.

but when $y = c$, $\Psi = 0 \therefore K = 0$

hence $y = c \sec \Psi \dots\dots\dots(5)$

which is the relation between Y and Ψ

4. Relation between x and Ψ :-

For any curve, we have $\frac{dx}{ds} = \cos \Psi$

$$\begin{aligned} \frac{dx}{d\Psi} &= \frac{dx}{ds} \cdot \frac{ds}{d\Psi} = \cos\Psi \cdot \frac{d}{ds} (c \tan\Psi) \\ &= \cos\Psi \cdot c \sec^2\Psi \\ &= c \sec \Psi \end{aligned}$$

integrating it we get

$$x = c \log(\sec\Psi + \tan\Psi) + A$$

where A is const.

but when $x = 0$, $\Psi = 0 \therefore A = 0$

hence $x = c \log(\sec\Psi + \tan\Psi)$,

which is the relation between x and Ψ

5. Relation between tension and ordinate from 1.21, we have:-

$$T \cos \Psi = T_0 \text{ and } T_0 = wc.$$

$$T = T_0 \sec \Psi = wc \sec \Psi$$

but $y = c \sec \Psi$

hence $T = wy$ (1)

which is the relation between T and y.

The relation (7) shows that the tension at any point of a Catenary is w times as the height of the point above the directrix.

6. Relation of curvature at any point of a Catenary:-

For a Catenary, we have

$$s = c \tan \Psi \quad \therefore P = \frac{ds}{d\Psi} = c \sec^2 \Psi.$$

Example 1: If $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity of the arc measured from the vertex upto the point P(x,y), prove that

$$\bar{x} = x - c \tan(\Psi/2), \bar{y} = \frac{1}{2} \left(\frac{c}{\cos \Psi} + x \cot \Psi \right)$$

Solution: The parametric equation of a Catenary are,

$$x = c \log(\sec \Psi + \tan \Psi) \text{ and } y = s \sec \Psi.$$

also for a Catenary, $s = c \tan \Psi$

$$\frac{ds}{d\Psi} = c \sec^2 \Psi$$

we get

$$\bar{x} = \frac{\int x ds}{\int ds} = \frac{\int_0^\Psi x \frac{ds}{d\Psi} d\Psi}{\int_0^\Psi \frac{ds}{d\Psi} d\Psi} \quad [\because \text{at the vertex } \Psi = 0]$$

$$= \frac{\int_0^\Psi c \log(\sec \Psi + \tan \Psi) c \sec^2 \Psi d\Psi}{\int_0^\Psi c \sec^2 \Psi d\Psi}$$

$$= \frac{c^2 [\{ \log(\sec \Psi + \tan \Psi) \cdot \tan \Psi \}_0^\Psi] - \int_0^\Psi \frac{1}{\sec \Psi + \tan \Psi} (\sec \Psi \tan \Psi + \sec^2 \Psi) \tan \Psi d\Psi}{c [\tan \Psi]_0^\Psi}$$

integrating the numerator by parts taking $\sec^2 \Psi$ as 2nd function

$$= \frac{c \left[\tan \Psi \log(\sec \Psi + \tan \Psi) - \int_0^\Psi \sec \Psi \tan \Psi \right]}{\tan \Psi}$$

$$= \frac{c [\tan \Psi \log(\sec \Psi + \tan \Psi) - \{ \sec \Psi \}_0^\Psi]}{\tan \Psi}$$

$$= \frac{x \tan \Psi - c (\sec \Psi - 1)}{\tan \Psi}$$

$$= \frac{x - c (\sec \Psi - 1)}{\tan \Psi} = x - \frac{c(1 - \cos \Psi)}{\sin \Psi}$$

$$= x - c \frac{2 \sin^2 \frac{1}{2} \Psi}{2 \sin^2 \frac{1}{2} \Psi \cos \frac{1}{2} \Psi} = x - c \tan \frac{1}{2} \Psi$$

$$\text{and } \bar{y} = \frac{\int y ds}{\int ds} = \frac{\int_0^\Psi y \frac{ds}{d\Psi} d\Psi}{\int_0^\Psi \frac{ds}{d\Psi} d\Psi} = \frac{\int_0^\Psi c \sec \Psi \cdot c \sec^2 \Psi d\Psi}{\int_0^\Psi c \sec^2 \Psi d\Psi}$$

$$= \frac{c^2 \left[\frac{1}{2} \sec \Psi \tan \Psi \right]_0^\Psi + \frac{1}{2} \int_0^\Psi \sec \Psi d\Psi}{c [\tan \Psi]_0^\Psi}$$

$$\left[\because \int \sec^n \theta d\theta = \frac{\sec^{n-2} \theta + \tan \theta}{n-1} - \frac{n-2}{n-1} \int \sec^{n-2} \theta d\theta \right]$$

$$C = \frac{\left[\frac{1}{2} \sec \Psi \tan \Psi + \frac{1}{2} \{ \log(\sec \Psi + \tan \Psi) \} \right]_0^\Psi}{\tan \Psi}$$

$$= \frac{1}{2 \tan \Psi} [c \sec \Psi \tan \Psi + c \log(\sec \Psi + \tan \Psi)]$$

$$= \frac{1}{2 \tan \Psi} [c \sec \Psi \tan \Psi + x] = \frac{1}{2} \left[\frac{c}{\cos \Psi} + x \cot \Psi \right]$$

Check your progress report

Q.1. Prove that the relation between s and y is

$$y^2 = c^2 + s^2$$

Q.2. In the Catenary, prove that $y^2 = P.C$, where s is the radius of curvature at any point whose ordinate is y.

$$\text{hint. } \rho = c \sec^2 \Psi = \frac{c^2 \sec^2 \Psi}{c} = \frac{y^2}{c}$$

Q.3. If the length of uniform chain, suspended between two points at the same level, is adjusted. So as the tension at the points of support is minimum for that particular span $2d$, show that the equation to determine c is

$$\coth \frac{d}{c} = \frac{d}{c}$$

hint, $T = wy = wc \cosh\left(\frac{d}{c}\right)$, where $x = d$. Now for T to be minimum, we must have $\frac{dT}{dc} = 0$ and $\frac{d^2T}{dc^2} > 0$

Q.4. Prove that the relation between x,y and s for the common Catenary is

$$y = c e^{\frac{x}{c}} - s$$

Q.5. Prove that tension at the lowest point of a Catenary is wc

3.8. Approximations to the common catenary

1. The cartesian equation of the common Catenary is.

$$y = c \cosh\left(\frac{x}{c}\right) = \frac{1}{2}c \left(e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right)$$

$$= \frac{c}{2} \left[\left\{ 1 + \frac{x}{c} + \frac{1}{2!} \left(\frac{x}{c}\right)^2 + \frac{1}{3!} \left(\frac{x}{c}\right)^3 + \frac{1}{4!} \left(\frac{x}{c}\right)^4 \dots \dots \dots \right\} \right. \\ \left. + \left\{ 1 - \left(\frac{x}{c}\right) + \frac{1}{2!} \left(\frac{x}{c}\right)^2 - \frac{1}{3!} \left(\frac{x}{c}\right)^3 + \frac{1}{4!} \left(\frac{x}{c}\right)^4 \dots \dots \right\} \right]$$

$$= c \left[1 + \frac{1}{2!} \left(\frac{x}{c}\right)^2 + \frac{1}{4!} \left(\frac{x}{c}\right)^4 \dots \dots \dots \right] \text{---(1)}$$

Now if $\frac{x}{c}$ is small then neglecting the power of $\frac{x}{c}$ higher than two the equation (1) reduces to

$$y = c \left[1 + \frac{1}{2!} \left(\frac{x}{c}\right)^2 \right] \text{ or } x^2 = 2c(y - c),$$

Which is the equation of a parabola of latus rectum $2c$ or $2 \frac{T_0}{w}$.

Thus if x is small compared to c , the common Catenary coincides very nearly with a parabola of latus rectum $2c$ or $2 \frac{T_0}{w}$ and vertex at the point $(0,c)$.

Example of a such case are the electric transmission wire and the telegraphic wires tightly stretched between the poles.

Besides such cases of tightly stretched string, even in the case of a common Catenary not tightly stretched if we consider the portion of the curve near the vertex, x is small compared to c .

2. When x is large i.e. at points far removed from the lowest point, $\frac{x}{c}$ is larger and so $e^{-\frac{x}{c}}$ becomes very small, hence

$y = c \cosh\left(\frac{x}{c}\right) = \frac{1}{2}c \left(e^{\frac{x}{c}} + e^{-\frac{x}{c}} \right)$ behaves as $y = \frac{1}{2}c e^{\frac{x}{c}}$, which is an exponential curve.

Hence at points far removed from the lowest point, a common Catenary behaves as an exponential curve.

3.9. Sag of tightly stretched wires

Consider a tightly stretched wire which appears nearly a straight line, as for example a telegraphic wire stretched tightly between the poles. Let A and B be two points in a horizontal line between which a wire is stretched tightly. Let C be the lowest point of the Catenary formed by the wire. Let W be the weight and ℓ the length of wire ACB. Also let T_0 be the horizontal tension at the lowest point C. The portion CA of the wire is in equilibrium under the action of the following forces.

- i. The tension T_0 acting horizontally at the point C.
- ii. The tension T at A acting along the tangent at A.
- iii. The weight $\frac{1}{2}W$ of the wire. CA acting vertically downwards through its centre of gravity G. Since the wire is tightly stretched, the distance of the centre of gravity G of the wire AC from the vertical line through A will be approximately equal to $\frac{1}{2}AC$ i.e. $\frac{1}{4}\ell$. Let k be the sag CD and a the span AB of the Catenary. Taking moments of the forces acting on the portion CA, about A, we get

$$T_0 k = \frac{1}{2}W \cdot \frac{1}{4}\ell.$$

$$T_0 = \frac{\ell W}{8k} \quad (1)$$

Now we have to calculate the increase in the length of the wire on account of the sag for a Catenary, we have

$$\begin{aligned} s &= c \sinh\left(\frac{x}{c}\right) = \frac{1}{2}c \left(e^{\frac{x}{c}} - e^{-\frac{x}{c}} \right) \\ &= \frac{c}{2} \left[\left\{ 1 + \frac{x}{c} + \frac{1}{2!} \left(\frac{x}{c}\right)^2 + \frac{1}{3!} \left(\frac{x}{c}\right)^3 + \frac{1}{4!} \left(\frac{x}{c}\right)^4 + \dots \right\} \right. \\ &\quad \left. + \left\{ 1 - \left(\frac{x}{c}\right) + \frac{1}{2!} \left(\frac{x}{c}\right)^2 - \frac{1}{3!} \left(\frac{x}{c}\right)^3 + \frac{1}{4!} \left(\frac{x}{c}\right)^4 - \dots \right\} \right] \\ &= c \left[\frac{x}{c} + \frac{1}{3!} \left(\frac{x}{c}\right)^3 + \frac{1}{5!} \left(\frac{x}{c}\right)^5 + \dots \right] \dots \dots \dots (2) \end{aligned}$$

The radius of curvature ρ of the Catenary is given by

$$\rho = c \sec^2 \psi$$

At the vertex, $\psi = 0$ and $\rho = c$ at the vertex. this shows that if the curve is flat near its vertex C so that ρ is large at the vertex, then C will be very long and

x will be small as compared to C . Thus for a tightly stretched wire $\frac{x}{c}$ is very small. Thus retaining only the first two forms in (2) , we get

$$s = c \left[\frac{x}{c} + \frac{1}{3!} \left(\frac{x}{c} \right)^3 \right] = x + \frac{x^3}{6c^2}$$

or $s - x = \frac{x^3}{6c^2}$

but $c = \frac{T_0}{w}$

$$s - x = \frac{w^2 x^3}{6T_0^2}$$

Now putting $x = \frac{1}{2}a$, where a is the span AB, we get $s - \frac{a}{2} = \text{arc CA} - \text{DA} = \frac{a^3 w^2}{48T_0^2}$

$\therefore$ total increase in the length of the wire due to sagging.
 $= \text{arc ABC} - \text{span AB} = 2S - a$

$$= \frac{a^3 w^2}{24T_0^2}$$

Some typical example:-

Example-1.

A uniform chain of length l is suspended from two points A,B in the same horizontal line. if the tension at A is n times that at the lowest point, show that the span AB is

$$\frac{l}{\sqrt{(n^2 - 1)}} \log \left\{ n + \sqrt{(n^2 - 1)} \right\}$$

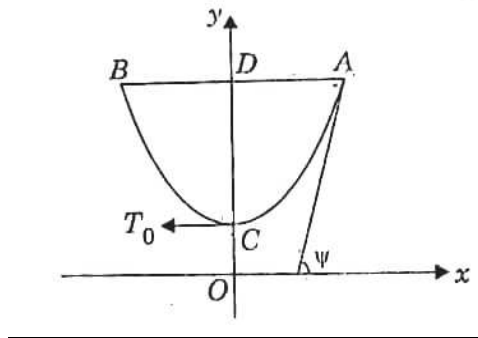


Fig.

Solution:-

Let T be the tension at the point A. It is given that

$$T = nT_0, \dots\dots\dots (1)$$

Where T_0 is the tension at the lowest point. If Ψ be the angle made by the tangent at A with the x-axis, then we know that

$$T_0 = T \cos \Psi$$

using it in (1), we find that

$$T = nT \cos \Psi, \text{ which gives } \sec \Psi = n$$

$$\text{hence } \tan \Psi = \sqrt{(\sec^2 \Psi - 1)} = \sqrt{(n^2 - 1)}$$

Further, we know that $s = c \tan \Psi$, and it is given that $2s = l$. therefore

$$s = \frac{l}{2} = c \tan \Psi = c \sqrt{(n^2 - 1)}, \text{ which gives } c = \frac{1}{2\sqrt{(n^2 - 1)}}$$

$$\text{Now span AB} = 2c \log(\sec \Psi + \tan \Psi)$$

$$= 2 \cdot \frac{l}{2\sqrt{(n^2 - 1)}} \log \left\{ n + \sqrt{(n^2 - 1)} \right\}$$

$$\text{i.e. span AB} = \frac{l}{\sqrt{(n^2 - 1)}} \log \left\{ n + \sqrt{(n^2 - 1)} \right\}$$

Example:2.

The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \left\{ \frac{1 + (1 + \mu^2)^{\frac{1}{2}}}{\mu} \right\}$$

Solution:-

For the maximum span, the point A and B of the chain must be in equilibrium.

Now the point A is in equilibrium under the action of the following forces:

- (i) Normal reaction R
- (ii) Force of friction μR and
- (iii) tension T .

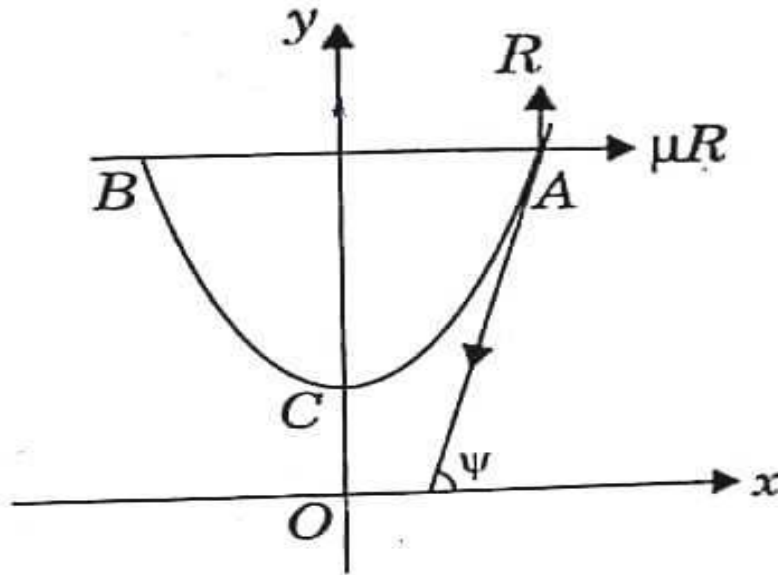


Fig.

Resolving the forces horizontally and vertically, we have

$$T \cos \Psi = \mu R \quad (1)$$

$$T \sin \Psi = R \quad (2)$$

Where Ψ is the angle made by the tangent at A with the x-axis.

Dividing (2) by (1), we obtain

$$\tan \Psi = \frac{1}{\mu}$$

$$\text{so that } \sec \Psi = (\tan^2 \Psi + 1)^{\frac{1}{2}} = \left(\frac{1}{\mu^2} + 1 \right)^{\frac{1}{2}} = \frac{(1 + \mu^2)^{\frac{1}{2}}}{\mu}$$

Now let l be the length of the chain ACB then using the formula $s = c \tan \Psi$, we obtain

$$\frac{1}{2}l = c \tan \Psi, \text{ which gives } c = \frac{1}{2}\mu l.$$

therefore,

$$\text{The maximum span } AB = 2c \log(\sec \Psi + \tan \Psi)$$

$$= 2 \times \frac{1}{2}\mu l \cdot \log \left\{ \frac{(1 + \mu^2)^{\frac{1}{2}}}{\mu} + \frac{1}{\mu} \right\}$$

$$= \mu l \cdot \log \left\{ \frac{1 + (1 + \mu^2)^{\frac{1}{2}}}{\mu} \right\}$$

$$\text{Hence } \frac{\text{maximum span}}{\text{length of the chain}} = \mu \log \left\{ \frac{1 + (1 + \mu^2)^{\frac{1}{2}}}{\mu} \right\}$$

Example.3:-

A heavy chain of length $2l$ has one end tied at A and other is attached to a small heavy ring. Which can slide on a rod and weight of the ring $m-n$ times. the weight of the chain, show that its greatest possible distance from A is

$$\frac{2l}{h} \log \left\{ \sqrt{(1 + h^2)} + h \right\}, \text{ where } \frac{1}{h} = H(2n + 1)$$

Solution:-

Suppose W be the weight of the chain so that the weight of the ring is nW . For the greatest possible distance of B from the fixed point A, the point B must be in equilibrium. Now the point B is in equilibrium under the action of the following forces.

- (i) Normal reaction R
- (ii) Force of friction μR
- (iii) Weight of the ring nW and
- (iv) Tension T

Resolving the forces horizontally and vertically, we get

$$T \cos \Psi = \mu R \quad \dots\dots\dots(1) \text{ and}$$

$$nW + T \sin \Psi = R \quad \dots\dots\dots(2)$$

where Ψ is the angle made by the tangent at B with the x-axis. using the value of R from (2) in (1), we get

$$\begin{aligned} T \cos \Psi &= \mu(nW + T \sin \Psi) \\ &= \mu(n \cdot 2lw + wl) \dots\dots\dots(3) \end{aligned}$$

where $w = \frac{W}{2l}$ is the weight per unit length of the chain and $T \sin \Psi = wl$.

Since $T \cos \Psi = T_0 = wc$, (3) reduces to

$$wc = \mu(2n + 1)lw, \text{ which gives } c = \mu(2n + 1)l.$$

But we are given that $\frac{1}{\chi} = \mu(2n + 1)$

$$\text{Therefore } c = \frac{l}{h} \quad \dots\dots\dots(4)$$

Now using the relation $s = c \tan \Psi$ at B, we have

$$l = c \tan \Psi \quad \text{i.e., } l = \frac{1}{\chi} \tan \Psi, \text{ using (4)}$$

$$\text{i.e. } \tan \Psi = \chi, \text{ so that } \sec \Psi = \sqrt{1 + \chi^2}$$

Hence the greatest possible distance of B from A

$$\begin{aligned} &= 2c \log(\sec \Psi + \tan \Psi) \\ &= \frac{2l}{h} \log \left\{ \sqrt{(1 + \chi^2)} + \chi \right\} \end{aligned}$$

Example.4:-

A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of water, while each of the extremities has an elevation above the surface. Show that the breadth of the river is nearly

$$\left(l - \frac{8k^2}{3l} \right)$$

Solution:-

Chain is tightly stretched over the river of breadth AB also, ACB is the measuring chain of length l the sag $CE=k$

Let the co-ordinates of the point B be (x, y)

therefore $\frac{l}{2} = c \sinh \frac{x}{c}$ at the point B , since $s = c \sinh \frac{x}{c}$

Expanding $\sinh \frac{x}{c}$ in the ascending power of $\frac{x}{c}$

we get

$$\frac{l}{2} = c \left[\frac{x}{2} + \frac{1}{3!} \left(\frac{x}{c} \right)^3 \dots \dots \right]$$

Neglecting higher powers of $\frac{x}{c}$, we get

$$\frac{l}{2} = x + \frac{x^3}{6c^2}, \text{ or } \frac{l-2x}{2} = \frac{x^3}{6c^2}, \text{ or } l - 2x = \frac{x^3}{3c^2}$$

$$\text{sag } EC=k=y-c=c \cosh \frac{x}{c} - c = c \left(\cosh \frac{x}{c} - 1 \right)$$

on expanding $\cosh \frac{x}{c}$ in ascending powers of $\frac{x}{c}$,

$$\text{we have } k = c \left[\left\{ 1 + \frac{1}{2!} \left(\frac{x}{c} \right)^2 + \frac{1}{4!} \left(\frac{x}{c} \right)^4 + \dots \dots \right\} - 1 \right]$$

Neglecting higher powers of $\frac{x}{c}$ other than $\left(\frac{x}{c} \right)^2$,

$$\text{we get } k = \frac{x^2}{2c} \text{ so that } c = \frac{x^2}{2k}.$$

on putting the value of c in equation (1) then we get

$$l - AB = \frac{4k^2}{3x} = \frac{4k^2}{3 \cdot \frac{l}{2}} = \frac{8k^2}{3l},$$

since chain is tightly stretched, we get taken $x = \frac{l}{2}$

therefore, $AB = l - \frac{8k^2}{3l}$, i.e. width of the river

$$l - \frac{8k^2}{3l}$$

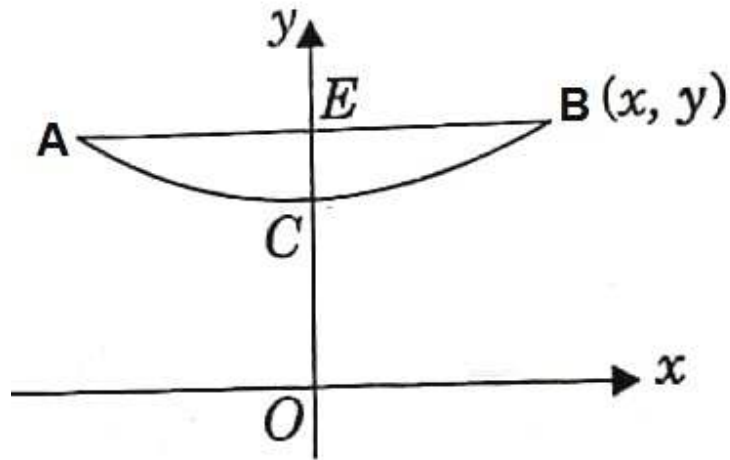


Fig.

Example.5:-

A telegraph wire is stretched between two poles at a distance 'a' meters a part says 'n' meters in the middle; prove that the tension at the ends is approximately

$$w \left(\frac{a^2}{8n} + \frac{7}{6} n \right)$$

Solution-

Applying $y = c \cosh \frac{x}{c}$ for A, we get

$$\begin{aligned} n + c &= c \cosh \frac{a}{2c} \\ &= c \left[1 + \frac{1}{2!} \frac{a^2}{4c^2} + \frac{1}{4!} \frac{a^4}{16c^4} + \dots \right] \\ &= c + \frac{a^2}{8c} + \frac{1}{24 \times 16c^3} \dots \end{aligned}$$

$$\text{or } n = \frac{a^2}{8c} + \frac{1}{24 \times 16c^3} \dots \quad (1)$$

since the wire is tightly stretched, c is very large, so that

$$n = \frac{a^2}{8c}, \text{ to a first approximation}$$

$$\text{i.e. } c = \frac{a^2}{8n}$$

putting the value of c in equation (1), we get

$$n = \frac{a^2}{8c} + \frac{a^4 \times 8 \times 8 \times 8 \cdot n^3}{24 \times 16 \times a^6}$$

$$\text{or } \frac{a^2}{8c} = n - \frac{4n^3}{3a^2} = n \left(1 - \frac{4n^2}{3a^2} \right)$$

$$\text{or } \frac{8c}{a^2} = \frac{1}{n} \left(1 - \frac{4n^2}{3a^2} \right)^{-1}$$

$$\text{or } c = \frac{a^2}{8n} \left[1 + \frac{4n^2}{3a^2} + \dots \right]$$

Expanding by binomial theorem and neglecting higher powers of $\frac{n}{a}$ because n is very small

$$= \frac{a^2}{8n} + \frac{1}{6}n$$

$\therefore$ Tension at the ends

$$= w(n + c) \quad [\because y = n + c \text{ for } A \text{ and } T = wy]$$

$$= w \left(n + \frac{a^2}{8n} + \frac{1}{6}n \right) = w \left(\frac{a^2}{8n} + \frac{7}{6}n \right) \text{ nearly}$$

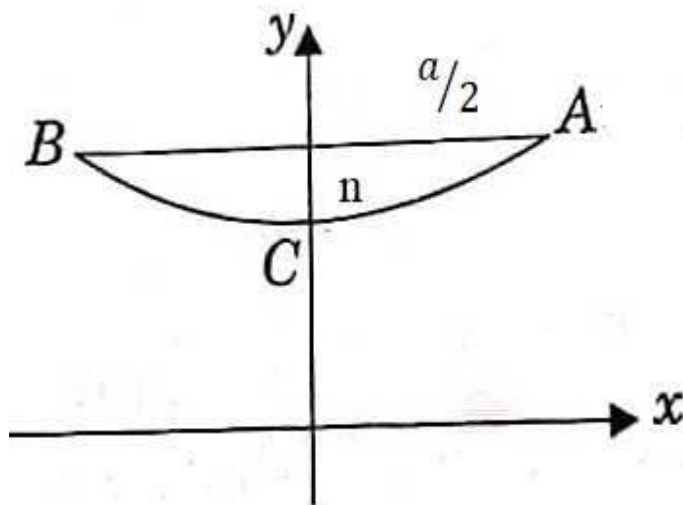


Fig.

3.10. Summary

Definition of common Catenary is discussed. Intrinsic equation of common Catenary and cartesian equation of common Catenary are discussed. Definitions related to Catenary are discussed. Some important relations for the common Catenary are discussed. Approximations of the common stretched wires are discussed. Example to be done with the help of according to their topic are also provided for better understanding.

3.11. Terminal Question

Q. 1. A telegraph wire is made of given material and such a length l is stretched between two parts distant d apart and of the same height as will produce the least possible tension at the posts show that

$$l = \frac{d}{\lambda} \sinh \lambda, \text{ where } \lambda \text{ is given by the equation } \lambda \tanh \lambda = 1$$

.....

Q.2. A uniform chain of length l and weight W is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of support is

$$\frac{K}{2L}W + \frac{l}{4k}W' + \frac{l}{8k}W$$

.....

Q.3. Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two thirds of the circumference of the pulley is

$$a \left\{ \frac{3}{\log(2 + \sqrt{3})} + \frac{4\pi}{3} \right\}$$

.....

Q.4. If the normal at any point P of a common Catenary meets the directrix at Cl then prove that $PQ = \rho$ (radius of curvature)

.....

Q. 5. A heavy uniform chain AB hangs freely under gravity with A fixed and B attached by a string BD to a fixed point D . At the same level as A . The length of the string and chain are such that the ends of the chain at A and B make angles of 60° and 30° respectively with the horizontal. prove that the ratio of these lengths are $\sqrt{3} : 1$

.....



Uttar Pradesh Rajarshi Tondon
open University

Bachelor of Science

UGMM -105

Mechanics - I

(Statics and Dynamics)

BLOCK 2

Dynamics

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Course Design Committee

Prof. Ashutosh Gupta, School of Computer and Information Science, UPRTOU, Prayagraj	Chairman
Prof. Sudhir Srivastav Dept. of Mathematics, DDU Gorakhpur University, Gorakhpur	Member
Prof. P. K. Singh Dept. of Mathematics, Allahabad University, Prayagraj	Member
Prof. Mona Khare Dept. of Mathematics, Allahabad University, Prayagraj	Member
Dr. A. K. Pandey Associate Professor, ECC Prayagraj	Member
Dr. Vikas Singh Academic Consultant, UPRTOU, Prayagraj	Member
Dr. S. S. Tripathi Academic Consultant, UPRTOU, Prayagraj	Member

Course Preparation Committee

Dr. Archana Shukla Assistant Professor (Dept. of Mathematics), Nehru gram bharti university, Prayagraj	Author (Unit – 1 -3, 7)
Ms. Samriddhi Saxena Assistant Professor (Dept. of Mathematics) Invertis University, Bareilly (U.P.)	Author (Unit – 4 - 6)
Dr. S. S. Tripathi Academic Consultant School of Science, UPRTOU, Prayagraj	Editor (Unit 1 - 7)
Prof. Ashutosh Gupta Director, School of Computer and Information Science, UPRTOU, Prayagraj	

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INTRODUCTION

This block is devoted to the study of *dynamics*, the part of mechanics which deals with analysis of bodies in motion. Dynamics include:

1. *Kinematics*, which is the study of the geometry of motion. Kinematics is used to relate displacement, velocity, acceleration and time without reference to the cause of motion. It deals with position in space as a function of time and has nothing to do with force, mass or energy.
2. *Kinetics*, which is the study of the relations existing between the forces acting on the body, the mass of the body and the motion of the body. Kinetics is used to predict the motion caused by the given forces or to determine the forces required to produce a given motion.

Chapter 1-3 are devoted to the *dynamics of particles*; in chapter 1 the *kinematics of particles* will be considered. The use of word *particles(or points)* does not mean that our study will be restricted to small corpuscles; rather, it indicates that in these chapters the motion of bodies – possibly large as cars, rockets or airplanes—will be considered without regard to their size. By saying that the bodies are analyzed as particles (or points), we mean that only their motion as entire unit will be considered; any rotation about their own mass center will be neglected. There are cases, however, when such a rotation is not negligible; the bodies cannot be considered as particles. Such a motion is considered in the study of *dynamics of rigid bodies*.

In this block following units will be discussed:

Unit 4: Motion in a plane.

Unit 5: Rectilinear motion.

Unit 6: Constrained motion.

Unit 7: Motion under central forces.

UNIT 4: MOTION IN A PLANE

Structure

4.1 Introduction

4.2 Objectives

4.3 Displacement and velocity

4.4 Acceleration

4.5 Scalar and vector quantities

4.6 Mean velocity

4.7 Mean acceleration

4.8 Solved examples

4.9 Resolved parts of velocity and acceleration in any fixed direction

4.10 Radial and transverse velocities and acceleration.

4.11 Solved examples

4.12 Tangential and normal velocities, acceleration

4.13 Solved examples

4.14 Self -learning questions.

4.15. Summary

4.16. Terminal questions

4.1. Introduction

To discuss velocity, acceleration, Mean velocity & Mean acceleration and to show the velocity and acceleration in any fixed direction To discuss the Radial velocity and Radial acceleration and also the tangential and normal acceleration.

4.2. Objectives

After reading this unit student should be able to –

- Define velocity and acceleration in a straight line and along a plane curve.
- understand the resolved part of the velocity and acceleration along the Radius vector and perpendicular to the radius vector.
- understand the concept of tangential velocity and acceleration and also the normal velocity and acceleration of the particle at a given point in a plane curve.
- To find the path of the moving particle if velocity and acceleration are given.

4.3. Displacement and velocity

Displacement: If a particle changes its position with respect to surrounding objects and at different times, occupies different positions, it is said to be in motion relative to those objects.

Suppose that a particle in motion is at point P at any instant t , and after a lapse of time Δt , it is at Q . The distance PQ is known as displacement of particle during the time interval Δt . The displacement of a moving particle is therefore a change of its position, this change is measured by the distance moved by a particle in a particular interval. Thus, displacement has both *magnitude and direction*.

Velocity: The velocity of a moving particle is the rate of displacement and has therefore a magnitude as well as direction. The magnitude of the velocity of a particle is called **speed**.

When constant, velocity is measured by the number of units of space described by the moving point in a unit of time. Suppose, in the first place that a point P is moving in a straight line then, its velocity if constant is given by the ratio

$$\frac{(s' - s)}{(t' - t)}$$

Where, s' and s stands for the distances of P from a fixed point O on the straight line at the instants of time t' and t .

When velocity is variable, the ratio $\frac{(s'-s)}{(t'-t)}$ defines *mean or average velocity* of P in the interval of time (t'-t) i.e., the velocity which, if kept constant for an interval of time (t'-t),

will enable P to describe the distance (s'-s).

Taking $s' = s + \Delta s$ and $t' = t + \Delta t$, so that Δs and Δt denote the corresponding increments of s and t, the **mean velocity** is defined by $\frac{\Delta s}{\Delta t}$

The limit of this ratio as Δt tends to zero, if it exists, is taken to define the **instantaneous velocity** or simply the **velocity** of P at any instant t. In notation of the differential calculus it is expressed by $\frac{ds}{dt}$

4.4. Acceleration

The rate of change of velocity of a moving point is called its acceleration. It has both magnitude as well as direction.

NOTE: The difference between velocity and acceleration is that while to find velocity we have to consider the change of position, to find acceleration we have to consider the change of velocity.

Let us consider a point P moving in a straight line. P has constant acceleration, if the change in velocity is the same in the same interval of time. Thus, if the velocity of P changes from v to v' in the interval of time (t'-t), the constant acceleration is equal to

$$\frac{(v'-v)}{(t'-t)}$$

If the acceleration is variable, the ratio $\frac{\Delta v}{\Delta t}$ where, Δv and Δt are corresponding increments in the velocity v and time t, is called the **mean or average acceleration** in the interval of time Δt . The limit of this ratio as Δt tends to zero, i.e. $\frac{dv}{dt}$ or $\frac{d^2s}{dt^2}$ is called the *acceleration of P at any instant t*.

4.5. Scalar and vector quantities

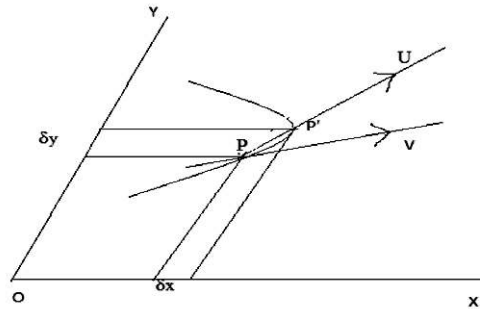
The quantities which do not depend upon direction and are completely determined by magnitude are called *scalar quantities or scalars*. eg. time, money, area and speed.

The quantities which have magnitude as well as direction are called *vector quantities or vectors*. Velocity and acceleration of a moving point are vectors, displacement of a moving point in any interval of time, is a

vector and it can be completely represented by a straight line joining the initial and final position of the point.

4.6. Mean velocity

Let us consider the general case of a point P moving in a curve (see fig) . Let P and P' be the positions of the point at the instant t and t+Δt where Δt need not be small yet.



Then the straight line PP' represents both magnitude and direction of displacement of P in the interval Δt. The **mean velocity** of P in the interval Δt may be defined as the ratio PP'/Δt in magnitude and PP' in direction. Then the directed straight line i.e. the vector PP'U which is drawn equal to PP'/Δt in length completely represents the mean velocity of P in this interval. This mean velocity (as in fig.) , simply signifies the velocity , which if kept constant for the interval Δt, will displace from point P to P'. As Δt tends to zero , the mean velocity represented by the vector PU tends , in general , to a definite limiting value

$$PV = \left(\lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta t} \right)$$

Which is taken to define the **velocity of P at the instant t**. The direction of PV is along the tangent to the curve P , for as Δt tends to zero, P' approaches to P and the chord PP' takes the position of the tangent at P.

Let s denote the arcual distance of P from a fixed point on the curve . If the arc PP' (=Δs) is small quantity of the first order , the difference between the chord PP' is small quantity of third order. Hence,

$$\lim_{\Delta s \rightarrow 0} \frac{PP'}{\Delta s} = 1$$

$$\therefore \lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{PP'}{\Delta s} \cdot \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

This shows that the complete velocity of P is given by s and its direction is along the tangent to the curve at P.

s is positive when s and t increases simultaneously and negative when s decreases as t increases.

Velocity like other vectors can be resolved into components. In two dimensions ,let the cartesian coordinates of P and P' be x,y and x+Δx , y+Δy respectively with reference to any axis, oblique or rectangular . The projections of the vector PP' on the axis are Δx and Δy,and therefore those of mean velocity PU are

$$\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t}$$

then in the limiting case , the component velocities along the coordinate axes at the instant t will

be

$$\frac{dx}{dt}, \frac{dy}{dt}$$

These denote the component velocities of P in the directions of the axes at the instant t , and are equal to the velocities of the projections of the moving point on the respective axes.In the case of rectangular axis

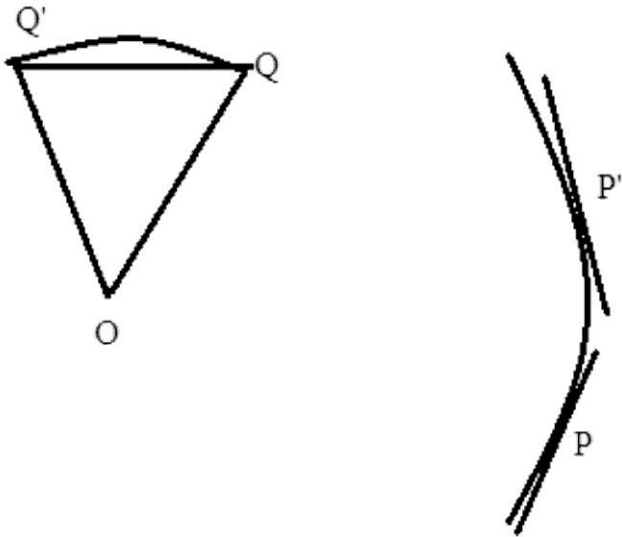
$$s^2 = x^2 + y^2 \text{ and } \tan \psi = y / x$$

Where, ψ is the angle which the direction of motion makes with the x axis.

4.7. Mean acceleration

Let us now consider the acceleration of a point P moving in a curve. In discussing velocity we had to consider the rate of change of position ; now we will consider rate of change of velocity.

If we draw any fixed point O a directed straight line OQ is drawn to represent the velocity of P both in magnitude and direction at any instant, the locus of Q as time varies, will be curve which is called **hodograph** of motion of P.



Let OQ and OQ' represent the velocities of P at any instant t and $t+\Delta t$. The components of OQ' are represented by OQ and QQ' and hence the vector OQ' completely represents the change in velocity of P in the interval Δt .

The **mean acceleration** of P in the interval Δt may be defined by the ratio

$\frac{OQ'}{\Delta t}$ in magnitude and by OQ' in direction. This mean acceleration

simply signifies an acceleration in the direction OQ' , which, if kept constant, for the interval Δt , it will produce the same change of velocity OQ' .

As Δt tends to zero, the limiting value if it exist, of the mean acceleration defines the **acceleration** of P at any instant t .

If we use Cartesian coordinates of OQ and OQ' on the axes will be $\dot{x}, \dot{y}$ and $\dot{x}+\Delta\dot{x}$, $\dot{y}+\Delta\dot{y}$ respectively (as explained in sec 1.4) Hence the projection of OQ' will be $\Delta\dot{x}$, $\Delta\dot{y}$ and those of mean projection in the interval Δt will therefore, be

$$\frac{\Delta\dot{x}}{\Delta t}, \frac{\Delta\dot{y}}{\Delta t}.$$

Hence, the components accelerations in the direction of the co-ordinate axes at the instant t are

$$\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2} \text{ or } \dots,$$

and are equal to the accelerations of projections of the moving point on the respective axes.

4.8. Solved examples

Example 1: Prove that , if a point moves with a velocity varying as any power not less than unity of its distance from a fixed point which it is approaching, it will take an infinite time to reach that point.

Let O be a fixed point and P be the position of the moving point at any instant t.

If OP=x, the magnitude of the velocity at P is λx^n where λ is a constant. Since the point is approaching O , it is infact moving in the direction of x decreasing while $\frac{dx}{dt}$ is positive in the direction of x increasing. Hence, we have,

$$\begin{aligned}\frac{dx}{dt} &= -\lambda x^n \\ \text{or } \frac{dx}{x^n} &= -\lambda dt \dots\dots\dots(i)\end{aligned}$$

When $n > 1$, integration gives,

$$\begin{aligned}-\frac{1}{(n-1)x^{n-1}} &= -\lambda t + c_1 \\ \frac{1}{(n-1)x^{n-1}} &= \lambda t - c_1 \dots\dots\dots(ii)\end{aligned}$$

Where, c_1 is constant of integration.

We see from (ii) that , as $x \rightarrow 0, t \rightarrow \infty$, i.e. the moving point will take an infinite time to reach O.

When $n=1$ the equation(i) becomes $\frac{dx}{x} = -\lambda dt$

Which on integration gives , $\log x = -\lambda t + c_2 \dots\dots\dots(iii)$

The equation (iii) shows that as $x \rightarrow 0, t \rightarrow \infty$, i.e. the moving point will take an infinite time to reach O.

EXAMPLE 2: If time t be a cubic function of x, express the acceleration as a function of x.

4.9. RESOLVED PARTS OF VELOCITY AND ACCELERATION IN ANY FIXED DIRECTION

The resolved parts of velocity and acceleration in any fixed direction are the rates of changes in that direction , we have by definition of derivative,

$$Res. Vel. = \lim_{\delta t \rightarrow 0} \frac{(res. disp. at _t + \delta t) - (res. disp. at _t)}{\delta t}$$

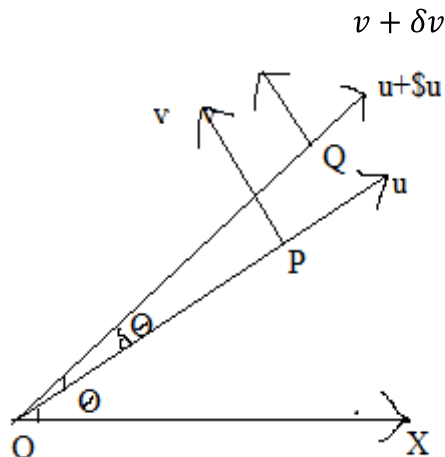
$$Res. Acc. = \lim_{\delta t \rightarrow 0} \frac{(res. vel. at _t + \delta t) - (res. vel. at _t)}{\delta t}$$

The resolved parts of velocity or acceleration in any two perpendicular directions, say, the axes x and y, are known, and the fixed directions makes an angle θ with the x-axis, then the resolved parts of velocity in the fixed direction = $\dot{x} \cos\theta + \dot{y} \sin \theta$ and the resolved part of acceleration = $\ddot{x} \cos\theta + \ddot{y} \sin\theta$.

4.10. RADIAL AND TRANSVERSE VELOCITIES AND ACCELERATION.

A point is moving in a plane curve, to express its component velocities and accelerations at any instant along and perpendicular to the radius vector drawn from a fixed point in the plane of the curve.

Take O as a pole and OX as the initial line. Let the position of the point at time t be P(r, θ) and at time t + δt , Q(r + δr , $\theta + \delta\theta$).



Then the displacement of the point in time δt along the radius vector OP

$$= OQ \cos \delta\theta - OP$$

$$= (r + \delta r) \cos \delta\theta - r$$

$$= (r + \delta r) \left[1 - \frac{(\delta\theta)^2}{2!} + \dots \right] - r$$

$$= \delta r + \text{quantities of smaller orders of magnitude.}$$

As the radial velocity is the rate of change of displacement along the radius vector, we have

$$\text{Radial velocity} = \lim_{\delta t \rightarrow 0} \frac{\delta r}{\delta t} = \frac{dr}{dt} \quad \dots\dots\dots(i)$$

The displacement of the point perpendicular to the radius vector OP in the direction θ increasing is equals to $(r + \delta r) \sin \delta \theta = r \delta \theta +$ quantities of small order in magnitude.

$$\text{Therefore, transversal velocity} = \lim_{\delta t \rightarrow 0} r \frac{\delta \theta}{\delta t} = r \frac{d\theta}{dt} \quad \dots\dots\dots(ii)$$

Now, suppose that the radial and transversal velocities at P at time t are u and v respectively, and that Q at time t + δt these velocities are u + δu and v + δv respectively.

$$\begin{aligned} \text{The change in velocity in time } \delta t \text{ along OP} &= \\ &= (u + \delta u) \cos \delta \theta - (v + \delta v) \sin \delta \theta - u \\ &= (u + \delta u) \left[1 - \frac{(\delta \theta)^2}{2!} + \dots \right] - (v + \delta v) \left[\delta \theta - \frac{(\delta \theta)^3}{3!} + \dots \right] - u \\ &= \delta u - v \delta \theta + \text{quantities of smaller order in magnitude} \end{aligned}$$

Therefore, the rate of change of velocity along OP i.e. **radial acceleration**

$$\begin{aligned} &= \lim_{\delta t \rightarrow 0} \frac{\delta u - v \delta \theta}{\delta t} \\ &= \frac{du}{dt} - v \frac{d\theta}{dt} \\ &= \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 \\ &= \ddot{r} - r \dot{\theta}^2 \end{aligned} \quad \dots\dots\dots(iii)$$

*As the ratios of quantities of second and higher orders to δt which is a small quantity of first order, tend to 0 as δt tends to 0.

Again, the change of velocity in time δt perpendicular to OP in the direction of θ increasing

$$\begin{aligned} &= (u + \delta u) \sin \delta \theta + (v + \delta v) \cos \delta \theta - v \\ &= (u + \delta u) \left[\delta \theta - \frac{(\delta \theta)^3}{3!} + \dots \right] + (v + \delta v) \left[1 - \frac{(\delta \theta)^2}{2!} + \dots \right] - v \\ &= u \delta \theta + \delta v + \text{quantities of smaller order in magnitude} \end{aligned}$$

Therefore ,**transverse acceleration**

$$\begin{aligned}
 &= \lim_{\delta t \rightarrow 0} \frac{u\delta\theta + \delta v}{\delta t} \\
 &= u \frac{d\theta}{dt} + \frac{dv}{dt} \\
 &= \frac{dr}{dt} \times \frac{d\theta}{dt} + \frac{d}{dt} \left(\frac{rd\theta}{dt} \right) \\
 &= 2\dot{r}\dot{\theta} + r\ddot{\theta} \\
 &= \frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) \dots \dots \dots (iv)
 \end{aligned}$$

by substituting the expression for u and v from (i) and (ii).

ALTERNATIVE METHOD:

Since, $x = r \cos\theta$, $y = r \sin\theta$

Differentiating once, both sides of these equations with respect to t (considering r and θ both variables) we get $\left(\frac{dx}{dt}\right)$ and $\left(\frac{dy}{dt}\right)$ in terms of r and θ and their differential coefficients.

Now, $\left(\frac{dx}{dt}\right)\cos\theta + \left(\frac{dy}{dt}\right)\sin\theta$ is the **radial velocity** and $\left(\frac{dy}{dt}\right)\cos\theta - \left(\frac{dx}{dt}\right)\sin\theta$ is the **transversal velocity**

Similarly, for radial and transverse acceleration, obtain $\left(\frac{d^2x}{dt^2}\right)$ and $\left(\frac{d^2y}{dt^2}\right)$ in terms of r and θ , and their differential coefficient with respect to t , and then evaluating the expression

$\left(\frac{d^2x}{dt^2}\right)\cos\theta + \left(\frac{d^2y}{dt^2}\right)\sin\theta, \left(\frac{d^2y}{dt^2}\right)\cos\theta - \left(\frac{d^2x}{dt^2}\right)\sin\theta$ we will get the **radial and transversal accelerations**.

COROLLARY 1: If a point moves in a circle of radius r= radius of circle (constant) then,

$$\text{Radial velocity} = \frac{dr}{dt} = 0 \quad , \quad \text{Transversal velocity} = r \left(\frac{d\theta}{dt} \right)$$

Radial acceleration $= -r \left(\frac{d\theta}{dt} \right)^2$ and Transversal acceleration $= r\ddot{\theta}$.

COROLLARY 2: If a particle move under a central force , the only acceleration on it is radial. Hence transversal acceleration $\frac{1}{r} \frac{d}{dt} (r^2 \dot{\theta}) = 0$

$$\therefore r^2 \dot{\theta} = h (constant)$$

$$\frac{1}{2} r^2 \delta \theta = \frac{h}{2} \delta t$$

$$\text{Integrating both sides ,we get } \int \frac{1}{2} r^2 d\theta = \frac{ht}{2}$$

The left hand expression stands for the area described by the radius vector in time t. Therefore,

$$\frac{\text{the area described}}{\text{the time taken}} = \text{constant.}$$

Hence, under a central force equal areas are swept by the radius vector in equal times. This is the case with planets which move under a central force towards the sun.

COROLLARY 3 : If the radial and transverse accelerations are denoted by f_1 and f_2 respectively, the rate of change of acceleration along the radius vector $= \frac{df_2}{dt} - f_2 \frac{d\theta}{dt}$ and along the perpendicular to the radius vector $= f_2 \frac{d\theta}{dt} + \frac{df_2}{dt}$.

NOTE: These results are obtained by replacing u and v by f_1 and f_2 respectively.

The main types of problems with which we shall be concerned are:

I To find path, when velocities, accelerations etc. are given.

Procedure: write down the equations , eliminate t and integrate.

II To find velocities, accelerations, etc. when the path, some velocity or acceleration are given.

Procedure: Write down the equation of the path and other conditions and differentiate.

4.11. Solved examples

Example 1: The velocities of a particle along and perpendicular to the radius from a fixed origin are λr and $\mu\theta$. Find the path and show that the accelerations along and perpendicular to the radius vector are $\lambda^2 r - \frac{\mu^2 \theta^2}{r}$ and $\mu\theta \left(\lambda + \frac{\mu}{r} \right)$.

Solution: Radial velocity $= \frac{dr}{dt} = \lambda r$ (i)

Transverse velocity $= r \frac{d\theta}{dt} = \mu\theta$ (ii)

Dividing (ii) by (i) we get,

$$r \frac{d\theta}{dt} = \frac{\mu\theta}{\lambda}$$

$$\frac{\mu dr}{\lambda r^2} = \frac{d\theta}{\theta}$$

Integrating , we get the equation of the path described as

$$-\frac{\mu}{r} = \log \theta + k \text{(iii)}$$

$$\text{Radial acceleration} = \ddot{r} - r\dot{\theta}^2 = \lambda^2 \dot{r} - r \frac{\mu^2 \theta^2}{r^2} \quad (\text{from (i) and (ii)})$$

But, $\ddot{r}$ from (i) is equal to $\lambda \dot{r}$. Therefore this acceleration equals

$$= \lambda^2 \dot{r} - \frac{\mu^2 \theta^2}{r}$$

$$\text{Transverse acceleration} = \frac{d}{dt}(r^2 \dot{\theta}) = 2\dot{r}\dot{\theta} + r\ddot{\theta} = 2\lambda r\dot{\theta} + r\ddot{\theta}$$

Now, differentiating (ii) we get , $r\ddot{\theta} + \dot{r}\dot{\theta} = \mu\dot{\theta}$

$$\text{Or,} \quad r\ddot{\theta} = \mu\dot{\theta} - \dot{r}\dot{\theta}$$

$$\mu\dot{\theta} - \lambda r\dot{\theta} = \dot{\theta}(\mu - \lambda r)$$

$$= \frac{\mu\theta}{r}(\mu - \lambda r)$$

$$\text{Hence, the transverse acceleration} = 2\lambda\mu\theta + \frac{\mu\theta}{r}(\mu - \lambda r)$$

$$= \mu \theta \left(\lambda + \frac{\mu}{r} \right)$$

Example 2: If the radial and transverse velocities of a point are always proportional to each other, and this holds for acceleration also, prove that its velocity will vary as same power of the radius vector.

Solution: We have, $r \frac{d\theta}{dt} = \lambda \frac{dr}{dt}$ where, λ is a constant

$$\text{Or, } r \frac{d\theta}{dt} = \lambda$$

$$\text{i.e., } r \frac{d\theta}{dr} = \tan \phi = \lambda \quad \dots\dots\dots(i)$$

$\therefore \phi$ is constant, and hence the curve is an equiangular spiral whose equation is $r = ae^{m\theta}$

where, $m = 1/\lambda$

Also, it is given that $\ddot{r} - r\dot{\theta} = \mu(2r\dot{\theta} - r\ddot{\theta}) \quad \dots\dots\dots(ii)$

Differentiating the equation $r\dot{\theta} = \lambda\dot{r}$, we have

$$r\ddot{\theta} + \dot{r}\dot{\theta} = \lambda\ddot{r},$$

$$\text{Or, } \lambda\ddot{r} = r\ddot{\theta} + \frac{r}{\lambda}\dot{\theta}^2 \quad (\text{substituting for } \dot{r})$$

Now, substituting for $\ddot{r}$ and $\dot{r}$ in (ii), we get

$$\frac{r}{\lambda^2}\dot{\theta} + \frac{r\ddot{\theta}}{\lambda} - r\dot{\theta}^2 = \mu \left[2\frac{r\dot{\theta}^2}{\lambda} + r\ddot{\theta} \right],$$

$$\text{Or, } \ddot{\theta}(m - \mu) = \dot{\theta}^2(2m\mu - m^2 + 1),$$

$$\text{Or, } \ddot{\theta} = \frac{(2m\mu - m^2 + 1)}{(m - \mu)} \dot{\theta}^2$$

This is of the form $\ddot{\theta} = k\dot{\theta}^2$, where k is a constant $\dots\dots(iii)$

$$\text{Put } \frac{d\theta}{dt} = z \text{ then, } \frac{d^2\theta}{dt^2} = \frac{dz}{d\theta} \times \frac{d\theta}{dt}$$

$$\text{Hence, } \frac{d\theta}{dt} \times \frac{dz}{d\theta} = kz \frac{d\theta}{dt} \quad (\text{by (iii)})$$

$$\text{Or, } \frac{dz}{d\theta} = kz$$

Integrating, we get $z = Ae^{k\theta}$, where A is a constant.(iv)

$$\begin{aligned} \text{Again, } v &= \frac{ds}{dt} = \frac{ds}{rd\theta} \times \frac{rd\theta}{dt} = r \frac{d\theta}{dt} \operatorname{cosec}(\phi) \\ &= \operatorname{cosec}(\phi).ae^{m\theta}.Ae^{k\theta} \quad (\text{From the equation of spiral and (iv)}) \\ &= A.a.\operatorname{cosec}(\phi).e^{(m+k)\theta} \end{aligned}$$

Now, since ϕ is constant and $r = ae^{m\theta}$, $v = Lx^n$ where L and n are constants. Therefore the velocity varies as some power of the distance.

Example 3: Show that the path of a point P, which possesses two constant velocities U and V, the first of which is in a fixed direction and second of which is perpendicular to the radius OP drawn from a fixed point, is a conic with focus at O and eccentricity U/V.

Solution: Let U be the velocity along OX and V, along the perpendicular to OP through P. Then, if P be the point (r, θ), velocity along OP = $\frac{dr}{dt} = U \cos \theta$;

and, velocity perpendicular to OP = $r \frac{d\theta}{dt}$

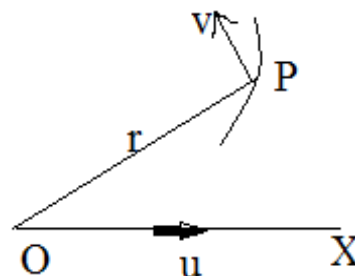
$$= V - U \sin \theta$$

Hence,

$$\frac{r \frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{V - U \sin \theta}{U \cos \theta}$$

$$\Rightarrow r \frac{d\theta}{dt} = \frac{V - U \sin \theta}{U \cos \theta} \frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{r} = \frac{U \cos \theta d\theta}{V - U \sin \theta}$$



Integration gives, $\log r = -\log(V - U \sin \theta) + \log c$

$$r(V - U \sin \theta) = c$$

Or,

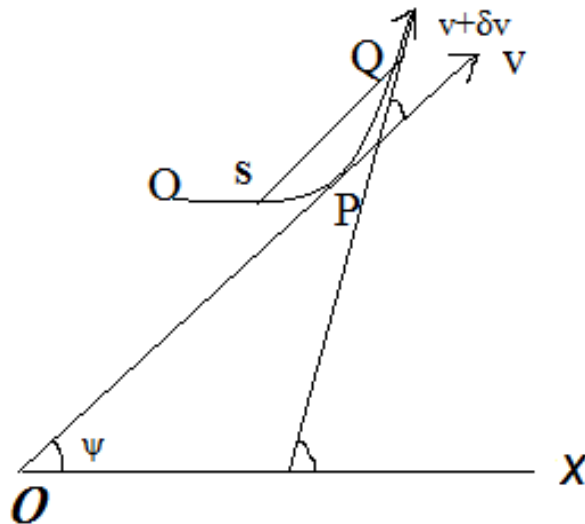
$$\frac{(c/V)}{r} = 1 - \frac{U}{V} \sin \theta$$

which is a conic with focus at O and eccentricity U/V.

Check your progress

- 1) A point describes a circle of radius 'a' with a uniform speed v . Take diameter as initial line and one end of this diameter as pole, and find, by actual differentiation, the radial and transverse accelerations of the point. Verify the result by resolution.
- 2) Find the path, radial and transversal accelerations of a moving point in terms of r and θ if its radial and transversal velocities be λr^2 and $\mu \theta^2$ respectively.
- 3) A ring which can slide on a thin long smooth rod rests at distance d from one end O . The rod is then set revolving uniformly about O in the plane. Show that in space the ring describes the curve $r = d \cosh \theta$.
- 4) If the angular velocity $\sqrt{3}\omega$ of a particle about the origin is constant and the rate of change of acceleration is directed wholly along the radius vector, prove that $r = a \cosh \omega t + b \sinh \omega t$, where, initially, the particle is at a distance 'a' with velocity ωb along the radius vector.
- 5) The angular velocity θ of a particle moving along a curve is constant. If the rate of change of resultant acceleration be perpendicular to the radius vector. Show that the equation of the curve is of the form $r = Ae^{\theta\sqrt{2}} + Be^{-\theta\sqrt{2}} + C$.

4.12. TANGENTIAL AND NORMAL VELOCITIES, ACCELERATIONS



Let P be the position of a particle moving in a curve at time t , and let the arcual distance of P from a fixed point O on the curve be S . Let the position of the particle at time $t + \delta t$ be Q , where $OQ = S + \delta S$. During this displacement the tangent must have turned through a small angle $\delta \psi$ about the tangent at P ,

where ψ is the angle which is tangent at P makes a fixed straight line in the plane.

This displacement of the particle in time δt along the tangent at P in the direction of S increasing = chord PQ $\cos \epsilon$, where ϵ is the angle which the chord PQ makes with the tangent at P.

Therefore, the **tangential velocity** , **which is**, *the rate of change of displacement along the tangent at P at time t is given by:*

$$\begin{aligned}
 &= \lim_{\delta t \rightarrow 0} \frac{PQ \cos \epsilon}{\delta t} \\
 &= \lim_{\delta t \rightarrow 0} \frac{PQ}{\delta t} \lim \cos \epsilon^* \\
 &= \frac{ds}{dt} \times 1 = \frac{ds}{dt} \quad \dots\dots\dots(i)
 \end{aligned}$$

$$\therefore \lim_{\delta t \rightarrow 0} \frac{PQ}{\delta t} = \frac{ds}{dt}$$

and as $\delta t \rightarrow 0$, Q approaches P, $\epsilon \rightarrow 0$ and so $\cos \epsilon \rightarrow 1$.

The displacement perpendicular to the tangent = $PQ \sin \epsilon = PQ \left(\epsilon - \frac{\epsilon^3}{3!} + \dots\dots\dots \right)$.

Therefore, **normal velocity** = $\lim_{\delta t \rightarrow 0} \frac{PQ \{ \epsilon - (\epsilon^3 / 3!) + \dots\dots\dots \}}{\delta t}$

$$\begin{aligned}
 &= \lim_{\delta t \rightarrow 0} \frac{PQ}{\delta t} \lim \{ \epsilon - (\epsilon^3 / 3!) + \dots\dots\dots \} \\
 &= \frac{ds}{dt} \times 0 = 0 \quad \dots\dots\dots(ii)
 \end{aligned}$$

Results (i) and (ii) were both implied in the definition of velocity as given in sec 1.4

Let v and $v + \delta v$ be respectively velocities along the tangents at P and Q. Therefore, the change of velocity in time δt along the tangent P = $(v + \delta v) \cos \delta \psi - v$

$$\begin{aligned}
 &= (v + \delta v) \left[1 - \frac{(\delta \psi)^2}{2!} + \dots\dots\dots \right] - v \\
 &= \delta v + \text{quantities of smaller order of magnitude}
 \end{aligned}$$

As the *tangential acceleration is the rate of change of velocity along the tangent*, we have

$$\text{Tangential acceleration} = \lim_{\delta t \rightarrow 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}.$$

We know that $v = \frac{ds}{dt}$.

Therefore, **tangential acceleration** = $\frac{d}{dx} \left(\frac{ds}{dt} \right) = \frac{d^2s}{dt^2} = \ddot{s}$

The tangential acceleration can also be expressed as : $\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{v dv}{ds}$
(iii)

The change of velocity perpendicular to the tangent at P measured inwards is

$$\begin{aligned} &= (v + \delta v) \sin \delta\psi \\ &= (v + \delta v) \left[\delta\psi - \frac{(\delta\psi)^3}{3!} + \dots \right] \\ &= v\delta\psi + \text{quantities of smaller order of magnitude} \end{aligned}$$

As the normal acceleration is the rate of change of velocity along the normal, we have normal acceleration along the inward normal

$$\begin{aligned} &= \lim_{\delta t \rightarrow 0} \frac{v\delta\psi}{\delta t} = v \frac{d\psi}{dt} \\ &= v \frac{d\psi}{ds} \times \frac{ds}{dt} \\ &= \frac{v^2}{\rho}, \end{aligned} \quad \dots(iv)$$

where ρ is the radius of curvature at the point P.

This shows that dv/dt is the complete acceleration only when ρ is infinite, i.e., only when the particle is moving in a straight line.

ALTERNATIVE METHOD: We have seen that v or ds/dt is the complete velocity of the particle at P to the curve, Therefore, if (x,y) denote the coordinates of P referred to fixed rectangular axes in the plane of the curve we may put

$$\frac{dx}{dt} = v \cos \psi \text{ and } \frac{dy}{dt} = v \sin \psi.$$

Differentiation of these equations with respect to t (considering both v and ψ constant) gives $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$.

Now evaluating ,

$$\frac{d^2x}{dt^2} \cos \psi + \frac{d^2y}{dt^2} \sin \psi \text{ and } \frac{d^2y}{dt^2} \cos \psi - \frac{d^2x}{dt^2} \sin \psi$$

we get tangential and normal accelerations at P.

COROLLARY: In the case of a circle, $s=a\theta$, where ‘a’ is the radius of circle .

Tangential velocity $=a\dot{\theta}$, the same as transverse velocity.

Normal velocity $=0$ =radial velocity.

Tangential acceleration $=a\ddot{\theta}$, the same as transverse acceleration.

Normal acceleration $=a\dot{\theta}^2$ = radial acceleration with the opposite sign.

NOTE: Radial acceleration is measured away from the centre and normal acceleration towards the centre .

4.13. Solved examples

Example 1: Prove that , if the tangential and normal accelerations of a particle describing a plane throughout the motion are the content the angle ψ through which the direction of motion turns in time t is given by

$$\psi = A \log(1 + Bt).$$

Solution: We have $\frac{d^2s}{dt^2} = \lambda$ (i)

and $\frac{v^2}{\rho} = \mu$ (ii)

where , λ and μ are constants.

Integrating eq (i) we have, $\frac{ds}{dt} = \lambda t + c$ where, c is a constant.

Substituting the value of ds/dt in (ii) , we get

$$\frac{(\lambda t + c)^2}{\rho} = \mu$$

Or
$$\begin{aligned} (\lambda t + c)^2 &= \mu \frac{ds}{d\psi} \\ &= \mu \frac{ds}{dt} \frac{dt}{d\psi} \\ &= \mu (\lambda t + c) \frac{dt}{d\psi} \\ \Rightarrow d\psi &= \frac{\mu dt}{(\lambda t + c)} \end{aligned}$$

Integrating this we have, $\psi + k = \frac{\mu}{\lambda} \log(\lambda t + c)$ where, k is an arbitrary const.

When $t = 0$ and $\psi = 0$, we get

$$k = \frac{\mu}{\lambda} \log c$$

$$\therefore \psi = \frac{\mu}{\lambda} \log \frac{\lambda t + c}{c}$$

$$\text{or } \psi = A \log(1 + Bt), \quad A \text{ and } B \text{ being const.}$$

Example 2: Find the intrinsic equation of the curve such that when a point moves on it with constant tangential acceleration and the magnitudes of the tangential velocity and the normal acceleration are in a constant ratio.

Solution: Let $\frac{d^2s}{dt^2} = k, \quad a \text{ const.}, \quad \dots\dots\dots(i)$

$$\frac{ds}{dt} = c \frac{v^2}{\rho}, \quad \text{where } c \text{ is a const.} \quad \dots\dots\dots(ii)$$

Now (ii) becomes $v = c \frac{v^2}{\rho}$

Or, $\rho = cv \quad \dots\dots\dots(iii)$

Writing $v \frac{ds}{dt}$ for $\frac{d^2s}{dt^2}$, and integrating (i) we get

$$v^2 = 2ks + A, \quad \text{where } A \text{ is const.} \quad \dots\dots\dots(iv)$$

Eliminating v between (iii) and (iv), we obtain

$$\rho = c\sqrt{(2ks + A)}$$

$$\text{or } \frac{ds}{d\psi} = c\sqrt{(2ks + A)}.$$

integrating this, we get

$$(c\psi + B) = \sqrt{(2ks + A)} / k$$

$$\text{or } (c\psi + B)^2 k^2 = 2ks + A$$

where, A,B,c and k are all constants.

This is required intrinsic equation of the curve.

4.14. Check your progress

1. A point moves in a plane in such a manner that its tangential and normal accelerations are always equal, and its velocity varies as $\exp\left(\tan^{-1} \frac{s}{c}\right)$, s being the length of arc of the curve measured from a fixed point. Find the path.

2. A particle moves in a catenary (intrinsic equation $s=c \tan \psi$), and the direction of its acceleration at any point makes equal angles with the tangent and the normal to the path at that point. If the speed at the vertex, where $\psi=0$, be u , show that the velocity and acceleration at any other point are given by

$$ue^{\psi} \text{ and } \frac{\sqrt{2}}{c} u^2 e^{2\psi} \cdot \cos^2 \psi.$$

3. If the time be quadratic function of the length of the path of a moving point, prove that the harmonic mean of the initial and final velocities is equal to the velocity at the middle point of the path and that the tangential retardation is proportional to the cube of velocity.
4. Prove that the acceleration of a point moving in a curve with uniform speed is $\rho \dot{\psi}^2$

4.15 Summary

Mean velocity and Mean acceleration has been explained. Resolved parts of velocity and acceleration in any fixed direction for example. Radial and transverse velocity and acceleration Also tangential and normal velocity and acceleration and problems related to above velocity and acceleration in given.

4.16 Terminal questions

1. Find intrinsic equation of a curve such that when a point moves on it with constant tangential acceleration. The magnitude of the tangential velocity and acceleration are in constant ratio.
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.....
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2. A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.
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3. If a particle moves such that the tangential acceleration is equal to the normal acceleration then discuss the motion.
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UNIT 5: RECTILINEAR MOTION

Structure

- 5.1. Introduction**
- 5.2. Objectives**
- 5.3. Simple harmonic motion**
- 5.4. Geometrical representation of simple harmonic motion**
- 5.5. Two simple harmonic motions of the same period and in the same straight line.**
- 5.6. Disturbed simple harmonic motion**
- 5.7. Solved examples**
- 5.8. Self-learning questions**
- 5.9. Motion of a particle under a force varying as a distance**
- 5.10. Attraction varying inversely as the square of the distance from a fixed point**
- 5.11. Motion of particle under gravity in a resisting medium**
- 5.12. Solved examples**
- 5.13. Self-learning questions**
- 5.14. Variable mass systems**
- 5.15. Rocket motion**
- 5.16. Summary**
- 5.17. Terminal questions**

5.1. Introduction

To discuss the motion of a particle when acceleration is variable ie Simple Harmonic motion in the case of horizontal motion and vertical motion. To discuss the motion of a particle Under a force varying as a distance square of the distance. To discuss the motion of a particle in resisting medium and motion when the mass varies.

5.2. Objectives

After reading this unit student should be able to –

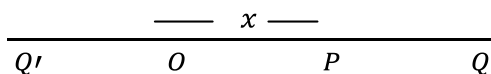
- Make the Differential equation of the given motion an then solved it.
- Make the Differential equation of the particle when there resistance to the motion then solved it.
- Right the equation of motion when the mass varies then solved it.

5.3. SIMPLE HARMONIC MOTION

A particle is said to execute simple harmonic motion if it moves in a straight line under the action of force, which varies as the distance of the particle from a fixed point in that line and is always directed towards the fixed point.

Let the fixed point on the line OX be O which is called the centre. Let the position of the particle at time t be P , where OP=x

Then the acceleration = $\frac{d^2x}{dt^2}$



The equation of motion is , $m \left(\frac{d^2x}{dt^2} \right) = -\lambda x$

.....(i)

Where λ , is a constant.

The Force λx has a negative sign before it , because it is directed towards

O, and the acceleration $\frac{d^2x}{dt^2}$ is in the direction of x increasing.

The equation (i) can be written as in the form , $\frac{d^2x}{dt^2} = -n^2 x$ (ii)

Where n is a real constant.

Suppose a particle starts from rest at a distance a from O . Multiply (ii) by $2(dx/dt)$, and integrate with respect to t .

$$\text{Then, } \left(\frac{dx}{dt}\right)^2 = -n^2 x^2 + A \dots\dots\dots(\text{iii})$$

Where A is a constant.

NOTE: The same result could have been obtained by writing $v \left(\frac{dv}{dx}\right)$ for

$\frac{d^2x}{dt^2}$ in (ii) and integrating with respect to x . For even thus, we get

$$v^2 = -n^2 x^2 + A \dots\dots\dots(\text{iv})$$

Equation (iii) and (iv) are identical.

at Q since $v=0$ when $x=a$ $A = n^2 a^2$. Therefore the equation (iii) becomes

$$\left(\frac{dx}{dt}\right)^2 = n^2 (a^2 - x^2)$$

$$\text{Or, } v = \frac{dx}{dt} = -n\sqrt{(a^2 - x^2)} \dots\dots\dots(\text{v})$$

The radial is taken in minus sign, because the velocity is directed towards O , while dx/dt is in the direction of x increasing. Equation (v) gives the velocity at any distance x from the origin.

Integrating (v), we get

$$\int \frac{dx}{-\sqrt{(a^2 - x^2)}} = n \int dt$$

$$\text{Or, } \cos^{-1}(x/a) = nt + B \quad \text{where } B \text{ is a constant}$$

When $x=a$, $t=0$ therefore $B=0$ and we get,

$$x = a \cos nt \dots\dots\dots(\text{vi})$$

This gives the position of particle at any time t .

The motion is such that the acceleration is greatest at the start and velocity is zero. The particle goes on assuming greater velocity, but the acceleration goes on decreasing as the particle approaches the centre O . When it reaches the centre, the velocity is maximum and the acceleration is zero. With this velocity

the particle describes the negative part of the line OX and the retardation sets in towards O, which destroys the velocity at a distance 'a' from O on the left side. The particle retraces its path.

NOTE: The results obtained above are true for both side of x-axis. As equation (ii) remains the same if we replace $-x$ by x in (ii)(i.e. equation of motion on negative side of x- axis).

If Q and Q^1 be the positions of instantaneous rest, then the particle goes from Q to Q^1 , returns again to Q and this oscillation is repeated over and over again. The time from Q to O, where $QO=a$, is given by (vi).

When $x=0$, $\cos(nt)=0$

Therefore, $nt=\pi/2$ or $t=\pi/2n$.

Similarly, the time from O to $Q^1 = \pi/2n$. Therefore, the time of one complete oscillation (A Q to Q^1 and back to Q) $= 2\pi/n$ which is independent distance from which the particle starts. OQ or OQ^1 is called the **Amplitude of the motion** and $2\pi/n$ is the **periodic time**.

The **frequency** is the number of complete oscillations in one second.

The general solution of the differential equation $\frac{d^2x}{dt^2} = -n^2x$

is $x = a \cos(nt + \epsilon)$, with $\dot{x} = -an \sin(nt + \epsilon)$ where, a and ϵ are arbitrary constants and can be determined by applying the initial conditions.

If the expression $x = a \cos(nt + \epsilon)$, is used for displacement in simple harmonic motion, the quantity ϵ is called EPOCH, and the angle $(nt + \epsilon)$ is called the ARGUMENT. While the phase of the motion is the time that has elapsed since the particle was at its maximum distance in the positive direction.

Evidently x is maximum at time t_0 where $nt_0 + \epsilon = 0$.

Hence, the phase at time $t = t - t_0 = t + \frac{\epsilon}{n} = \frac{nt + \epsilon}{n}$.

COROLLARY: If the particle, instead of starting from rest at Q, is projected with a velocity V in the positive direction, the equation of motion remains the same. But to determine constant of integration in (iv), we have to use the initial conditions $dx/dt=V$, when $x=a$ and then constant A becomes

$$V^2 + n^2 a^2, \text{ and } dx/dt = \sqrt{\{V^2 + n^2(a^2 - x^2)\}}$$

Which shows that the velocity vanishes when $x = \sqrt{\{(V^2 + n^2 a^2) / n^2\}} = b$ (say).

The motion is again simple harmonic with an amplitude b instead of a and $dx / dt = -n\sqrt{(b^2 - x^2)}$

On integrating, we get $\cos^{-1}(x / b) = nt + B$

When $t=0$, $x=a$ $B = \cos^{-1}(a / b)$

$$\therefore nt = \cos^{-1}(x / b) - \cos^{-1}(a / b)$$

This gives t , when x is known.

5.4. GEOMETRICAL REPRESENTATION OF SIMPLE HARMONIC MOTION

If a particle describes a circle with constant angular velocity, the foot of the perpendicular from the particle on a diameter moves with simple harmonic motion.

Let the equation of circle be $x^2 + y^2 = a^2$ and let the x -axis be taken as diameter on which the perpendicular from the moving point is dropped. Let the position of the moving point be $P(x,y)$, its acceleration along the tangent is zero because the angular velocity is constant, but the acceleration along $PO = v^2 / a = \omega^2 a$, where ω is the angular velocity of OP . $\therefore v = a\omega$. The component of acceleration along $OX = -a\omega^2 \cos \theta = -\omega^2 a(x / a) = -\omega^2 x$. This will be the acceleration of the foot of perpendicular from P on the x -axis. Hence, $\frac{d^2 x}{dt^2} = -\omega^2 x$ is the equation of motion of the foot of perpendicular.

5.5. To compound two simple harmonic motions of the same period and in the same straight line.

The displacements of this kind are given by $a \cos(nt + \epsilon)$ and $b \cos(nt + \epsilon')$

$$\begin{aligned} \text{So that,} \quad x &= a \cos(nt + \epsilon) + b \cos(nt + \epsilon') \\ &= A \cos(nt + E) \end{aligned}$$

Where A and E are easily determined. That is, the result of compounding two simple harmonic motions of the same period and in the same straight line is a

simple harmonic motion of the same period whose amplitude and epoch are known.

5.6. Disturbed simple harmonic motion

We shall consider only two simple cases:

CASE 1: Suppose the disturbing force produces constant acceleration f . Then

the equation of motion becomes
$$\frac{d^2x}{dt^2} = -n^2x + f$$

Whose solution is $x = \frac{f}{n^2} + A \cos nt + B \sin nt$

Where A and B are two arbitrary constants, to be determined by applying initial conditions.

CASE 2: Suppose the disturbing force is periodic one, proportional to $\cos pt$;

then the equation of motion becomes
$$\frac{d^2x}{dt^2} = -n^2x + f \cos pt$$

Putting $x = A \cos pt$,

we find this equation has a solution $x = \frac{f \cos pt}{n^2 - p^2}$,

And the complete solution is
$$x = a \cos(nt + \epsilon) + \frac{f \cos pt}{n^2 - p^2},$$

the two terms representing the free and forced oscillations. If n and p are nearly equal, the amplitude of forced oscillations is large. If $n = p$, the equation of motion is

$$\frac{d^2x}{dt^2} + n^2x = f \cos nt.$$

The particular integral of this equation is $x = \left(\frac{f}{2n}\right) t \sin nt$, as can be seen by substitution.

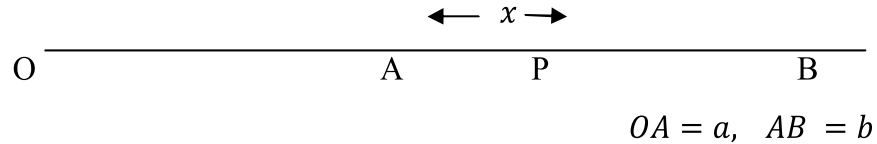
NOTE: This solution corresponds to the forced oscillation and shows that the amplitude will increase indefinitely with time.

5.7. Solved examples

Example 1: One end of an elastic string whose modulus of elasticity is λ and whose natural length is 'a', is tied to a fixed point on smooth horizontal table and the other end is tied to a mass 'm' lying on the table. The particle is pulled to a distance where the extension of the string becomes 'b' and then let go; describe the character of the motion and show that the period of one complete oscillation is $2\left(\pi + \frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}$.

Solution: Let O be a fixed point and OA the natural length of the string. Let AB be the extension 'b' of the string. Take A as the origin and measure x along AB. Then the equation of motion of mass 'm' when at a distance 'x' from A is

$$m \left(\frac{d^2 x}{dt^2} \right) = -T,$$



because, the only force on the mass is the T tension towards A.

But $T = \left(\frac{\lambda}{a} \right) x$, (by Hooke's law)

Hence, $m \frac{d^2 x}{dt^2} = - \left(\frac{\lambda}{a} \right) x. \quad \dots\dots\dots(i)$

The motion is simple harmonic. The equation (i) can be written as in the form

$$v \left(\frac{dv}{dx} \right) = - \left(\frac{\lambda}{am} \right) x$$

Integrating with respect to 'x' we get

$$v^2 = - \left(\frac{\lambda}{am} \right) x^2 + c$$

When $x=b, v=0 \quad c = \left(\frac{\lambda}{am} \right) b^2$

Therefore, $v^2 = \left(\frac{\lambda}{am} \right) (b^2 - x^2) \quad \dots\dots(ii)$

The velocity on reaching A for which $x=0$ is,

$$v = b \sqrt{\left(\frac{\lambda}{am} \right)}$$

Now the string assumes its natural state and the tension ceases to act on the particle, which proceeds towards O with constant velocity $b \sqrt{\left(\frac{\lambda}{am} \right)}$ and describes the distance 2a with this velocity. When it reaches A', the string again begins to be stretched and the tension comes in. The velocity $b \sqrt{\left(\frac{\lambda}{am} \right)}$ is destroyed by the tension in the distance A'B' exactly equal to AB in which it was produced, and the particle retraces its path in the same manner. The

motion is simple harmonic for the amplitude BA or A'B', and the period is $2\pi\sqrt{\left(\frac{ma}{\lambda}\right)}$. The distance 2a is described with a uniform velocity $b\sqrt{\left(\frac{\lambda}{am}\right)}$.

Therefore the period of one complete oscillation for this distance $= \frac{4a}{b\sqrt{(\lambda/am)}} = \frac{4a}{b}\sqrt{\left(\frac{ma}{\lambda}\right)}$

So, the time of one complete oscillation i.e. from B to B' and back again

$$= 2\left(\pi + \frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Example 2: A particle P moves in a straight line OCP, being attracted by a force $m\mu$. PC is always directed towards C, while C moves along OC with a constant acceleration f . If initially C was at rest at the origin O and P was at a distance 'c' from O moving with velocity V, prove that the distance of P from O at any time t is

$$\left(\frac{f}{\mu} + c\right) \cos\sqrt{(\mu)}t + \frac{V}{\sqrt{\mu}} \sin\sqrt{(\mu)}t - \frac{f}{\mu} + \frac{1}{2}ft^2.$$

Solution: Measure 'x' along OCP. If at any time, P is at a distance 'x' from C, then the equation of the relative motion of P with respect to C is

$$\frac{d^2x}{dt^2} = -\mu x - f \quad \dots\dots\dots(i)$$

Or
$$\frac{d^2x}{dt^2} = -\mu\left\{x + \frac{f}{\mu}\right\}$$

The motion is simple harmonic, of period $\frac{2\pi}{\sqrt{\mu}}$, about a moving centre always at a distance $\frac{f}{\mu}$ behind C.

Put
$$x + \frac{f}{\mu} = z$$

Then
$$\frac{d^2z}{dt^2} = -\mu z \quad \dots\dots\dots(ii)$$

The solution of this equation is
$$z = A \cos \{\sqrt{(\mu)}t + B\} \quad \dots\dots\dots(iii)$$

When $t = 0, x = c$ and $z = c + f/\mu$

Therefore
$$c + \frac{f}{\mu} = A \cos B \quad \dots\dots\dots(iv)$$

Also
$$\frac{dz}{dt} = \frac{dx}{dt} = -\sqrt{\mu}A \sin \{\sqrt{(\mu)}t + B\} \quad \dots\dots\dots(v)$$

When $t = 0$, $\frac{dx}{dt} = V$

$$\therefore V = -\sqrt{\mu} A \sin B \dots\dots\dots(vi)$$

$$\text{Now, (iii) gives } x + \frac{f}{\mu} = A \cos\sqrt{(\mu)}t \cos B - A \sin\sqrt{(\mu)}t \sin B$$

Substituting the values of $A \cos B$ and $A \sin B$ from (iv) and (vi) we get,

$$x = -\frac{f}{\mu} + \left(\frac{f}{\mu} + c\right) \cos\sqrt{(\mu)}t + \frac{V}{\sqrt{\mu}} \sin\sqrt{(\mu)}t \dots\dots\dots(vi)$$

This is the relative displacement of P with respect to C. Now the displacement of C in time $t = \frac{1}{2}ft^2$.

$\therefore$ the absolute displacement of P i.e., measured from a fixed point O in time $t =$ the relative displacement of P + the absolute displacement of C

$$= \frac{1}{2}ft^2 - \frac{f}{\mu} + \left(\frac{f}{\mu} + c\right) \cos\sqrt{(\mu)}t + \frac{V}{\sqrt{\mu}} \sin\sqrt{(\mu)}t.$$

Example 3: A light weight string of natural length l and modulus of elasticity λ is hung by one end, and to the other is tied a particle of mass 'm', find the motion:

- i) If t_1 and t_2 be the periods corresponding to two different attached weights and c_1, c_2 the statical extensions due to these weights, prove that

$$g = 4\pi^2 \frac{(c_1 - c_2)}{(t_1^2 - t_2^2)}$$

Solution: Let O, the point of suspension be taken as origin and x is measured downwards. Let the position of the weight at any time t be x below O. The equation of motion is

$$m \left(\frac{d^2x}{dt^2} \right) = mg - T \dots\dots\dots(i)$$

for the only forces acting on the mass m are its weight and the tension T . But, by Hooke's law

$$T = \left(\frac{\lambda}{l} \right) (x - l)$$

$$\text{Therefore, the eq (i) becomes } m \frac{d^2x}{dt^2} = - \left(\frac{\lambda}{l} \right) (x - l) + mg \dots\dots\dots(ii)$$

In the equilibrium position, the tension balances the weight and $\frac{d^2x}{dt^2} = 0$. Hence the extension in the equilibrium position is

$$(x - l) = \frac{g l m}{\lambda} \dots\dots\dots\text{(iii)}$$

The equation (ii) can be written as

$$\frac{d^2 x}{dt^2} = - \left(\frac{\lambda}{m l} \right) \left\{ x - l \frac{(\lambda + m g)}{\lambda} \right\}$$

$$\text{Or} \quad \ddot{\xi} = - \left(\frac{\lambda}{m l} \right) \xi$$

$$\text{Where} \quad \xi = \left(x - l \frac{(\lambda + m g)}{\lambda} \right)$$

This shows that the motion is simple harmonic about the equilibrium position $\xi=0$ or

$$x = l \left\{ \frac{(\lambda + m g)}{\lambda} \right\} \text{ and , the periodic time}$$

$$T = 2\pi \sqrt{\frac{l m}{\lambda}} \dots\dots\dots\text{(v)}$$

- i) If c_1 and c_2 be the extensions due to the weight w_1 and w_2 respectively, and let t_1, t_2 the corresponding periodic times, we have

$$c_1 = \frac{w_1 l}{\lambda} \quad , \quad c_2 = \frac{w_2 l}{\lambda} \quad \text{(from (iii))}$$

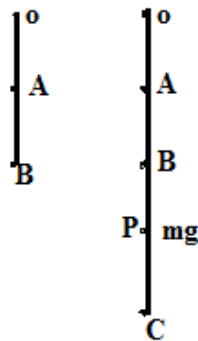
$$t_1 = 2\pi \sqrt{\frac{l w_1}{g \lambda}} \quad , \quad t_2 = 2\pi \sqrt{\frac{l w_2}{g \lambda}} \quad \text{(from (v))}$$

$$\text{Hence,} \quad g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$$

$$\text{Or} \quad g = 4\pi^2 \frac{(c_1 - c_2)}{(t_1^2 - t_2^2)} .$$

Example 4: One end of an elastic string of natural length 'a' is fixed and the other end is fastened by a particle heavy enough to stretch the string to double its length. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle returns to this point in time $2 \left(\frac{2\pi}{3} + \sqrt{3} \right) \sqrt{\frac{a}{g}} .$

Solution: Let OA be the natural length of the string to which a weight mg is attached at its end A.



On account of this weight, the string stretches to double its length i.e. $OB=2a$. The particle is taken to the point C, where $OC=4a$ and then released. Let P be the position of the particle at time t where $OP=x$. Then the equation of motion is

$$m\ddot{x} = mg - T \quad \dots\dots\dots(i)$$

Where, T is the tension in the string and is given by Hooke's law as,

$$T = \frac{\lambda(x-a)}{a} \quad \dots\dots\dots(ii)$$

λ , being modulus of elasticity.

Since, on account of the weight mg , the string stretches upto B and remains in equilibrium till it is stretched further upto C, we have, from (i)

$$0 = mg - T \quad \text{or} \quad mg = T.$$

$$\text{Then (ii) gives, } mg = \frac{\lambda}{a}(2a - a) \quad \text{or} \quad \lambda = mg \quad \dots\dots(iii)$$

$$\text{From (i), (ii) and (iii), we have} \quad m\ddot{x} = mg - \frac{mg}{a}(x - a)$$

$$\text{Or} \quad \ddot{x} = -\frac{g}{a}(x - a - a) = -\frac{g}{a}(x - 2a)$$

Putting $x-2a=\xi$ (iv) becomes, $\xi'' = -\frac{g}{a}\xi$ which shows that motion is simple harmonic.

The centre of oscillation is given by $\xi=0$ or $x=2a$, i.e. the point B. The period of motion is $\therefore 2\pi\sqrt{\frac{a}{g}}$.

$$\text{The solution of (iv) is } x - 2a = a_1 \cos \sqrt{\frac{g}{a}} t + b_1 \sin \sqrt{\frac{g}{a}} t \quad \dots\dots(v)$$

$$\text{And} \quad \dot{x} = -a_1 \sqrt{\frac{g}{a}} \sin \sqrt{\frac{g}{a}} t + b_1 \sqrt{\frac{g}{a}} \cos \sqrt{\frac{g}{a}} t$$

When $t=0$, $x=4a$ and $\dot{x}=0$, therefore $a_1 = 2a, b_1 = 0$. Then (v) becomes

$$x - 2a = 2a \cos \sqrt{\frac{g}{a}} t \quad \dots\dots\dots(\text{vi})$$

And $\dot{x} = -2a \sqrt{\frac{g}{a}} \sin \sqrt{\frac{g}{a}} t \quad \dots\dots\dots(\text{vii})$

The motion will be simple harmonic from C to A and since the string would become slack immediately after the particle would move above A, the motion above A will be that against gravity . If t_1 be the time from C to A, we have from (vi)

$$a - 2a = 2a \cos \sqrt{\frac{g}{a}} t_1$$

Or $-\frac{1}{2} = \cos \sqrt{\frac{g}{a}} t_1$

Or $\cos \frac{2\pi}{3} = \cos \sqrt{\frac{g}{a}} t_1$

i.e. $t_1 = \frac{2\pi}{3} \sqrt{\frac{a}{g}}$

and if u be the velocity at A , we have, $u = -2a \sqrt{\frac{g}{a}} \sin \left[\sqrt{\frac{g}{a}} \frac{2\pi}{3} \sqrt{\frac{a}{g}} \right] =$
 $-2a \sqrt{\frac{g}{a}} \frac{\sqrt{3}}{2} = -\sqrt{3}ag$

the negative sign shows that the velocity at A is towards O.

If u becomes 0 in time t_2 , $0 = ut_2 - g$

Or $t_2 = \frac{\sqrt{3ag}}{g} = \frac{\sqrt{3a}}{g}$

Thus the time from C to A and back to C $= 2(t_1 - t_2)$

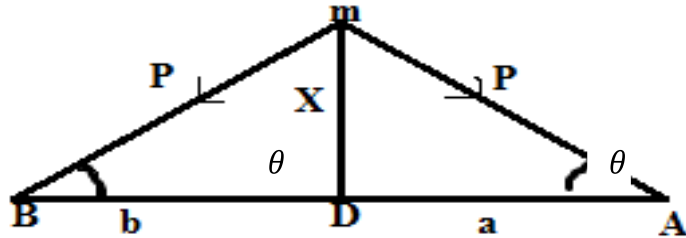
$$= 2 \left[\frac{2\pi}{3} \sqrt{\frac{a}{g}} + \frac{\sqrt{3a}}{g} \right]$$

$$= 2 \sqrt{\frac{a}{g}} \left(\frac{2\pi}{3} + \sqrt{3} \right)$$

Example 5: A particle m is attached to a light wire which is stretched tightly between two fixed points with a tension P . If a, b be the distances of the particle from the two ends , prove that the period of small transverse oscillation

of m is $2\pi\sqrt{\left\{\left(\frac{m}{P}\frac{ab}{(a+b)}\right)\right\}}$. Also, prove that for a wire of given length, the period is longest when the particle is attached to the middle point.

Solution :



When a particle is given a small transversal displacement x , so that the change in P is negligible fraction of P , then the components of tension P on either side of m , perpendicular to the line joining the fixed points are $P\frac{x}{\sqrt{a^2+x^2}}$ and $P\frac{x}{\sqrt{b^2+x^2}} = P \cos \theta$ & $P \cos \theta$

Hence the equation of motion is $m\frac{d^2x}{dt^2} = -Px\left[\frac{1}{a}\left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}} + \frac{1}{b}\left(1+\frac{x^2}{b^2}\right)^{-\frac{1}{2}}\right]$

Or, $\frac{d^2x}{dt^2} = -\frac{P}{m}x\left[\frac{1}{a} + \frac{1}{b}\right]$ if, x^2 be neglected

The motion is simple harmonic with a period $2\pi\frac{\sqrt{abm}}{(a+b)P}$. It is obvious that if $(a+b)$ be constant, ab is greatest when $a=b$. Therefore, the period is maximum when $a=b$.

5.8. Check your progress

1. A heavy particle is attached to an extensible string to a fixed point from which the particle is allowed to fall freely, when the particle is in its lowest position the string is of twice its natural length. Prove that the modulus is four times the weight of the particle and find the time during which the string is extended beyond its natural length.
2. A body is attached to one end of an inextensible string and the other end moves in a vertical line with simple harmonic motion of amplitude a , making n complete oscillations per second. Show that the string will not remain tight during the motion unless $n^2 < g/(4\pi^2 a)$.

3. A particle is suspended at the end of an elastic string of negligible mass. The upper end of the string has simple harmonic vertical oscillation. Find the motion of the particle.

5.9. MOTION OF A PARTICLE UNDER A FORCE VARYING AS DISTANCE

A particle moves in a straight line under the action of a force which varies as its distance from a fixed point in the line, and always away from that point. We will determine the motion;

Let the line be OX and let O be the fixed point in it. Let the position of particle at time t be P, where $OP=x$. Then the equation of motion is:

$$\frac{d^2x}{dt^2} = n^2x \quad \dots\dots\dots(i)$$

Here, the force and the acceleration are both away from O and hence, x is positive.

Multiplying both sides of (i) by $2(dx/dt)$ and integrating we get,

$$\left(\frac{dx}{dt}\right)^2 = n^2x^2 + A. \quad \dots\dots\dots(ii)$$

Let the velocity be zero at $x=a$, then $A = -n^2a^2$, and equation (ii) becomes

$$\frac{dx}{dt} = n\sqrt{x^2 - a^2} \quad \dots\dots\dots(iii)$$

the positive sign before the radical being taken, because the velocity is away from O in the direction of x increasing.

The equation (iii) gives the velocity at any distance x . Integrating (iii), we get

$$B + \int \frac{dx}{\sqrt{x^2 - a^2}} = \int n dt$$

Or
$$\log \frac{x + \sqrt{x^2 - a^2}}{a} + B = nt \quad \dots\dots\dots(iv)$$

When $t=0$, $x=a \therefore B=0$ and so (iv) becomes

$$\log \frac{x + \sqrt{x^2 - a^2}}{a} = nt \quad \dots\dots\dots(v)$$

Equation (v) can be written as in the form

$$x + \sqrt{x^2 - a^2} = ae^{nt}$$

Or,
$$x - \sqrt{x^2 - a^2} = ae^{-nt}$$

$\therefore x = \left(\frac{a}{2}\right)(e^{nt} + e^{-nt}) = a \cosh nt \quad \dots\dots\dots(vi)$

And
$$x = \left(\frac{an}{2}\right)(e^{nt} - e^{-nt})a \sinh nt \quad \dots\dots\dots(vii)$$

These equations show that if t increases continuously, x and $\ddot{x}$ both increases continuously. The particle is always moving away from O with a continuously increasing velocity.

5.10. ATTRACTION VARYING INVERSELY AS THE SQUARE OF THE DISTANCE FROM A FIXED POINT

The differential equation of motion in this case is

$$\frac{d^2x}{dt^2} = v \left(\frac{dv}{dx}\right) = -\frac{\mu}{x^2} \quad \dots\dots\dots(i)$$

Integrating equation (i) with respect to x , we get

$$v^2 = \frac{2\mu}{x} + C.$$

Suppose the particle the particle starts from rest at a distance ‘a’ from the centre of the force , which is taken to be origin here. Then v=0, when x=a so that C=-2μ/a

$\therefore v^2 = 2\mu \left(\frac{1}{x} - \frac{1}{a}\right)$

Or
$$v = \frac{dx}{dt} = -\sqrt{2\mu} \sqrt{\left(\frac{1}{x} - \frac{1}{a}\right)} \quad \dots\dots\dots(ii)$$

because, the particle is moving towards the origin.

Integrating (ii) we have,
$$\frac{\sqrt{2\mu}}{a}(t) = -\int \frac{\sqrt{x}}{\sqrt{a-x}} dx$$

Put $x=a \cos^2\theta$ then,
$$dx = -2a \cos\theta \sin\theta d\theta$$

And
$$\begin{aligned} -\int \frac{\sqrt{x}}{\sqrt{a-x}} dx &= \int 2a \cos^2\theta d\theta \\ &= a \int (1 + \cos 2\theta) d\theta \\ &= a \left[\theta + \frac{\sin 2\theta}{2} \right] \end{aligned}$$

$\therefore t = \sqrt{\frac{a}{2\mu}} \left[a \cos^{-1} \sqrt{\frac{x}{a}} + \sqrt{(ax - x^2)} \right] \quad \dots\dots\dots(iii)$

the constant of integration vanishes since $t=0$, when $x=a$.

It follows from this that the time taken by the particle in reaching the centre where $x = 0$ is given by

$$\sqrt{\frac{2\mu}{a}} t = a \left[\frac{\pi}{2} + \right] = \frac{a\pi}{2}$$

Or

$$t = \frac{\pi a^{\frac{3}{2}}}{2\sqrt{2\mu}}$$

It follows from (ii) that the velocity is infinite when $x=0$, i.e., the particle reaches the centre with an infinite velocity. Now the particle proceeds to describe the negative part of x-axis, attraction being directed towards the origin. The law remaining the same, the velocity of the particle is destroyed in the same time in which it was generated on the positive side of the x-axis, and the particle comes out to an instantaneous rest at a distance 'a' from O on the left side. It then retraces its path, passes through O and comes out to an instantaneous rest at the point at which it started. The motion is one of oscillation and the period of one complete oscillation is

$$\frac{\sqrt{2\pi a^{\frac{3}{2}}}}{\sqrt{\mu}}$$

NOTE: The equation of motion does not remain the same on the negative side of x-axis.

COROLLARY1: Suppose the particle falls from rest at infinity on the earth surface, then from (ii)

$$v^2 = 2\mu \left(\frac{1}{x} - \frac{1}{a} \right), \quad \text{where, } a \text{ is infinite}$$

Or

$$v^2 = \frac{2\mu}{x}$$

To determine μ we use the fact that the acceleration on the earth's surface is g , which must be equal to μ/r^2 , where, r is the radius of earth's surface.

Therefore,

$$\mu = gr^2$$

Hence,

$$v^2 = \frac{2gr^2}{r} \quad \text{on earth's surface,}$$

Or $v = \sqrt{2gr}$ (About 11.2 km per second) towards the centre of earth. = the velocity due to a fall through the height equal to the radius of the earth's surface with a constant acceleration g .

Note: This is also the velocity with which a body must be projected so that it shall never return.

COROLLARY 2: Suppose the particle goes to the centre of the earth through a fixed hole. Now, when the particle is inside the earth's surface, the attraction towards the centre varies directly as the distance from the centre, and the equation of motion is

$$\frac{d^2x}{dt^2} = -\lambda x$$

When $\lambda r = g$ on the earth's surface i.e. $v \left(\frac{dv}{dx} \right) = - \left(\frac{g}{r} \right) x$

Or
$$v^2 = - \left(\frac{gx^2}{r} \right) + C$$

When $x=r$,
$$v = \sqrt{2gr} \quad (\text{by cor. 1})$$

Therefore
$$C = 2gr + gr = 3gr$$

So,
$$v^2 = - \left[\left(\frac{gx^2}{r} \right) - 3gr \right]$$

When particle reaches the centre, $x=0$ and $v^2 = 3gr$.

Corollary 3: The time of falling from a height h above the surface to the surface of earth is given by (iii) as
$$\sqrt{\frac{r+h}{2g}} \left\{ \frac{r+h}{r} \sin^{-1} \sqrt{\frac{h}{r+h}} + \sqrt{\frac{h}{r}} \right\} \quad r \text{ being earth's radius.}$$

5.11. MOTION OF PARTICLE UNDER GRAVITY IN A RESISTING MEDIUM

When a particle passes through a medium like air, it experiences a certain amount of resistance which generally, increases as the velocity increases; and which may therefore, be supposed as some function of velocity. No definite law of resistance is known, but in certain cases the best approximations are obtained by assuming the resistance to be varying as the square of velocity; while in others the best results are obtained by assuming the resistance to be varying as some other power of the velocity.

In the case of particle falling under gravity in a resisting medium the velocity will never exceed a definite limit. Suppose the law of resistance to be kmv^n , then the downward acceleration is $(g - kv^n)$ and this vanishes when the velocity $= (g/k)^{1/n}$. Then, this is the maximum velocity possible and is called **terminal or limiting velocity**. It is on this account that one cannot tell the height from which the raindrop fall by observing their velocity on reaching the

ground. The raindrops, after acquiring their terminal velocity, continue to fall with a more or less constant velocity.

1. *A particle falls under constant force of gravity in a resisting medium whose resistance varies as the square of velocity; we will find the motion*

Measuring x downwards from the position of rest of the particle, the equation of motion is

$$v \left(\frac{dv}{dx} \right) = g - kv^2 \quad \left(\text{terminal velocity} = \frac{\sqrt{g}}{k} \right)$$

Or
$$\frac{dv^2}{dx} + 2kv^2 = 2g \quad \dots\dots\dots(i)$$

Which is linear in v

Solving the differential equation, we have,

$$v^2 e^{2kx} = \left(\frac{g}{k} \right) e^{2kx} + C$$

Now, since $v=0$, when $x=0$, $C=-g/k$

Therefore
$$v^2 = \left(\frac{g}{k} \right) (1 - e^{-2kx}) \quad \dots\dots\dots(ii)$$

Now
$$\frac{dx}{dt} = \sqrt{\left\{ \left(\frac{g}{k} \right) (1 - e^{-2kx}) \right\}}$$

So that
$$t = \left\{ \frac{1}{\sqrt{kg}} \right\} \cosh^{-1} e^{kx} \quad \dots\dots\dots(iii)$$

From which we obtain
$$x = \left(\frac{1}{k} \right) \log \cosh \sqrt{kg} t \quad \dots\dots\dots(iv)$$

2. *A particle is projected upwards in a medium the resistance of which varies as the square of the velocity; we will find the motion.*

Measuring x upwards from the point of projection, the equation of motion is

$$v \frac{dv}{dx} = -g - kv^2 = -g \left(1 + \frac{v^2}{a^2} \right) \quad \dots\dots\dots(i)$$

Where, $a^2 = \frac{g}{k}$.

$$\begin{aligned} \therefore \frac{2gx}{a^2} &= - \int \frac{2v}{v^2 + a^2} dv \\ &= -\log (v^2 + a^2) + C_1. \end{aligned}$$

If V be the velocity of projection we have, $v=V$, when $x=0$ then $C_1 = \log (V^2 + a^2)$

$$\therefore \frac{2gx}{a^2} = \log [(V^2 + a^2)/(v^2 + a^2)] \dots\dots\dots(ii)$$

Again, from (i) $\frac{dv}{dt} = -g \left(1 + \frac{v^2}{a^2}\right)$

$$\therefore -\frac{gt}{a^2} = \int \frac{dv}{v^2 + a^2}$$

$$= \frac{1}{a} \tan^{-1} \frac{v}{a} + C_2$$

Now, when $t=0$, $v=V$, then, $C_2 = -\frac{1}{a} \tan^{-1} \frac{V}{a}$

$$\therefore \frac{gt}{a} = \tan^{-1} \frac{V}{a} - \tan^{-1} \frac{v}{a} \dots\dots\dots(iii)$$

5.12. Solved examples

Example 1: A particle is projected with velocity U vertically upwards under gravity in a medium whose resistance varies as the square of the velocity, prove that the particle will return to the point of projection with a velocity $UV/\sqrt{(U^2 + V^2)}$, where V is the terminal velocity in the medium.

Solution: If the resistance $= kv^2$, then the terminal velocity $V = \sqrt{\frac{g}{k}}$

The height h to which the particle will rise in its upward motion is given by (ii)(in 2.2)

i.e $e^{-2kh} = \frac{V^2}{U^2 + V^2} \dots\dots\dots(i)$

now the particle returns to the point of projection with the velocity v given by eq (ii) (in 2.1)

i.e. $v^2 = V^2(1 - e^{-2kh}) \dots\dots\dots(ii)$

eliminating h between (i) and (ii) we get,

$$v^2 = V^2 \left(1 - \frac{V^2}{U^2 + V^2}\right)$$

$$v = \frac{UV}{(U^2 + V^2)^{1/2}}$$

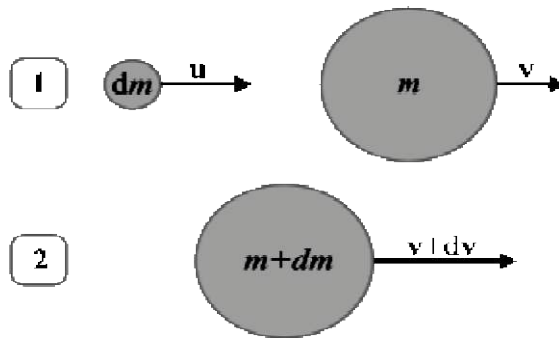
5.13. Self Learning Questions

1. A car of mass 'm', moves on a level road, the power of the engine being constant equal to λ . The frictional resistance is proportional to the speed. The steady speed at which the car travel is C . find the (x, v) relation where, x is the space described and v is the velocity.

2. A heavy particle is projected vertically upwards in a medium, the resistance of which varies as the square of the velocity. It has a kinetic energy k in its upward path at a given point; when it passes the same point on the way down. Show that its loss of energy is $k^2/(k_1+k_2)$, where k_1 is the limit to which its energy approaches in its downward course.
3. A particle starts from rest at A and moves towards a centre of force O. if the time to any position P varies inversely as the distance AP moved through .show that the attraction towards O varies as the cube of AP.

5.14. VARIABLE MASS SYSTEMS

Let us consider a mass dm with relative velocity u is about to collide with the body of mass m and velocity v . After a time dt , at instant 2, both particles move as one body.



Coupling of two moving carriages

Consider two train carriages of mass m_1 , m_2 moving on the same track with speeds U_1 and U_2 , where $U_1 > U_2$ (see figure 1). When they catch each other up they couple together to make a single coupled pair of carriages that moves with speed V . Find V .

Solution: We have conservation of momentum:

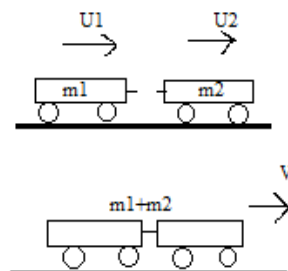


Figure 1: Two carriages coupling together into a single mass

momentum before = momentum after

$$m_1 U_1 + m_2 U_2 = (m_1 + m_2) V$$

So that

$$V = (m_1 U_1 + m_2 U_2) / (m_1 + m_2)$$

Thus the coupled carriages move with the speed of the centre of mass of the uncoupled carriages.

Before colliding the total kinetic energy is $T_b = \frac{1}{2} m_1 U_1^2 + \frac{1}{2} m_2 U_2^2$

and after colliding and joining the kinetic energy is

$$T_a = \frac{1}{2} (m_1 + m_2) \left(\frac{m_1 U_1 + m_2 U_2}{m_1 + m_2} \right)^2$$

Hence the change in kinetic energy $\Delta T = T_a - T_b$ is given by;

$$\Delta T = \frac{1}{2} (m_1 + m_2) \left(\frac{m_1 U_1 + m_2 U_2}{m_1 + m_2} \right)^2 - \frac{1}{2} m_1 U_1^2 - \frac{1}{2} m_2 U_2^2$$

On solving we get, $\Delta T = -\frac{m_1 m_2 (U_1 - U_2)^2}{2(m_1 + m_2)} < 0$

Thus energy is lost in the collision.

5.15. ROCKET MOTION

A rocket of mass m emits mass backwards at speed u relative to the rocket at a constant rate k . Ignoring gravity and air resistance find its speed v at time t if at $t = 0$ it has speed v_0 and mass $M_0 = M + m_0$, where m_0 is the amount of fuel for burning

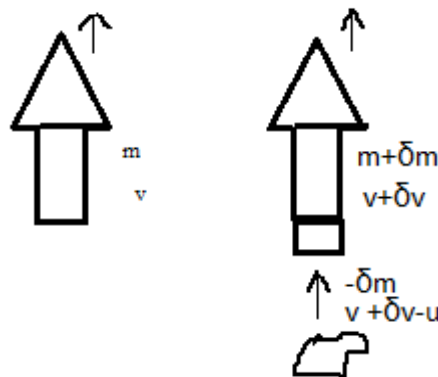


Fig. Rocket burning fuel that is ejected at velocity u relative to the rocket

Therefore, rate of change in momentum

$$\begin{aligned}
 &= \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \{ (m + \delta m)(v + \delta v) + (-\delta m)(v + \delta v - u) - mv \} \\
 &= \lim_{\delta t \rightarrow 0} \frac{1}{\delta t} \{ \delta m v + m \delta v + \delta m \delta v - \delta m v + u \delta m - \delta m \delta v \} \\
 &= m \frac{dv}{dt} + u \frac{dm}{dt}.
 \end{aligned}$$

There is no external force acting so the rate of change of momentum is zero:

$$m \frac{dv}{dt} = -u \frac{dm}{dt}$$

But $dm/dt = -k$ is the constant loss of fuel mass. Thus, $m(t) = M_0 - kt = M + m_0 - kt$ for $t < m_0/k$ (after which there is no more fuel to burn).

$$\frac{dv}{dt} = ku / (M_0 - kt)$$

which integrates to give $v(t) = -u \log(M_0 - kt) + K$

Since $t = 0$ we have $v = v_0$,

we obtain
$$v(t) = v_0 - u \log \left(1 - \frac{k}{M_0} t \right), \quad t \leq m_0/k$$

5.16. Summary

Simple Harmonic motion and geometrical representation a simple his described. Disturb Simple Harmonic motion his explain. Motion the particle attached to an elastic string in horizontal case and vertical case is discussed. Motion of a particle under a force varying inversely as the square of dissents. Motion of particle under gratify in a resisting medium.

5.17. Terminal questions

1. A particle whose means is m is acted upon by a force $m\mu \left(x + \frac{a^4}{x^3} \right)$ towards the origin if it start from rest at distance a , show that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$

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2. A particle rest equilibrium under the attraction of to centre of forces which attract directly at the distance. There attraction per unit mass at unit distances being μ & μ' . The particle is Slightly displaced towards one of then. Show that time of small oscillation is $\frac{2\pi}{\sqrt{\mu+\mu'}}$

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3. Two bodies of masses M and M' are attached to the lower of an elastic string who is upper end is fixed and hangs at rest. If M' falls off show that distance of M from the upper end of the string in time t is $a + b + c \cos\left(\sqrt{\frac{g}{b}} t\right)$

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UNIT – 6 : CONSTRAINED MOTION

6.1. Introduction

6.2. OBJECTIVES

6.3. CONSTRAINED MOTION

6.4 MOTION IN A VERTICAL CIRCLE

6.5. CYCLOIDAL MOTION

6.6. CYCLOIDAL PENDULUM

6.7. CYCLOIDAL PROPERTIES

6.8. SUMMARY

6.9. TERMINAL QUESTIONS

6.1. INTRODUCTION

To discuss the motion when a particle is compelled to move on given smooth plane curve under the action of given force. As the Particular case we shall discuss the motion of a particle in .

1. A circle whose plane is vertical.
2. In a cycloid whose plane is vertical and vertex is upwards or down wards.

6.2. OBJECTIVES

After reading this unit student should be able to –

- Write the Differential equation of the particle and to get the velocity, tension in the string or reaction on the curve.
- To find the point where the particle leaves the curve when dropped from the top of the curve.

6.3. CONSTRAINED MOTION

Constrained motion is a desired thing in the machines. The movements of links in the machine should be constrained for a successful use of it.

The most distinctive characteristic of a machine is that the parts are interconnected and guided in such a way that their motions relative to one another are constrained. Relative to the block, for example, the piston of a reciprocating engine is constrained by the cylinder to move on a straight path; points on the crankshaft are constrained by the main bearings to move on circular paths; no other forms of relative motion are possible.

On some machines the parts are only partially constrained. If the parts are interconnected by springs or friction members, the paths of the parts relative to one another may be fixed, but the motions of the parts may be affected by the stiffness of the springs, friction, and the masses of the parts.

If all the parts of a machine are comparatively rigid members whose deflections under load are negligible, then the constraint may be considered complete and the relative motions of the parts can be studied without considering the forces that produce them. For a specified rotational speed of the crankshaft of a reciprocating engine, for example, the corresponding speeds of points on the connecting rod and the piston can be calculated. The determination of the displacements, velocities, and accelerations of the parts of a machine for a prescribed input motion is the subject matter of kinematics of machines. Such calculations can be made without considering the forces involved, because the motions are constrained.

In a mechanical Pair when an element of the pair has a definite motion with respect to the other element, relative motion between the elements is called constrained motion.

TYPES OF CONSTRAINED MOTION

Constrained motion is categorized in three types given below.

1. Completely constrained motion.

In a pair of two elements if the direction of the motion of these two elements is fixed or definite on the application of the force in the pair, the motion is called completely constrained motion. The motion of two elements with respect to each other can be linear or rotary.

The examples of constrained motion are shaft in a hole with collar, cylinder Movement in a piston, a Prism moving in a hollow prism.

2. Incompletely constrained motion.

If the element of a pair can have more than one type of motion in the pair with respect to the each other then, the motion is referred as incompletely constrained motion.

The shaft in a hollow part is a good example of incompletely constrained motion. The shaft can have two types of motion in a hollow bush. It can rotate and can slide in the hole. These two motions are independent of each other inside the hole.

The motion between elements mainly depends on the type and direction of the force applied in the pair. According to applied force, the element chooses its motion.

3. Successfully constrained motion.

In a pair, if there are more than one direction of moment of an element and we want that element to move in a single direction then we have to apply the external effort in the system. The time until the external load is not applied the motion would be incompletely constrained motion. But when we apply an external load to make the element move in a single direction the motion will become successfully constrained motion.

Foot step bearing is best example of this motion. In a foot step bearing when the shaft rotates, it has the tendency to move in upward direction. But with an external load this upward movement can be controlled or eliminated and desired motion and direction can be achieved.

6.4. MOTION IN A VERTICAL CIRCLE

A particle is attached to the end of a tight inextensible string of length l , whose upper end is tied to a fixed point O , describes the circle in a vertical plane. we will determine the motion

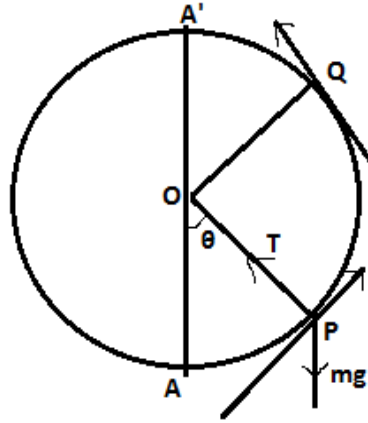


Fig. 1

Let P be the position of the particle at time t . The effective forces are $m \left(\frac{d^2 s}{dt^2} \right)$ and $m(v^2/l)$ along the tangent and the normal respectively. The impressed forces on the particle are mg and T , the tension of the string.

Therefore,
$$m \left(\frac{d^2 s}{dt^2} \right) = -mg \sin \theta \quad \dots\dots\dots(i)$$

$$\left(\frac{mv^2}{l} \right) = T - mg \cos \theta \quad \dots\dots\dots(ii)$$

Since $s=l\theta$, equation (i) becomes

$$l \frac{d^2 \theta}{dt^2} = -g \sin \theta.$$

Integrating this equation, we get,

$$l \left(\frac{d\theta}{dt} \right)^2 = 2g \cos \theta + C \quad \dots\dots\dots(iii)$$

Suppose the particle is projected with velocity u from the lowest point A of the circle, so that when $\theta=0$ the velocity $l\dot{\theta} = u$.

So,
$$\frac{u^2}{l} = 2g + C$$

Therefore,
$$C = \frac{u^2}{l} - 2g$$

$$l^2 \left(\frac{d\theta}{dt} \right)^2 = 2gl \cos \theta + u^2 - 2gl$$

UGMM-105/114 Or,
$$v^2 = u^2 - 2gl(1 - \cos \theta) \quad \dots\dots\dots(iv)$$

NOTE: Equation (iv) is recognized as the energy equation , for $\left(\frac{m}{2}\right)(v^2 - u^2) =$ the work done against gravity from A to P, which is $-mg$ (height raised) $= -mg(1 - \cos \theta)$

Hence from (ii) we have,

$$T = m \left\{ \frac{v^2}{l} + g \cos \theta \right\}$$

$$= m \left[\frac{u^2}{l} - 2g(1 - \cos \theta) + g \cos \theta \right] \dots\dots\dots(v)$$

Equations (iv) and (v) are quite sufficient to give the velocity of the particle and the tension of the string at any point on the circular path.

From (iv) the velocity vanishes ,

when $u^2 = 2gl(1 - \cos \theta)$

or $\cos \theta = 1 - \left(\frac{u^2}{2gl}\right) \dots\dots\dots(vi)$

from (v) $T=0$, when

$$\cos \theta = \frac{1}{3} \left\{ 2 - \frac{u^2}{gl} \right\} \dots\dots\dots(vii)$$

Various cases will arise:-

1. If v vanishes before T is zero , *the motion is oscillatory.*
That is, if the value of θ given by (vi) is less than that of θ given by (vii),
i.e. $1 - \left(\frac{u^2}{2gl}\right) > \frac{1}{3} \left\{ 2 - \frac{u^2}{gl} \right\}$
or, $u^2 < 2gl$
we have the oscillatory motion.
In this case the particle never rises above the diameter of the circle.
2. If v and T vanish simultaneously , $u^2 = 2gl$ and the particle rises up to the horizontal diameter of the circle.
3. If T vanishes before v , the condition is $1 - \left(\frac{u^2}{2gl}\right) < \frac{1}{3} \left\{ 2 - \frac{u^2}{gl} \right\}$
i.e. $u^2 > 2gl$
In this case the particle rises above the horizontal diameter, and either leaves the curve at a point where T is zero, or goes right round the circle if T does not vanish anywhere up to the highest point.
In 4 below it is shown that if $u^2 < 5gl$, T is bound to vanish somewhere .
Hence, the condition for leaving the circle is that
 $u^2 > 2gl$ but $< 5gl$
4. In order that a particle may go round the circle, neither the velocity nor the tension should vanish at the highest point for which $\theta=\pi$.

If T is not to be zero, when $\theta = \pi$, u^2 should exceed $5gl$, from (v).
Also, v cannot vanish before T if $u^2 > 2gl$.

Hence, the condition for making complete revolution is that u^2 should not be less than $5gl$, when $u^2 = 5gl$, T vanishes at the highest point but the velocity does not and so the circular motion persists, and when $u^2 > 5gl$, neither v nor T vanishes at the highest point.

$$\text{The tension at the lowest point} = m \left\{ \frac{u^2}{l} + g \right\}$$

(from v)

Hence, if $u^2 > 5gl$, $T^2 > 6mg$, i.e., the tension at the lowest point must be greater than $6mg$, if the particle is to go right round the circle.

These results equally apply to the motion of a particle moving inside a smooth vertical circle. We only have to replace T by the reaction of the curve.

If the motion is outside the circle we have to replace T by $-R$ and see where R becomes zero, that will give us the point at which the particle leaves the curve.

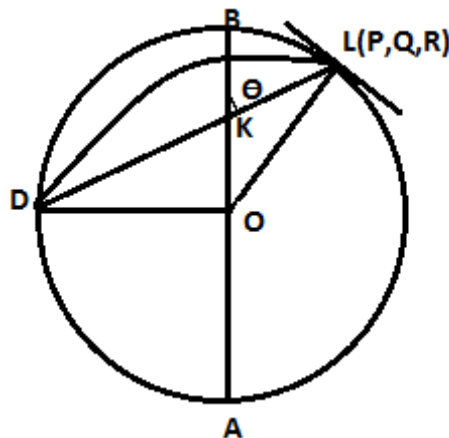
If the restraining curve be a smooth circular tube, the particle can never leave the circular path and hence the condition for a complete revolution is that v should not vanish when $\theta = \pi$.

$$\text{i.e.} \quad u^2 > 4gl.$$

NOTE: It is evident that if the particle rises above the horizontal diameter, it will either leave the circular path or go right round it. If it does not rise so high, the motion is oscillatory

Example 1 : Determine the point D , where the particle after leaving the circular path cuts it again.

Solution : Suppose the particle leaves the circle at L and describes a parabola subsequently, which cuts the circle again at D . The direction of the particle at L is a tangent to both the curves, the circle and the parabola. Therefore the circle and the parabola touch each other at L .



Also if the direction of the velocity at L is reversed , the subsequent path would cut the circle at L as the free parabolic path lies inside the vertical circle and is symmetrical about a vertical line. Hence at L there is a contact of the second order between the two curves, and the three points P, Q , R common to both the curves may be considered to be coincident there. Therefore the circle and the parabola intersect in four points P, Q, R and D, of which the first three are coincident at L. As the common chords of a parabola and the cutting circle are equally inclined to the axis of the parabola , PQ and DR are equally inclined to the vertical line OB; for this parabola has its axis vertical. But PQ is a tangent to the circle as well as to the parabola at L. Hence the tangent at L and DL are each making an angle θ with a vertical line AOB.

Now we wish to prove that, the arc BD = three times the arc LB.

Let DL cut OB at K. Then , $\angle DKB = \pi - \theta$; $\angle BOL = \frac{1}{2}\pi - \theta$

because ,OL is perpendicular to the tangent L.

$$\begin{aligned}\text{Therefore, } \angle OLK &= \angle ODK \\ &= \theta - \left(\frac{1}{2}\pi - \theta\right) \\ &= 2\theta - \frac{1}{2}\pi\end{aligned}$$

$$\begin{aligned}\text{Hence , } \angle DOK &= \angle DKB - \angle ODK \\ &= (\pi - \theta) - \left(2\theta - \frac{1}{2}\pi\right) \\ &= \frac{3}{2}\pi - 3\theta = 3\left(\frac{1}{2}\pi - \theta\right) \\ &= 3 \text{ times } \angle BOL\end{aligned}$$

Hence, the arc BD = 3 times the arc LB.

Example 2: Assuming that the particle goes right round the circle , prove that the sum of the tensions at the ends of any diameter is constant.

Solution : The tension at any point P = $m\left\{\frac{u^2}{l} - 2g + 3g \cos \theta \right\}$
(from v)

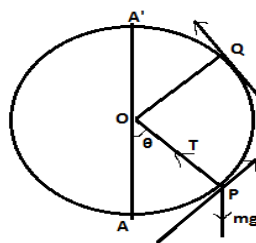
And the tension at the point which is diametrically opposite to P will be,

$$m\left\{\frac{u^2}{l} - 2g + 3g \cos (\theta + \pi) \right\}$$

The sum of these two tensions = $2m\left(\frac{u^2}{l} - 2g\right)$, which is constant.

Example 3: A heavy particle , hanging vertically from a fixed point by a light inextensible string of length l , is stuck by a horizontal blow which imparts to it a velocity $2\sqrt{gl}$; prove that the sting becomes slack when the particle has risen to a height $2/3(l)$ above that fixed point and find the height of the highest point of the parabola subsequently described.

Also find the initial velocity in order that the particle after leaving its circular path may pass through the centre of the circle described.



Solution : With the same notation as in sec. 1.1 and the figure above we have,

$$l^2 \left(\frac{d\theta}{dt} \right)^2 = 2gl \cos \theta + C \quad \dots\dots\dots(i)$$

When $\theta = 0$, $l\dot{\theta}$ is the initial velocity, viz., $2\sqrt{gl}$.

$$\therefore 4gl = 2gl + C$$

$$\text{Or, } C = 2gl.$$

$$\therefore l^2 \left(\frac{d\theta}{dt} \right)^2 = 2gl(1 + \cos\theta) \quad \dots\dots\dots(ii)$$

Now, we have to find that the value of θ for which the string becomes slack i.e. for which T is zero.

From (ii) of sec 1.1, T vanishes when,

$$\frac{v^2}{l} = -g \cos \theta \quad \dots\dots\dots(iii)$$

Substituting the value of v^2 in (iii) from (ii), we get

$$2g(1 + \cos\theta) = -g \cos \theta$$

$$\text{Or } 3 \cos \theta = -2$$

$$\text{Or } \cos \theta = -2/3 \quad \dots\dots\dots(iv)$$

It follows that θ lies between $\frac{1}{2}\pi$ and π . If A' be the highest point of the circle and Q such that

$$\cos AOQ = -2/3$$

Then

$$\cos A'OQ = -\cos AOQ = 2/3$$

$$\text{And the height of } Q \text{ above } O \text{ is } l \cos A'OQ = \frac{2}{3}l$$

The velocity at Q is given by

$$v^2 = -lg \cos \theta = \frac{2}{3}lg$$

$$\text{Or } v = \sqrt{\frac{2}{3}}lg \quad \dots\dots\dots(v)$$

and its direction is along the tangent to the circle at Q .

The inclination of the tangent at Q to the horizontal = the inclination of the normal OQ to the vertical OA' which is equal to $(2/3)l$.

Now we have the case of a projectile whose initial velocity is $\sqrt{\frac{2}{3}}lg$ and whose inclination to the horizontal is $\cos^{-1} \frac{2}{3}$ and so the height of the vertex of the parabolic path Q is

$$\frac{\left(\frac{2}{3}lg \times \frac{5}{9}\right)}{2g} = \frac{5l}{27}$$

Therefore , the height of the vertex of the parabola above the fixed point O is

$$= \frac{2}{3} l + \frac{5}{27} l = \frac{23}{27} l.$$

For second part,

let the initial velocity be u and let the particle rise to a point Q, where $\angle A'OQ = \theta$,

let v be the velocity at Q, where the sting becomes slack.

Then the height of Q above the point of projection A = $l(1 + \cos\theta)$.

Therefore $\frac{1}{2} mv^2 = \frac{1}{2} mu^2 - mg(1 + \cos\theta)$, (by principle of energy)

Or $v^2 = u^2 - 2lg (1 + \cos\theta)$ (vi)

Since the string becomes slack at Q,

$$m(v^2/l) = 0 - mg \cos (\pi - \theta)$$

For $\angle AOQ = \pi - \theta$ or $v^2 = lgc \cos \theta$(vii)

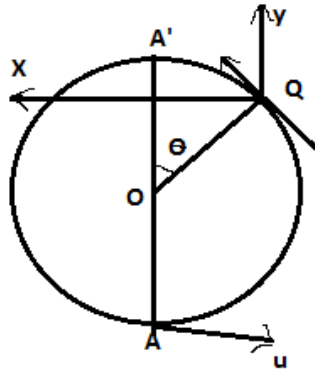


Fig. 2

Now the particle leaves the circular path at Q and describes a parabola with the initial velocity v and the inclination θ to the horizon.

Taking Q as the origin and QX and QY as the axes, the equation of the parabola subsequently described is

$$y = x \tan \theta - \frac{gx^2}{2v^2 \cos^2 \theta}$$

The coordinates of O , the centre , with reference to the axes chosen are $(l \sin \theta, -l \cos \theta)$.

Hence, if the parabola path passes through O,

$$-\cos\theta = \sin\theta \tan\theta - \frac{1}{2} \frac{gl \tan^2\theta}{v^2},$$

Or $2v^2(\sin\theta \tan\theta + \cos\theta) = gl \tan^2\theta$

Or $v^2 = \frac{gl \sin^2\theta}{2 \cos \theta}$ (viii)

Now from (vi),(vii) and (viii) , by eliminating v, we get the values of u and θ .

From (vii) and (viii)

$$lg \cos \theta = \frac{gl \sin^2\theta}{2 \cos \theta}$$

Or $2 \cos^2 \theta = \sin^2 \theta = 1 - \cos^2 \theta$

Or $\cos \theta = 1/\sqrt{3}$ (ix)

From (vi) and (vii) substituting the value of $\cos \theta$, we get

$$\frac{lg}{\sqrt{3}} = u^2 - 2 \lg \left(1 + \frac{1}{\sqrt{3}}\right)$$

Or $u^2 = \lg(2 + \sqrt{3})$ (x)

This gives the initial velocity of projection.

Also $\angle AOQ = \pi - \theta$, where $\theta = \cos^{-1} \left(\frac{1}{\sqrt{3}}\right)$

Therefore, $\angle AOQ = \cos^{-1} \left(\frac{-1}{\sqrt{3}}\right)$

Hence, if the initial velocity u be such that $u^2 = \lg(2 + \sqrt{3})$, the string becomes slack when it has described an angle $\cos^{-1} \left(\frac{-1}{\sqrt{3}}\right)$ and the subsequent path of the particle passes through O .

Example 4: A heavy bead slides on a smooth fixed circular wire of radius 'a'. If it be projected from the lowest point with the velocity just sufficient to carry it to the highest point. Prove that the radius through the bead, in time t , will turn through an angle

$$2 \tan^{-1} \left(\sinh \sqrt{\frac{g}{a}} t \right)$$

Solution: With the notation and the first figure of sec 1.1, we have, by principle of energy,

$$v^2 = u^2 - 2ag(1 - \cos \theta) \quad \text{.....(i)}$$

Where,

$$u^2 = 4ag.$$

$$v^2 = a^2 \left(\frac{d\theta}{dt} \right)^2 = 2ag(1 + \cos \theta) \quad \text{...(ii)}$$

Hence

$$\frac{d\theta}{dt} = \sqrt{\left\{ 2 \frac{g}{a} (1 + \cos \theta) \right\}}$$

Or

$$\frac{d\theta}{dt} = 2 \sqrt{\left(\frac{g}{a} \right) \cos \frac{\theta}{2}}$$

Therefore

$$\sqrt{\frac{g}{a}} t = \int_0^\theta \frac{d\theta}{2 \cos \frac{1}{2} \theta}$$

Put $1/2\theta = \phi$.

$$\sqrt{\frac{g}{a}} t = \int_0^{2\phi} \frac{d\phi}{\cos \phi}$$

$$\sqrt{\frac{g}{a}} t = \left[\log \tan \left(\frac{\pi}{4} + \frac{\phi}{2} \right) \right]_0^{2\phi} = \left[\log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) \right]_0^\theta$$

$$\sqrt{\frac{g}{a}} t = \log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) \quad \text{.....(iii)}$$

Again,

$$\log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) = \log \frac{\cos \left(\frac{1}{4} \theta \right) + \sin \left(\frac{1}{4} \theta \right)}{\cos \left(\frac{1}{4} \theta \right) - \sin \left(\frac{1}{4} \theta \right)}$$

$$\log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) = \log \frac{1 + \sin \left(\frac{1}{2} \theta \right)}{\cos \left(\frac{1}{2} \theta \right)}$$

$$\log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) = \log \left(\sec \frac{\theta}{2} + \tan \frac{\theta}{2} \right)$$

$$\log \tan \left(\frac{\pi}{4} + \frac{\phi}{4} \right) = \sinh^{-1} \left(\tan \frac{\theta}{2} \right)$$

Therefore

$$\sqrt{\frac{g}{a}} t = \sinh^{-1} \left(\tan \frac{\theta}{2} \right)$$

Or

$$\tan \frac{\theta}{2} = \{ \sinh t \sqrt{\frac{g}{a}} \}$$

Or

$$\theta = 2 \tan^{-1} \{ \sinh t \sqrt{\frac{g}{a}} \}$$

If $\theta = \pi$, equation (iii) gives

$$t \sqrt{\frac{g}{a}} = \log \tan \frac{1}{2} \pi, \text{ which is infinite.}$$

Hence, t is infinite. That is, the bead will take an infinite time to reach the highest point.

Example 5: A particle is placed at the highest point of a smooth vertical circle of radius 'a' and is allowed to slide down starting with a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one-third of the radius. Find the position of the directrix and the focus of parabola subsequently described and show that its latus rectum is $16a/27$.

Solution : Let O be the centre and A the highest point of the circle. Let P be a position of the particle, where $\angle AOP = \theta$.

Then, by principle of energy, we get,

$$v^2 = 2ag(1 - \cos \theta)$$

Where, v is the velocity of the particle at P.

$$\text{Or} \quad a^2 \left(\frac{d\theta}{dt} \right)^2 = 2ag(1 - \cos \theta) \quad \dots\dots\dots(i)$$

Also if R be the reaction of the curve and m be the mass of the particle,

$$\frac{mv^2}{a} = mg \cos \theta - R \quad \dots\dots\dots(ii)$$

Or

$$R = mg \cos \theta - 2mg(1 - \cos \theta) \text{ (by i)}$$

$$R = mg(3 \cos \theta - 2) \quad \dots\dots\dots(iii)$$

The particle leaves the curve when $R = 0$, i.e., when $\cos \theta = \frac{2}{3}$. Therefore the depth of the point at which the particle leaves the curve below A

$$= a(1 - \cos \theta) = \frac{1}{3} a, \text{ (since, } \cos \theta = \frac{2}{3} \text{)}$$

The velocity with which the particle leaves the curve is given by (i)

i.e.

$$v^2 = 2ag(1 - \cos \theta) = \frac{2}{3} ag \quad \dots\dots\dots(iv)$$

The direction of this velocity is along the tangent to the circle and hence it makes an angle $\cos^{-1} \frac{2}{3}$ with the horizontal, for the normal makes an angle $\theta = \cos^{-1} \frac{2}{3}$ with the vertical.

Let Q be the point where the particle leaves the curve. Then the velocity at Q is that due to a fall from the height A. hence the directrix of the parabola subsequently described is a horizontal line through A.

The horizontal component of velocity at Q = $v \cos \theta$ $\angle AOQ = \frac{2}{3}v$.

Therefore , the latus rectum == $\frac{2v^2 \cos^2}{g}$

Check your progress

1. A heavy particle of weight W , attached to a fixed point by a light weight inextensible string describes a circle in a vertical plane. The tension of the string has values mW and nW respectively when the particle is at the highest and the lowest path . Show that $n=m+6$.
2. A particle is hanging from affixed point O by means of a light string of length 'a'. There is a smooth nail at O' in the same horizontal line with O at a distance $b(<a)$ from O. Find the minimum velocity with which the particle should be projected from the lowest position in order that if it may make a complete revolution round the nail without the string becoming slack.
3. A particle moves under gravity in a vertical circle, sliding down the convex side of a smooth circular arc . If its initial velocity is that due to a fall to the starting point from a height h above the centre , show that it will fly off the circle when at a height $\frac{2}{3}h$ above the centre.
4. A particle at the end of a string of length l , the upper end of which is fixed , is projected horizontally with a velocity which is k times that due to fall through l. If the particle leaves the circular path , Show that it does so at a height $\frac{1}{3}l(1+2k^2)$ above the lowest point. Also find the greatest height the particle attains during its entire motion.
5. A heavy particle slides under gravity down the inside of a smooth circular tube held in a vertical plane. It starts from the highest point with the velocity $\sqrt{2ga}$, where a is the radius of circle. Prove that when in the subsequent motion the vertical component of its acceleration is a maximum, the pressure on the curve is equal to twice the weight of the particle.

6.5. CYCLOIDAL MOTION

Suppose a particle of mass m is moving on a vertical smooth plane Cycloid whose equation is

$$s = 4a \sin \varphi. \quad \& \text{Vertex downwards :}$$

Let the particle be at P in time t

Resolving along the tangent we have,

$$\frac{md^2 s}{dt^2} = -mg \sin \varphi.$$

Or
$$\frac{d^2 s}{dt^2} = -g \sin \varphi. = -g \frac{s}{4a}$$

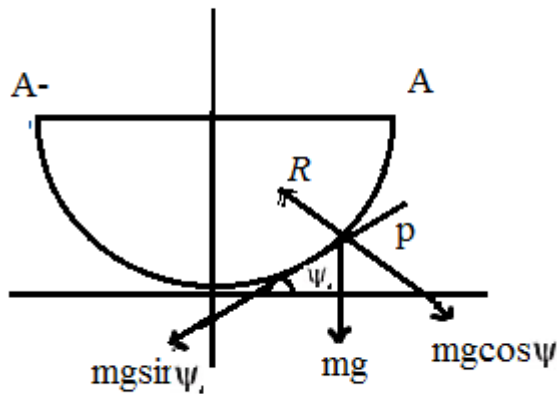
This represents a simple harmonic motion . The periodic time (T) is

$$T = 4\pi \sqrt{\frac{a}{g}}$$

which is independent of the amplitude. From whatever point the particle may be allowed to fall down , the periodic time remains the same.

Such a motion is called *Isochronous Motion*.

If a particle starts from P , it passes the lowest point , rises on the other side to a point P' such that AP'=AP and then retraces its path.



This motion is oscillatory between P and P' , wherever P may be chosen on the arc.

6.6. CYCLOIDAL PENDULUM

If the bob of the pendulum can be made to describe the arc of a cycloid , the periodic time will be the same whatever be the extent of swing.

Let $s = 4a \sin \varphi$, be the equation of cycloid.

Then in Differential calculus it is proved that *the evolute of the cycloid consists of two halves of an equal cycloid.*

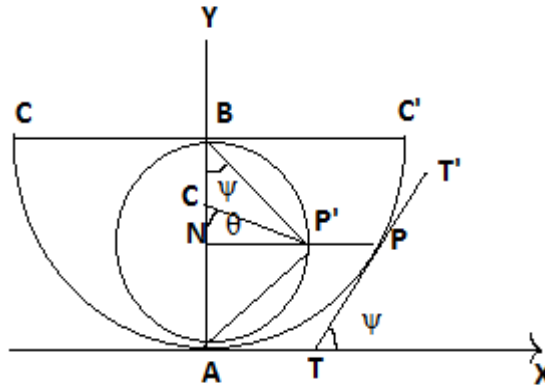
Hence, the evolute of the arc CAC' consists of the arcs CA' and C'A', which are respectively the left and the right halves of a cycloid equal to CAC', and so the

$$\text{arc } CA' = \text{arc } C'A' = \text{arc } CA$$

Now, AA' = twice the height of C above A, i.e., 4a. Let the bob be at P and let the upper end of the string of length 4a be attached to A'. When the bob is let fall, the string is always normal to the lower cycloid and the tangent to one of the arcs CA' and A'C'.

Hence P describes the involute of the curve consisting of the arcs CA' and C'A'. That is, P moves on the cycloid CAC'. The periodic time is independent of the extent of swing. The system is called a *cycloidal pendulum*.

6.7. CYCLOIDAL PROPERTIES



If the tangent at the vertex be taken as the x-axis and the axis of cycloid be taken as y-axis, the equation of the curve in the Cartesian coordinates is

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta), \quad \text{where, } \theta =$$

0

gives the vertex.

Then

$$dx = a(1 + \cos \theta) d\theta \quad \text{and} \quad dy = a \sin \theta d\theta$$

Therefore

$$\tan \varphi = \frac{dy}{dx} = \frac{a \sin \theta}{a(1 + \cos \theta)} = \tan \frac{\theta}{2}$$

Therefore,

$$\varphi = \frac{\theta}{2} \quad \dots\dots\dots(i)$$

Also

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2(1 + \cos \theta)^2 + a^2 \sin^2 \theta = 4a^2 \cos^2 \frac{\theta}{2}$$

Therefore
$$\frac{ds}{d\theta} = 2a \cos \frac{\theta}{2}$$

Or
$$ds = 2a \cos \frac{\theta}{2} d\theta = 4a \cos \varphi d\varphi \quad \text{from (i)}$$

Hence
$$s = 4a \sin \varphi$$
 , the constant of integration vanishing.

This is intrinsic equation of the curve.

Drop PN perpendicular from P on the axis of the cycloid cutting the generating circle placed at the vertex P'. If θ be the angle turned through by the circle in reaching P , $\angle P'CA = \theta$ and $\angle P'BA = \frac{1}{2} \theta$, which is equal to the angle P'AX.

But we have proved that
$$\varphi = \frac{1}{2} \theta.$$

Hence AP' is parallel to the tangent at P. Also, the chord $AP' = 2a \sin \frac{1}{2} \theta$

$$= 2a \sin \varphi$$

$$= \frac{1}{2} (4a \sin \varphi)$$

$$= \frac{1}{2} (\text{arc } AP)$$

Now ,
$$\rho = \frac{ds}{d\varphi} = 4a \cos \varphi = 4a \cos \frac{1}{2} \theta$$
 which is twice the chord BP'.

Since the tangent at P is parallel to the chord AP' the normal at P is parallel to the chord BP' which is perpendicular to AP'. Therefore, the radius of curvature at P is parallel to the chord BP' and is twice this chord.

Again ,
$$PN = x = a(\theta + \sin \theta) ,$$

$$P'N = P'C \sin \theta = a \sin \theta$$

Therefore ,
$$PP' = a\theta = \text{arc } AP'$$

Lastly,
$$y = a(1 - \cos \theta) = 2a \sin^2 \frac{1}{2} \theta = 2a \sin^2 \varphi = \frac{s^2}{8a}$$

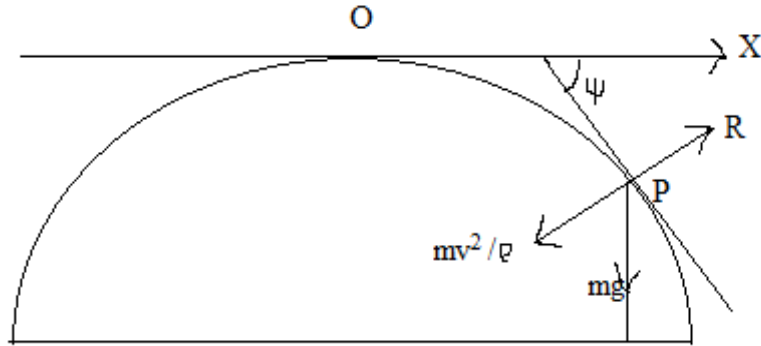
Therefore ,
$$s = \sqrt{8ay}$$

These results can be summed up as follows :

1. The tangent at P is parallel to the chord AP' and the arc AP is twice the total chord AP'.
2. The radius of curvature at P is parallel to the chord BP' and is twice its length.
3. The distance PP' is equal to the arc AP'.
4. $s = \sqrt{8ay}$

Example 1: A particle is allowed to fall down the arc of a smooth cycloid, whose axis is vertical and vertex upwards. It starts with a negligible velocity from the highest point. Prove that it will leave the curve, when it has fallen through half the vertical height of the cycloid. Also find the latus rectum of the parabola subsequently.

Solution :



Taking the vertex as the pole and the tangent at it as the initial line , the equation of cycloid is

$$s = 4a \sin \varphi$$

Then $m \frac{d^2 s}{dt^2} = mg \sin \varphi$ (i)

And $\frac{mv^2}{\rho} = mg \cos \varphi - R$ (ii)

From (i) $v \frac{dv}{ds} = g \sin \varphi = \frac{gs}{4a}$

Therefore $v^2 = \frac{gs^2}{4a}$ (iii)

the constant of integration vanishes , since s and v vanish together.

Also $\rho = \frac{ds}{d\varphi} = 4a \cos \varphi$

Therefore the equation (ii) gives

$$\frac{R}{m} = g \cos \varphi - \frac{v^2}{4a \cos \varphi}$$

$$\frac{R}{m} = g \cos \varphi - \frac{g \times 16a^2 \sin^2 \varphi}{4a \times 4a \cos \varphi}$$

$$\frac{R}{m} = g \left(\cos \varphi - \frac{\sin^2 \varphi}{\cos \varphi} \right) \text{(iv)}$$

The particle will leave the curve when R vanishes. That is , when

$$\cos \varphi - \frac{\sin^2 \varphi}{\cos \varphi} = 0$$

or
$$\frac{\cos 2\varphi}{\cos \varphi} = 0$$

or
$$\varphi = \frac{\pi}{4}$$

that is , the tangent to the curve at the point where the particle leaves it, is inclined to the horizon at an angle of 45° .

Now, ψ being $\frac{1}{4} \pi$, the depth $y=a$, since $y = a(1 - \cos \varphi)$ i.e. the particle leaves the curve when it has fallen through half the vertical height of the cycloid .Also the velocity at the time of leaving is $\sqrt{2ag}$,

and its horizontal component $= \sqrt{2ag} \cos \frac{1}{4} \pi = \sqrt{ag}$

The length of the latus rectum of the parabola subsequently described is

$$\frac{2ag}{g} = 2a$$

Note: The vertical component of the velocity is $\sqrt{2ag} \sin \frac{1}{4} \pi = \sqrt{ag}$.

The depth of the horizontal plane through the base of the cycloid below the point where the particle leaves the curve is a .

Therefore the time t of striking this horizontal plane after leaving the curve is given by

$$a = \sqrt{ag}t + \frac{1}{2}gt^2$$

i.e.
$$t = (\sqrt{3} - 1)\sqrt{\frac{a}{g}}$$

Example 2: Two particles are let drop the cusp of a cycloid down the curve at an interval of time T . Prove that they will meet at a time

$$2\pi\sqrt{\frac{a}{g}} + \frac{1}{2}T$$

Where, a is the radius of the generating circle.

Solution: Let a particle of mass m , starting from rest at the cusp, slide down the cycloid. Suppose at an instant t when it is at $P(x,y)$, its velocity is v . Then, by energy equation,

$$\frac{1}{2}mv^2 - 0 = mg(2a - y)$$

Or
$$v^2 = 2g\left(2a - \frac{y^2}{8a}\right)$$

$$\text{Or} \quad \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a} (16a^2 - s^2)$$

$$\text{Or} \quad \frac{ds}{dt} = -\sqrt{\left\{\frac{g}{4a} (16a^2 - s^2)\right\}}$$

the sign before the radical being negative , because s decreases as t increases.

$$\int_{4a}^s \frac{-ds}{\sqrt{(16a^2 - s^2)}} = \int_0^t \sqrt{\frac{g}{4a}} t$$

Integration gives,

$$\left[\cos^{-1} \frac{s}{4a}\right]_{4a}^s = \sqrt{\frac{g}{4a}} t$$

$$\text{Or} \quad \cos^{-1} \frac{s}{4a} = \sqrt{\frac{g}{4a}} t$$

$$\text{Or} \quad s = 4a \cos \sqrt{\frac{g}{4a}} t$$

Suppose that the two particles meet after a time t measured after the first particle started. Then the second particle moves for the time t-T. Therefore, in order that the two particles meet , we must have,

$$s = 4a \cos \sqrt{\frac{g}{4a}} t = 4a \cos \left[\sqrt{\frac{g}{4a}} (t - T) \right]$$

$$\text{Or} \quad \sqrt{\frac{g}{4a}} t = 2\pi - \left[\sqrt{\frac{g}{4a}} (t - T) \right]$$

$$\text{Or} \quad 2\sqrt{\frac{g}{4a}} t = 2\pi + \sqrt{\frac{g}{4a}} T$$

$$\text{Or} \quad t = \sqrt{\frac{a}{g}} \left[2\pi + \frac{1}{2} \sqrt{\left(\frac{g}{a}\right) T} \right]$$

$$\text{Or} \quad t = 2\pi \sqrt{\frac{a}{g}} + \frac{1}{2} T.$$

SLQ

1. A heavy particle slides down a smooth cycloid starting from rest at the cusp , the axis being vertical and vertex downward . Prove that the magnitude of the acceleration of the particle is equal to g at every point of its path and that when it arrives at the vertex the pressure on the curve is equal to twice the weight of the particle.
2. Show that a cycloid can be a free path for a particle acted on each point by a constant force parallel to the corresponding radius of the generating circle, this circle being placed at the vertex.

3. A particle falls down a cycloid under its own weight . If it starts from a point on the cycloid whose height above the vertex is h, with a velocity due to fall through h. Show that it will attain the same height as it started from, first in time

$$\frac{s\pi}{\{2\sqrt{2gh}\}}$$

where s denotes the arcual distance between the vertex and the point of start.

4. A particle is moving on a smooth curve under gravity and its velocity varies as the arcual distance from the highest point. Prove that the curve must be a cycloid.

6.8. SUMMARY

Constrained motion and types of Constrained motion is explained. Motion is a vertical circle from the lowest point and from the highest point. Motion of a particle in a cycloid whose plane is vertical and vertex downwards and also when the vertex is upwards.

6.9. TERMINAL QUESTIONS

1. A particle moves along the arc of a cycloid in such a manner that the tangent at it rotates with constant angular velocity. Show that the acceleration of the moving point is constant in magnitude.

.....

2. A particle slides down the arc of a smooth cycloid whose axis is vertical & vertex lowest prove that the time of falling down the first half of the vertical height is equal to the time of falling down the second half.

.....

3. A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc show that time of reaching the vertex is

$$2\sqrt{\frac{a}{g}} \tan^{-1} \left[\frac{\sqrt{4ag}}{V} \right]$$

.....

Unit - 7 : Motion under central forces, central orbit conservation of angular momentum, areal velocity, Kepler's laws of motion, differential equation to the path of a particle.

Structure

- 7.1. Introduction
- 7.2. Objectives
- 7.3. Motion under central forces and Central orbit
- 7.4 . Differential equation of a central orbit
- 7.5. Angular momentum or moment of momentum and areal velocity
- 7.6 . Kepler's laws of motion
- 7.7. Velocity from Infinity
- 7.8 Velocity in a Circle
- 7.9. Motion under central forces
- 7.10. Given the Law of Force, to Find the Orbit
- 7.11. Solved example
- 7.12. Summery
- 7.13. Terminal questions

7.1. Introduction

We discussed motion under central forces and central orbit and we have discussed orbital velocity is related to conservation of angular velocity and we discussed hyperbolic and parabolic orbits and there are many theorem related to the central force and central orbits and some important result and also discussed kepler's law of motion and related to important topic and question.

7.2. Objectives

After reading this unit students should be able to

- Define and evaluate the motion under central forces and determine the period of an elliptical orbital.
- Describe how orbital velocity is related to conservation of angular momentum.
- Understand kepler's law of motion.
- Describe the conic sections and how they relate for.
- Understand differential equation of the path of a particle.

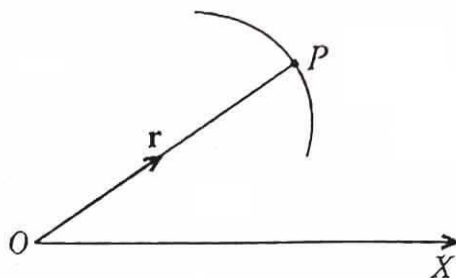
7.3. Motion under central forces and Central orbit

A force whose line of action always passes through a fixed point is called a central force. The fixed point is known as the centre of force.

Central orbit path is described by a particle moving under the action of a central force is known as central orbit and it is a plane curve.

Theorem: A central orbit is always a plane curve.

Solve. Take the central of force O as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\overrightarrow{OP} = \vec{r}$. Then $\frac{d^2\vec{r}}{dt^2}$ is the expression for the acceleration vector of the particle at the point P . Since the particle moves under the action of a central force with centre at O , therefore the only force acting on the particle at P is along the line OP or PO so the acceleration vector of P is parallel to the vector $\overrightarrow{OP}$.



$$\therefore \frac{d^2 \vec{r}}{dt^2} \text{ is parallel to } \vec{r} \Rightarrow \frac{d^2 \vec{r}}{dt^2} \times \vec{r} = 0$$

$$\Rightarrow \frac{d^2 \vec{r}}{dt^2} \times \vec{r} + \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0 \quad \left[\because \frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt} = 0 \right]$$

$$\Rightarrow \frac{d}{dt} \left(\frac{d\vec{r}}{dt} \times \vec{r} \right) = 0$$

$\Rightarrow \frac{d\vec{r}}{dt} \times \vec{r} = \text{a constant vector} = \vec{h}$, say taking dot product of both sides of (1) with the vector $\vec{r}$, we get

$$\vec{r} \cdot \left(\frac{d}{dt} \times \vec{r} \right) = \vec{r} \cdot \vec{h}$$

But the left hand member is scalar triple product involving two equal vectors, and so it vanishes,

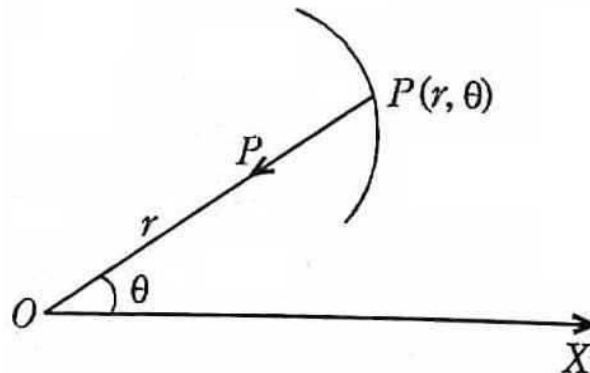
$$\vec{r} \cdot \vec{h} = 0$$

Which shows that $\vec{r}$ is always perpendicular to a constant vector $\vec{h}$. Thus the radius vector OP is always perpendicular to a fixed direction. and hence lies in a plane. Therefore the path P is a plane curve.

7.4. Differential equation of a central orbit

Example 1 : A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane to obtain the differential equation of the path.

Solve. Let a particle move in a plane with an acceleration P which is always directed to a fixed point O in the plane. Take the centre O force O as the pole. Let OX be the initial line and (r, θ) be the co-ordinates of the position P of the moving particle at any instant t .



Since the acceleration of the particle is always directed towards the pole O, Therefore the particle has only the radial acceleration and the transverse component of the acceleration of the particle is always zero. So the equation of motion of the particle at the point P is the radial acceleration. i.e.

$$\frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = -P \quad (1)$$

The negative sign has been taken because the radial acceleration P is in the direction of r decreasing and the transverse acceleration i.e.

$$\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = 0 \quad (2)$$

from(2) we get $\frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = 0$

integrating we get

$$r^2 \frac{d\theta}{dt} = \text{constant} = h \quad (3)$$

Let $r = \frac{1}{u}$

now from (3), we get

$$\frac{d\theta}{dt} = \frac{h}{r^2} = hu^2$$

also $\frac{dr}{dt} = -\frac{1}{u^2} \frac{du}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = -\frac{1}{u^2} \frac{du}{d\theta} u^2 h = -h \frac{du}{d\theta}$

and $\frac{d^2r}{dt^2} = -h \frac{d^2u}{d\theta^2} \cdot \frac{d\theta}{dt} = -h \frac{d^2u}{d\theta^2} (u^2 h) = -h^2 u^2 \frac{d^2u}{d\theta^2}$

Substituting in (1), we get-

$$-h^2 u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} \cdot (u^2 h)^2 = -P$$

or $h^2 u^2 \frac{d^2u}{d\theta^2} + h^2 u^2 = P$

or $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2 u^2} \quad (4)$

Which is the differential equation of a central orbit in polar form referred to the centre of force as the pole.

Pedal form:- If P is the length of the perpendicular drawn from the origin upon the tangent at the point P, we get

$$\frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2$$

$$\text{But} = \frac{1}{r} \cdot \text{Therefore} \frac{du}{d\theta} = \frac{-1}{r^2} \frac{dr}{d\theta}$$

$$\text{i.e.} \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2$$

$$\text{so } \frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \text{-----}(5)$$

Differential both sides of (5) with respect to θ we get

$$\frac{-2}{p^3} \frac{dp}{d\theta} = 2u \frac{du}{d\theta} + 2 \frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2} = 2 \frac{du}{d\theta} \left(u + \frac{d^2u}{d\theta^2}\right)$$

$$\text{or } \frac{-1}{p^3} \frac{dp}{d\theta} = \frac{du}{d\theta} \frac{p}{h^2 u^2}$$

$$\text{or } \frac{-1}{p^3} \cdot \frac{dP}{dr} \frac{dr}{d\theta} = \left(\frac{-1}{r^2} \frac{dr}{d\theta}\right) \left(\frac{p}{h^2 u^2}\right) \quad \left\{ \because \frac{du}{d\theta} = \frac{-1}{r^2} \frac{dr}{d\theta} \right\}$$

$$\text{or } \frac{1}{p^3} \cdot \frac{dP}{dr} = \frac{1}{r^2} \cdot \frac{P}{h^2 u^2} = u^2 \cdot \frac{P}{h^2 u^2} = \frac{P}{h^2}$$

$$\text{or } P = \frac{h^2}{p^3} \cdot \frac{dP}{dr}$$

Which is the differential equation of a central orbit in pedal form.

Check your progress report

Question1. To derive the differential equation of the path in terms of P and r (pedal equation) where P is the perpendicular length from O upon the tangent at P.

Question2. Show that the law of force when the orbit is $r = a \sin \theta$, is

$$h^2 \left[\frac{2n^2 a^2}{r^5} - \frac{n^2 - 1}{r^3} \right]$$

Question3. Prove that in the following cases of the central orbits:-

(I) The curve $r = A \cos n\theta + B \sin n\theta$

$$f = \frac{\mu}{r^{2n+3}}$$

(II) Circle, Pole on the circumference:

$$r = a \cos \theta, \quad f = \frac{\mu}{r^5}$$

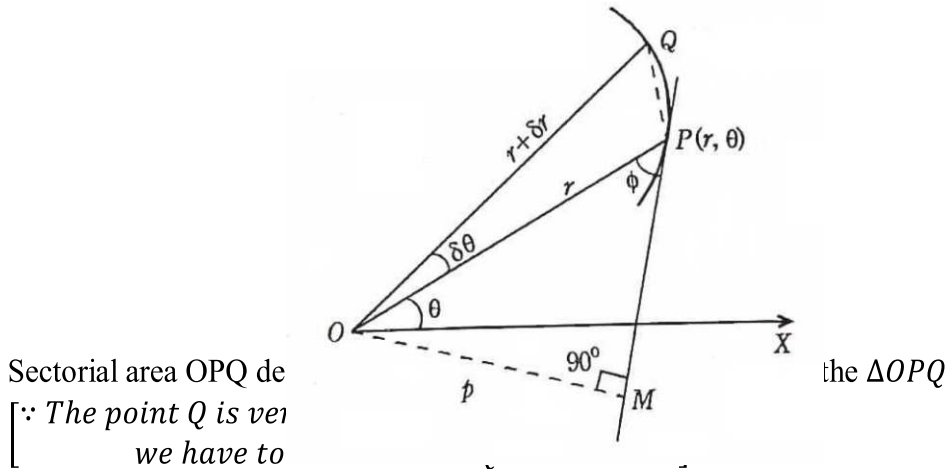
(III) Cardioid: $r = a(1 + \cos \theta), f = \frac{\mu}{r^4}$

7.5. Angular momentum or moment of momentum and areal velocity

The expression $r^2 \left(\frac{d\theta}{dt} \right)$ is called the angular momentum or the moment of momentum about the pole O of a particle of unit mass moving in plane curve since in a central orbit the angular momentum is conserved.

In every central orbit, the sectorial area traced out by the radius vector to the centre of force increase uniformly per unit of time and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Take the centre of force O as the pole and OX as the initial line. Let $P(r, \theta)$ and $\theta(r + \delta r, \theta + \delta \theta)$ be the position of a particle moving in a central orbit at times t and $t + \delta t$ respectively.



$$= \frac{1}{2} OP \cdot OQ \sin \angle POQ$$

$$= \frac{1}{2} r(r + \delta r) \sin \delta \theta$$

$\therefore$ rate of description of the sectorial area

$$= \lim_{\delta t \rightarrow 0} \frac{\text{sectorial area } OPQ}{\delta t}$$

$$= \lim_{\delta t \rightarrow 0} \frac{1}{2} r(r + \delta r) \sin \delta \theta / \delta t$$

$$= \lim_{\delta t \rightarrow 0} \frac{1}{2} r(r + \delta r) \cdot \frac{\sin \delta \theta}{\delta \theta} \cdot \frac{\delta \theta}{\delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h \quad \left[\because r^2 \frac{d\theta}{dt} = h \right]$$

Thus the rate of description of the sectorial area is constant and is equal to $\frac{h}{2}$

The rate of description of the sectorial area is also called areal velocity of the particle about the fixed point O.

Again for a central orbit, we get

$$r^2 \frac{d\theta}{dt} = h$$

$$r^2 \frac{d\theta}{ds} \frac{ds}{dt} = h \text{ or } r^2 \frac{d\theta}{ds} \cdot v$$

$$\left[\because \frac{ds}{dt} = v \text{ (i.e. the linear velocity)} \right]$$

But from differential calculus, we get

$$r \frac{d\theta}{ds} = \sin \phi$$

Where ϕ is the angle between the radius vector and the tangent

$$r^2 \frac{d\theta}{ds} = r \sin \phi = P$$

Where P is the length of the perpendicular drawn from the pole O on the tangent at P

putting $r^2 \left(\frac{d\theta}{ds} \right) = P$ in (2), we get

$$vP = h$$

or
$$v = \frac{h}{P} \text{ ————— (3)}$$

$$v \propto \frac{1}{P}$$

i.e. The linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.

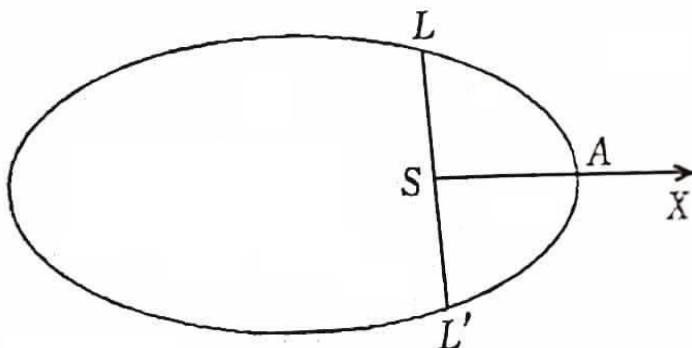
from(3), we have
$$v^2 = \frac{h^2}{P^2}$$

$$\text{But } \frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 = u^2 + \left(\frac{du}{d\theta} \right)^2$$

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right] \text{ ————— (4)}$$

Elliptic orbit (Focus as the centre of force)

A particle moves in an ellipse under a force which is always directed towards its focus:- to find



(i) The law of force

(ii) The velocity at any point of its path

(iii) The periodic time.

We know that the polar equation of an ellipse referred to its focus ---- as pole is.

$$\frac{l}{r} = 1 + e \cos \theta$$

or $u = \frac{1}{l} + \frac{e}{l} \cos \theta$ _____(1)

where $u = \frac{1}{r}$ differentiating, _____(1)

we get $\frac{du}{d\theta} = \frac{-e}{l} \sin \theta$ and $\frac{d^2u}{d\theta^2} = \frac{-e}{l} \cos \theta$

(i) Law of force:-

we know that the differential equation of a central orbit as pole is

$$\frac{p}{h^2 u^2} = \frac{d^2 u}{d\theta^2} + u$$

Where P is the central acceleration assumed to be attractive.

Now here $P = h^2 u^2 \left[u + \frac{d^2 u}{d\theta^2} \right]$

$$= h^2 u^2 \left[\frac{1}{l} + \frac{e}{l} \cos \theta - \frac{e}{l} \cos \theta \right]$$

Substituting for u and $\frac{d^2 u}{d\theta^2}$

$$= \frac{h^2 u^2}{l} = \frac{h^2 / l}{r^2} = \frac{\mu}{r^2} \text{-----} (2)$$

$$\text{where } \mu = \frac{h^2}{l} \text{ or } h^2 = \mu l \text{-----} (3)$$

$$\therefore P \propto \frac{1}{r^2} \quad \text{From (2)}$$

Hence the acceleration varies inversely as the square of the distance of the particle from the focus also the force is attractive because the value of P is positive.

(ii) Velocity:-

We know that the velocity in a central orbit is given by

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right]$$

$$\therefore \text{here } v^2 = h^2 \left[\left(\frac{1}{l} + \frac{e}{l} \cos \theta \right)^2 + \left(\frac{-e}{l} \sin \theta \right)^2 \right]$$

$$= h^2 \left[\frac{1}{l^2} + \frac{2e}{l^2} \cos \theta + \frac{e^2}{l^2} \right]$$

$$= \frac{h^2}{l} \left[\frac{1 + e^2}{l} + \frac{2e \cos \theta}{l} \right]$$

$$= \mu \left[\frac{1 + e^2}{l} + 2 \left(\mu - \frac{1}{l} \right) \right]$$

$$= \mu \left[2u - \frac{1 - e^2}{l} \right] = \mu \left[\frac{2}{r} - \frac{1 - e^2}{l} \right]$$

If $2a$ and $2b$ are the lengths of the major and minor axis of the ellipse we get

$l = \text{the semi-latus rectum}$

$$\frac{b^2}{a} = \frac{a^2(1 - e^2)}{a} = a(1 - e^2)$$

$$\therefore \frac{(1 - e^2)}{l} = \frac{1}{a}$$

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \text{-----} (4)$$

Which gives the velocity of the particle at any point of its path.

Equation (4) shows that the magnitude of the velocity at any point of the path depends only on the distance from the focus and that it is independent of the direction of the motion. also $v^2 < \frac{2\pi}{r}$

(iii) Periodic time:-

We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $h/2$. Let T be the time period for one complete revolution i.e. the time taken by the particle in describing the whole of the ellipse. The sectorial area traced in describing the whole arc of the ellipse is equal to the whole area of the ellipse.

$$T \left(\frac{h}{2} \right) = \text{the whole arc of the ellipse} = \pi ab$$

$$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{ul}} \quad [\because h^2 = \mu l]$$

$$T = \frac{2\pi ab}{\sqrt{\left\{ \mu \left(\frac{b^2}{a} \right) \right\}}} \quad \left[\because l = \frac{b^2}{a} \right]$$

$$\text{or} \quad T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

i.e. The time period for one complete revolution is proportional to $a^{3/2}$ a being semi-major axis.

Hyperbolic and parabolic orbits:-

(i) Hyperbolic orbit:- In the case of hyperbola, we get $e > 1$

$$\text{also} \quad l = \frac{b^2}{a} = \frac{a^2(e^2 - 1)}{a} = a(e^2 - 1)$$

from equation (3), we have $P = \frac{\mu}{r^2}$ where $h^2 = \frac{\mu}{l}$

[Note this result does not depend upon the value of e]

Also proceeding as in establishing the result (4), we get here

$$v^2 = \mu \left[\frac{2}{r} + \frac{e^2 - 1}{l} \right]$$

or
$$v^2 = \mu \left[\frac{2}{r} + \frac{1}{a} \right]$$

Note that here $v^2 > \frac{2\mu}{r}$

(ii) Parabolic orbit:- In this case $e = 1$ proceeding as in equation (3) we get here

$$P = \frac{\mu}{r^2} \text{ and } v^2 = \frac{2\mu}{r}$$

7.6. Kepler's laws of motion

The great astronomer kepler, after many years of hard work, discovered following three laws connecting the motion of the various planets about the sun

- (1) Each planet describes an ellipse having the sun at one of its foci.
- (2) The areas described by the radius drawn from the planet to the Sun are in the sam orbit, proportional to the times of describing them.
- (3) The squares of the periodic times of various planets are proportional to the cubes of the major axes of their orbits.

Deductions from Kepler's laws:-

(1) In this chapter we have already proved in elliptic orbit (Focus as the centre of force) that if the orbit is an ellipse under an acceleration towards one of its foci, then the law of acceleration is that of the inverse square of the distance from the focus (Centre of force). Hence from the first Kepler's law we conclude that the acceleration of each planet towards the sun varies inversely as the square of its distance from the Sun.

(2) According to the Kepler's second law the rate of description of sectorial area is constant which is true only in central orbits. Hence from the second Kepler's law we conclude that the acceleration of the planet and therefore the force on it is directed towards the Sun.

(3) The periodic time T of a closed central orbit under inverse square law is given by

$$T = \frac{\text{area of the ellipse (i.e. the curve described)}}{\text{rate of description of the sectorial area.}}$$

$$= \frac{\pi ab}{\frac{1}{2}h} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu l}} = \frac{2\pi ab}{\sqrt{\left\{\mu \left(\frac{b^2}{a}\right)\right\}}}$$

$$= \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}}$$

But according to the Kepler's third law $T^2 \propto a^3$ from which it follows that μ is constant. Hence from the Kepler's third law we conclude that 'the absolute acceleration μ is the same for all the planets.'

Some important Geometrical properties of an ellipse:-

In this adjoining figure S and H are two foci of the ellipse and C is the centre of the ellipse.

1. The product of the perpendiculars drawn from the foci on the tangent at any point of an ellipse is constant and is equal to the square of the semi minor axis of the ellipse i.e.

$$SM.HN = b^2$$

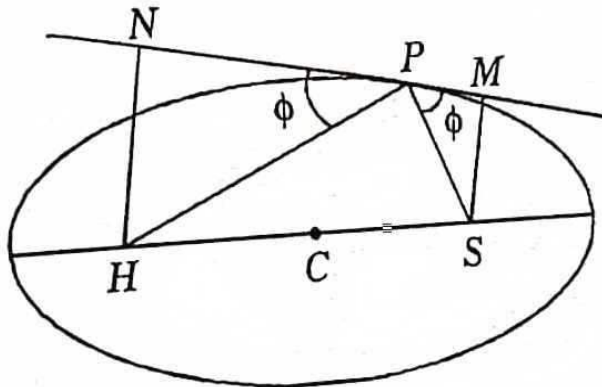
2. The sum of the focal distance of any point on an ellipse is equal to the $2a$, where $2a$ is the length of the major axis of the ellipse. Thus

$$SP + HP = 2a$$

3. The length of the latus rectum of an ellipse is $2\left(\frac{b^2}{a^2}\right)$ where

$$b^2 = a^2(1 - e^2)$$

e being the eccentricity of the ellipse.



A'

4. The tangent at any point P of an ellipse is equally to the focal radii SP and HP of the point.

7.7. Velocity from Infinity

In connection with the central orbits by the phrase ‘*velocity from infinity at any point*’ we mean the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the action of an attractive force in accordance with the law associated with the orbit.

Let us consider a particle falls from rest from infinity in a straight line under the action of a central attractive acceleration P directed toward the centre of force O .

Let Q be the position of the particle at any time t , where $OQ = r$.

Let suppose is the velocity of the particle at Q . The expression for acceleration at the point Q is $v(dv/dr)$.

The equation of motion of the particle at the point Q is

$$v \frac{dv}{dr} = -P.$$

[*–ive sign has been taken because the acceleration*

P is in the direction of r decreasing]

or $v dv = -P dr$. _____(1)

Let V be the velocity acquired in falling from rest at infinity to a point distant a from the centre of force O . Then integrating (1) from infinity to the point $r=a$, we get

$$\int_0^V v dv = - \int_{\infty}^a P dr$$

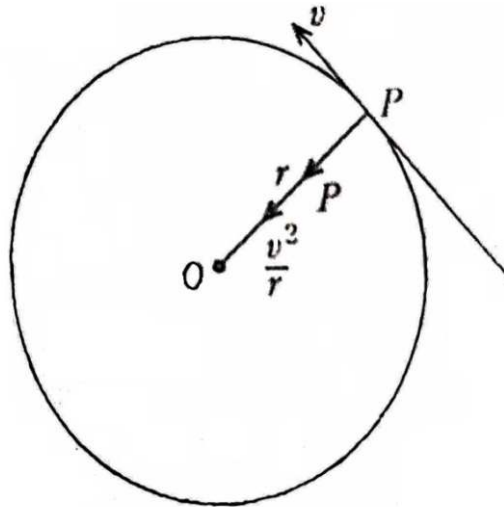
or $\frac{1}{2} V^2 = - \int_{\infty}^a P dr$

or $V^2 = -2 \int_{\infty}^a P dr, \quad \textbf{(Remember)}$

which gives the velocity from infinity at a distance a from the centre of force while moving under the central acceleration P associated with the orbit.

7.8. Velocity in a Circle

The phrase '*velocity in a circle*' at any point of a central orbit means the velocity necessary to describe a circle, passing through that point and with centre at the centre of force, while moving under the action of the pole. Let P be the centre of force O as the pole. Let P be the central acceleration, directed towards O , at any point P of the orbit where $OP=r$. Let v is the velocity in a circle at P . Then v is the velocity at the point P of a particle which moves, under the same central acceleration P , in a circle with centre at O . But for a circle with centre at the pole O , the radius vector OP is also normal to the circle at P . Therefore the central radial acceleration P



= the inward normal acceleration v^2/p

i.e. $p = v^2 / r$ [$\because$ for the circle, $p = r$]

$\therefore v^2 = rP$ **(Remember)**

Thus while moving under a central attractive acceleration P , the velocity V in a circle at a distance a from the centre of force is given by

$$V^2 = a.[P]_{r=a}.$$

7.9. Motion under central forces

1. **Apse** : *An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a maximum or minimum value.*

(Lucknow 2007, 10 ; Rohilkhand 07; Avadh 09)

2. **Apsidal Distance** : *The length of the radius vector at an apse is called an apsidal distance.*

(Lucknow 2007, 10 ; Rohilkhand 07; Avadh 09)

3. **Apsidal Angle** : *The angle between two consecutive apsidal distance is called an apsidal angle.*

(Lucknow 2007, 10 ; Rohilkhand 07; Avadh 09)

7.10. Given the Law of Force, to Find the Orbit

For solving such a problem we substitute the given expression for P in the differential equation

$$h^2 \left[\frac{d^2 u}{d\theta^2} + u \right] = \frac{P}{u^2}$$

We know that referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{d^2 u}{d\theta^2} + u = \frac{P}{h^2 u^2}.$$

where P is the central acceleration assumed to be attractive.

From the given equation of the orbit we can easily calculate u and $d^2 u / d\theta^2$ and substituting their values in (1) we can determine P . Thus we find the law of force. If the value of P is positive, the force is attractive and if the value of P is negative, the force is repulsive.

Case . *The equation of the orbit being given in the pedal form (p,r) (Age 2007)*

The differential equation of a central orbit in (p,r) form is

$$\frac{h^2}{p^3} \frac{dp}{dr} = P.$$

From the given equation of the orbit in (p,r) form, we can find out dp / dr and then substituting its value in (2) we can determine P .

7.11. Solved example

Example. 1:- If the central orbit is $r^n = a^n \cos n\theta$

$$a^n u^n \cos n\theta = 1$$

Taking log on both sides we get

$$a^n \log a + n \log u + \log n\theta = \log 1$$

Differential this we get

$$\frac{n}{u} \frac{du}{d\theta} = n \tan \theta \text{ or } \frac{du}{d\theta} = u \tan n\theta$$

Differentiating again, we get

$$\begin{aligned} \frac{d^2 u}{d\theta^2} &= \tan n\theta \frac{du}{d\theta} + u \sec^2 n\theta \\ &= u \tan^2 n\theta + u \sec^2 n\theta \\ &= u [\tan^2 n\theta + \sec^2 n\theta] \end{aligned}$$

$$\begin{aligned} \text{so that } \frac{d^2 u}{d\theta^2} + u &= u [1 + \tan^2 n\theta + \sec^2 n\theta] \\ &= u [\sec^2 n\theta + \sec^2 n\theta] \\ &= u (n+1) \sec^2 n\theta = a^{2n} u^{2n+1} (n+1) \\ &= (n+1) a^{2n} u^{2n+1} \end{aligned}$$

$$\text{In a central orbit, we get } \frac{f}{h^2 u^2} = \frac{d^2 u}{d\theta^2} + u$$

$$\text{Therefore } \frac{f}{h^2 u^2} = (n+1) a^{2n} u^{2n+1} \quad \text{or}$$

$$f = (n+1) h^2 a^{2n} u^{2n+3}$$

$$\text{i.e. } f = \frac{(n+1) h^2 a^{2n}}{r^{2n+3}}$$

Example. 2: If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the Sun, Prove that

$$(1-e)v_1 = (1+e)v_2$$

Solution: The path of a planet is an ellipse with the Sun at its focus. Therefore the velocity v of the planet at a distance r from the focus S

(the Sun) is given by $v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$

Let v_1 and v_2 be the velocities of the planet at the points A and A'

which are nearest and farthest from the Sun at S , then A ,

$$r = SA = CA - CS = a - ae, v = v_1$$

and at A' , $r = SA' = CS + CA' = ae + a, v = v_2$

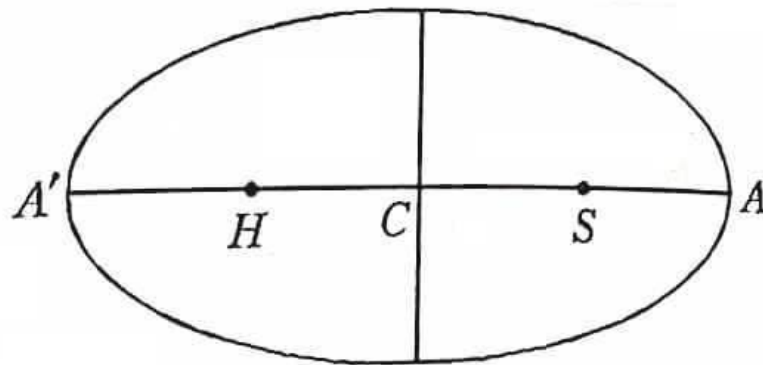
Substituting these values in (1), we get

$$v_1^2 = \mu \left(\frac{2}{a - ae} - \frac{1}{a} \right) = \mu \left\{ \frac{2 - (1 - e)}{a(1 - e)} \right\} = \frac{\mu(1 + e)}{a(1 - e)}$$

$$v_2^2 = \mu \left(\frac{2}{ae + a} - \frac{1}{a} \right) = \mu \left\{ \frac{2 - (1 + e)}{a(1 + e)} \right\} = \frac{\mu(1 - e)}{a(1 + e)}$$

Dividing, we get $\frac{v_1^2}{v_2^2} = \frac{(1+e)^2}{(1-e)^2}$ or $\frac{v_1}{v_2} = \frac{1+e}{1-e}$

or $(1 - e)v_1 = (1 + e)v_2$



Example. 3:- Find the law of force towards the pole under which the curve

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$
 is described

Solution:- The equation of the given central orbit is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1 \text{ _____ (1)}$$

(1) Is the pedal equation of an ellipse referred to the focus as pole differentiating both sides of (1) with respect to 'r' we get

$$-\frac{2b^2}{p^3} \frac{dp}{dr} = -\frac{2a}{r^2} \text{ or } \frac{h^2}{p^3} \frac{dp}{dr} = \frac{a}{b^2} \frac{h^2}{r^2}$$

$$\therefore P = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{ah^2}{b^2} \cdot \frac{1}{r^2}$$

Thus $P \propto \frac{1}{r^2}$ i.e. the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Example. 4:- If w be the angular velocity of a planet at the nearer end of the major axis, prove that its period is

$$\frac{2\pi}{w} \sqrt{\left\{ \frac{1+e}{(1-e)^3} \right\}}$$

Solution:- For the motion of a planet in a central orbit there is no transversal component of the acceleration. Thus

$$\frac{1}{r} \frac{d}{dt} \left(r^2 \frac{d\theta}{dt} \right) = 0$$

integrating it, we get

$$r^2 \frac{d\theta}{dt} = h(\text{constant}) \text{ _____ (1)}$$

Here $r = a - ae$, since the planet is at the nearer end of the major axis. Therefore, from (1) we get

$$(a - ae)^2 w = h \text{ _____ (2)}$$

Further, $h^2 = \frac{\mu b^2}{a}$

$$= \frac{\mu a^2 (1 - e^2)}{a} = \mu a (1 - e^2) \text{ _____ (3)}$$

from (2) and (3), we get

$$a^4(1 - e)^4 w^2 = \mu(1 - e^2)$$

$$\text{or } \mu = a^3(1 - e)^3 \frac{w^2}{1 + e}$$

therefore,

$$T = \frac{2\pi}{\sqrt{\mu}} \frac{a^3}{2} = \frac{2\pi}{w} \sqrt{\left\{ \frac{1+e}{(1-e)^3} \right\}}$$

Example. 5:- Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to that point from rest at a distance from the Sun equal to the length of major axis.

Solution:- We know that the velocity at a planet at any point of its orbit is given by

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right)$$

Further, If the particle falls to the Sun under inverse square law from a distance $2a$ to the distance r (measured from the Sun), then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$$

Multiplying both sides by $2 \frac{dx}{dt}$ and then integrating, we get

$$\left(\frac{dx}{dt} \right)^2 = -2 \int_{2a}^r \mu / x^2 dx = 2\mu \left[\frac{1}{x} \right]_{2a}^r$$

$$v_1^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \text{ ————— (2)}$$

When v_1 is the velocity at the point distant r from the Sun

from (1) and (2) we get

$$v = v_1$$

Example. 6 : A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a , show that equation to its path is $r \cos (\theta / \sqrt{2}) = a$.

Solution:- Here the central acceleration varies inversely as the cube of the distance

$$i.e \quad P = \frac{\mu}{r^3} = \mu u^3, \text{ where } \mu \text{ is a constant.}$$

If V is the velocity for a circle of radius a , then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

or $V = \sqrt{(\mu/a^2)}.$

$\therefore$ the velocity of projection $v^1 \sqrt{2V} = \sqrt{(2\mu/a^2)}.$

The differential equation of the path is

$$h^2 \left[u \frac{d^2 u}{d\theta^2} \right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both side by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right] = \mu u^2 + A.$$

where A is a constant.

But initial when $r = a$ i.e., $u = \frac{1}{a}, \frac{du}{d\theta} = 0$ (at an apse),

$$\text{and } v = v_1 = \sqrt{(2\mu/a^2)}$$

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2 \left[\frac{1}{a^2} \right] = \frac{\mu}{a^2} + A,$$

$\therefore$ $h^2 = 2\mu$ and $A = \mu/a^2.$

Substituting the values of h^2 and A in (1), we have

$$2\mu \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right] = \mu u^2 + \frac{\mu}{a^2}$$

or $2 \left(\frac{du}{d\theta} \right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2 u^2}{a^2}$

$$\text{or} \quad \sqrt{2}a \frac{du}{d\theta} = \sqrt{(1 - a^2u^2)}$$

$$\text{or} \quad \frac{d\theta}{\sqrt{2}} = \frac{a \, du}{\sqrt{(1 - a^2u^2)}}$$

Intergrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$, where B is a Constant.

But initially, when $u = \frac{1}{a}$, $\theta = 0$.

$$\therefore B = \sin^{-1} 1 = \frac{1}{2}\pi.$$

$$\therefore \left(\frac{\theta}{\sqrt{2}}\right) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$$

$$\text{or} \quad au = \frac{a}{r} = \sin\left\{\frac{1}{2}\pi + \left(\frac{\theta}{\sqrt{2}}\right)\right\}$$

$$\text{or} \quad a = r \cos\left(\frac{\theta}{\sqrt{2}}\right),$$

which is the required equation of the path.

7.12. Summary

Definition of central force and central orbit and A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane to obtain the differential equation of the path is described and elliptic orbits, angular, momentum and areal velocity are discussed and Kepler's law of motion and some important geometrical properties of in ellipse are also discussed. Example to be done with the help of LHL and RHL are also provided for better understanding.

7.13. Terminal Question

Q. 1. A particle describes a circle pole on its circumference, under a force P to the pole. Find the law of force.

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Q.2. A particle describes the curve $r = a \sin n\theta$ under a force P to the pole. Find the law of force.

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Q.3. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$ it is projected with velocity V at a distance R. Show that its path is rectangular hyperbola if the angle of projection is

$$\sin^{-1} \frac{\mu}{VR \left(V^2 - \frac{2\mu}{R} \right)^{1/2}}$$

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Q.4. If the velocity of the earth at any point of its orbit, assumed to be circular, were increased by about one-half, prove that it would describe a parabola about the Sun as focus.

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Q. 5. find the law of force towards the pole under which the following curves are described.

- (i) $au = e^{n\theta}$
- (ii) $r = ae^{\theta \cot \alpha}$

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Notes