U. P. RajarshiTandon Open University, Prayagraj

Master of Science/

Master of Arts

PGMM-107N/MAMM-107N
Complex Analysis

U. P. RajarshiTandon
Open University

Master of Science
PGMM -107N
Complex Analysis

Block: 1 Function of a Complex Variables

Block: 2 Conformal Mappings

Block: 3 Complex Integration and Series

Block: 4 The Calculus of Residues

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PGMM–107N:COMPLEX ANALYSIS

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Syllabus

PGMM-107N/MAMM-107N:

Complex Analysis

Block-1: Function of Complex Variables

Unit-1: Complex Numbers

Introduction, Graphical representation, exponential functions of complex numbers, circular and hyperbolic functions of complex numbers.

Unit-2: Analytic Functions

Introduction, single and multivalued functions, analytic functions, Cauchy Riemann equations, Polar form of C-R equation, Derivative of w in polar form.

Unit-3: Harmonic Functions

Orthogonal system, Laplace equations, harmonic functions, conjugate functions, Milne's Thomson method.

Block-2: Conformal Mappings

Unit-4: Introduction to Conformal Mappings

Mappings or Transformations, Jacobian of a transformations, ordinary and critical points, conformal mappings, some general transformations: Translation, Rotation, Magnification, Inversion.

Unit-5: Bilinear Transformations

Bilinear transformations, Fixed points of the Bilinear Transformations, Some important properties of Bilinear Transformations, Special conformal transformations, Joukowski's transformation and Schwarz-Christoffel transformation.

Block-3: Complex Integration and Series

Unit-6: Introduction to Complex Integration

Complex integration, Arc, Regular Arc, Contour, Connected and Non-connected region, simple and multi connected region

Unit-7: Cauchy's Theorem

Cauchy Fundamental theorem, and Cauchy-Goursat theorem.

Unit-8: Cauchy's Integral Formula and its Applications

Cauchy's integral formula, An extension of the Cauchy's integral formula, Cauchy'sintegral formula for derivative, Cauchy'sinequality.

Unit-9: Important Theorems in Complex Integration

Morera's Theorem, Liouville's theorem and its applications, The fundamental theorem of Algebra, Maximum modulus principle.

Unit-10: Series

Introduction, Entire functions, Taylor's theorem and its applications, Laurent's Theorem anditsapplications.

Block-4: The calculus of residues

Unit-10: Singularities and Residues

Zero of analytic functions, singularities of analytic functions, isolated and non-isolated singularities, removable singularities, pole, residue at pole, Cauchy's Residue theorem.

Unit-11: Applications of Complex Integration-I

Evaluation of real definite integrals, Integration round the unit circle, Jorden's inequality, Jordan's lemma, Evaluation of integrals in which no poles lie on the real axis.

Unit-12: Applications of Complex Integration-II

Introduction, Evaluation ofintegrals in which poles lie on the real axis, integral involved many valued functions.

Unit-14: Meromorphic Functions

Meromorphic functions, Mittag Leffler's expansion theorem, number of poles and zeroes of a meromorphic function, Principle of argument, Rouche's theorem.

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Complex Analysis

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Block

1 Function of Complex Variables

Unit- 1
Complex Numbers

Unit- 2
Analytic Functions

Unit- 3
Harmonic Functions

Block-1

Function of Complex Variables

Complex analysis have a great importance to solve many engineering problems such as the flow of electricity, electrostatics, forced vibrations of a dynamical systems, aerodynamic, fluid dynamic, elasticity, heat flow, potential theory, electromagnetic theory and many other sciences. It is an essential part of the mathematical background of engineers, scientists and physicits etc.

Functions of complex variables are widely used in physics, engineering, and mathematics to describe a variety of phenomena, including fluid flow, electromagnetic fields, and quantum mechanics. The complex plane is a graphical representation of complex numbers, where the real part is plotted along the horizontal axis and the imaginary part along the vertical axis. This allows complex numbers to be visualized as points in a plane, with addition and multiplication of complex numbers corresponding to geometric operations.

Functions of complex variables are functions that take complex numbers as inputs and produce complex numbers as outputs. These functions can be thought of as mappings from the complex plane to itself. Just as with real-valued functions, complex functions can be continuous, differentiable, and have other properties, but they often exhibit more intricate behavior due to the nature of complex numbers.

The first unit covers the fundamental principles of complex numbers, operations with complex numbers, and their graphical representations. In the second unit we shall discuss with the analytic function, singularities, continuity, differentiability, Cauchy Riemann Equation and their applications.Thirt unit deals with the Orthogonal system, Laplace equations, harmonic functions, conjugate functions and Milne’s Thomson method.

UNIT-1: Complex Numbers

Structure

1.1 Introduction

The first mathematician who introduced the symbol i with property i^2=-1 was Eular (1707-1783). In 1806, Swiss mathematician Jean-Robert Argand wrote a book on the geometric representation of complex number. Complex numbers are numbers that extend the idea of the real numbers. They have both a real part and an imaginary part and are written in the form $a+bi$, where a and b are real numbers and i is the imaginary unit, which satisfies the equation i^2=-1.In this form, a is the real part of the complex number, and bi is the imaginary part. Complex numbers can be added, subtracted, multiplied, and divided, much like real numbers. They are used in many areas of mathematics, physics, and engineering, particularly in the study of electrical circuits, quantum mechanics, and signal processing.

This unit deals with the basic theory of complex numbers, operations involving complex numbers and their graphical representations.In the real number theory, there is no real number whose square is a negative real number and no real number which satisfies the equation $x^2+1=0$. To find solution of this type of equation, a new type of number was introduced known as imaginary number, denoted by 'i' or 'j' with property i^2 (or j^2)=-1. In electronics and electrical theory, we denote the imaginary number with j, because i use for current.

1.2 Objectives

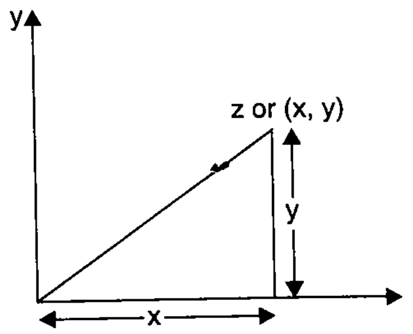
After reading this unit the learner should be able to understand about the:

- Introduction and Graphical representation of Complex Numbers
- Exponential functions of complex numbers
- Circular and hyperbolic functions of complex numbers

1.3 Graphical Representation of Complex Numbers

To make the real and imaginary numbers part of one system, a new number $a+ib$ was introduced, which is called ***complex number*** and denoted by $\boldsymbol{z}(=a+ib)$, where a is the real partand b is the imaginary part of the complex number z, denoted by $\mathbf{Re}(z)=a$ and $\mathbf{Im}(z)=b$.z also can be written as $z=(x, y)$.

The method of representing complex numbers by points in a plane is due to Argand and is known as Argand's diagram.



1.4 Properties of Complex Numbers

1.4.1Conjugate of complex number:

If $\boldsymbol{z}=a+ib$ then $\bar{z}=a-ib$ is known as conjugate of the complex number.

1.4.2 Addition of two complex number:

Consider $\boldsymbol{z_1}=a_1+ib_1$ and $\boldsymbol{z_2}=a_2+ib_2$.

Then we have

$$z_1+z_2= (a_1+ib_1) + (a_2+ib_2) = (a_1+a_2) +i\ (b_1+b_2).$$

1.4.3Subtraction of two complex number:

Consider $z_1=a_1+ib_1$ and $z_2=a_2+ib_2$.

Then we have

z_1-z_2= (a_1+ib_1) - (a_2+ib_2)

= (a_1-a_2) +i (b_1-b_2).

1.4.4 Equality of complex number:

Consider $\mathbf{z_1}$=a_1+ib_1 and $\mathbf{z_2}$=a_2+ib_2.

If z_1=$z_2$then a_1=$a_2$and b_1= b_2.

1.4.5 Multiplication of two complex number:

Consider z_1= a_1+ib_1 and z_2=a_2+ib_2.

Then we have

z_1z_2= (a_1+ib_1)(a_2+ib_2)

= (a_1a_2-b_1b_2) +i (a_1b_2+a_2b_1).

1.4.6 Division of two complex number:

Consider z_1= a_1+ib_1 and z_2=a_2+ib_2.

Then we have

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$$

$$= \frac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)}$$

$$= \left(\frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} \right) + i \left(\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2} \right)$$

1.4.7 Modulus of complex number:

It is denoted by

$$|z| = |a + ib| = \sqrt{a^2 + b^2} = r \text{, where } r^2 = a^2 + b^2.$$

1.4.8 Argument of complex number:

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then we have

$$z_1 z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$

and $\quad \arg(z_1+z_2) = \theta_1 + \theta_2.$

If $z = 0$ then arg z is not defined.

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$$

and $\quad \arg(z_1) - \arg(z_2) = \arg(z_1/z_2).$

Note: $\arg z = \theta = \tan^{-1} \dfrac{b}{a}$.

1.5 Euler's theorem and De Moivre's theorem

In term of the polar co-ordinates (r,θ), the variables a and b are

$$a = r \cos\theta,\ b = r \sin\theta,$$

then the complex number z is written as

$$z = r\cos\theta + i\, r\sin\theta = re^{i\theta},$$

where $e^{i\theta}= \cos\theta+i\sin\theta$ and $r^2=a^2+b^2$, $\tan\theta=b/a$.

The Euler's theorem is

$$e^{i\theta}= \cos\theta+i\sin\theta.$$

The De Moivre's theorem is

$$(\cos\theta+i\sin\theta)^m = \cos m\theta+i\sin m\theta.$$

Note. (i) $|z| = |\overline{z}|$.

(ii) $|z|^2 = z\overline{z}$.

(iii) $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$.

(iv) $\overline{(z_1 \pm z_2)} = \overline{z_1} \pm \overline{z_2}$.

(v) $|z_1 \pm z_2| \le |z_1| + |z_2|$.

(vi) $|z_1 \pm z_2| \ge ||z_1| - |z_2||$.

(vii) The cube roots of unity is 1, ω, ω^2,

where $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$.

(viii) $1+\omega+\omega^2 =0$ and $1.\omega.\omega^2 =1$, *i.e.*, $\omega^3=1$.

(ix) The three cube roots of unity are the vertices of an equilateral triangle inscribed in the circle $|z| = 1$.

(x) The fourth roots of unity are 1, i, -1, $-i$.

(xi) $\text{Log}(x+iy)=2n\pi i+\log(x+iy)$

and $\log(z)=\log|z|+i\arg(z)$.

(xii) $|x|=x$, if $x>0$; $|x|=-x$, if $x<0$; $|x|=0$, if $x=0$.

Check your Progress

1. What do you mean by Complex number?

2. State the Euler's and De Moivre's theorem.

1.6 Exponential functions of complex numbers

The exponential formula is

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\dots\dots\dots+\frac{x^n}{n!}+\dots\dots$$

Consider $\mathbf{z}=a+ib$, using equation (1), we have

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\dots\dots\dots\dots+\frac{z^n}{n!}+\dots\dots$$

Let us consider $z=r\,e^{i\theta}$

and $e^z=e^{x+iy}$

$$=e^x.e^{iy}$$

$$=e^x.(\cos y+i\sin y)$$

Similarly, we have

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and $$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

1.7 Circular functions of Complex Numbers

The Euler's theorem is

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \ldots.(1)$$

and $$e^{-i\theta} = \cos\theta - i\sin\theta \qquad \ldots.(2)$$

From equations (1) and (2), we get

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

or $$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Putting $\theta = z$, we get

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \ldots.(3)$$

Subtracting from equations (1) and (2), we get

$$e^{i\theta} - e^{-i\theta} = -2i\sin\theta$$

or $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Putting $\theta = z$, we get

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \ldots.(4)$$

From equations (3) and (4), we get

$$\text{tanz} = \frac{\sin z}{\cos z}$$

$$= \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

$$= -i\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$$

1.8 Hyperbolic functions of Complex Numbers

Consider z is any real or complex number, then we have

$$\sinh z = \frac{e^{z} - e^{-z}}{2} \qquad \ldots.(1)$$

and

$$\cosh z = \frac{e^{z} + e^{-z}}{2} \qquad \ldots.(2)$$

From equations (1) and (2), we get

$$\text{tanhz} = \frac{\sinh z}{\cosh z}$$

$$= \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

and $$\coth z = \frac{\cosh z}{\sinh z}$$

$$= \frac{e^{z} + e^{-z}}{e^{z} - e^{-z}}$$

and $$\operatorname{sech} z = \frac{1}{\cosh z}$$

$$= \frac{2}{e^{z} + e^{-z}}$$

and $$\operatorname{cosech} z = \frac{1}{\sinh z}$$

$$= \frac{2}{e^{z} - e^{-z}}.$$

Note: 1. $\cosh^2 x - \sinh^2 x = 1.$

2. $\operatorname{sech}^2 x + \tanh^2 x = 1.$

3. $\coth^2 x - \operatorname{cosech}^2 x = 1.$

4. $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y.$

5. $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y.$

6. $\cosh^2 x = \frac{1}{2}(1 + \cosh 2x).$

7. $\sinh^2 x = \frac{1}{2}(\cosh 2x - 1).$

8. $\sinh 2x = 2 \sinh x \cosh x.$

9. $\cosh 2x = \cosh^2 x + \sinh^2 x.$

10. $\sinh x + \sinh y = 2\sinh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}.$

11. $\sinh x - \sinh y = 2\sinh\dfrac{x-y}{2}\cosh\dfrac{x+y}{2}.$

12. $\cosh x + \cosh y = 2\cosh\dfrac{x+y}{2}\cosh\dfrac{x-y}{2}.$

13. $\cosh x - \cosh y = 2\sinh\dfrac{x+y}{2}\sinh\dfrac{x-y}{2}.$

1.9 InverseHyperbolic functions of Complex Numbers

Consider $\mathrm{z} = \sinh u$, then we have

$$u = \sinh^{-1} z \qquad \text{....(1)}$$

i.e.,*u* is known as a sine hyperbolic inverse of z.

$$\mathrm{z} = \sinh u$$

$$= \frac{e^u + e^{-u}}{2}$$

or $$2z = e^u + e^{-u}$$

or $$2z = e^u + \frac{1}{e^u}$$

or $$2ze^u = e^{2u} + 1$$

or $$e^{2u} - 2ze^{u} - 1 = 0$$

or $$e^{u} = \frac{2z \pm \sqrt{4z^2 + 4}}{2}$$

or $$e^{u} = z \pm \sqrt{z^2 + 1}$$

or $$u = \log\left(z \pm \sqrt{z^2 + 1}\right)$$

From equation (1), we get

$$\sinh^{-1} z = \log\left(z \pm \sqrt{z^2 + 1}\right)$$

Similarly, we have

$$\cosh^{-1} z = \log\left(z \pm \sqrt{z^2 - 1}\right)$$

$$\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

$$\coth^{-1} z = \frac{1}{2}\log\left(\frac{z+1}{z-1}\right)$$

$$\operatorname{sech}^{-1} z = \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$$

$$\operatorname{cosech}^{-1} z = \log\left(\frac{1+\sqrt{z^2+1}}{z}\right).$$

1.10 Relation between Circular and Hyperbolic function of Complex Numbers

Consider z is any real or complex number, then we have

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \text{....(1)}$$

and $$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{....(2)}$$

Putting $\theta = iz$, equations (1) and (2), we get

$$\cos(iz) = \frac{e^{i^2 z} + e^{-i^2 z}}{2}$$

$$= \frac{e^{-z} + e^{z}}{2}$$

$$= \cosh z \qquad \text{....(3)}$$

and $$\sin(iz) = \frac{e^{i^2 z} - e^{-i^2 z}}{2i}$$

$$= \frac{e^{-z} - e^{z}}{2i}$$

$$= -\frac{e^{z} - e^{-z}}{2i}$$

$$= -\frac{1}{i}\sinh z$$

$$= i\sinh z \qquad \text{....(4)}$$

From equations (3) and (4), we get

$$\tan(iz) = i\tanh z$$

Similarly, we have

$$\cot(iz) = -i\coth z$$

$$\sec(iz) = \text{sech}\, z$$

$$\text{cosec}(iz) = -i\,\text{cosech}\, z$$

1.11 Summary

If $\boldsymbol{z} = a+ib$ then $\bar{z} = a-ib$ is known as conjugate of the complex number.

If $\boldsymbol{z_1} = a_1+ib_1$ and $\boldsymbol{z_2} = a_2+ib_2$ then

$$z_1+z_2 = (a_1+ib_1) + (a_2+ib_2) = (a_1+a_2) + i\,(b_1+b_2)$$

and $\quad z_1-z_2 = (a_1+ib_1) - (a_2+ib_2) = (a_1-a_2) + i\,(b_1-b_2).$

and $\quad z_1z_2 = (a_1+ib_1)(a_2+ib_2) = (a_1a_2-b_1b_2) + i\,(a_1b_2+a_2b_1).$

and $\quad \dfrac{z_1}{z_2} = \dfrac{a_1+ib_1}{a_2+ib_2} = \dfrac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)} = \left(\dfrac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right) + i\left(\dfrac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)$

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then we have $z_1z_2 = r_1 r_2 e^{i\theta_1} e^{i\theta_2}$ and arg $(z_1+z_2) = \theta_1+\theta_2$.

The Euler's theorem is $e^{i\theta} = \cos\theta + i\sin\theta$.

The De Moivre's theorem is $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$.

1.12 Terminal Questions

Q.1. What do you means by Complex Numbers?

Q.2. Write the Euler's theorem.

Q.3. Explain the De Moivre's theorem.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.

2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.

3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.

4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.

5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.

6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)

7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT- 2: ANALYTIC FUNCTIONS

Structure

2.1 Introduction

A function of a complex variable is one that accepts a complex number as an input and produces a complex number as an output. These functions can be visualized as transformations from the complex plane to itself, where each point in the input plane corresponds to a point in the output plane.Functions of complex variables often exhibit behavior that differs from functions of real variables. For instance, they can be infinitely differentiable, even if they are not simple polynomials. This characteristic is termed analyticity.

Complex analysis, which focuses on the study of functions of complex variables, is a diverse and significant field of mathematics with applications across various scientific and engineering disciplines.Function of a complex variable plays an important role in fluid mechanics, heat transfer and field theory.In this unit, we deal with the function of complex variable, single and multivalued functions,continuity, differentiability, analytic function, Cauchy-Riemann equation in cartesian and polar form with their applications.

2.2 Objectives

After studying this unit, the learner will be able to understand the:

- Function of a complex variable
- IUniform or Single valued functions and multivalued functions
- Continuity and differentiability
- analytic function
- Cauchy-Rieman equations in Cartesian and polar form

2.3 Function of a complex Variable

A variable w is said to be a function of a complex variablez if, to every value of z in a certain domain D, there correspond one or more definite of w. Thus if w is a function of z, it is written as

$$w = f(z)$$

If $w = u + iv$ andz = $x + iy$

Then both w and v are function of x and y, so we write u as $u(x,y)$ and v as $v(x,y)$, for example let

$$w = z^3$$

$$= (x + iy)^3$$

$$= x^3 + 3x^2iy + 3xi^2y^2 + i^3y^3$$

$$= (x^3 - 3xy^2) + i\,(3x^2y - y^3)$$

$$= u + iv$$

i.e., $u = x^2 - 3xy^2$ and $v = 3x^2y - y^3$

Henceu and v are function of x and y so when w is a function of z and z is $x+iy$, then w is $u+iv$ and u and v both are function of x and y.

2.4 Uniform or Single Valued Functions

If w takes only one value for each value of z in the region D, then w is said to be a single valued function of z.

For example, we have $w = 2z + 3$ $\{\because z = x+iy)\}$

For point (1, 2) , we hace

$$w = 2\,(1+2i) + 3$$

$$= 2+4i + 3$$

$$= 5+4i$$

Here function$w$ is single valued.

2.5 Multi-Valued Functions or Many Valued Function

If w takes two or more values for some values or all values of z in the region D, then w is said to be multi-valued function of z.

For example, we have $w^2 = z$

$= x + iy$

If we take $x = 5$ and$y = 12$.

Then $w^2 = 5 + 12i$

$= (3 + 2i)^2$

i.e.,$w= \pm\ (3 + 2i)$

Here function$w$ is multi-valued function.

Check your Progress

1. What do you mean by Function of aComplex variable?

2. Define the uniform or multivalued function.

2.6 Continuity

The function $f(z)$ of a complex variable z is said to be continuous at the point z_0, if given any positive number ε, we can find a number δ such that

$$|f(z) - f(z_0)| < \varepsilon$$

For all points of a domain D satisfy $|z - z_0| < \delta$.

In general, δ depends on both ε and z_0.If it is possible to find a numberη(ε) independent of z_0 such that

$|f(z) - f(z_0)| < \varepsilon$

hold for every pair of points (z, z_0) of a domain D which satisfy $|z - z_0| < \eta$, then the function $f(z)$ is said to be uniformed continuous.

or

A function $f(z)$ is said to be continuous at z = a if given $\varepsilon>0$, $\exists\delta>0$ such that

$|f(z) - f(z_0)| < \varepsilon$

Whenever $0<|z - z_0|<\delta$ it follows from the above definition that $f(z)$ will be continuous at $z = a$ if

$$\lim_{z \to a} f(z) = f(a)$$

Note: A function $f(z)$ is said to be continuous in a domain D if it is continuous an every point of D.

2.7 Differentiability

If a function $f(z)$ is single valued in a domain D, then the derivative of $f(z)$ at $z = a$ is denoted by $f'(a)$ and is defined as

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}, \quad \text{where} \quad h = \delta z$$

provided that the limit exist and is independent of the path along which $h \to 0$. In this case we also say that $f(z)$ is differentiable at $z = a$.

or

We have $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x_0 + h) - f(x_0)}{h}$

$$x_0 + h \to x$$

$$\Rightarrow \qquad h \to x - x_0$$

$$\Rightarrow \qquad \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

For $\varepsilon>0$, $\left|\dfrac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \varepsilon$ and $|x - x_0| < \delta$

Differential of $f(z)$ at the point z_0 is given by

$$f'(z)_{z\to z_o} = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

provided that the limit on the right hand side exist uniquely.

Examples

Example.1. Discuss the differentiability of the function

$$f(z)=\begin{cases}\dfrac{x^2y^5(x+iy)}{x^4+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}.$$

Solution: We have

$$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$$

$$=\lim_{z\to 0}\frac{1}{z}\left[\frac{x^2y^5(x+iy)}{x^4+y^{10}}\right]$$

$$=\lim_{z\to 0}\frac{x^2y^5}{x^4+y^{10}} \qquad \text{..... (1)}$$

If $z\to 0$ along any radius vector $y = mx$, then we have

$$f'(0)=\lim_{x\to 0}\frac{x^2(mx)^5}{x^4+(mx)^{10}}$$

$$=\lim_{x\to 0}\frac{m^5x^3}{1+m^{10}x^6}=0 \qquad \text{..... (2)}$$

Now if $z\to 0$ along $y^5 = x^2$, then we have

$$f'(0) = \lim_{x \to 0} \frac{x^2(x^2)}{x^4 + (x^2)^2} = \frac{1}{2}. \qquad \text{….. (3)}$$

These different values along $y=mx$ and $y^5=x^2$form equations (2) and (3) shows that the function $f(z)$ is not differentiable at z=0.

Example.2. Given that $f(z) = \frac{x^3 y(y-ix)}{x^6 + y^2}$, $z \neq 0$, **f(0) = 0 prove that** $\frac{f(z) - f(0)}{z} \to 0$ **as z→0 along any radius vector but not as z→0 in any manner.**

Solution: It is given that

$$f(z) = \frac{x^3 y(y-ix)}{x^6 + y^2}, \; z \neq 0, f(0) = 0 \qquad \text{…. (1)}$$

If $z \to 0$ along any radius vector $y = mx$, then we have

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{1}{z} \left[\frac{x^3 y(y-ix)}{x^6 + y^2} \right]$$

$$= \lim_{z \to 0} \frac{1}{z} \left[\frac{x^3 y(-ix - i^2 y)}{x^6 + y^2} \right]$$

$$= \lim_{z \to 0} \frac{1}{z} \left[\frac{-ix^3 y(x+iy)}{x^6 + y^2} \right]$$

$$= \lim_{z \to 0} \frac{1}{z} \left[\frac{-ix^3 y.z}{x^6 + y^2} \right]$$

$$= \lim_{z \to 0} \left[\frac{-ix^3 y}{x^6 + y^2} \right] \qquad \text{….. (2)}$$

As $z \to 0$ this implies$x+iy \to 0$.

Along radius vector $y=mx \Rightarrow x+imx \to 0$ *i.e.,* $x(1+im) \to 0 \Rightarrow x \to 0$.

$$= \lim_{x \to 0} \left[\frac{-ix^3 (mx)}{x^6 + (mx)^2} \right]$$

$$= \lim_{x\to 0}\left[\frac{-ix^4 m}{x^2\left\{x^4+m^2\right\}}\right]$$

$$= \lim_{x\to 0}\left[\frac{-ix^2 m}{x^4+m^2}\right]\to 0$$

Now if $z\to 0$ along $y = x^3$, then we have

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\left[\frac{-ix^3 y}{x^6+y^2}\right] \text{[Using equation (2)]}$$

As $z\to 0$ this implies $x+iy\to 0$

Along $y = x^3$ $\Rightarrow x + ix^3\to 0$

i.e., $x(1+ix^2)\to 0$ $\Rightarrow x\to 0$.

Therefore, we have

$$= \lim_{x\to 0}\frac{-ix^3.x^3}{x^6+(x^3)^2}$$

$$= \lim_{x\to 0}\frac{-i}{2}$$

$$= \frac{-i}{2}$$

These different values $y=mx$ and $y=x^3$ shows that function $f(z)$ is not differentiable.

Example.3. Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Solution. Let us consider $f(z)$ is differentiable at $z = z_0$. Therefore first we show that $f(z)$ is continuous at $z = z_0$.

i.e.,

$$f'(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}, \quad where \quad h=\delta z$$

$$= \lim_{h\to 0}\frac{f(z_0-h)-f(z_0)}{-h}$$

Assuming, this limit exist and is unique. We have

$$f'(z_0) = \frac{f(z_0+h) - f(z_0)}{h}$$

and $$f'(z_0) = \frac{f(z_0-h) - f(z_0)}{-h} + \varepsilon'$$

where ε and ε' as $h \to 0$

This implies $$h\ f'(z_0) = f(z_0+h) - f(z_0) + h\varepsilon$$

and $$-h\ f'(z_0) = f(z_0-h) - f(z_0) + h\varepsilon'$$

Taking $h \to 0$, we get

$$0 = \lim_{h\to 0}\left[f(z_0+h) - f(z_0)\right]$$

and $$0 = \lim_{h\to 0}\left[f(z_0-h) - f(z_0)\right]$$

This implies

$$\lim_{h\to 0} f(z_0+h) = f(z_0)$$

$$\lim_{h\to 0} f(z_0-h) = f(z_0)$$

i.e., $f(z)$ is continuous at $z=z_0$.

Now suppose $f(z)$ is continuous everywhere but not differentiable everywhere except at $z = 0$.

Since $f(z)$ is continuous then we have

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{|z_0+\Delta z|^2 - |z_0|^2}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{(z_0+\Delta z)(\overline{z_0}+\Delta\overline{z}) - z_0.\overline{z_0}}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{z_0\Delta\overline{z} + \Delta z(\overline{z_0}+\Delta\overline{z})}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\left[z_0 . \frac{\Delta\overline{z}}{\Delta z} + \overline{z_0} + \Delta\overline{z}\right] \qquad \text{.....(2)}$$

Here two cases arise:

Case I. Consider $\Delta z \to 0$ along real axis *i.e.*, $\Delta y = 0$

This implies $\Delta \bar{z} = \Delta z = \Delta x$

and $\Delta x \to 0$ as $\Delta z \to 0$

Using equation(2), we have

$$f'(z_0) = \lim_{\Delta x \to 0} (z_0 + \overline{z_0} + \Delta x)$$

$= z_0 + \overline{z_0}$. ……. (3)

Case II. Consider $\Delta z \to 0$ along imaginary axis *i.e.*, $\Delta x = 0$

This implies $\Delta z = i\Delta y$ and $\Delta \bar{z} = -i\Delta y$ and $\Delta y \to 0$ as $\Delta z \to 0$

Using equation (2), we have

$$f'(z_0) = \lim_{\Delta y \to 0} \left[z_0 \frac{-i\Delta y}{i\Delta y} + \overline{z_0} - i\Delta y) \right]$$

$= -z_0 + \overline{z_0}$. …. (4)

Using equations (3) and (4), we see that $f'(z_0)$ along the two paths are different except at $z = 0$.

Hence the function $f(z)$ is non-differentiable for any non-zero value of z.

2.8 Analytic Function

A function $f(z)$ which is single-valued and differentiable at any point z_0 is said to be *analytic* at the point z_0.

If the function $f(z)$ is analytic at every point z of a certain domain D then $f(z)$ is said to be analytic in D and $f(z)$ is also called regular or holomorphic function.

Note. Regular and holomorphic are also sometimes used as synonyms for analytic.

2.9 Cauchy-Riemann Equations

A necessary condition for $w = f(z) = u(x, y)+iv(x, y)$ to be analytic in a domain D is that u and v satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{at every point of D} \qquad \ldots. (1)$$

If these partial derivatives of equation (1) are also continuous, then Cauchy-Riemann equations are sufficient conditions for $f(z)$ to be analytic in D.

Necessary Conditions for *f*(*z*) to be analytic

We have $z = x+iy$

$\Rightarrow \Delta z = \Delta x+i\Delta y$.

Let $w = f(z) = u(x, y)+iv(x, y)$ be analytic function inside a region D. It means differentiable of w exist at any point of this region *i.e.*,

$$\frac{dw}{dz} = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} \qquad \ldots. (2)$$

exist and unique along whatever path Δz may across to zero.

Now, we take $\Delta z\to 0$ along two path x-axis and y-axis.

Along x-axis, there is no change in y*i.e.*, $\Delta y = 0$ therefore $\Delta z = \Delta x+i\Delta y = \Delta x$.

Using equation (2), we have

$$\frac{dw}{dz} = \lim_{\Delta x\to 0} \frac{\Delta w}{\Delta x} \qquad \ldots.(3)$$

And when the motion is parallel to y-axis (or along y-axis) then there is no change in x. Therefore$\Delta x = 0$ thus we have

$\Delta z = \Delta x + i\Delta y = i\Delta y.$

From equations (2) and (3), we have

$$\frac{dw}{dz} = \lim_{\Delta y \to 0} \frac{\Delta w}{i\Delta y} \qquad \text{....(4)}$$

Now by equations (3) and (4), we have

$$\frac{dw}{dz} = \frac{\partial w}{\partial x} = \frac{\partial w}{i\partial y} \qquad \text{....(5)}$$

We have $w = u + iv$

$$\Rightarrow \frac{\partial w}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and $$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$$

$$\Rightarrow \qquad \frac{\partial w}{i\partial y} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

Using equation (5), we have

$$\frac{\partial w}{\partial x} = \frac{\partial w}{i\partial y}$$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{i\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{i\partial u}{i^2\partial y} + \frac{\partial v}{\partial y}$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{.... (6)}$$

Equating real and imaginary parts of equation (6), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or $u_x = v_y$ and $u_y = -v_x$

These are the necessary condition for a function to be analytic.

Sufficient Condition

The sufficient condition of f(z) to be analytic assume the existence and continuity of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$.

The sufficient condition for the function $f(z)$ to be analytic required other the continuity four partial derivatives of u and v. Since u is a function of x and y.

So we have $u = u(x, y)$

$\Rightarrow u+\Delta u = u(x+\Delta x, y+\Delta y)$

and $\partial u = u(x+\partial x, y+\partial y) - u(x, y)$

$\Rightarrow \quad = u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y) + u(x+\Delta x, y) - u(x, y)$

$$\Delta u = \Delta y\left[\frac{u(x+\Delta x, y+\Delta y) - u(x+\Delta x, y)}{\Delta y}\right] + \Delta x\left[\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x}\right]$$

$$\Delta u = \Delta y . u_y(x+\Delta x, y+\theta_1 \Delta y) + \Delta x u_x(x+\theta_2 \Delta x, y), \quad 0<\theta_1<1,\ 0<\theta_2<1 \qquad \ldots\ldots (1)$$

$$\Delta u = \Delta y\,(u_y + \varepsilon_1) + \Delta x\,(u_x + \varepsilon_2)$$

and $\Delta v = \Delta y\,(v_y + \varepsilon_3) + \Delta x\,(v_x + \varepsilon_4)$, $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \to 0$ as $\Delta z \to 0$

Using Cauchy Riemann equations, we have

$$\Delta u = \Delta y\,(-v_x + \varepsilon_1) + \Delta x\,(u_x + \varepsilon_2)$$

and $\Delta v = \Delta y\,(u_x+\varepsilon_3)+\Delta x\,(v_x+\varepsilon_4)$

or $i\Delta v = i\Delta y\,(u_x+\varepsilon_3)+i\Delta x\,(v_x+\varepsilon_4)$

On adding last two equation and simplify, we get

$$\Delta u+i\Delta v = (\Delta x+i\Delta y)(u_x+iv_x)+\eta\Delta x+\eta'\Delta y \qquad \text{where } \eta=(\varepsilon_2+i\varepsilon_4) \text{ and } \eta'=(\varepsilon_1+i\varepsilon_3)$$

Dividing by $\Delta x+i\Delta y$, we get

$$\Rightarrow \quad \frac{\Delta u+i\Delta v}{\Delta x+i\Delta y}=(u_x+iv_x)+\eta\frac{\Delta x}{\Delta z}+\eta'\frac{\Delta y}{\Delta z}$$

$$\Rightarrow \frac{\Delta w}{\Delta z}=(u_x+iv_x)+\eta\frac{\Delta x}{\Delta z}+\eta'\frac{\Delta y}{\Delta z}$$

$$\left|\eta\frac{\Delta x}{\Delta z}+\eta'\frac{\Delta y}{\Delta z}\right|\le|\eta|+|\eta'|\to 0 \left\{\because \quad \left|\frac{\Delta x}{\Delta z}\right|\prec 1, \quad \left|\frac{\Delta y}{\Delta z}\right|\prec 1\right\}$$

Thus $$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$$

Hence $\dfrac{dw}{dz}$ exist because $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$ exists.

Check your Progress

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1. What do you mean by analytic function?

2. State the Cauchy-Riemann equations.

Examples

Example.4. Show that the function $e^x(\cos y+i \sin y)$ is analytic and find its derivative.

Solution. It is given that

$$f(z)= e^x (\cos y+i \sin y) \qquad ..... (1)$$

We have $w = f(z)$

$$\Rightarrow \quad u+iv = e^x (\cos y+i \sin y) \qquad ..... (2)$$

Equating the real and imaginary part from equation (2), we get

$$u = e^x \cos y \qquad ..... (3)$$

and $$v = e^x \sin y \qquad .... (4)$$

Differentiate partially equations (3) and (4) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad .... (5)$$

$$\frac{\partial u}{\partial y} = - e^x \sin y \qquad .... (6)$$

$$\frac{\partial v}{\partial x} = e^x \sin y \qquad .... (7)$$

$$\frac{\partial v}{\partial y} = e^x \cos y \qquad .... (8)$$

From equations (5), (6), (7) and (8) we see that C-R equations are satisfied. Thus the necessary and sufficient conditions for $f(z)$ to be analytic are satisfied.

Hence the given function $e^x (\cos y+i \sin y)$ is analytic.

and $\dfrac{dw}{dz} = f'(z) = \dfrac{\partial u}{\partial x} + i\dfrac{\partial v}{\partial x} = e^x \cos y + ie^x \sin y.$

Example.5. Prove that an analytic function with constant real part is constant.

Solution. It is given that $f(z)$ is analytic.

We have $f(z) = u + iv$ ....(1)

And also given u=c (constant) ....(2)

To show that the function $f(z)$ is constant.

Using equation (2), we have

$$\frac{\partial u}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0 \qquad .... (3)$$

Using Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad and \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} = 0 \qquad .... (4)$$

i.e.,v is constant.

Using equations (2) and (4), we get

$$f(z) = u + iv = \text{constant.}$$

Hence an analytic function with constant real part is constant.

Example.6. Show that the function *f(z) = sin x cosh y + i cos x sinh y* is continuous as well as analytic everywhere.

Solution. It is given that

$$f(z)=u+iv$$

$$= \sin x \cosh y + i \cos x \sinh y \qquad \text{..... (1)}$$

Equating the real and imaginary parts of equation (1), we get

$$u = u(x, y) = \sin x \cosh y \qquad \text{.... (2)}$$

$$v = v(x, y) = \cos x \sinh y \qquad \text{.... (3)}$$

Now we have $u(x+0, y+0) = \lim\limits_{\substack{h\to 0\\ k\to 0}} u(x+h\ , y+k)$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \sin(x+h) \cosh(y+k)$$

$$= \sin x \cosh y \qquad \text{.... (4)}$$

and $u(x-0, y-0) = \lim\limits_{\substack{h\to 0\\ k\to 0}} u(x-h\ , y-k)$

$$= \lim_{\substack{h\to 0\\ k\to 0}} \sin(x-h) \cosh(y-k)$$

$$= \sin x \cosh y \qquad \text{..... (5)}$$

From equation (4) and (5), we see that

$u(x, y) = u(x+0, y+0)$

$= u(x-0, y-0)$

i.e., $u(x, y)$ is continuous function.

Similarly, we get

$v(x, y) = v(x+0, y+0)$

$= v(x-0, y-0)$. ..... (6)

Hence u and v are continuous function of x and y.

Now differentiate partially equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = \cos x \cosh y \qquad \text{.... (7)}$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y \qquad \text{.... (8)}$$

$$\frac{\partial v}{\partial x} = - \sin x \sinh y \qquad \text{.... (9)}$$

$$\frac{\partial v}{\partial y} = \cos x \cosh y \qquad \text{.... (10)}$$

Here $f(z)$ is continuous and Cauchy-Riemann equation are satisfy. Hence the given function is analytic.

Example.7. Prove that the function $f(z) = xy+iy$ is everywhere continuous but not analytic.

Solution. It is given that

$$f(z) = u + iv$$

$$= xy + iy \qquad \text{…. (1)}$$

Equating the real and imaginary parts of equation (1), we get

$$u = xy \qquad \text{…. (2)}$$

and $$v = y \qquad \text{…. (3)}$$

Now we have $u(x+0, y+0) = \lim\limits_{\substack{h\to 0\\ k\to 0}} u(x+h\ , y+k)$

$$= \lim_{\substack{h\to 0\\ k\to 0}} (x+h)(y+k)$$

$$= xy \qquad \text{…. (4)}$$

and $u(x-0, y-0) = \lim\limits_{\substack{h\to 0\\ k\to 0}} u(x-h\ , y-k)$

$$= \lim_{\substack{h\to 0\\ k\to 0}} (x-h)\,(y-k)$$

$$= xy \qquad \text{..... (5)}$$

From equation (4) and (5), we see that

$u(x, y) = u(x+0, y+0)$

$\qquad = u(x-0, y-0)$

i.e., $u(x, y)$ is continuous function.

Similarly, we get

$$v(x, y) = v(x+0, y+0)$$

$$= v(x-0, y-0). \qquad \text{..... (6)}$$

Hence u and v are continuous function of x and y.

Now differentiate partially equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = y \qquad \text{.... (7)}$$

$$\frac{\partial u}{\partial y} = x \qquad \text{.... (8)}$$

$$\frac{\partial v}{\partial x} = 0 \qquad \text{.... (9)}$$

$$\frac{\partial v}{\partial y} = 1 \qquad \text{.... (10)}$$

Here $f(z)$ is continuous butCauchy Riemann equations are not satisfied. Hence the given function is not analytic.

Example.8. If $f(z)$ is an analytic function with constant modulus, then it is constant.

Solution: Let $f(z) = u + iv$ be an analytic function, then it is satisfied the Cauchy-Riemann equation *i.e.*, $u_x = v_y$ and $u_y = -v_x$

We have $$|f(z)| = |u + iv|$$

$$= |u^2 + v^2|^{1/2} = c \qquad \text{[constant given]}$$

$$\Rightarrow \quad u^2 + v^2 = c^2 \qquad \text{..... (1)}$$

Differentiating equation (1) with respect to x and y respectively, we get

$$2uu_x + 2vv_x = 0, \text{ where } u_x = \frac{\partial u}{\partial x} \quad etc \qquad \text{.... (2)}$$

and $$2uu_y + 2vv_y = 0 \qquad \text{..... (3)}$$

or $$uu_x + vv_x = 0 \qquad \text{.... (4)}$$

and $$uu_y + vv_y = 0 \qquad \text{..... (5)}$$

Squaring and adding equations (4) and (5), we have

$$u^2(u_x^2 + u_y^2) + v^2(v_x^2 + v_y^2) = 0$$

or $$u^2(u_x^2 + u_x^2) + v^2(v_x^2 + v_x^2) = 0 \qquad \text{\{Using Cauchy-Riemann equation\}}$$

or $$(u^2 + v^2)(u_x^2 + v_x^2) = 0$$

or $$c^2(u_x^2 + v_x^2) = 0$$

or $$u_x^2 + v_x^2 = 0$$

This implies $u_x = 0 = v_x$

We have $$\frac{d}{dz}(f(z)) = f'(z)$$

$$= \frac{dw}{dz}$$

$$= \frac{\partial w}{\partial z}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$= 0+i.0$

$= 0$

$$\frac{df(z)}{dz} = 0$$

On integrating, we get

$f(z)$ = constant.

Example.9.Show that the function $f(z) = e^{-z^{-4}}$ **,** $z=0$ *and* $f(0)=0$ **is not analytic at** $z=0$**. Although Cauchy-Riemann equations are satisfied at the point.**

Solution. It is given that $f(z) = u + iv$

$= e^{-z^{-4}}$

$= e^{-(x+iy)^{-4}}$ .... (1)

Now we have $-(x+iy)^{-4} = -\frac{1}{(x+iy)^4}\cdot\frac{(x-iy)^4}{(x-iy)^4}$

$$= -\frac{x^4 - 6x^2y^2 + y^4 - 4ixy(x^2 - y^2)}{(x^2+y^2)^4}$$

$$= -\frac{(x^4 - 6x^2y^2 + y^4) - 4ixy(x^2 - y^2)}{(x^2+y^2)^4}$$

Consider $A = -\frac{x^4 - 6x^2y^2 + y^4}{(x^2+y^2)^4}$

and $$B = \frac{4ixy(x^2 - y^2)}{(x^2 + y^2)^4}$$

and $$-(x + iy)^{-4} = A + iB$$

Now by equation (1), we have

$f(z) = e^{A+iB}$

$= e^A (\cos B + i \sin B)$ …. (2)

Equating the real and imaginary parts of equation (2), we get

$u = e^A \cos B$ …. (3)

and $v = e^A \sin B$ …. (4)

$$u = e^{-(x^4 - 6x^2y^2 + y^4)/(x^2+y^2)^2} \cos\left\{\frac{4xy(x^2 - y^2)}{(x^2 + y^2)^2}\right\}$$

$$v = e^{-(x^4 - 6x^2y^2 + y^4)/(x^2+y^2)^2} \sin\left\{\frac{4xy(x^2 - y^2)}{(x^2 + y^2)^2}\right\}$$

Now $u(x, 0) = e^{-x^{-4}}$ $v(x, 0) = 0$

$u(0, y) = e^{-y^{-4}}$ $v(0, y) = 0$

$u(0, 0) = 0$ $v(0, 0) = 0.$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x}$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{e^{-x^{-4}}}{x}$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{1}{x} \left[\frac{1}{1 + \frac{1}{x^4} + \frac{1}{2x^8} + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \right]$$

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \left[\frac{1}{x + \frac{1}{x^3} + \frac{1}{2x^7} + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \right]$$

$$= \frac{1}{0 + \infty + \infty + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}$$

$$= 0$$

Similarly, we get

$\frac{\partial v}{\partial x} = 0$, $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial u}{\partial y} = 0$

Thus Cauchy-Riemann equations are satisfied.

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z \to 0} \frac{1}{z} e^{-z^{-4}} \qquad \dots\dots (5)$$

Now we assume that, $\qquad z = r\, e^{i\pi/4}$

$$z^{-4} = \left[r\, e^{i\pi/4} \right]^{-4}$$

$$= r^{-4}\, e^{-i\pi}$$

$$= r^{-4} (\cos\pi - i \sin\pi)$$

$$= r^{-4}$$

Now using equation (5), we have

$$\lim_{r\to 0}\frac{e^{r^{-4}}}{\frac{r}{\sqrt{2}}(1+i)}=\lim_{r\to 0}\frac{\sqrt{2}e^{1/r^4}}{r(1+i)}$$

$$=\lim_{r\to 0}\frac{\sqrt{2}}{r(1+i)}\left[1+\frac{1}{r^4}+\ldots\ldots\ldots\ldots\ldots\right]$$

$$=\lim_{r\to 0}\frac{\sqrt{2}}{(1+i)}\left[\frac{1}{r}+\frac{1}{r^5}+\ldots\ldots\ldots\ldots\ldots\right]$$

$$=\lim_{r\to 0}\frac{\sqrt{2}}{(1+i)}.\infty$$

$$=\infty.$$

i.e., it show that $f'(0)$ does not exist.

Hence the given function *f(z)*= $e^{-z^{-4}}$ is not analytic at *z*=0.

Example.10. Prove that the function $f(z)=\dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ z\neq 0,\ f(0)=0$ is continuous and that Cauchy Riemann equation that are satisfied at the origin yet *f'(z)* does not exist at *z=0.*

Solution. It is given that

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2},\ z\neq 0,\ f(0)=0 \qquad \ldots.\ (1)$$

We have $w = f(z)$

$= u + iv$

$$=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

Here $u = (x^3 - y^3)/(x^2 + y^2)$

and $v = (x^3 - y^3)/(x^2 + y^2)$

Now test the continuity of u, v at $z = 0$. We change u, v to polar coordinates.

$u = r\,(\cos^3\theta - \sin^3\theta)$, $v = r\,(\cos^3\theta + \sin^3\theta)$

As $z \to 0$ and $r \to 0$

Evidently $\lim\limits_{r\to 0} u = 0 = \lim\limits_{r\to 0} v$

$\Rightarrow \lim\limits_{r\to 0} u = 0$

i.e., $f(z)$ is continuous at $z = 0$.

Hence $f(z)$ is continuous everywhere.

Now we have $u(x, 0) = x$, $u(0 , y) = -y$

$v(x, 0) = x$, $v(0, y)$, $v(0,0) = 0 = u(0,0)$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x\to 0} \frac{x}{x} = 1.$$

Similarly, we get

$\dfrac{\partial u}{\partial y} = -1$, $\dfrac{\partial v}{\partial x} = 1$, and $\dfrac{\partial v}{\partial y} = 1$.

Hence the Cauchy-Reimann equation are satisfied.

Now again we have

$$f'(z) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{f(z) - 0}{z}$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x^2 + y^2)(x + iy)}$$

Let $z \to 0$ along the path $y = x$, then

$$f'(0) = \lim_{z \to 0} \frac{(x^3 - x^3) + i(x^3 + x^3)}{(x^2 + x^2)(x + ix)}$$

$$= \frac{i.2x^3}{2x^3(1+i)}$$

$$= \frac{i}{(1+i)}$$

Let $z \to 0$ along x axis, then we have

$$f'(0) = \lim_{z \to 0} \frac{(x^3 - 0) + i(x^3 + 0)}{(x^2 + 0)(x + i.0)}$$

$$= \frac{x^3(1+i)}{x^3}$$

$$= (1+i)$$

Since the values of $f'(0)$ are not unique along the path $y = x$ and x axis.

Thus $f'(0)$ does not exist *i.e.,f*(z) is not analytic at $z = 0$.

2.10 Polar Form of Cauchy-Riemann Equations

Theorem. If $f(z) = u + iv$ is an analytic function and $z = re^{i\theta}$ *where u, v, r ,* θ are all real, show that the Cauchy-Riemann equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Derive the necessary and sufficient conditions for *f(z)* to be analytic in polar coordinates.

Proof. We know that

$$x = r\cos\theta \qquad \text{….. (1)}$$

and $$y = r\sin\theta \qquad \text{….. (2)}$$

From equation (1) and (2), we get

$$r^2 = x^2 + y^2 \qquad \text{….. (3)}$$

and $$\theta = \tan^{-1} y/x \qquad \text{….. (4)}$$

From equation (3), we have

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\Rightarrow \qquad \frac{\partial r}{\partial x} = \frac{x}{r} = \frac{r\cos\theta}{r} = \cos\theta$$

and $$2r\frac{\partial r}{\partial y} = 2y$$

$$\Rightarrow \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta$$

Now from equation (4), we have

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r}$$

and $$\frac{\partial \theta}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{x}{x^2 + y^2} = \frac{\cos\theta}{r}$$

Now we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial u}{\partial r}(\cos\theta) + \frac{\partial u}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$= \frac{\partial u}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} \qquad \text{….. (5)}$$

and $$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial u}{\partial r}(\sin\theta) + \frac{\partial u}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$= \frac{\partial u}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} \qquad \text{….. (6)}$$

and $$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$= \frac{\partial v}{\partial r}(\cos\theta) + \frac{\partial v}{\partial \theta}\left(-\frac{\sin\theta}{r}\right)$$

$$= \frac{\partial v}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{….. (7)}$$

and $$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial v}{\partial r}(\sin\theta) + \frac{\partial v}{\partial \theta}\left(\frac{\cos\theta}{r}\right)$$

$$= \frac{\partial v}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{..... (8)}$$

We know that Cauchy-Riemann equation is

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Using equations (5), (6), (7) and (8), the Cauchy-Riemannequation is

$$\frac{\partial u}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{..... (9)}$$

and $$\frac{\partial u}{\partial r}\sin\theta + \frac{\cos\theta}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}\cos\theta + \frac{\sin\theta}{r}\frac{\partial v}{\partial \theta} \qquad \text{..... (10)}$$

Multiply equation (9) by $\cos\theta$, equation (10) by $\sin\theta$ and adding we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \text{..... (11)}$$

Again multiplying equation (9) by $\sin\theta$, equation (10) by $\cos\theta$ and substracting, we get

$$\frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \text{..... (12)}$$

The above equations (11) and (12) are known as the polar forms of the Cauchy Riemann equations.

2.11 Derivatives of *w* in Polar Form

Prove that $$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}.$$

Proof. For the movement of along *x*-axis when there is no change in *y*, therefore$\Delta y = 0$.

Then we have

$\Delta z = \Delta x$.

Now we have

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\left(\frac{\partial r}{\partial x}\right) + \frac{\partial w}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right)$$

$$= \frac{\partial w}{\partial r}(\cos\theta) + \frac{\partial w}{\partial \theta}\left(-\frac{\sin\theta}{r}\right) \qquad \left[\therefore \ \frac{\partial r}{\partial x} = \cos\theta \ \text{ and } \ \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}\right]$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta} \qquad \text{..... (1)}$$

Now we have

$w = u + iv$

Then we have

$$\frac{\partial w}{\partial \theta} = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r} \qquad \left[\therefore \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \ \text{ and } \ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\right] \text{..... (2)}$$

and $\quad \dfrac{\partial w}{\partial r} = \dfrac{\partial u}{\partial r} + i\dfrac{\partial v}{\partial r}$

$$= \frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta} \qquad \left[\therefore \ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \ \text{ and } \ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}\right] \text{..... (3)}$$

Now from equations (1) and (2), we have

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\left(-r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)$$

$$= \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\left(i^2 r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}\right)$$

$$= \frac{\partial w}{\partial r}\cos\theta - \sin\theta\left(i^2 \frac{\partial v}{\partial r} + i\frac{\partial u}{\partial r}\right)$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\sin\theta\left(i \frac{\partial v}{\partial r} + \frac{\partial u}{\partial r}\right)$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\sin\theta\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$= \frac{\partial w}{\partial r}\cos\theta - i\sin\theta\frac{\partial w}{\partial r}$$

$$= \frac{\partial w}{\partial r}\left(\cos\theta - i\sin\theta\right)$$

$$= e^{-i\theta}\frac{\partial w}{\partial r}.$$

Now from equations (1) and (3), we have

$$\frac{dw}{dz} = \frac{\partial w}{\partial r}\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{i}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= \frac{1}{r}\cos\theta\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= \frac{1}{r}\cos\theta\left(-i^2\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= -\frac{i}{r}\cos\theta\left(i\frac{\partial v}{\partial \theta} + \frac{\partial u}{\partial \theta}\right) - \frac{\sin\theta}{r}\frac{\partial w}{\partial \theta}$$

$$= -\frac{i}{r}\cos\theta\left(\frac{\partial u}{\partial\theta} + i\frac{\partial v}{\partial\theta}\right) - \frac{\sin\theta}{r}\frac{\partial w}{\partial\theta}$$

$$= -\frac{i}{r}\cos\theta\frac{\partial w}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial w}{\partial\theta}$$

$$= -\frac{i}{r}\frac{\partial w}{\partial\theta}\left(\cos\theta + \frac{1}{i}\sin\theta\right)$$

$$= -\frac{i}{r}\frac{\partial w}{\partial\theta}\left(\cos\theta - \frac{i^2}{i}\sin\theta\right)$$

$$= \frac{\partial w}{\partial\theta}(\cos\theta - i\sin\theta)$$

$$= e^{-i\theta}\frac{\partial w}{\partial\theta}.$$

2.12 Summary

A variable *w* is said to be a function of a complex variablez if, to every value of z in a certain domain D, there correspond one or more definite of *w*.

If *w* takes only one value for each value of *z* in the region D, then w is said to be a single valued function of *z*.

If *w* takes two or more values for some values or all values of *z* in the region D, then *w* is said to be multi-valued function of *z*.

A function *f*(*z*) is said to be continuous at *z* = a if given ε>0, ∃δ>0 such that $|f(z) - f(z_0)|$<ε whenever 0< $|z - z_0|$<δ it follows from the above definition that *f*(*z*) will be continuous at *z* = *a* if $\lim_{z\to a} f(z) = f(a)$.

If a function $f(z)$ is single valued in a domain D, then the derivative of $f(z)$ at $z = a$ is denoted by $f'(a)$ and is defined as $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ provided that the limit exist and is independent of the path along which $h \to 0$.

A function $f(z)$ which is single-valued and differentiable at any point z_0 is said to be analytic at the point z_0.

The Cauchy-Riemann equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The polar form of Cauchy-Riemann equation are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \ \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

The derivatives of w in polar form are

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r} = -\frac{i}{r}e^{-i\theta}\frac{\partial w}{\partial \theta}.$$

2.12 Terminal Questions

Q.1.Explain theanalytic function.

Q.2. State theCauchy Riemann equations.

Q.3. Give an example of a function which is differentiable at a point, but it is not analytic there.

Q.4. If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ and $f(z)$ subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$.

Answers

3. $f(z) = |z|^2$, at $z = 0$.

4. $f(z) = \cot\left(\frac{z}{2}\right) + \frac{1}{2}(1 - i)$.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.

2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.

3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.

4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.

5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.

6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)

7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT- 3: HARMONIC FUNCTIONS

Structure

3.1 Introduction

Harmonic functions are important in physics and engineering because they arise in many problems involving waves, heat conduction, and electrostatics. Harmonic functions are functions that satisfy Laplace's equation, which is a partial differential equation. In two dimensions, Laplace's equation states that the sum of the second-order partial derivatives of a function with respect to its variables is equal to zero. The Laplace equation is fundamental in potential theory and is used to describe phenomena such as the distribution of temperature in a conductive medium, the electrostatic potential in a region with no charge, and the flow of an incompressible, inviscid fluid. Solutions to the Laplace equation are known as harmonic functions.

These functions play a crucial role in mathematical analysis and have many important properties, such as the mean value property and the maximum principle, which are widely used in various branches of science and engineering. In this unit we shall discuss about orthogonal system, Laplce equation, Harmonic function, Conjugate function and Milne’s Thomson method.

3.2 Objectives

After studying this unit, the learner will be able to understand the:

- Orthogonal System and Laplace Equation
- Harmonic function and their applications
- Conjugate function and their solution procedure
- Milne’s Thomson method and their applications

3.3 Orthogonal System

An orthogonal system typically refers to a set of vectors in a vector space that are pairwise orthogonal. Orthogonality in this context means that the vectors are perpendicular to each other, or more precisely, their dot product is zero.

Two curves are said to be orthogonal to each other when they intersect at right angles at each of their points of intersection.

Examples

Example.1. To show that if $w = f(z) = u + iv$ be an analytic function $z = x + iy$, then the family of curve $u(x, y) = C_1$ and $v(x, y) = C_2$ are orthogonal system.

Solution. We have $u(x, y) = C_1$ …. (1)

and $x = \phi(t)$ and $y = \psi(t)$ are the function of variable *t*.

From equation (1) we have

$$\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = \frac{d}{dt}C_1 \quad \text{…. (2)}$$

or $$\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} = 0$$

or $$\frac{\partial u}{\partial x}\frac{dx}{dt} = -\frac{\partial u}{\partial y}\frac{dy}{dt}$$

or $$\frac{dy}{dt}\Big/\frac{dx}{dt} = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}$$

or $$\left(\frac{dy}{dx}\right)_u = -\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y} \quad \text{…. (3)}$$

Similarly, we get

$$\left(\frac{dy}{dx}\right)_v = -\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y} \qquad \text{.... (4)}$$

Now from equation (3) and (4), we have

$$\left(\frac{dy}{dx}\right)_u\left(\frac{dy}{dx}\right)_v = \left(\frac{\partial u}{\partial x}\bigg/\frac{\partial u}{\partial y}\right)\left(\frac{\partial v}{\partial x}\bigg/\frac{\partial v}{\partial y}\right)$$

It is given that the given function is analytic, therefore the given function satisfied the Cauchy-Riemann equations. Thus we have

$$\left(\frac{dy}{dx}\right)_u\left(\frac{dy}{dx}\right)_v = \frac{\left(\frac{\partial u}{\partial x}\right)\left(-\frac{\partial u}{\partial y}\right)}{\left(\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}\right)} = -1$$

Hence the given family of curves $u(x, y) = C_1$ and $v(x, y) = C_2$ are orthogonal system.

3.4 Laplace Equation

The Laplace equation is a second-order partial differential equation named after Pierre-Simon Laplace.The Laplace equation arises in many areas of physics and engineering, including electrostatics, fluid dynamics, and heat conduction. It describes systems where the value of *u* at any point is the average of the values of *u* in its neighborhood. This property is often referred to as the "harmonic" property of solutions to the Laplace equation.

The equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is known as Laplace equation.

Examples

Example.2. If f(z) isa regular function of z then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

Solution. We have f(z)=u+iv

$\Rightarrow \qquad |f(z)| = \sqrt{u^2+v^2}$

$$\Rightarrow \qquad |f(z)|^2 = u^2+v^2 = F(x,y).$$

$$\frac{\partial F}{\partial x} = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x},$$

$$\frac{\partial^2 F}{\partial x^2} = 2\left[u\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 + v\frac{\partial^2 v}{\partial x^2} + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

Similarly, we have

$$\frac{\partial^2 F}{\partial y^2} = 2\left[u\frac{\partial^2 u}{\partial y^2} + \left(\frac{\partial u}{\partial y}\right)^2 + v\frac{\partial^2 v}{\partial y^2} + \left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 2\begin{bmatrix} u\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 \\ +v\dfrac{\partial^2 v}{\partial x^2} + \left(\dfrac{\partial v}{\partial x}\right)^2 \end{bmatrix} + 2\begin{bmatrix} u\dfrac{\partial^2 u}{\partial y^2} + \left(\dfrac{\partial u}{\partial y}\right)^2 \\ +v\dfrac{\partial^2 v}{\partial y^2} + \left(\dfrac{\partial v}{\partial y}\right)^2 \end{bmatrix} \qquad \text{...(1)}$$

Since f(z) is analytic function so u and v satisfied the laplace equations,

i.e., $\qquad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

and $\qquad \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0.$

From equation (1), we have

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

or $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)F = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right] \qquad \text{….(2)}$$

Now we have f(z)=u+iv

$$\Rightarrow \qquad f'(z)=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}$$

$$\Rightarrow \qquad |f'(z)|^2=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2 \qquad \ldots(3)$$

From equations (2) and (3), we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)F=4|f'(z)|^2$$

or $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2.$$

3.5 Harmonic Function

Harmonic functions can be represented as the real or imaginary part of a complex analytic function.In complex analysis, a function is analytic if and only if it is harmonic (in the real and imaginary parts).

Any function of *x* and *y* which possess continuous partial derivatives of the first and second order and satisfies the Laplace equation is known as harmonic function.

Note.1. If *w*=*f*(*z*)=*u*+*iv* is an analytic function of *x* and *y* then *u* and *v* are known as harmonic functions.

2. If the harmonic functions*u* and *v* satisfy the Cauchy Riemann equation then the function is analytic.

3.Let v be the conjugate function of u, then

$$\begin{aligned} dv &= \frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy \\ &= \left(-\frac{\partial u}{\partial y}\right)dx+\left(\frac{\partial u}{\partial x}\right)dy \qquad [\text{by Cauchy Riemann equation}] \end{aligned}$$

Examples

Example.3. Show that the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ **is harmonic function.**

Solution. We know that if any function satisfies the Laplace equation is known as harmonic function.It is given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ ….(1)

Differentiate partially equation (1) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \qquad \text{..... (2)}$$

and $$\frac{\partial u}{\partial y} = -6xy - 6y \qquad \text{..... (3)}$$

Now differentiate partially the above equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial^2 u}{\partial x^2} = 6x + 6 \qquad \text{..... (4)}$$

and $$\frac{\partial^2 u}{\partial y^2} = -6x - 6 \qquad \text{..... (5)}$$

From equations (4) and (5), we get $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Hence the given function u is harmonic function.

Check your Progress

1. What do you mean by orthogonal system?

2. Define the Laplace equation and harmonic function.

3.6 Conjugate Function

If $w=f(z)=u+iv$ is analytic function and if u and v satisfy the Laplace equation

$$\nabla^2 u = \nabla^2 v = 0$$

then u and v are known as conjugate harmonic functions of the conjugate function.

Examples

Example.4. ***Find the analytic function w=u+iv, if the real part is given by $e^x \cos y$.*** **Solution.** It is given that

$u = e^x \cos y$ ..... (1)

Differentiate partially equation (1) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad ..... (2)$$

and $$\frac{\partial u}{\partial y} = -e^x \sin y \qquad ..... (3)$$

Now differentiate partially equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \qquad ..... (4)$$

and $$\frac{\partial^2 u}{\partial y^2} = -e^x \cos y \qquad ..... (5)$$

From equations (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic function.

Let v be the conjugate function of u, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy. \qquad [\text{by Cauchy Riemann equation}]$$

$$= \left\{ e^x \sin y\right\} dx + \left\{ e^x \cos y\right\} dy$$

$$= d(e^x \sin y)$$
$$= e^x \sin y + c.$$

Therefore

$$v = e^x\, sin\, y + c.$$

Now we have

$$w = u + iv$$

$$= e^x \cos y + i(e^x \sin y + c)$$

$$= e^x (\cos y + i \sin y) + ic$$

$$= e^x e^{iy} + ic$$

$$= e^z + c_1.$$

Example.5.Show that the function u=e^x($x$cos$y$-$y$sin$y$) is harmonic function. Determine its harmonic conjugate.

Solution. It is given that

u=e^x($x$cos$y$-$y$sin$y$) …. (1)

Differentiate partially equation (1) with respect to x and y respectively, we get

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x \cos y \qquad \text{..... (2)}$$

and
$$\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) \qquad \text{..... (3)}$$

Now differentiate partially equations (2) and (3) with respect to x and y respectively, we get

$$\frac{\partial^2 u}{\partial x^2} = e^x(x\cos y - y\sin y) + e^x \cos y + e^x \cos y \qquad \text{..... (4)}$$

and
$$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y - \cos y + y\sin y) \qquad \text{..... (5)}$$

From equations (4) and (5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Hence u is harmonic function.

Let v be the conjugate function of u, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy. \qquad [\text{by Cauchy Riemann equation}]$$

$$= \left\{-e^x(-x\sin y - \sin y - y\cos y)\right\}dx$$
$$+ \left\{e^x(x\cos y - y\sin y) + e^x \cos y\right\}dy$$

$$= d(xe^x \sin y + ye^x \cos y)$$

$$= xe^x \sin y + ye^x \cos y + c.$$

Hence $v = xe^x \sin y + ye^x \cos y + c.$

Example.6. Find the analytic function w=u+iv, if u=e^{2x}(xcos2y-ysin2y).

Solution. It is given that

u=e^{2x}(xcos2y-ysin2y) …. (1)

Differentiate partially equation (1) with respect to$x$ and y respectively, we get

$$\frac{\partial u}{\partial x} = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}\cos 2y \quad \text{…. (2)}$$

and $$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y) \quad \text{…. (3)}$$

Using Cauchy Riemann equations, we have

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$$

and $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2e^{2x}(x\cos 2y - y\sin 2y) + e^{2x}\cos 2y$$

Let v be the conjugate function of u, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy. \quad [\textit{by CR equation}]$$

$$= \begin{Bmatrix} e^{2x}(2x\sin 2y + \sin 2y \\ +2y\cos 2y) \end{Bmatrix} dx + \begin{Bmatrix} 2e^{2x}(x\cos 2y - y\sin 2y) \\ +e^{2x}\cos 2y \end{Bmatrix} dy$$

$$= d(xe^{2x}\sin 2y + ye^{2x}\cos 2y)$$

$$= xe^{2x}\sin 2y + ye^{2x}\cos 2y + c.$$

Hence $v = xe^{2x} \sin 2y + ye^{2x} \cos 2y + c.$

Now we have

$$w = u + iv$$

$$= e^{2x}(x\cos 2y - y\sin 2y) + i(xe^{2x}\sin 2y + ye^{2x}\cos 2y + c)$$

$$= ze^{2z} + ic_1.$$

3.7 Milne's Thomson Method

Milne's Thomson method, also known as the method of images, is a technique used in the study of electrostatics to simplify certain boundary value problems. It was developed by Sir John William Strutt, also known as Lord Rayleigh, and is named after Sir Edward Milne.Milne's Thomson method is particularly useful for solving problems involving conductors and dielectrics, especially in simple geometries such as spheres, cylinders, and planes. It allows one to find the electric field and potential in regions that would otherwise be difficult to analyze using direct methods.

Milne's Thomson method is used to determine the function *f*(*z*) when one conjugate function is given.

Let us consider $z = x + iy$ then $\bar{z} = x - iy$.

Therefore we have

$$x = \frac{z+\bar{z}}{2}, \text{ and } y = \frac{z-\bar{z}}{2i}. \qquad \text{.....(1)}$$

Now we have

$$w = f(z)$$

$$= u + iv$$

$$= u(x, y) + iv(x, y) \qquad \text{.....(2)}$$

From equations (1) and (2), we get

$$w = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + iv\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) \qquad \text{.....(3)}$$

If we take $z = \overline{z}$, then from equation (3), we have

$$w = u(z, 0) + iv(z, 0) \qquad \text{.....(4)}$$

From equation (2) we have

$$w = u(x, y) + iv(x, y)$$

or $$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

$$= \phi_1(x, y) - i\phi_2(x, y)$$

$$= \phi_1\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - i\phi_2\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

At $z = \overline{z}$, we have

$$\frac{dw}{dz} = \phi_1(z, 0) - i\phi_2(z, 0)$$

or $$w = \int \left(\phi_1(z, 0) - i\phi_2(z, 0)\right) dz$$

Now we have

$$\frac{dw}{dz} = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= \left(\frac{\partial u}{\partial x}\right)_{(z,0)} - i\left(\frac{\partial u}{\partial y}\right)_{(z,0)}.$$

Examples

Example.7. If $u-v=(x-y)(x^2+4xy+y^2),\ \ f(z)=u+iv$ **is analytic function of z=x+iy, find f(z) in terms of z.**

Solution. It is given that

$$u-v=(x-y)(x^2+4xy+y^2)$$

$$=x^3+3x^2y-3xy^2-y^3 \qquad \text{....(1)}$$

Differentiate partially equation (1) with respect to$x$ and y respectively, we get

$$\frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=3x^2+6xy-3y^2$$

and $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=3x^2-6xy-3y^2$$

Using the Cauchy Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$, we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=3x^2+6xy-3y^2 \qquad \text{....(2)}$$

and $$\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}=3x^2-6xy-3y^2 \qquad \text{....(3)}$$

Solving equations (2) and (3), we get

$$\frac{\partial u}{\partial x}=6xy,\quad \frac{\partial u}{\partial y}=3(x^2-y^2)$$

Using Milne's Thomson method, we have

$$\frac{dw}{dz} = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} - i\left(\frac{\partial u}{\partial y}\right)_{(z,0)}$$

$$= (6xy)_{(z,0)} - i\left(3(x^2 - y^2)\right)_{(z,0)}$$
$$= -3iz^2.$$

Integrating, we get

$$w = -i\,z^3 + c.$$

3.8 Summary

Two curves are said to be orthogonal to each other when they intersect at right angles at each of their points of intersection.

The equation $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$ is known as Laplace equation.

If $w=f(z)=u+iv$ is an analytic function of x and y then u and v are harmonic functions.

If the harmonic function u and v satisfy the Cauchy Riemann equation then function is analytic.

Let v be the conjugate function of u, then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy \qquad [\text{by Cauchy Riemann equation}]$$

If $w=f(z)=u+iv$ is analytic function and if u and v satisfy the Laplace equation $\nabla^2 u = \nabla^2 v = 0$ then u and v are known as conjugate harmonic functions of the conjugate function.

Let $z = x + iy$ then $\bar{z} = x - iy$.

Therefore we have $x = \frac{z+\bar{z}}{2}$, and $y = \frac{z-\bar{z}}{2i}$.

and we have $\frac{dw}{dz} = \left(\frac{\partial u}{\partial x}\right)_{(z,0)} - i\left(\frac{\partial u}{\partial y}\right)_{(z,0)}$.

3.9 Terminal Questions

Q.1. Wrhat do you mean by Harmonic function.

Q.2. State the Laplace equation.

Q.3. Explain the Milne's Thomson method**.**

Q.4. Show that the function u = cos x cosh y is harmonic and find its harmonic conjugate.

Q.5.Find the regular function w = u + iv, where u = e^{-x} [(x^2 – y^2) cos y + 2 xy sin y].

Q.6. Show that the function u = sin x cosh y + 2 cos x sinh y + x^2 – y^2 + 4xy satisfies Laplace's equation and determine the corresponding analytic function f(z) = u + iv.

Q.7. Prove that the function u = e^x (x cos y – y sin y) satisfies Laplace's equation and find the corresponding analytic function f(z) = u + iv.

Q.8.If $u - v = e^x(\cos y - \sin y),\ \ f(z) = u + iv$ is analytic function of $z = x+iy$, find $f(z)$ in terms of z.

Answer

4. $v = -\sin x \sinh y + c.$

5. $f(z) = z^2 e^{-z} + c.$

6. $f(z) = (1-2i)(\sin z + z^2) + c.$

7. $f(z) = ze^z + c.$

8. $f(z) = e^z + c.$

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

Master of Science

PGMM -104N

Complex Analysis

U. P. RajarshiTandon Open University

Block

2

Conformal Mappings

Unit- 4

Introduction to Conformal Mapping

Unit- 5

Bilinear Transformations

Block-2

Conformal Mapping

Conformal mappings are important in various fields, including fluid dynamics, where they are used to map flow fields and analyze the behavior of fluids. They are also used in the study of complex functions and the behavior of complex numbers. Conformal mapping is a technique used in complex analysis to map one region of the complex plane to another in such a way that angles between curves are preserved. In other words, if you have a function $f(z)$ that maps points from one complex plane to another, and if $f'(z) \neq 0$ for all z in the domain, then f is said to be conformal at z.Conformal mappings can be used to transform difficult boundary value problems into simpler ones. By mapping a complex domain to a simpler domain where the problem is easier to solve, mathematicians and scientists can gain insights into the original problem.

Conformal mappings are used in engineering applications, such as in the design of electronic circuits and the analysis of heat transfer. They can help engineers optimize designs and understand the behavior of complex systems. Conformal mapping is a powerful tool with applications in a wide range of fields. Its ability to preserve angles and simplify complex problems makes it a valuable tool for researchers and engineers alike. Conformal mappings have applications in physics, particularly in the study of electromagnetism and fluid dynamics. They can be used to simplify the mathematical models used to describe physical phenomena.

In the fourth unit, we shall discussed abouttransformation or mapping, ordinary and critical point, conformal mapping, some general transformations: translation, rotation, magnification and inversion. Bilinear Fixed points of the Bilinear Transformations, some important properties of Bilinear Transformations, special conformal transformations, Joukowski's transformation and Schwarz-Christoffel transformation are discussed in unit fifth.

UNIT-4: Introduction to Conformal Mappings

Structure

4.1 Introduction

Conformal mapping has a great importance in the theory of fluid flow, potential and solving their problems. In this unit we discuss basic theory of conformal mapping and mainly concerned with certain geometric question, although our initial approach will be analytic but can be easily converted into a geometric one. Conformal mapping is used to map complicated regions conformally on to simpler, standard regions such a circular disc, strip and half plane for which the boundary value problems are easier.

In this unit, we shall deals about the transformation, ordinary and critical point, conformal mapping, some general transformations: translation, rotation, magnification and inversion.

4.2 Objectives

After reading this unit the learner should be able to understand about:

- Transformation or mapping
- Ordinary point and critical points
- Jacobian of a transformation
- Conformal mapping and its applications
- Some general transformation: translation, rotation, magnification and inversion.

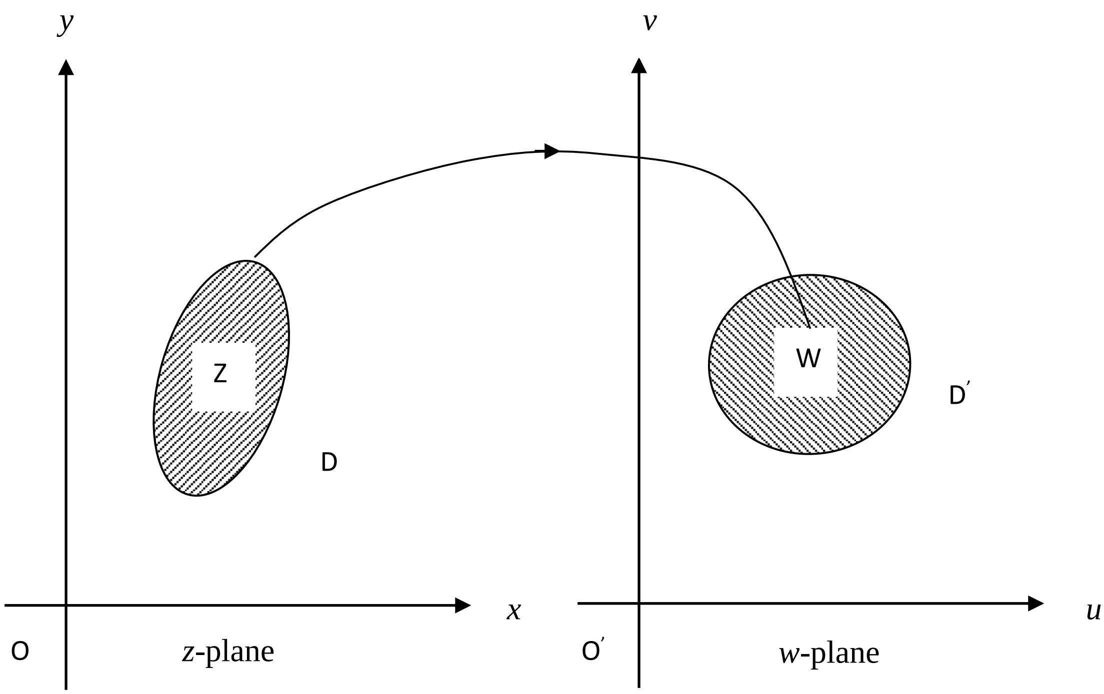
4.3 Transformation or Mapping

Consider the function of a complex variable z

$$w = f(z)$$

$$= u(x, y) + iv\,(x, y)$$

which involves the four real value $x, y, u(x, y), v(x, y)$. Here a four dimensional region needed for their geometrical representation, whichis not possible due to four dimensional region.Therefore we take two complex plane, one for $z = x + iy$ and other for $u + iv$. If the point (x, y) on a curve c in the z-plane get transformed or mapped to point of an image curve c' in the w-plane. Then the function $w = f(z)$ is called a transformation or mapping from z-plane.



If the corresponding to each point of z-plane, there is a unique point of w-plane and conversely then such mapping is called one-one mapping.

4.4 Ordinary and Critical Point

Consider the function of a complex variable z

$$w = f(z)$$

Those points in which $\frac{dw}{dz} \neq 0$ are known as ordinary points. And those points are known as critical point in which $\frac{dw}{dz}$ provided the values 0 or ∞.

4.5 Jacobian of a Transformation

Consider the function of a complex variable z

$$w = f(z)$$

$$= u(x, y) + iv(x, y) \qquad \ldots(1)$$

which maps a closed region D of the z-plane into a closed D of the *w*-plane. Consider Δw denote respectively areas of these regions. Then it can be easily shown that if *u, v* are continuously differentiable, we have

$$\lim_{\Delta z \to} \frac{\Delta w}{\Delta z} = \left| \frac{\partial(u,v)}{\partial(x,y)} \right| \ldots(2)$$

The determinate

$$\frac{\partial(u\ v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \ldots(3)$$

is known as the Jacobian of the transformation of equation (1).

Examples

Example.1. Find the image of the point *z = 2 + 3i* under the transformation *w = f(z) = z(z – 2i).*

Solution. It is given that

$$w = f(z)$$

$$= z(z - 2i) \qquad \ldots(1)$$

Putting $z = 2 + 3i$ in equation (1), we get

`

$$f(2 + 3i) = (2 + 3i)(2 + 3i - 2i)$$

$$= (2 + 3i)(2 + i)$$

$$= 4 + 6i - 3 + 2i$$

$$= 1 + 8i.$$

Example.2. Find all the pre-image in the z-plane under the transformation *w = z(z – 2i)*= -l.

Solution. It is given that

$$w = z(z - 2i) = -1 \qquad \ldots(1)$$

Solving equation (1), we have

$$z(z - 2i) = -1$$

or $$z^2 - 2iz + 1 = 0$$

or $$z = \frac{2i \pm \sqrt{-4-4}}{2}$$

or $$z = \frac{2i \pm 2\sqrt{2i}}{2}$$

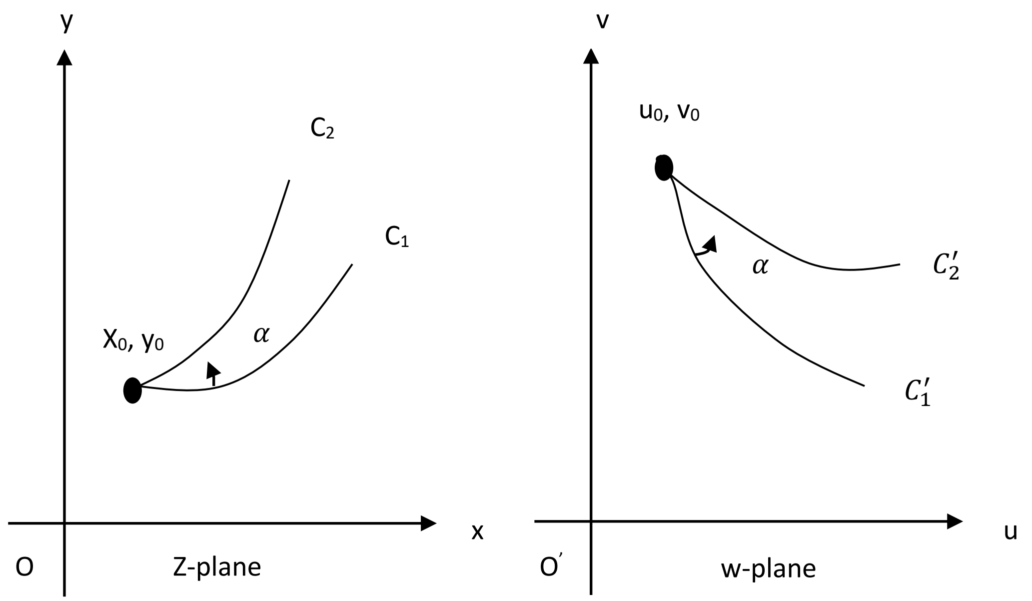
or $$z = i\left(1+\sqrt{2}\right).$$

Hence the pre-image of $w = -1$ is $z = i\left(1+\sqrt{2}\right)$.

4.6 Conformal Mapping

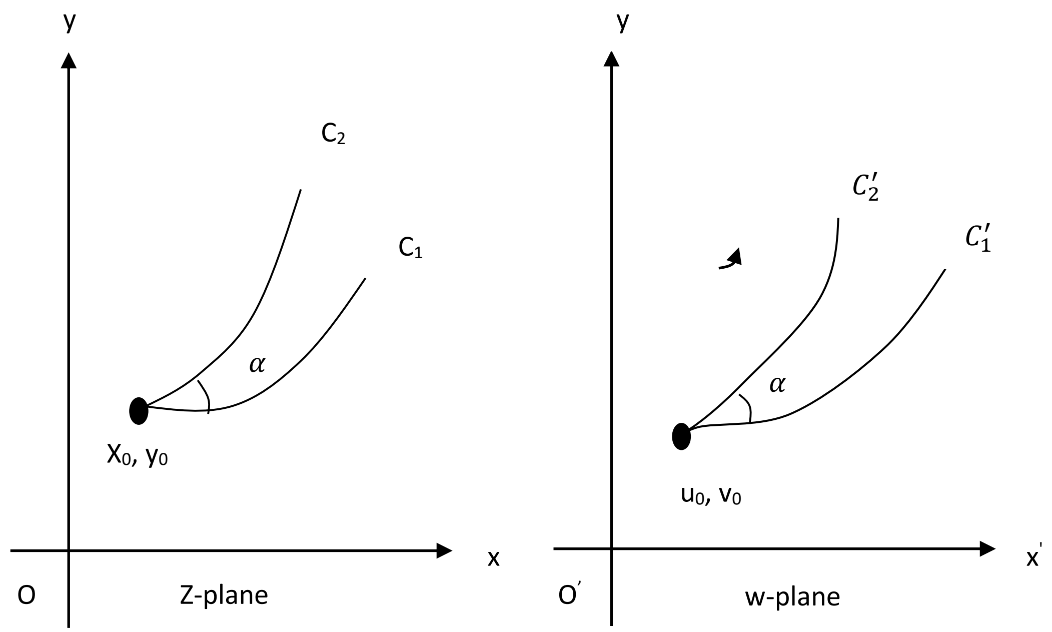
Consider the transformation

$$u = u(x, y), v = v(x, y) \qquad \ldots(1)$$



maps the two curves C_1, C_2 of z-plane on the two curve C_1', C_2' of w-plane.

If the transformation is such that the angle at (x_0, y_0) between C_1, and C_2 is equal to the angle at (u_0, v_0) between C_1' and C_2' both in magnitude and sense (direction), is known as the conformal transformation at (x_0, y_0). A mapping in which preserves the magnitude of angles but not in sense (direction) is known as the isogonal transformation.



(Isogonal transformation)

Check your Progress

1. What do you mean by transformation or mapping?

2. Define the ordinary and critical point.

3. Explain the conformal mapping.

4.7 Some General Transformation

The linear transformation is defined by

$$w = \alpha z + \beta \quad ...(1)$$

Where $\alpha \neq 0$, and β are arbitrary complex constants.

Linear transformation preserves circles i.e., a circle in the z-plane under linear transformation maps to a circle in the w-plane. Some special cases of linear transformation are:

4.7.1 Translation

Consider the linear transformation is

$$w = \alpha z + \beta \quad ...(1)$$

Putting $\alpha = 1$ in the given equation (1), we get

$$w = z + \beta \quad ...(2)$$

If $z = x + iy$, $\beta = a + ib$ and $w = u + iv$ then by equation (2), we have

$$u + iv = x + iy + a + ib$$

or

$$u + iv = (x + a) + i(y + b) \quad ...(3)$$

Equating the real and imaginary parts of both sides of equation (3), we get

$$u = x + a$$

and $$v = y + b.$$

Examples

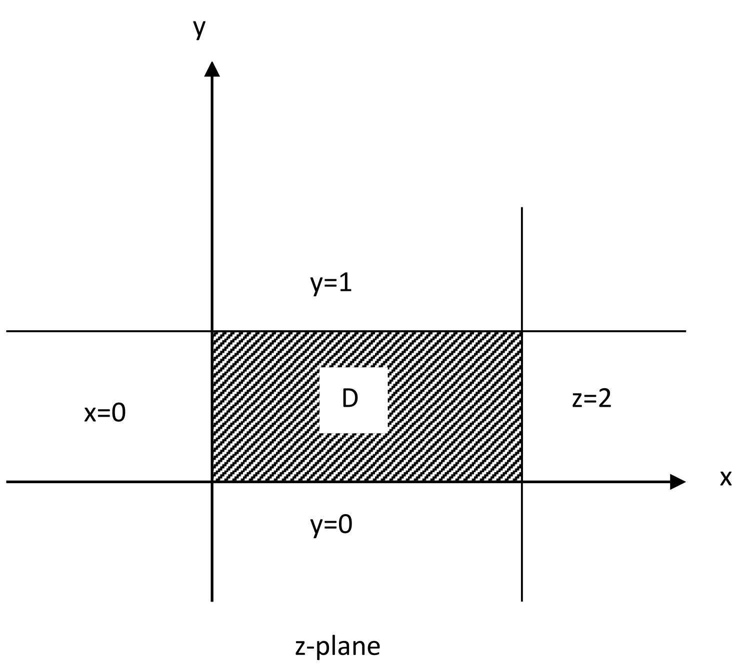
Example.3. Let a rectangular domain D be bounded by $x = 0.$, $y = 0$, $x = 2$, $y = 1$. Determine the region D' of w-plane into which D is mapped under the transformation $w = z + (1 - 2i)$.

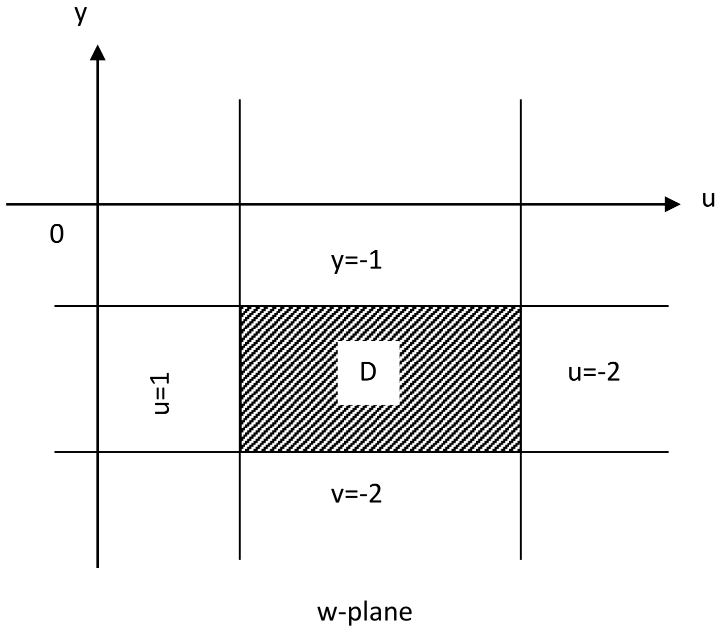
Solution. It is given that

$$w = z + (1 - 2i) \qquad \ldots(1)$$

In region d, z-plane bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 1$.

In region D', w-plane





$w = z + (1 - 2i)$

or $\quad u + iv = (x + iv)(1 - 2i)$

$= (x + 1) + i(y - 2)$

$u = x + 1, v = y - 2$

When $\quad x = 0,\ u = 1$

When $\quad x = 2,\ u = 3$

When $y = 0,\ v = -2$

When $\quad y = 1,\ v = -1.$

Example 4. Consider the transformation $w = z + (1 - i)$ and determine the region D' of the w-plane corresponding to the rectangular region D in the z-plane bounded by $x = 0$, $y = 0$, $x = 1$ and $y = 2$.

Solution. It is given that

$w = z + + (1 - i)$ ....(1)

In region D, z-plane bounded by $x = 0$,

$y = 0, x = 1$ and $y = 2$

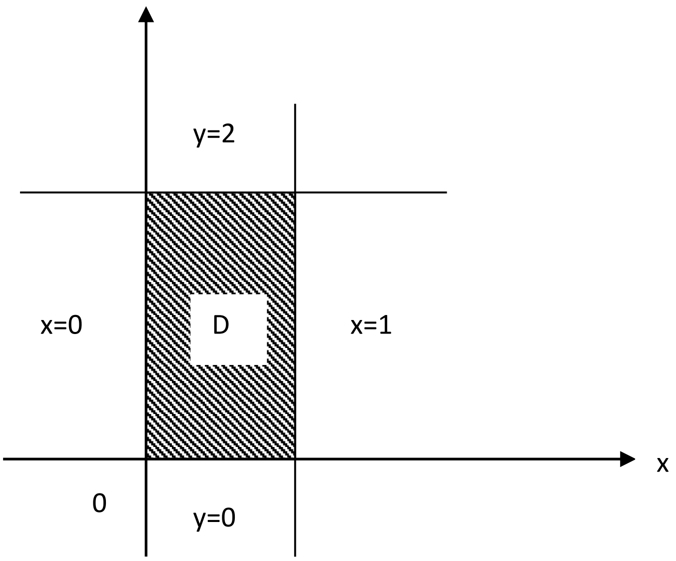
In region D', w-plane

$w = z + (1 - i)$

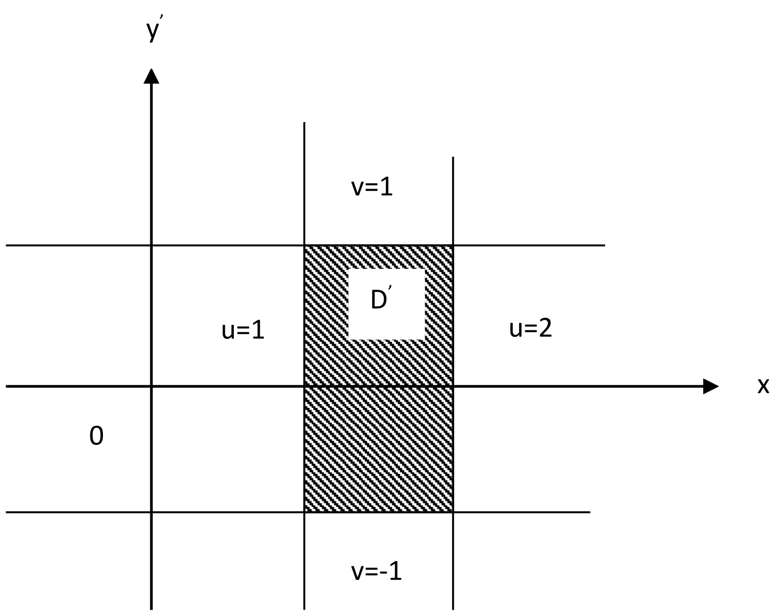
$$u + iv = (x + iy) + (1 - i)$$

$$= (x + 1) + I(y - 1)$$

$\Rightarrow \quad u = x + 1, v = y - 1$



z-plane



w-plane

When $x = 0, \; u = 1$

When $x = 1, \; u = 2$

When $y = 0, \; v = -1$

When $y = 2, \; v = 1.$

4.7.2 Rotation

Consider the linear transformation is

$$w = \alpha z + \beta \ldots(1)$$

Putting $\alpha = e^{i\theta}$ and $\beta = 0$ in equation (1), we get

$$w = z\, e^{i\theta}$$

which rotates z through an angle θ in anticlockwise in w-plane.

Examples

Example.5. Given the transformation $w = ze^{i\pi/4}$ and determine the region D' of *w*-plane corresponding to the triangular refion D bounded by the lines *x*=0, *y* = 0 and *x*+*y* = 1 in *z*- plane.

Solution. It is given that

$$w = ze^{i\pi/4} \qquad \ldots(1)$$

$$\Rightarrow \qquad u + iv = (x + iv)\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= x\left(\frac{1+i}{\sqrt{2}}\right) + iy\left(\frac{1+i}{\sqrt{2}}\right)$$

$$= \left(\frac{x-y}{\sqrt{2}}\right) + iy\left(\frac{x+y}{\sqrt{2}}\right)$$

Equating the real and imaginary parts both sides, we get

$$u = \frac{x-y}{\sqrt{2}} \qquad \ldots(2)$$

and $$v = \frac{x+y}{\sqrt{2}} \qquad \ldots(3)$$

Putting x = 0 in the equations (2) and (3), we get

$$u = -\frac{1}{\sqrt{2}}y, \quad v = \frac{1}{\sqrt{2}}y$$

$\Rightarrow$ $$u = -v$$

Now putting y = 0 in equation (2), and (3), we get

$$u = \frac{1}{\sqrt{2}}y, \quad v = \frac{1}{\sqrt{2}}y$$

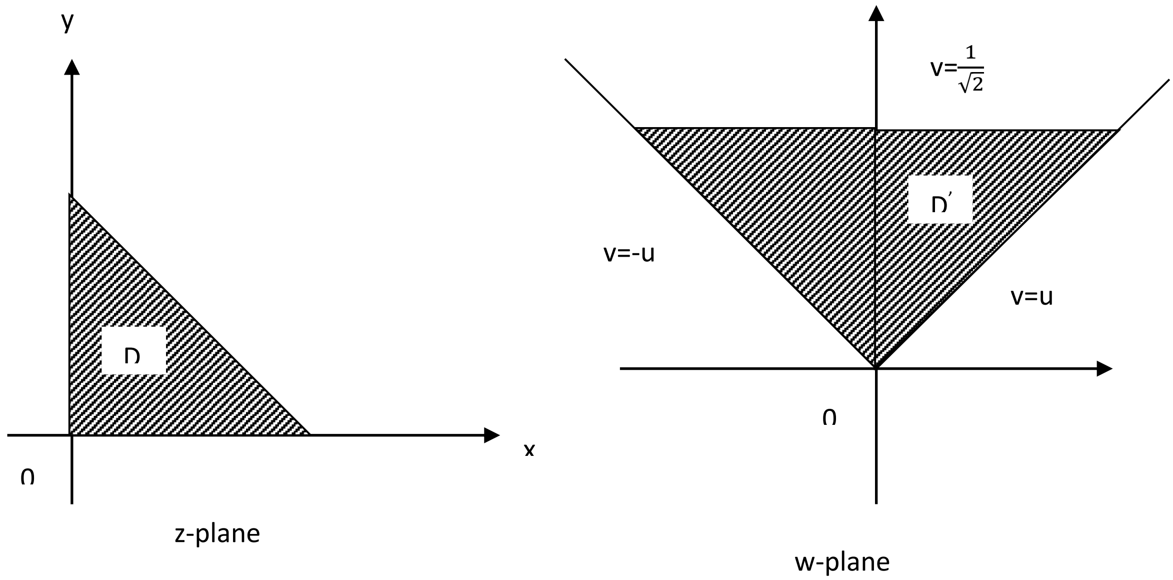
$\Rightarrow$ $$u = v$$

Again putting x + y = 1 in equation (3), we get

$$v = \frac{1}{\sqrt{2}}$$

Therefore the triangular region D in z-plane is mapped on a triangular region D' of w-plane bounded by the lines

$$u = -v, \; u = v, \; v = \frac{1}{\sqrt{2}}$$



4.7.3 Magnification

Consider the linear transformation is

$$w = \alpha z + \beta \quad ...(1)$$

Putting $\beta = 0$ in equation (1), we get $w = \alpha z$

which show that the figure in w-plane magnified α-times the size of the figure in z-plane

Examples

Example.6.Determine the region D' in w-plane on the transformation of rectangular region enclosed by $x = 1$, $y = 1$, $x = 2$, and $y = 2$ in the z-plane. The transformation is $w = 2z$.

Solution. It is given that

$$w = 2z \quad \ldots(1)$$

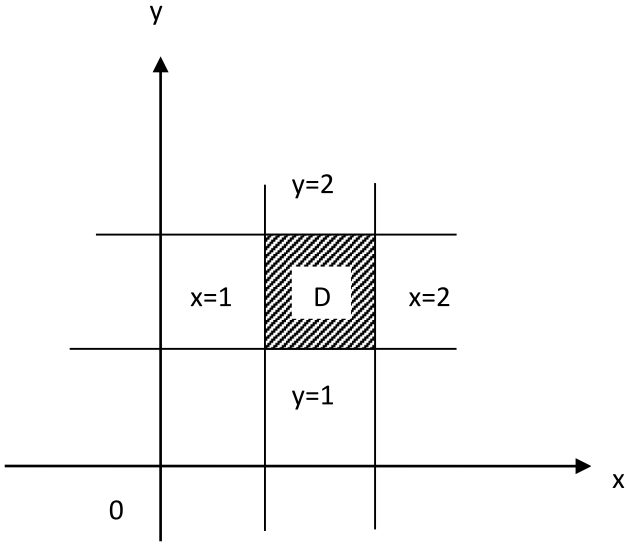
$$u + iv = 2(x + iv)$$

Equating real and imaginary parts both sides, we get

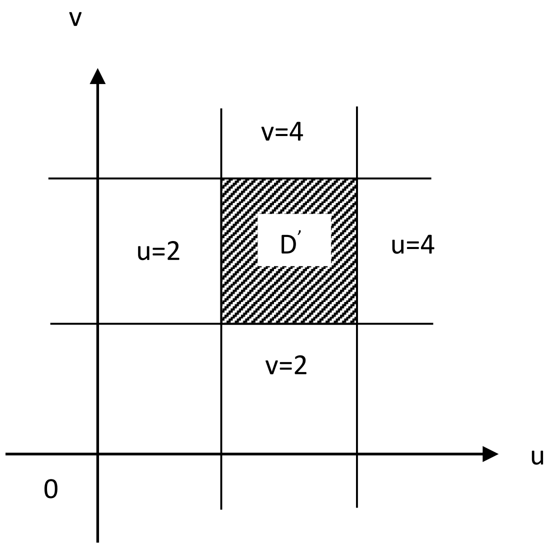
$$U = 2x, \;\; v = 2y$$

When $x = 1$ then $u = 2$ when $y = 1$ then $v = 2$

When $x = 2$ then $u = 4,$ when $y = 2$ then $v = 4.$



z-plane



w-plane

Example 7. Consider the transformation *w = 2z* and determine the region D' of the w-plane into which the triangular region D enclosed by the lines *x = 0, y = 0, x + y = 1* in the z-plane is mapped under this transformation.

Solution. It is given that

$$w = 2z \qquad \ldots(1)$$

$$\Rightarrow \qquad u + iv = 2(x + iy)$$

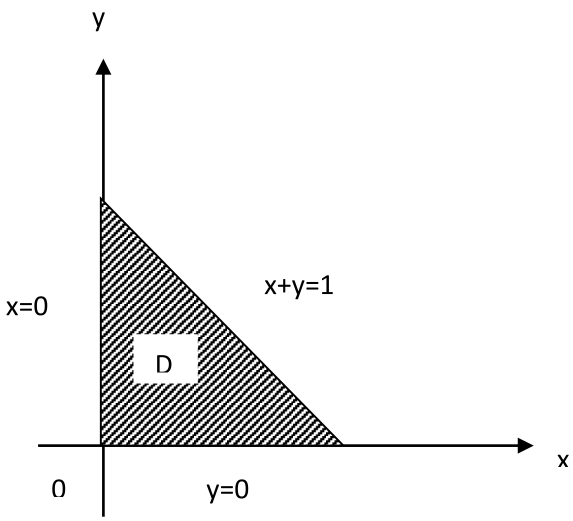
Equating the real and imaginary parts both sides, we get $u = 2x$

and $v = 2y$

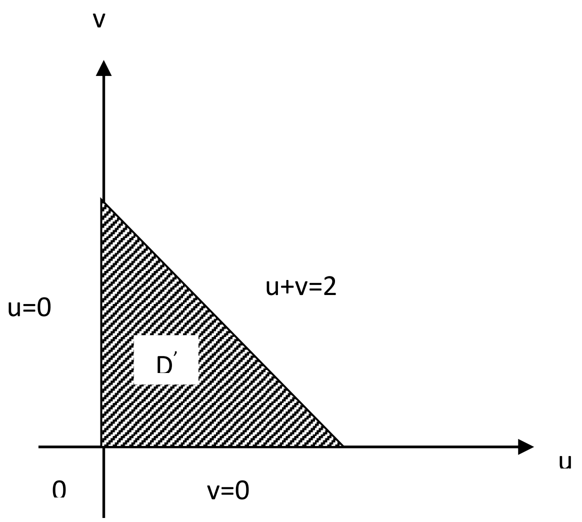
When $x = 1$ then $u = 0$

When $y = 0$ then $v = 0$

When $x + y = 1$ then $u + v = 2.$



z-plane



w-plane

4.7.4 Rotation and Magnification

Consider the linear transformation is

$$w = \alpha z + \beta \quad ...(1)$$

Putting $\beta = 0$ in equation (1), we get

$w = \alpha z$, where $\alpha$is complex number …(2)

If $\rho e^{i\varphi} = w = \rho^{i\varphi}$ and $\alpha = ae^{i\gamma}$, then equation (1), we have

$$\rho e^{i\varphi} = ae^{i\gamma} . re^{i\theta} = are^{i(\theta + \gamma)} \quad \text{where } \rho = \text{are and } \varphi = \theta + \gamma$$

which shows that the transform $w = \alpha z$ corresponding to a rotation together with magnification.

Check your Progress

1. What do you mean by rotation and magnification?

Examples

Example.8. Determine the region D' of the w-plane corresponding to the region D z-plane bounded by *x = 0, x = 2, y = 0, y = 3,* under the transformation *w = (1 + i) z.*

Solution. It is given that

$w = (1 + i)z$ …(1)

In region D, z-plane bounded by $x = 0$, $x = 2$, $y = 0$ and $y = 3$.

In region D', w-plane

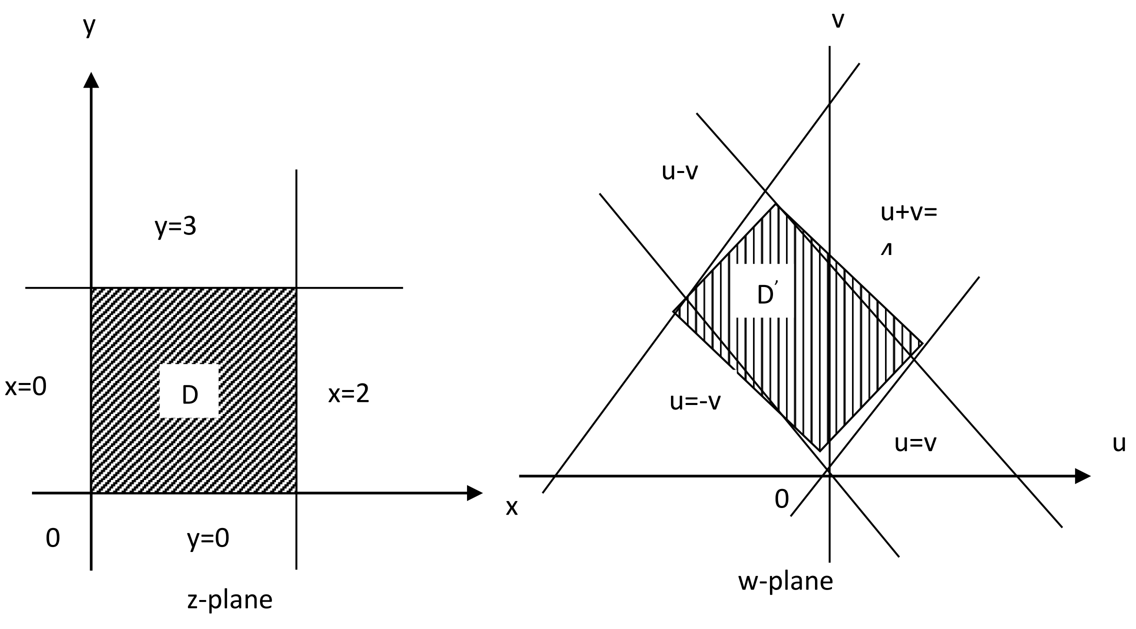
$$w = (1 + i)z$$

$$u + iv = (1 + i)(x + iv)$$

$$= (x + ix) + (iy - y)$$

$$= (x - y) + i(x + y)$$

$$\Rightarrow \quad u = (x - y), \; v = (x + y)$$



When $x = 0$ $\Rightarrow$ $u = -y, v = y$

$\Rightarrow$ $u = -v$

When $y = 0$ $\Rightarrow$ $u = x,$ $v = x$

$\Rightarrow$ $u = v$

When $x = 2$ $\Rightarrow$ $u = 2,$ $v = 2 + y$

$\Rightarrow$ $u - v = -6$

When $y = 3$ $\Rightarrow$ $u = x - 3$ $v = x + 3$

$\Rightarrow$ $u - v = -6.$

4.7.5 Inversion

We have $$w = \frac{1}{z}\left(z \neq 0\right) \qquad \ldots(1)$$

If $w = \rho e^{i\varphi}$ and $z = re^{i\theta}$ then by equation (1), we have

$$\rho e^{i\varphi} = \frac{1}{re^{i\theta}}$$

or $$\rho e^{i\varphi} = \frac{1}{r}e^{-i\theta}$$

where $\rho = \frac{1}{r}$ and $\phi = -\theta$.

which show that the figures in z-plane are mapped upon the reciprocal figures in w-plane.

Example.9.Find the image of the infinite strips $\frac{1}{4} < y < \frac{1}{2}$ under the transformation w = $\frac{1}{z}$.

Solution. It is given that

$$w = \frac{1}{z} \qquad \ldots(1)$$

$$\Rightarrow \qquad u + iv = \frac{1}{x+iy} = \frac{x - iy}{x^2 + y^2}$$

Equating the real and imaginary parts both sides, we get

$$u = \frac{x}{x^2 + y^2}$$

and $$v = \frac{y}{x^2 + y^2}$$

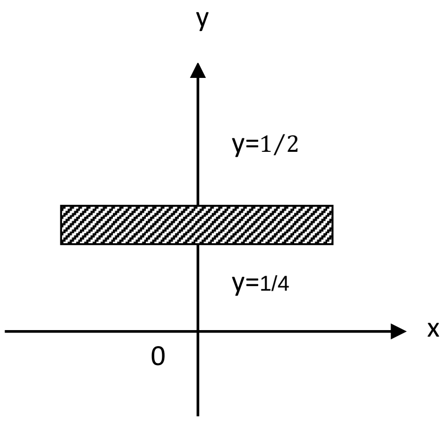
$$\Rightarrow \qquad \frac{u}{v} = -\frac{x}{y}$$

or $$x = -\frac{uy}{v}$$

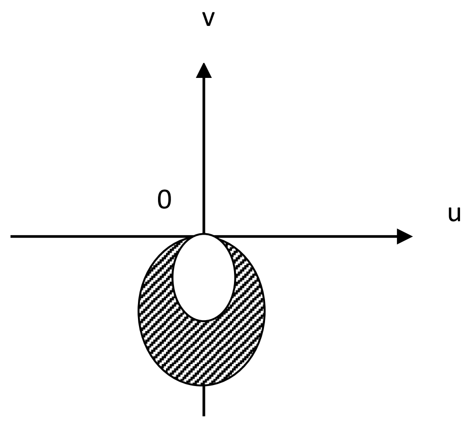
so that $$v = -\frac{y}{x^2 + y^2} = \frac{y}{\left(-\frac{uy}{v}\right) + y^2}$$

or $$v = -\frac{v^2}{\left(u^2 + v^2\right) y}$$

or $$y = -\frac{v}{u^2 + v^2}$$



z-plane



w-plane

If $y = \frac{1}{4}$ then we have

$$-\frac{v}{u^2+v^2} = \frac{1}{4}$$

$\Rightarrow$ $$u^2 + v^2 + 4v = 0$$

or $$u^2 + (v+2)^2 = 4.$$

If $y = \frac{1}{2}$ then we have

$$-\frac{v}{u^2+v^2} = \frac{1}{2}$$

$\Rightarrow$ $$u^2 + v^2 + 2v = 0$$

or $$u^2 + (v+1)^2 = 1.$$

4.8 Summary

Consider the function of a complex variable z is $w = f(z)$. Those points in which $\frac{dw}{dz} \neq 0$ are known as ordinary points. And those points are known as critical point in which $\frac{dw}{dz}$ provided the values 0 or ∞.

Consider the transformation $u = u(x, y)$, $v = v(x, y)$ which maps the two curves C_1, C_2 of z-plane on the two curve C_1', C_2' of w-plane.If the transformation is such that the angle at (x_0, y_0) between C_1, and C_2 is equal to the angle at (u_0, v_0) between C_1' and C_2' both in magnitude and sense (direction), is known as the conformal transformation at (x_0, y_0).

The linear transformation is defined by

$$w = \alpha z + \beta.$$

4.9 Terminal Questions

Q.1. Explain the ordinary and critical points.

Q.2. Write a short note on Conformal mapping.

Q.3. What do you mean by transformation.

Q.4. Find the image of the rectangular $x = 0, y = 0, x = 1 \; and \; y = 2$ in z-plane under the map $w = (1 + i)z + (2 - i)$.

Answer

4. Image in w-plane is rectangular bounded by $u + v = 1, u = v = 3, u + v = 3 \; and \; v - u = 1$.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-5: BILINEAR TRANSFORM

Structure

5.1 Introduction

The bilinear transform is a mathematical transformation used in signal processing and control system theory to convert continuous-time systems into discrete-time systems. It is particularly useful for designing digital filters based on analog prototypes. The bilinear transform preserves stability, meaning that if the continuous-time system is stable, the discrete-time system obtained by the transformation will also be stable. However, it introduces frequency warping, which must be compensated for in filter design to achieve the desired frequency response.

Bilinear transform is a fundamental tool for converting between continuous-time and discrete-time domains, enabling the analysis and design of systems in both domains.The bilinear transform is important in several areas of signal processing and control system design for the frequency mapping, stability preservation, time domain analysis, control system design and digital signal processing.

5.2 Objectives

After reading this unit the learner should be able to understand about

- the Bilinear (or Mobius) Transformation
- Invariant or Fixed points of the Bilinear Transformation
- Some Properties of Bilinear Transformation
- Special Conformal Transformations
- Joukowski's Transformation $\left(w = z + \frac{1}{z}\right)$
- Schwarz-Christoffel Transformation

5.3 Bilinear (or Mobius) Transformation

The transformation of the form

$$w = \frac{az + b}{cz + d}$$

in which a, b, c and d are complex constants and $ad - bc \neq 0$ is called a bilinear (or Mobius) transformation.

The constant $ad - bc$ is known as the determinant of the transformation (1).

Equation (1) may be written as

$$cwz + dw - az - b = 0$$

which is linear both in z and w.

By equation (2), we have

$$z(cw - a) + dw - b = 0$$

or $$z = \frac{-dw + b}{cw - a}$$

which is inverse transformation of equation (1) and also a bilinear transformation.

Suppose w_1 and w_2 be the points corresponding z_1 and z_2 in equation (1) then

$$w_2 - w_1 = \frac{az_2 + b}{cz_2 + d} - \frac{az_1 + b}{cz_1 + d}$$

$$= \frac{(ad - bc)(z_2 - z_1)}{(cz_1 + d)(cz_2 - d)}$$

$\therefore$ $$w_2 - w_1 = 0 \quad if \quad ad - bc = 0$$

This shows that w is constant or meaningless.

Note. (i) We have $w = \dfrac{az + b}{cz + d}$

or $\qquad w = \dfrac{a}{c} \cdot \dfrac{z + \dfrac{b}{a}}{z + \dfrac{d}{c}}$

if $\qquad z = -\dfrac{b}{a}$ then w = 0

if $\qquad$ z = $-\dfrac{d}{c}$ then w = ∞

(ii) We have $\quad$ w = $-\dfrac{dw + b}{cw - a}$

or $\qquad$ z = $-\dfrac{d}{c} \cdot \dfrac{w - \dfrac{b}{d}}{w - \dfrac{a}{c}}$

if $\quad$ w = $\dfrac{b}{d}$ then we havez = 0.

if $\quad$ w = $\dfrac{a}{c}$ then we have $z = \infty$.

(iii) The transformation $w = \dfrac{az + b}{cz + d}$ is said to be normalized if $ad - bc = 1$.

(iv) The transformation $w = \dfrac{az + b}{cz + d}$ is said to be conformed if $ad - bc \neq 0$ then $\dfrac{dw}{dz} \neq 0$.

(v) We have $w = f(z)$ and $\dfrac{dw}{dz} = f'(z)$.

(vi) We have $w = \dfrac{az + b}{cz + d}$.

Differentiating with respect to z both sides, we get

$$\frac{dw}{dz} = \frac{(cz + d)(a) - (az + b)(c)}{(cz + d)^2}$$

$$= \frac{ad - bc}{(cz + d)^2}$$

The implies $\frac{dw}{dz} = 0 \ \ if \ \ z = \infty$

and $\frac{dw}{dz} = \infty \ \ if \ \ z = -\frac{d}{c}$

Hence the points z = ∞ and $z = -\frac{d}{c}$ are called critical points where the conformed property does not hold goods

(vii) A mapping $w = f(z)$ is conformal at point of z_1 where f(z) is analytic and $f'(z_1) = 0$.

Examples

Example.1. To show that $w = \frac{4z + 5}{2z + 7}$ **is not a bilinear transformation.**

Solution. It is given that

$$w = \frac{4z + 5}{2z + 7} \qquad \ldots(1)$$

We know that the bilinear transformation

$$w = \frac{az + b}{cz + d} \qquad \ldots(2)$$

Comparing equations (1) and (2), we get

$a = 4$, $b = 5$, $c = 2$, $d = 7$.

For bilinear transformation, we have

$$ad - bc \neq 0$$

$\Rightarrow$ $(4)(7) - (5)(2) \neq 0$

or $28 - 10 \neq 0$

This shows that *w* is a bilinear transformation.

Example.2. To show that $w = \dfrac{7z+4}{14z+8}$ **is not a bilinear transformation.**

Solution. It is given that

$$w = \frac{7z+4}{14z+8} \qquad \ldots(1)$$

We know that the bilinear transformation

$$w = \frac{az+b}{cz+d} \qquad \ldots(2)$$

Comparing equations (1) and (2), we get

$$a = 7,\ b = 4,\ c = 14,\ d = 8.$$

For bilinear transformation, we have

$$ad - bc \neq 0$$

$\Rightarrow$ $(7)(8) - (4)(14) = 0$

This shows that *w* is not a bilinear transformation.

Example.3. Given are $w = \dfrac{\alpha z+\beta}{\gamma z+\delta},\ \alpha\delta - \beta\gamma \neq 0.$ **To determine the critical point of w.**

Solution. It is given that

$$w = \frac{az + \beta}{\gamma z + \delta},\ \alpha\delta - \beta\gamma \neq 0. \qquad \ldots(1)$$

Differentiating with respect to z both sides of equation (1), we get

$$\frac{dw}{dz} = \frac{(\gamma z + \delta)(\alpha) - (\alpha z - \beta)(\gamma)}{(\gamma z + \delta)^2}$$

$$= \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2} \qquad \ldots(2)$$

Using equation (2), we have

$$\frac{dw}{dz} = 0 \quad \text{if} \quad z = \infty.$$

Example.4. To show that *w = z* is a bilinear transformation.

Solution. It is given that

$$w = z = \frac{z}{1} \qquad \ldots(1)$$

We know that the bilinear transformation

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \ldots(2)$$

Comparing equations (1) and (2), we get

$a = 1$, $b = 0$, $c = 0$, $d = 1$.

For bilinear transformation, we have

$$ad - bc = (1)\,(1) - (0)\,(0)$$

$$= 1 \neq 0.$$

Hence $w = z$ is a bilinear transformation.

5.4 Invariant or Fixed points of the Bilinear Transformation

The bilinear transformation is

$$w = \frac{az + b}{cz + d}, \quad ad - bc \neq 0 \qquad \ldots(1)$$

or $$cz^2 + zd - az - b = 0$$

or $$cz^2 - (a - d)z - b = 0$$

or $$z = \frac{(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2c} \qquad \ldots(2)$$

The roots of equation (2) are known as the invariant or fixed points of the bilinear transformation of equation (1).

Note. (i) If there are distinct points z_1 and z_2 then the transformation may be put in the normal form

$$\frac{w - z_1}{w - z_2} = k\,\frac{z - z_1}{z - z_2}$$

and if there is only one invariant point z_1 then the transformation may be put in the normal form

$$\frac{1}{w - z_1} = \frac{1}{z - z_1} + k\,.$$

Check your Progress

1. What do you mean by Bilinear transformation?

2. Define the invariant.

5.5 Some Properties of Bilinear Transformation

(i) A bilinear transformation maps circles into circles.

The bilinear transformation is

$$w = \frac{az+b}{cz+d},\ ad-bc \neq 0$$

or
$$w = \frac{a}{c}\left(\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right)$$

$$= \frac{a}{c}\left\{1+\frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right\}$$

$$= \frac{a}{c}\left\{1+\frac{\frac{bc-ad}{ac}}{z+\frac{d}{c}}\right\}$$

$$= \frac{a}{c}+\frac{bc-ad}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

Letting $w_1 = z + \frac{d}{c}$,

$w_2 = \frac{1}{w_1}$,

$w_3 = \frac{bc-ad}{c^2}\, w^2$

Then $w = \frac{a}{c} + w_y$

Here each of these transformation is one or other of the standard transformation

$$w = z + \alpha,\ w = \frac{1}{z},\ w = \beta z$$

and under each of these a circle. Therefore the bilinear transformation maps circles into circles.

Note. (i) $w = z + \alpha$ represents a translation

(ii) $w = \frac{1}{z}$ represents an inversion

(iii) $w = \beta z$ represents a dilation.

(ii) A bilinear transformation preserves cross-ratio of four points.

The bilinear transformation is

$$w = \frac{az+b}{cz+d},\ ad-bc \neq 0 \qquad \ldots(1)$$

Suppose the points z_1, z_2, z_3, z_4 of the z-plane maps onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation (1), we have

$$w_1 = \frac{az_1+b}{cz_1+d},\ w_2 = \frac{az_2+b}{cz_2+d},$$
$$w_3 = \frac{az_3+b}{cz_3+d},\ w_4 = \frac{az_4+b}{cz_4+d}$$

Now $$w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(az_1+b)(cz_2+d)-(az_2+b)(cz_1+d)}{(cz_1+d)(cz_2+d)}$$

$$= \frac{(ad-bc)}{(cz_1+d)(cz_3+d)}(z_2-z_3) \qquad \ldots(2)$$

Similarly $$w_2 - w_3 = \frac{(ad-bc)}{(cz_2+d)(cz_3+d)}(z_2-z_3) \qquad \ldots(3)$$

$$w_3 - w_4 = \frac{(ad-bc)}{(cz_3+d)(cz_4+d)}(z_3 - z_4) \qquad \ldots(4)$$

$$w_4 - w_l = \frac{(ad-bc)}{(cz_4+d)(cz_1+d)}(z_4 - z_1) \qquad \ldots(5)$$

Using equation (2), (3), (4) and (5) we have

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

or $\quad (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$.

Hence the cross-ratio of four points is invariant under bilinear transformation.

Examples

Example.5. Determine the two bilinear transformations whose fixed points are 1 and 2.

Solution. The bilinear transformation is

$$w = \frac{az+b}{cz+d}, \text{ and } ad - bc \neq 0 \qquad \ldots(1)$$

For fixed point we put w = z, by equation (1)

$$z = \frac{az+b}{cz+d}$$

or $\quad cz^2 + dz - az - b = 0$

or $\quad cz^2 + z(d-a) - b = 0$

or $\quad z^2 - z\frac{(a-d)}{c} - \frac{b}{c} = 0 \qquad \ldots(2)$

It is given that the fixed points are 1 and 2, therefore we have

$$(z-1)(z-2)=0$$

or $$z^2 - 3z + 2 = 0 \qquad \ldots(3)$$

Equating the coefficient of z and constant in the equations (2) and (3), we get

$$\frac{a-d}{c} = 3 \quad \text{and} \quad -\frac{b}{c} = 2$$

$\Rightarrow$ $$b = -2c \quad \text{and} \quad d = a - 3c$$

Putting the values of *b* and *d* in the equation (1), we get

$$w = \frac{az - 2c}{cz + a - 3c}$$

Taking $a = 2$, $c = -1$ and $a = 1$, $c = -1$, we have

$$w = \frac{2z+2}{-z+5} \quad \text{and} \quad w = \frac{z+2}{-z+4}$$

Example.6. Find the bilinear transformation which transforms $z=\infty$, i, 0 into w=0, i, ∞ respectively.

Solution. Consider the given points $z_1=\infty$, $z_2=i$, $z_3=0$ and $z_4=z$ which maps into the points $w_1=0$, $w_2=i$, $w_3=\infty$, $w_4=w$.

The bilinear transformation of four cross-ratio is

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} \qquad \ldots(1)$$

or $$\frac{(o-i)(\infty-w)}{(i-\infty)(w-0)} = \frac{(\infty-i)(0-z)}{(i-0)(z-\infty)}$$

or $$\frac{-i(\infty-w)}{w(i-\infty)} = \frac{-z(\infty-i)}{i(z-\infty)}$$

or $$\frac{-1}{wz} = \frac{(i-\infty)(\infty-i)}{(z-\infty)(\infty-w)}$$

or $$\frac{-1}{wz}=1$$

or $$wz=-1.$$

$$\left[\because \quad \frac{(i-\infty)(\infty-i)}{(z-\infty)(\infty-w)}=\lim_{m\to\infty}\frac{(i-n)(n-i)}{(z-n)(n-w)} \right.$$
$$\left. =\lim_{n\to\infty}\frac{\left(\frac{i}{n}-1\right)\left(1-\frac{i}{n}\right)}{\left(\frac{z}{n}-1\right)\left(1-\frac{w}{n}\right)}=\frac{(-1).1}{(-1).1}=1\right]$$

Example.7. Find the transformation which maps the points -1, *i* 1, of the z-plane onto 1, *i*, -1, of the w-plane respectively,

Solution. Consider the given points z_1 = -1, z_2 = i, z_3=1 and z_4 =z which maps onto the points w_1=1, w_2= i, w_3= -1 and w_4 = w.

The bilinear transformer of four points cross-ratio is

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}=\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} \qquad \ldots(1)$$

or $$\frac{(1-i)(-1-w)}{(i+1)(w-1)}=\frac{(-1-i)(1-z)}{(i-1)(1+z)}$$

or $$\frac{(1-i)(1+w)}{(1+i)(w-1)}=\frac{(1+i)(1-z)}{(i-1)(z+1)}$$

or $$\frac{(w+1)}{(w-1)}=\frac{(1-z)}{(1+z)}.\frac{(1+i)^2}{(i-z)(1-i)}$$

or $$\frac{(w+1)}{(w-1)}=\frac{(1-z)}{(1+z)}$$

Applying componendo and dividend, we get

$$\frac{2w}{2}=\frac{2}{-2z}$$

or $$w = -\frac{1}{z}$$

which is the required bilinear transformation.

Example.8. Find the bilinear transformation which transforms *z* = 0, 1, 2 into *w* = *i*, 0, 1 respectively.

Solution. Consider the given points $z_1 = 0$, $z_2 = 1$, $z_3 = 2$ and $z_4 = z$ maps into the points $w_1 = i$, $w_2 = 0$, $w_3 = 1$, $w_4 = w$.

The bilinear transformation of four cross-ratio is

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \ldots(1)$$

or $$\frac{(i-0)(1-w)}{(0-1)(w-i)} = \frac{(0-1)(2-z)}{(1-2)(z-0)}$$

or $$\frac{i(w-1)}{(w-i)} = \frac{2-z}{z}$$

or $$\frac{2}{z} - 1 = \frac{i(w-1)}{(w-i)}$$

or $$\frac{2}{z} = \frac{i(w-1)}{(w-i)} + 1$$

or $$\frac{2}{z} = \frac{(w-i)+iw-i}{(w-i)}$$

or $$\frac{2}{z} = \frac{w(1+i)-2i}{w-i}$$

or $$2w - 2i = wz(1+i) - 2iz$$

or $$2w - wz(1+i)] = 2iz + 2i$$

or $$w[2 - z(1+i)] = 2i(1-z)$$

or $$w = \frac{2i(1-z)}{2-z(1+i)}$$

which is the required bilinear transformation.

Example.9: Find the bilinear transformation which maps the points z = 0, -1,∞into the points w = -1, -2-i, $i$respectively.

Solution. Consider the given points z_1= 0, z_2 = -1, z_3 = $\infty$and z_4 = z maps into the points w_1=-1, w_2 = -2-i, w_3 =$i$and w_4 = w.

The bilinear transformation of our points cross-ratio is

$$\frac{(w_1 - w_2)\,(w_3 - w_4)}{(w_2 - w_3)\,(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \qquad \ldots(1)$$

or $$\frac{(-1+2+i)(i-w)}{(-2-i-i)(w+1)} = \frac{(0+1)\,(\infty - z)}{(-1-\infty)(z-0)}$$

or $$-\frac{1}{2}\frac{(1+i)(i-w)}{(1+i)(w+1)} = -\frac{(\infty - z)}{(\infty+1)\,z}$$

or $$\frac{(i-w)}{(w+1)} = \frac{2}{z}\,.1$$

or $$(i- w)z = 2\,w + 1)$$

or $$iz - wz - 2w - 2 = 0$$

or $$w = \frac{iz-2}{2+z}$$

which is the required transformation.

Note: $$\left[\because \quad \frac{\infty - z}{\infty + 1} = \lim_{n\to\infty}\frac{n-z}{n+1} = \lim_{n\to\infty}\frac{\left(1-\frac{z}{n}\right)}{\left(1+\frac{1}{n}\right)} = \frac{\left(1-\frac{z}{\infty}\right)}{\left(1+\frac{1}{\infty}\right)} = \frac{1-0}{1+0} = 1\right]$$

Example.10. Determine the fixed points and the normal formed of the bilinear transformation $w = \dfrac{z}{z-2}$.

Sol. It is gven that the bilinear transformation

$$w = \frac{z}{z-2}$$

To find fixed point we take $w = z$ in equation (1), we have

$$z = \frac{z}{z-2}$$

or $$z^2 - 2z - z = 0$$

or $$z^2 - 3z = 0$$

or $$z(z-3) = 3$$

or $$z = 0, 3$$

For determine the normal form, we have

$$w - 3 = \frac{z}{z-2} - 3$$

or $$w - 3 = \frac{-2z+6}{z-2}$$

or $$\frac{w-0}{w-3} = \frac{\frac{z}{z-2}}{-\frac{2z+6}{z-2}}$$

or $$\frac{w-0}{w-3} = -\frac{1}{2}\frac{z}{z-3}$$

or $$\frac{w-0}{w-3} = -\frac{1}{2}\frac{z-0}{z-3}$$

or $$\frac{w-0}{w-3} = k\frac{z-0}{z-3}\frac{1}{2} \qquad \left[\text{Here } k = -\frac{1}{2} = \frac{1}{2}e^{i\pi}\right]$$

which is the required normal form.

5.6 Special Conformal Transformations

(i) $w = e^z$

We have $u + iv = e^{x+iv}$

or $u + iv = e^z (\cos y + i \sin y)$

Equating the real and imaginary parts both sides, we get

$$u = e^x \cos y$$

and $v = e^x \sin y$

Again we have $w = e^z$

$$\rho e^{i\varphi} = e^x e^{iv}$$

$$\rho = e^x \text{ and } \phi = y.$$

Hence $x = \log \rho$ and $\phi = y$.

Examples

Example.11. Find the image and draw a rough sketch of the mapping of the region $1 \le x \le 3$ under the mapping $w = e^z$.

Sol. Given the mapping $w = e^z$ …(1)

Let $w = \rho e^{i\varphi}$ …(2)

By equation (1), we have

$$w = e^z = e^{z + iv} \qquad \text{...(3)}$$

From equations (2), and (3), we have

$$\rho e^{i\varphi} = e^{x+iv}$$

$$or \;\; \rho^{i\varphi} = e^x \,.\, e^{iv}$$

Equating the real and imaginary parts both sides, we get

$$\rho = e^x$$

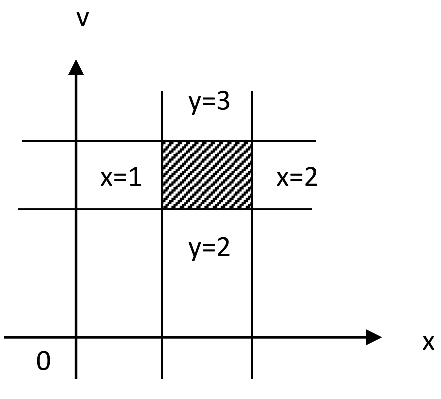
$$and \; e^{i\varphi} = e^{iv}$$

(a) $1 \leq x$, *then* $\rho = e^1$ *represents a circle of radius* $e^1 = 2$.

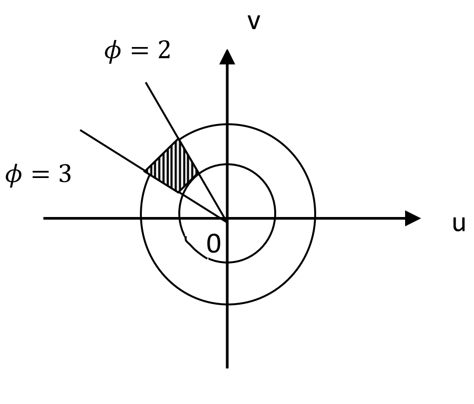
(b) $x = 2$, *then* $\rho = e^2$ *represents a circle of radius* $e^2 = 7.4$.

(c) $y = 2$, *then* $\varphi = 2$ *represents a radial line making an angle of 2 radian with the x-axis.*

(d) $y = 3$, *then* $\varphi = 3$ *represents a radial line making an angle of 3 radians with the x-axis.*



z-plane



w-plane

(ii) $\quad w = \cosh z \qquad \ldots(1)$

$\Rightarrow \qquad u + iv = \cosh(x + iy)$

$$\left| \begin{array}{l} \cos\, ix = \cos y \\ \sin\, ix = i \sinh y \end{array} \right.$$

$$= \cos i\,(x + iy)$$

$$= \cos ix \cos y + \sin ix \sin y$$

$$= \cosh x \cos y + i \sinh y \sin y$$

Equating the real and imaginary parts both sides, we get

$$u = \cosh x \cos y \qquad \ldots(2)$$

$$v = \sinh y \sin y \qquad \ldots(3)$$

Eliminating x form equations (2) and (3), we get

$$\Rightarrow \quad \frac{u^2}{\cos^2 y} - \frac{v^2}{\sin^2 y} = 1 \qquad \Big|\{\cosh^2 x - \sinh^2 x = 1\} \qquad \ldots(4)$$

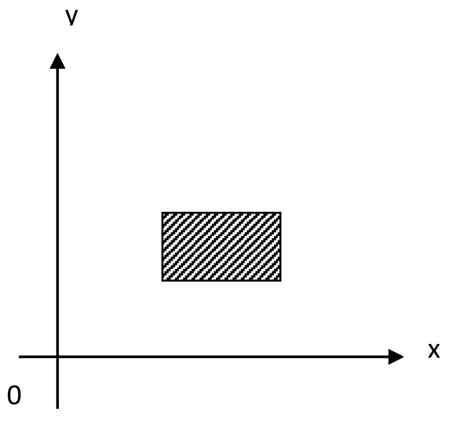
Now eliminating y from equations (2) and (3), we get

$$\Rightarrow \quad \frac{u^2}{\cosh^2 x} - \frac{v^2}{\sinh^2 x} = 1 \qquad \Big|\{\cos^2 y + \sin^2 y = 1\} \qquad \ldots(5)$$

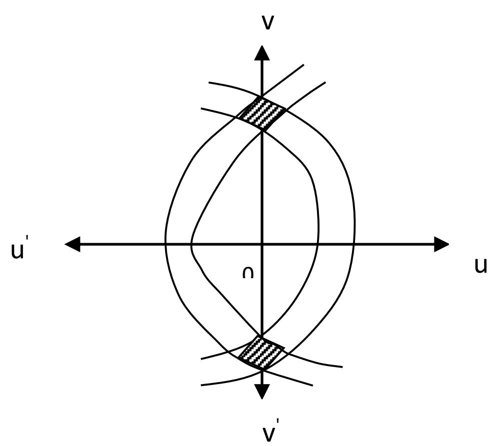
(i) Put $y = a$ in the equation (4), we get

$$\frac{u^2}{\cos^2 a} - \frac{v^2}{\sin^2 a} = 1 \quad \{ i.e, hyperbola\} \qquad \ldots(6)$$

Which shows that the above lines parallel to x-axis in the z- plane map into hyperbola in the w-plane.



z-plane



w-plane

(ii) Put $x = b$ in the equation (5), we get

$$\frac{u^2}{\cosh^2 a} - \frac{v^2}{\sinh^2 a} = 1 \; \{ i.e, ellipse \} \qquad ...(7)$$

which shows that the above lines parallel to y-axis in the z-plane map lines into ellipse in the w-plane.

The rectangular $a \leq x \leq b,\ c \leq x \leq d$ in the z-plane transforms into the shaded portion in the w- plane.

5.7 Joukowski's Transformation $\left(w = z + \frac{1}{z} \right)$

We have $$w = z + \frac{1}{z} \qquad \ldots(1)$$

Differentiating equation (1) with respect to z, we get

$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

$$= \frac{z^2 - 1}{z^2} \qquad \ldots(2)$$

which is non zero except $z = \pm 1$, so transformation is not conformal at $z = \pm 1$.

From equation (1), we have

$$w = z + \frac{1}{z}$$

$$\Rightarrow \qquad u + iv = w = z + \frac{1}{z} \qquad \{z = r\,(\cos\theta + i\sin\theta\}$$

$$u + iv = r(\cos\theta + i\sin\theta) + \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

Equating the real and imaginary parts both sides, we get

$$u = \left(r + \frac{1}{r}\right)\cos\theta \qquad \ldots(2)$$

and $$v = \left(r - \frac{1}{r}\right)\sin\theta \qquad \ldots(3)$$

Eliminating θ from equations (2) and (3), we get

$$\frac{u^2}{\left(r + \frac{1}{r}\right)^2} + \frac{v^2}{\left(r - \frac{1}{r}\right)^2} = 1 \qquad \left\{\cos^2\theta + \sin^2\theta = 1\right\} \qquad \ldots(4)$$

Now eliminating *r* from equations (2) and (3), we get

$$\frac{u^2}{4\cos^2\theta} - \frac{v^2}{4\sin^2\theta} = 1 \qquad \left\{\therefore \left(r + \frac{1}{r}\right)^2 - \left(r - \frac{1}{r}\right)^2 = 4\right\} \qquad \ldots(5)$$

5.8 Schwarz-Christoffel Transformation

Schwarz-Christoffel Transformation determines the specific function which conformally maps (bounded or unbounded) polygons to upper half plane and consequently to any region into which the half plane can be transformed such as unit dask.

Formula of this transformation is

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1}(z - x_2)^{\frac{\alpha_2}{\pi} - 1} \ldots\ldots (z - x_n)^{\frac{\alpha_2}{\pi} - 1}$$

or $$w = A\int (z - x_1)^{\frac{\alpha_1}{\pi} - 1}(z - x_2)^{\frac{\alpha_2}{\pi} - 1} \ldots\ldots (z - x_n)^{\frac{\alpha_2}{\pi} - 1} \quad dz + B$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$, are the interior angles of the polygon of the *w*- plane with verticies $A_1, A_2 \ldots\ldots, A_n$ are mapped into the points $x_1, x_2, \ldots\ldots, x_n$ on the real axis of the *z*-plane and A and B are complex constants.

Examples

Example.12. Find the bilinear transformation which maps the points *z = 1, i, -1* onto the points *w= i,0, -i,* Also find the invariant points of this transformation and image of $|z| < 1$.

Solution. Consider the given points $z_1 = 1, z_2 = i, z_3 = -1$ and $z_4 = z$ maps onto the points $w_1 = i, w_2 = 0, w_3 = -i$ and $w_4 = w$.

The bilinear transformation of four points cross-ratio is

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \quad \ldots(1)$$

or
$$\frac{(i-0)(-i-w)}{(0+i)(w-1)} = \frac{(1-i)(-1-z)}{(i+1)(z-1)}$$

or
$$\frac{(w+i)}{(w-1)} = \frac{(1-i)(z+1)}{(1+i)(z-1)}$$

Applying componendo and dividendo, we get

$$\frac{2w}{2i} = \frac{(1-z)(z+1)+(1+i)(z-1)}{(1-i)(z+1)-(1+i)(z-1)}$$

or
$$\frac{w}{i} = \frac{(z-zi+1-i)+(z+zi-1-i)}{(z-zi+1-1)-(z+zi-1-i)}$$

or
$$w = \frac{2i(z-1)}{2(1-zi)} \quad \ldots(3)$$

or $$w=\frac{1+iz}{1-iz}$$

which is the required bilinear transformation

To find the invariant points of the given transformation, we put $w = z$ in equation (3),

$$w=\frac{1+iz}{1-iz}$$

or $$z - iz^2 = 1 + iz$$

or $$z - iz^2 - iz - 1 = 0$$

or $$z = \frac{(1-i)\pm\sqrt{(1-i)^2-4i}}{2i}$$

$$z = \frac{(1-i)\pm\sqrt{(-1+1-2i)-4i}}{2i}$$

$$=\frac{1}{2}[(1+i)]\pm\sqrt{6i} \qquad \ldots(4)$$

which is the required invariant points. Now equation (3) may be written as

$$w - iwz = 1 + iz$$

or $$z(-iw-i) = 1-w$$

or $$z = \frac{1-w}{-i(w+1)}$$

or $$z = 1,\ \frac{1-w}{1+w}$$

$\therefore$ $$1 > |z| = \left|\frac{(1-w)}{(1+w)}\right|$$

or $$|1+w| > |i|\ |1-w| = 0$$

or $\quad (1 + u^2) + v^2 > (1 - u)^2 + v^2$

or $\quad 2u > -2u$

or $\quad 4u > 0$

or $\quad u > 0$

or $\quad Re(w) > 0$

Hence the interior of the unit circle in the z-plane is mapped onto the entire half of the w-plane to the right of the imaginary axis.

Example.13. Show that the transformation $w = z + \frac{a^2 - b^2}{4^2}$, a and b are real transforms the circle of radius $\frac{1}{2}(a+b)$, with center at (0, 0) in z-plane into an ellipse having a and b as semi-axis in w-plane.

Solution. It is given that the transformation

$$w = z + \frac{a^2 - b^2}{4z} \ldots (1)$$

Also given the circle specification is

$$|z| = \frac{1}{2}(a+b)$$

$$\Rightarrow z = \frac{1}{2}(a+b)e^{i\theta} \ldots (2)$$

Using equations (1) and (2), we have

$$w = \frac{1}{2}\left(a+b\right)e^{i\theta} + \frac{a^2-b^2}{4} - \frac{1}{\frac{1}{2}\left(a+b\right)e^{i\theta}}$$

$$= \frac{1}{2}\left(a+b\right)e^{i\theta} + \frac{\left(a-b\right)}{2}\ e^{i\theta}$$

$$= a\left\{\frac{e^{i\theta}\ +e^{i\theta}}{2}\right\} + b\left\{\frac{e^{i\theta}\ +e^{i\theta}}{2}\right\}$$

$$= a\ \cos\theta + i\ b\ \sin\ \theta$$

$$\Rightarrow \qquad u+iv \quad = a\ \cos\theta + i b \sin\theta$$

$$\therefore \qquad u \quad = a\ \cos\theta, \quad v = b\sin\theta$$

$$\Rightarrow \qquad \frac{u^2}{a^2} + \frac{v^2}{b^2} = 1$$

Which is an ellipse having *a, b* as semi-axis in w-plane

Example.14. Determine the invariant points of the transformation $w = \frac{z-1}{z+1}$**.**

Solution. It is given that the transformation

$$w = \frac{z-1}{z+1} \qquad \text{....(1)}$$

To find the invariant points of the transformation, we put w = z in equation (1), then

$$z = \frac{z-1}{z+1}$$

or $\qquad z^2 + z - z + 1 = 0$

or $\qquad z^2 + 1 = 0$

or $\qquad z = \pm\ i$

which are the required invariant points.

Example.15. Show that under the transformation $w=\dfrac{z-1}{z+1}$, real axis in the z-plane is mapped into the circle |w| = 1. Which portion of the z-plane corresponds to the interior of the circle?

Solution. It is given that

$$w=\frac{z-1}{z+1}$$

We have $$|w| = \left|\frac{z-1}{z+1}\right|$$

$$= \left|\frac{x+iy-1}{x+iy+1}\right|$$

$$= \left|\frac{x+1(y-1)}{x+1(y+1)}\right|$$

$$= \frac{\left[x^2+(y-1)^2\right]^{1/2}}{\left[x^2+(y+1)^2\right]^{1/2}}$$

For real axis in z-plane (i.e., y=0) transforms into

$$|w| = \frac{\left[x^2+1\right]^{1/2}}{\left[x^2+1\right]^{1/2}} = 1$$

Therefore the real axis in the z-plane is mapped into the circle |w| =1.

For interior of the circle i.e., {|1| < 1|}

or $$\frac{\left[x^2+(y-1)^2\right]^{1/2}}{\left[x^2+(y+1)^2\right]^{1/2}} = 1$$

or $$x^2 + (y-1)^2 < x^2 + (y+1)^2$$

or $$-4y < 0$$

or $y > 0$

Hence the upper half of the z-plane corresponds to the interior of the circle |w| < 1.

Example.16. Prove that the bilinear transformation $w = \frac{iz + 2}{4z+1}$ **, transformation the real axis in z-plane into a circle in the w-plane. Find the centre and radius of the circle and the point in the z-plane which corresponds to the centre of the circle**

Solution. The given bilinear transformation is

$$w = \frac{iz + 2}{4z+1} \quad \ldots(1)$$

From equation (1), we have

$$4wz + wi = iz + 2$$

or $$z(4w - i) = 2 - wi$$

or $$z = \frac{2 - wi}{4w - i} \quad \ldots(2)$$

The real axis in z-plane (i.e., y = 0), then we have

$$y = 0$$

or $$2iy = 0$$

or $$z - \bar{z} = 0$$

Using equation (2), we have

$$\frac{2 - wi}{4w - i} - \frac{2 + \overline{w}i}{4\overline{w} + i} = 0$$

or $$(2\text{-}wi)\left(4\overline{w} + i\right) - \left(2w + \overline{w}i\right)\left(4w - i\right) = 0$$

or $$8\overline{w} - 4iw\overline{w} + 2i + w - 8w - 4iw\overline{w} + 2i - \overline{w} = 0$$

$$or \qquad -8iw\overline{w} + 7(\overline{w} - w) + 4i = 0$$

$$or \qquad 8i(u^2 + v^2) - 14iv + 4i = 0 \qquad \{\because w = u + iv, \overline{w} = u - iv\}$$

$$or \qquad u^2 + v^2 + \frac{7v}{4} - \frac{1}{2} = 0$$

which is the equation of a circle in w-plane with centre at $\left(0, -\frac{7}{8}\right)$.

This implies $\quad w = 0 - \dfrac{7i}{8} = -\dfrac{7i}{8}$

and radius $= \sqrt{(0)^2 + \left(-\frac{7}{8}\right)^2 - \left(-\frac{1}{2}\right)}$

$$= \sqrt{\frac{49}{64} + \frac{1}{2}}$$

$$= \sqrt{\frac{81}{64}} = \frac{9}{8}$$

Put $w = \frac{7i}{8}$ in the equation (2), we get

$$z = \frac{2 - i\left(\frac{-7i}{8}\right)}{4\left(\frac{-7i}{8}\right) - i}$$

$$= \frac{2 - \frac{7}{8}}{-i\left(\frac{7}{2} + 1\right)}$$

$$= \frac{\left(\frac{9}{8}\right)}{-i\left(\frac{9}{2}\right)} = \frac{1}{4}i.$$

Hence the real axis in z-plane is transformed onto a circle of w-plane whose centre is at $w = -\frac{7i}{8}$, radius $= \frac{9}{8}$, and the centre of the circle corresponds to the point $z = \frac{i}{4}$ in the z-plane.

Example.17. Show that the transformation $w=\dfrac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ into straight line 4u + 3 = 0.

Solution. The given the transformation

$$w=\frac{2z+3}{z-4} \qquad \ldots(1)$$

From equation (1), we have

$$wz-4w=2z+3$$

or $$z(w-2)=3+4w$$

or $$z=\frac{4w+3}{w-2} \qquad \ldots(2)$$

so that $$\bar{z}=\frac{4\bar{w}+3}{\bar{w}-2} \qquad \ldots(3)$$

Given equation of the circle in z-plane

$$x^2+y^2-4x=0$$

or $$z\bar{z}-2(z+\bar{z})=0$$

or $$\left(\frac{4w+3}{w-2}\right)\left(\frac{4\bar{w}+3}{\bar{w}-2}\right)-\left(\frac{4w+3}{w-2}+\frac{4\bar{w}+3}{\bar{w}-2}\right)=0$$

or $$\frac{(4w+3)(4\bar{w}+3)}{(w-2)(\bar{w}-2)}-2\frac{\{(4w+3)(\bar{w}-2)(4\bar{w}+3)(w-2)\}}{(w-2)(\bar{w}-2)}=0$$

or $$16w\bar{w}+12(w+\bar{w})+9-2[4w\bar{w}+3w\bar{w}-8w-6+4w\bar{w}+3w-8\bar{w}-6]=0$$

or $$22(w+\bar{w})+33=0 \text{ or } 2(w+\bar{w})+3=0$$

or 4u + 3 = 0, which is the equation of a straight line in w-plane.

5.9 Summary

The transformation of the form $w = \dfrac{az+b}{cz+d}$ in which a, b, c and d are complex constants and $ad - bc \neq 0$ is called a bilinear (or Mobius) transformation. The constant $ad - bc$ is known as the determinant of the transformation.

The transformation $w = \dfrac{az+b}{cz+d}$ is said to be normalized if $ad - bc = 1$.The transformation $w = \dfrac{az+b}{cz+d}$ is said to be conformed if $ad - bc \neq 0$ then $\dfrac{dw}{dz} \neq 0$.

A bilinear transformation maps circles into circles.A bilinear transformation preserves cross-ratio of four points. The bilinear transformation of four cross-ratio is

$$\frac{(w_1 - w_2)(w_3 - w_4)}{(w_2 - w_3)(w_4 - w_1)} = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}.$$

5.10 Terminal Questions

Q.1. What do you mean by Bilinear Transform.

Q.2. To show that bilinear transformation maps circles into circles.

Q.3. To show that bilinear transformation preserves cross-ratio of four points.

Q.4.Determine the following transformation is bilinear transformation or not:

(i) $w = \frac{(1+i)z+1}{2z+(1+i)}$

(ii) $w = \frac{(2+3i)z+i}{-13iz+(2-3i)}$

(iii) $w = 2z$

(iv) $w = \frac{z+i(1-i)}{iz+(2-i)}$

Q.5. Find the bilinear transformation which maps the points $z = -2,\ 0,\ 2$ into the points $w = 0,\ i,\ -i$ respectively.

Q.6. Find the fixed points and the normal form of the following bilinear transformation:

(i) $w = \frac{z}{2-z}$ (ii) $w = \frac{z-1}{z+1}$

Q.7. In the transformation z= $\frac{i-w}{i+w}$ show that the positive half of the w-plane given by $v \geq 0$ corresponds to the circle $|z| \leq 1$ in the z-plane.

Answer

4.(i) No

ii) No

(iii) Yes

(iv) Yes

5. $w = i\frac{(2+z)}{(2-3z)}$.

6. (i) $0,\ 1;\ \frac{w-0}{w-1} = \frac{1}{2}\frac{z-0}{z-1}$.

(ii) $i,\ -i;\ \frac{w-i}{w+i} = -i\frac{z-i}{z+i}$.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

Master of Science
PGMM -104N
Complex Analysis

U. P. RajarshiTandon
Open University

Block 3 Complex Integration and Series

Block-3

Complex Integration and Series

Complex analysis has significant utility in applied mathematics. In 1813, the French mathematician S.D. Poisson investigated integration in the complex plane. Complex integration is extremely important in the study of functions of a complex variable. In 1831, Cauchy discovered his famous formula, the Cauchy integral formula. Cauchy began developing his theorem, the Cauchy fundamental theorem, in 1825. These contributions are foundational in the field of complex analysis and have had a profound impact on mathematics.Complex integration plays a significant role in complex geometry, where it is used to study complex manifolds and algebraic varieties. It contributes to understanding the topology and geometry of complex spaces. In number theory, complex integration is applied in studying complex multiplication of elliptic curves and the distribution of prime numbers. Hence the complex integration's versatility and power make it a valuable tool in mathematics, science, and engineering, with applications in fluid dynamics, control theory, complex geometry, and number theory.

In complex analysis, certain integrals are path-independent, meaning the integral's value depends only on the endpoints of the path. This property is crucial for many applications. In complex analysis, series play a crucial role in understanding the behavior of complex functions. Series in complex analysis are used for various purposes, including representing functions, solving differential equations, and studying the properties of complex functions. They provide powerful tools for analyzing complex functions and understanding their behavior.

In the sixth unit, we delve into the foundational concepts of complex integration and their properties. The seventh unit covers Cauchy's fundamental theorem and the Cauchy-Goursat theorem, along with their applications. Unit eight explores Cauchy's integral formula, an extension of the formula, the formula for derivatives, and Cauchy's inequality. Unit nine provides detailed discussions on Morera's theorem, Liouville's theorem and its applications, the fundamental theorem of Algebra, and the Maximum modulus principle. Lastly, in unit ten, we will examine Entire functions, Taylor's theorem and its applications, as well as Laurent's theorem and its applications.

UNIT-6: Introduction to Complex Integration

Structure

6.1 Introduction

Complex integration involves integrating functions that output complex numbers along curves in the complex plane. It's an extension of real-valued integration to complex functions. The key concept is to parameterize a curve in the complex plane and then integrate the complex function along this curve with respect to this parameter. The most common type of complex integration is the line integral, where you integrate a complex-valued function $f(z)$ along a curve C in the complex plane. There are several important results in complex integration, such as Cauchy's integral theorem and Cauchy's integral formula, which are fundamental in complex analysis. These results establish connections between the values of a complex function in a region and its values on the boundary of that region, with applications in mathematics and physics.

Complex integration is essential in complex analysis, a branch of mathematics with applications in physics, engineering, and signal processing.

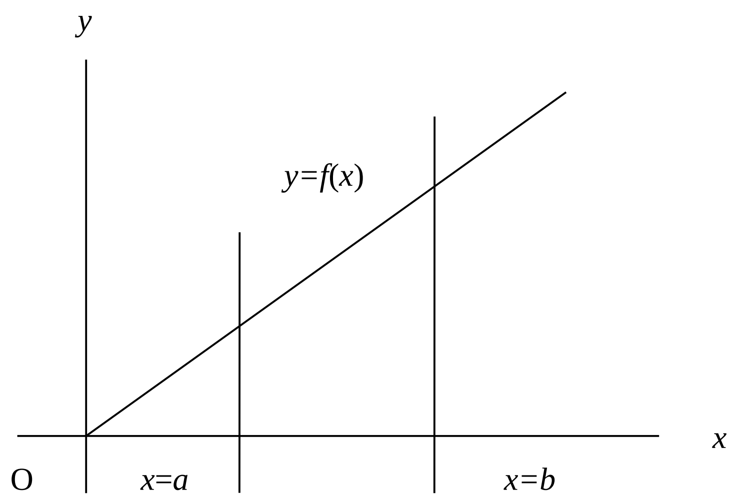
6.2 Objectives

After reading this unit the learner should be able to understand about:

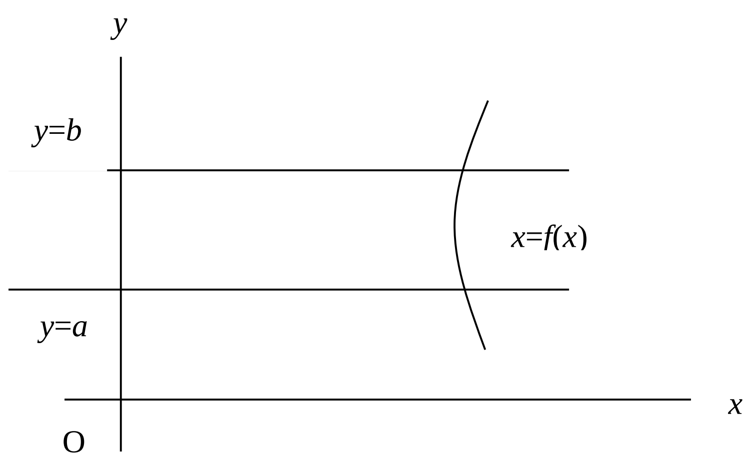
- the Complex Integration and their applications
- the arc and multiple point on an arc and Jordan arc
- the Contour and simple closed contour
- the Rectifiable curve and its length and integration
- the Connected region and non-connected region
- the Simple and multiple connected region
- some properties of complex integration

6.3 Complex Integration

(i) We know that for the definite integral $\int_a^b f(x)dx$. The path of integration is always along $x = a$ to $x = b$ or integral a to b.

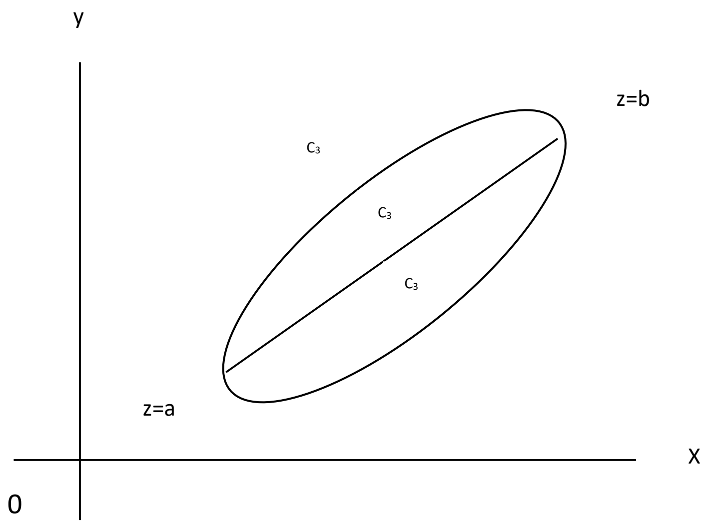


(ii) We know that for the definite integral $\int_a^b f(y)\, dy$. The path of integration is always along y-axis from $y = a$ to $y = b$.



(iii) But in the integral of complex function $\int_a^b f(z)\, dz$, the value of integral depends upon the curve along with z which assumes values from a to b. But if the curve like C_1, C_2 and C_3 along with z takes values from

a to b are regular curve then along whatever curve z vary along a to b or in the other words whatever may be the path in case of regular curve, integral will have to same values.



6.4 Arc and Multiple point of an arc

An are C is a set of point $z = (x, y)$ in the complex plane such that

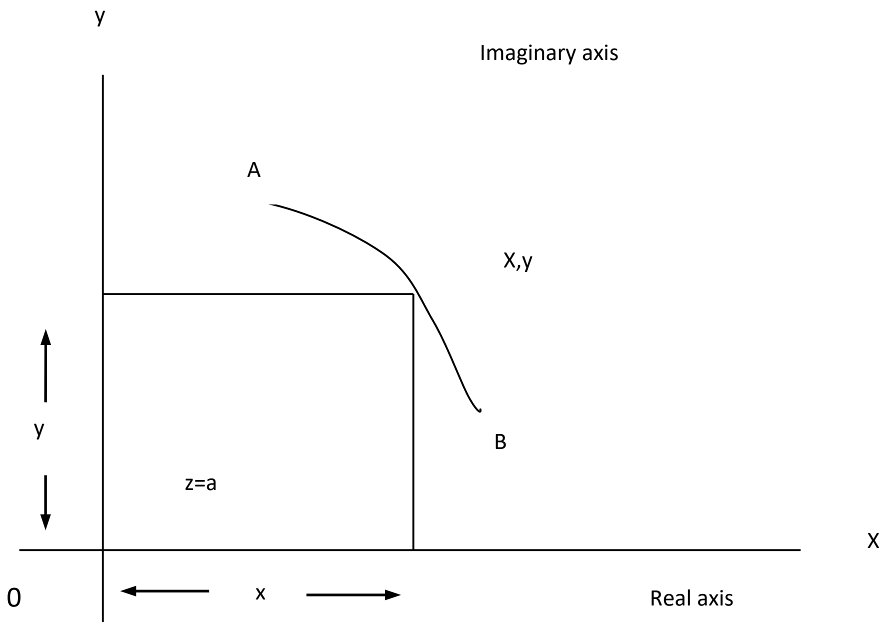
$$x = x(t), \qquad y = y(t), a \leq t \leq b$$

where $x(t)$ and $y(t)$ are continuous functions of real parameter 't' thus an are is represented by

$z(t) = x(t) + i\, y(t), a \leq t \leq b$, for the curve

$x^2 + y^2 = a^2$ i.e., parametric equation, where

$$x = a \cos t, \quad y = a \sin t; \quad 0 \leq t \leq 2\pi.$$



Multiple point of an Arc

Consider $z = \phi(t) + i\psi(t) = x + iy$

then we have $x = \phi(t)$ and y=$\psi(t)$

if these equations are satisfied by more than one values of t then the point z on (x, y) is called a multiple point of arc

$z = sin^2t + i\, cos^2\, t$, where $x = sin^2\, t,\ y = cos^2\, t\ \ 0 \le t \le 2\pi$

Put $z = i$ when t= 0

$z = i$ when $t = \pi$

$z = i$ when $t = 2\pi$

$z = i$ is satisfied more than one value of t.

So $z = i$ is multiple point. $0 \le t \le 2\pi$.

Note. A continuous curve have no multiple point.

6.5 Jordan Arc

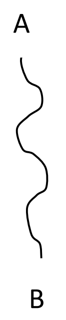
An are $z(t) = x(t) + i\,y(t)$, $a \le t \le b$ is said to be (Regular) smooth arc if its $Z'(t)$

(i) exists (ii) is non zero and (iii)continuous in $[a, b]$.

Note: A continuous are without multiple point (or arc) is called a Jordan curve or arc.

6.6 Contour and Simple Closed Contour

Contour is a Jordan curve consisting of continuous chain of finite number of regular arcs if A be the starting point of first arc and B is the last point of last arc then integrating along such a curve is written as $\int_A^B f(z)dz$.

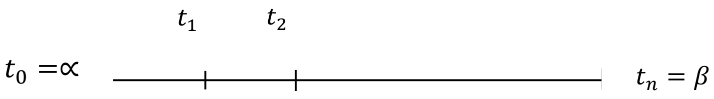


A contour C is said to be simple closed contour if its ending points coincide.For example,circle, boundary of a triangle or rectangle.

Note:A contour is a continuous chain of a finite number of regular arcs. A contour is an arc consisting of a finite number of smooth arcs joining end to end.

6.7 Rectifiable Curve and Its Length

Let the equation of the arc of the curve be $x = \phi(t)$ and $y = \psi(t)$, where $\alpha \le t \le \beta$. Subdivide the interval (α, β) by points $t_0, t_1, t_2 \ldots. t_n$ by the points $P_0, P_1 \ldots.\ P_n$. The length of two polynomials line $P_0, P_1 \ldots\ P_{n-1}, P_n$ is the sum of length of the line $P_0, P_1, P_2, \ldots\ P_{n-1}, P_n$.



Let the points corresponding to $P_0, P_1, \ldots P_n$ be $z_0, z_1, z_2 \ldots z_n$ having coordinates $(x_0, y_0)(x_1, y_1) \ldots (x_n, y_n)$ then the length

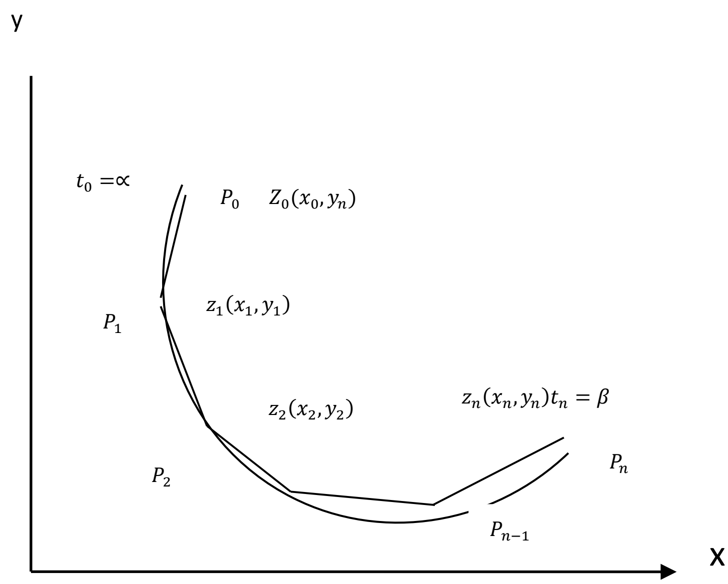
$$P_0P_1 \ldots \ldots \ldots P_n = P_0P_1 + P_1P_2 + \ldots + P_{n-1}P_n$$

$$= \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$+ \ldots \ldots \ldots \ldots + \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$$

$$= \sum_{r=1}^{n} [(x_r - x_{r-1})^2 + (y_r - y_{r-1})^2]^{1/2}$$

The value of the sum depends upon the mode of subdivision and is called length of the inscribed polynomial. If the arc is such that this sum (the length of all inscribed polygons) have a finite upon bound i for all modes of subdivision then the curve is said to be Rectifiable and i is the length.



Check your Progress

1. What do you mean by complex integration?

2. Define the Jorden arc.

3. Explain the contour and simple closed contour.

6.8 Definition of Integration

Let a function $f(z)$ of a complex variable z be continuous in a domain D. Let $z = a$ and $z = b$ be two points in this domain.

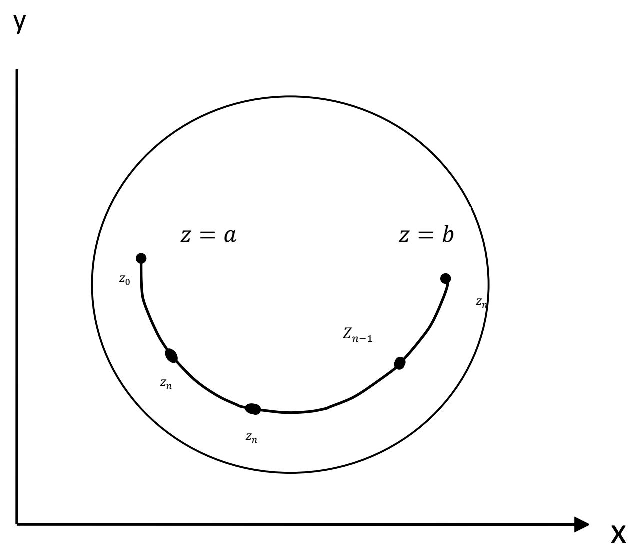
Let C be a curve from $z = a$ to $z = b$ and C is completely in the domain D so that $f(z)$ is continuous on C.

Let $a = z_0$ and $b = z_n$ we take a partition

$f = \{a = z_0, z_1, z_2, \ldots\ z_r, \ldots\ z_{n-1}, z_n = b\}$ of this curve C

from the sum $\sum_{r=1}^{n} (z_r - z_{r-1})\, f(\xi_r)$ where $z_{r-1} \le \xi_r \le z_r$

i.e., $(z_1 - z_0)\, f(\xi_1) + (z_2 - z_1)\, f(\xi_2) + \ \ldots$



The limit of this sum, if it exists uniquely for any path joining $z = a$ to $z = b$ lying in D whatever may be the way of partition is called the integral of $f(z)$ over C from a to b and is written as

$$\int_c f(z)\, dz \text{ or } \int_a^b f(z)\, dz$$

This $$\int_a^b f(z)dz = \lim_{n\to\infty} \sum_{r=0}^{n} (z_r - z_{r-1})\ f(\xi_r).$$

6.9 Connected and Non-Connected Region

A region D is said to be a connected region if any two points of region D can be connected by a curve which lies entirely with in the region. (Fig. 1)

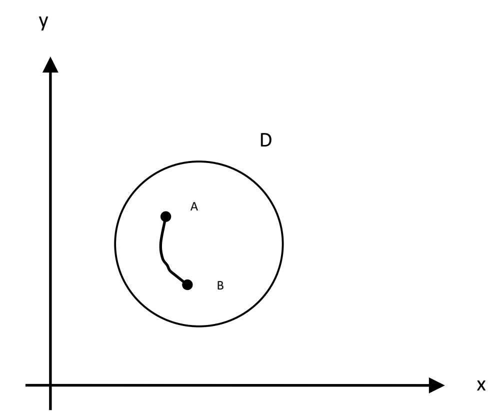


Fig. 1

The region in Fig. 2 is not connected because the curve joining z_1, z_2 lies outside the region.

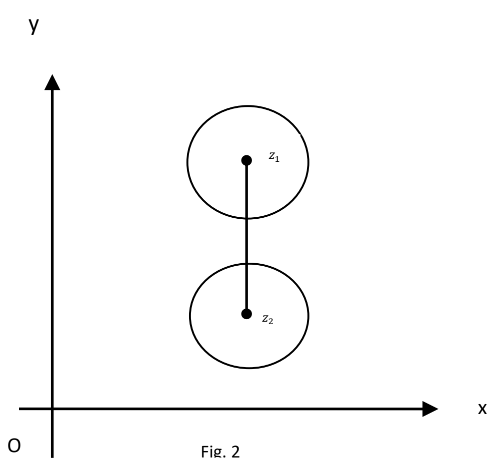


Fig. 2

6.10 Simple and MultipleConnected Region

A connected region *D* is said to be simply connected if all the interior points of closed curve *C* draw in the region *D* are points of the region *D.*

Fig. 4 is not simple connected region.

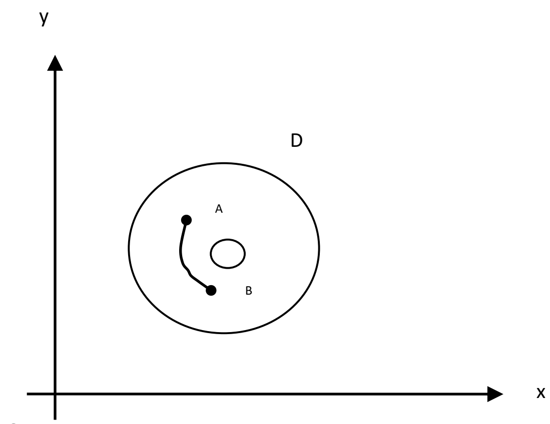


Fig. 3

A region in which every closed curve can be shrunk to a point without passing out of the region is called a simply connected region otherwise it is said to be multiconnected region.

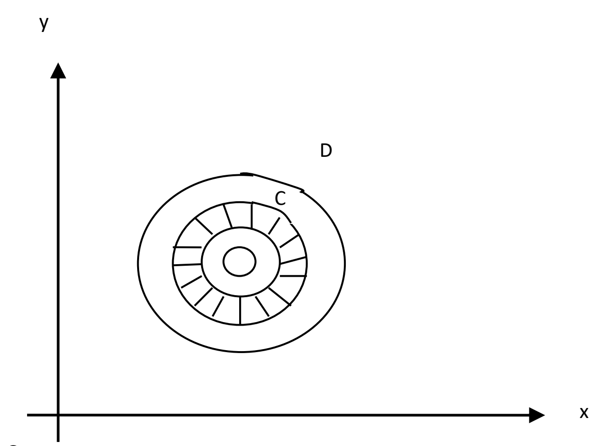
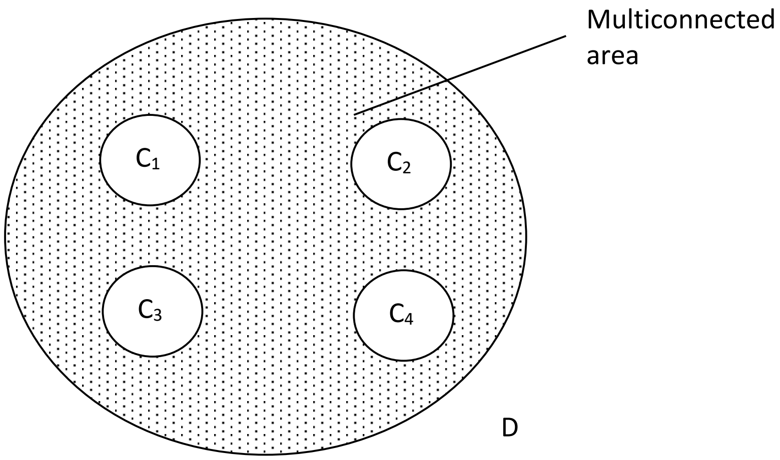


Fig. 4

If all the points of area bounded by two or more closed curve drawn in the region D area of points of region D then region is said to be multiconnected region.



If $C_1, C_2, \dots \dots . C_n$ are drawn in region D and if all the points of area lying in the closed curve $C_1, C_2, C_3, C_4, C_5, C_6$ .... and so on i.e. the area which is interion to C and exterial to the other curves $C_1, C_2, \dots$ .... and so on the region area of the point region D then region D is said to be multiconnected region.

6.11 Some properties of Complex Integration

$$(i) \int_{-C} f(z)dz = -\int_{C} f(z)dz$$

$$(ii) \left|\int_{-C} f(z)dz\right| \leq \int_{C} |f(z)||dz|$$

$$(iii) \int_{c_1+c_2} f(z)dz = \int_{c_1} f(z)dz + \int_{c_2} f(z)dz$$

$$(iv) \int_{c} f(z)dz = \int_{c} f(z)\frac{dz}{dt}dt = \int_{c} f(z)z'(t)dt$$

$$(v) \int_{c} k\, f(z)dz = k\int_{c} f(z)dz$$

$(vi) \int_c \{f_1(z) + f_2(z)\}dz = \int_c f_1(z)dz + \int_c f_2(z)dz$

(vii)$|e^{i\theta}| = 1$

$(viii) \int_c f(z)dz = 0$ and $\int_c z\, f(z)dz = 0$

$\int_c z^2\, f(z)dz = 0$ where C is closed contour

(ix) To prove property (ii) $\left|\int_c f(z)dz\right| \leq \int_L |f(z)||dz|$

Proof. We have

$$R\left[C \int_L f(z)dz\right] = R\left[\int_L C\, f(z)dz\right]$$

Since C is arbitrary, we may set $C =^{-i\theta}$, where θ is real but arbitrary

$$\text{The } R\left[e^{-i\theta} \int_L f(z)dz\right] = R\left[\int_L e^{-i\theta} f(z)dz\right] = \int R[e^{-i\theta} f(z)dz]$$

$$\leq \int_L |e^{-i\theta} f(z)dz|$$

[since real part of complex number cannot exceed its absolute value]

$$= \int_L |f(z)||dz|$$

Again since θ is arbitrary, we may set $\theta = arg \int_L f(z)dz$

Then $\int_L f(z)dz = \left|\int_L f(z)dz\right| e^{-i\theta}$

Left hand side of equation (1) $R\left[e^{-i\theta}\left|\int_L f(z)dz\right|e^{i\theta}\right]=\left|\int_L f(z)dz\right|$

Hence $\left|\int_L f(z)dz\right| \leq \int_L |f(z)||dz|$.

Examples

Exampe 1. $\boldsymbol{Evaluate\ \int_C \frac{1}{z}\,dz}$**, where C is circle** $\boldsymbol{z=z^{i\theta}, 0\leq\theta\leq\pi}$

Sol, We have $I=\int_C \frac{1}{z}\,dz$ …(1)

$$z=e^{i\theta}\Rightarrow dz=e^{i\theta}d\theta$$

$$=iz\,d\theta$$

$$\Rightarrow\frac{dz}{z}=i\,d\theta$$

Using equation (1), we have

$$I=\int_0^{\pi} i\,d\theta = i[\theta]_0^{\pi}$$

$$=i[\pi-0]$$

$$=\pi i.$$

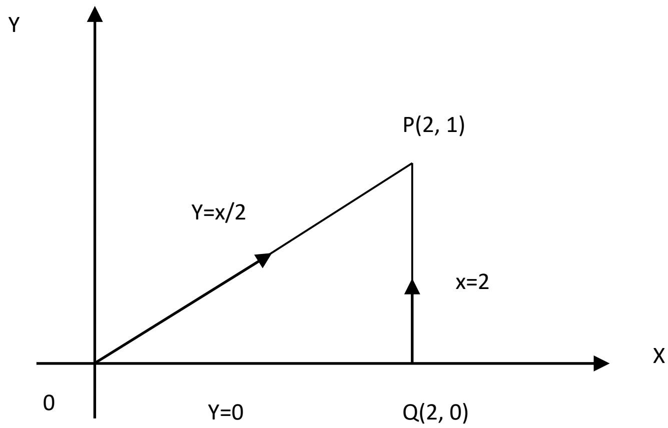
Example. 2. $\boldsymbol{Evaluate\ \int_0^{2+i}(\bar{z})^2\,dz}$**, along**

(i) the line $\boldsymbol{y=x/2}$

(ii) the real axis from x=0 to 2 and then vertically to 2+i.

Sol. We have

$$I=\int_0^{2+i}(\bar{z})^2\,dz \qquad \ldots(1)$$



(i) Along the line OP,

$$y = \frac{x}{2} \Rightarrow x = 2y,$$

$$z = x + iy$$

$$z = (2+i)y$$

$$\bar{z} = (2-i)y$$

$$dz = (2+i)dy$$

By equation (1), we have

$$I = \int_0^{2+i} (\bar{z})^2\, dz = \int_0^1 (2-i)^2 y^2 . (2+i) dy$$

$$= 5(2-i)\left[\frac{y^3}{3}\right]_0^1 = \frac{5}{3}(2-i).$$

(ii) Now along OQ, $z = x,\ \ \bar{z} = x\ \Rightarrow dz = dx$

and along QP, $z = 2 + iy,\ \ \bar{z} = 2 - iy\ \Rightarrow dz = i\, dy$

By equation (1), we have

$$\therefore \qquad \mathrm{I}=\int_{OQ} (\bar{z})^2\, dz + \int_{QP} (\bar{z})^2\, dz = \int_0^2 x^2\, dx + \int_0^1 (2-iy)^2\, i\, dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + \int_0^1 [4y + (4-y^2)i]dy$$

$$= \frac{8}{3} + 4.\frac{1}{2} + \left[4(1) - \frac{1}{3}\right] i = \frac{1}{3}(14+11i).$$

Example 3. Integrate z^2 along the straight line *OM* and also the path *OLM* consisting of two straight line segments *OL* and *OM* where *O* is the origin, L is the point Z =3 and M the point $z = 3 + i$. Hence show that the integral of z^2along the closed path *OLMO* is zero.

Sol. We have $z^2 = (x+iy)^2$

$$= x^2 - y^2 + 2ixy$$

On the line OM, $x = 3y\ so$

$$dx = 3dy$$

Also $y - axis$ from 0 to 1 as z moves to OM from 0 to M

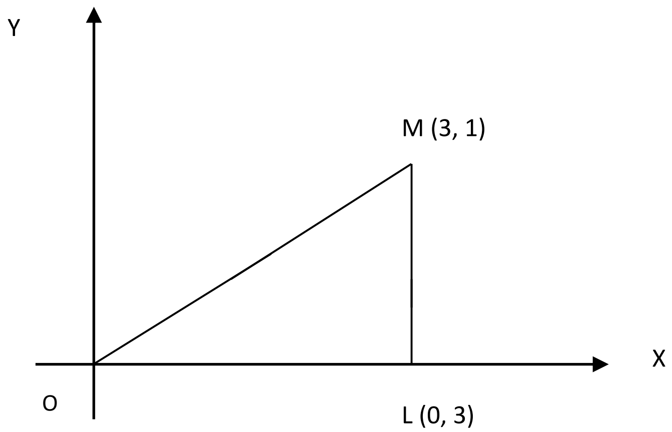
$$\therefore \quad I_1 = \int_{OM} z^2\, dz = \int_{OM} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_0^1 (9y^2 - y^2 + 6iy^2)(3dy + idy)$$

$$= \int_0^1 (8+6i)(3+i)y^2 dy$$

$$= \tfrac{1}{3}(8+6i)(3+i) = \tfrac{1}{3}(18+26i) = 6 + \tfrac{26}{3}i.$$

Now we consider the integral of z^2 along the path OLM. We observe that on OL, y=0 so that dy = 0 and along LM $x = 3$ so that $dx = 0$. Also z moves along OL from 0 to L, x varies from 0 to 3 and as z moves along LM from L to M, y varies from 0 to 1.



Hence

$$I_2 = \int_{OLM} x^2\, dz = \int_{OL} z^2\, dz + \int_{LM} z^2\, dz$$

$$= \int_{OL} (x^2 - y^2 + 2ixy)(dx + idy) + \int_{LM} (x^2 - y^2 + 2ixy)(dx + idy)$$

$$= \int_0^3 x^2\, dx - \int_0^1 (9 - y^2 + 6iy)i\, dy \qquad \{\text{y=0 in (i), (ii)}x = 3\}$$

$$= \left\{\frac{x^3}{3}\right\}_0^3 + i\left\{9y - \frac{y^3}{3} + 6i\frac{y^2}{2}\right\}_0^1$$

$$= 9 + i\left\{9 + \frac{1}{3} + 3i\right\}$$

$$= 9 + \frac{26}{3}i - 3$$

$$= 6 + \frac{26}{3}i$$

$$Thus\ I_1 = I_2 \text{ in this case}$$

Also since $\sum_{OM} z^2\, dz = 6 + \frac{26}{3}i$ we have $z^2\, dz = -\left(6 + \frac{26}{3}i\right)$

Hence $\int_{OLMO} z^2\,dz = \int_{OLM} z^2\,dz + \int_{MO} z^2\,dz = 0$ by equations (2) and (3).

Example 4. Find the value of the integral $\boldsymbol{\int_0^{1+i} (x - y + ix^2)\,dz}$

(a) **Along the straight line from $\boldsymbol{z = 0\ to\ z = 1 + i}$**

(b) **Along the real axis from $\boldsymbol{z = 0\ to\ z = 1}$**

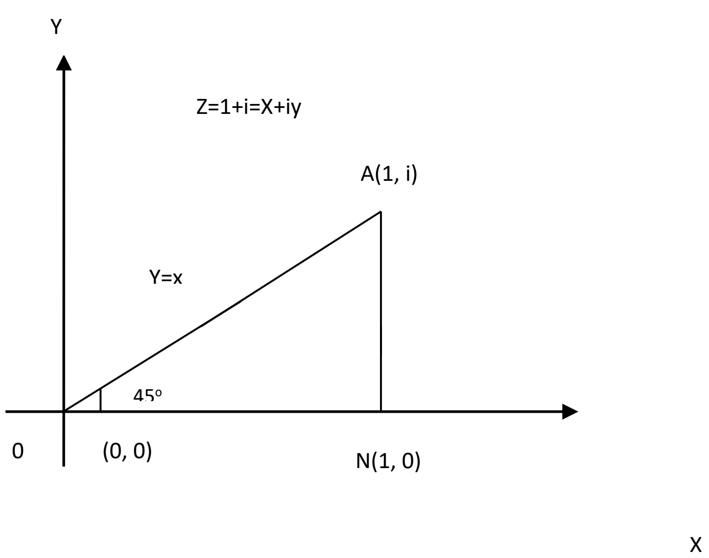
and then along a line parallel to the imaginary axis from $\boldsymbol{z = 1\ to\ z = 1 + i}$.

Sol. Let the point A(1, i) correspond to $z = 1 + i$ and N correspond to $z = 1 = x + iy$

$\Rightarrow \quad x = 1, \quad y = 0$

therefore N is (1, 0) OA is line from $z = 0\ to\ z = 1 + i$.

Hence line $OA\ is\ y = x$



(a) For integral along $OA \to y = x$

$$z = x + iy = x + ix$$

$$\Rightarrow \qquad dz = (1 + i)dx$$

Integral along OA $\int_0^A (x - y + ix^2)dz = \int_0^1 (ix^2)dz$

$$= i\int_0^A x^2\,dz = i\int_0^1 (1+i)x^2\,dx \qquad \{\text{dz=(1+i) d}x\}$$

$$= i(1+i)\left\{\frac{x^3}{3}\right\}_0^1$$

$$= i(1+i)\left\{\frac{1}{3}-0\right\}$$

$$= \frac{(i-1)}{3}$$

(b) Now for integral along $ON \rightarrow y = 0$

$$z = x + ix \Rightarrow x + i0 = x \Rightarrow dz = dx$$

Integral along ON $\int_0^N (x - y + ix^2)dz = \int_0^1 (x - 0 + ix^2)dx \qquad \{y = 0, \quad dx = dz\}$

$$= \int_0^1 (x + ix^2)dx = \left\{\frac{x^2}{2} + i\frac{x^3}{3}\right\}_0^1 = \frac{1}{2} + \frac{i}{3}.$$

Now integral along $NA \rightarrow x = 1$

$$z = x + iy = 1 + iy \Rightarrow dz = idy$$

$$\int_N^A (x - y + ix^2)dz = \int_N^A (1 - y + i1^2)\,idy \qquad \{x = 1, \quad dz = idy\}$$

$$= \int_0^1 (1 + i - y)idy = \int_0^1 (i - 1 - y)dy$$

$$= \left\{(i-1)y - \frac{iy^2}{2}\right\}_0^1$$

$$= (i-1) - \frac{i}{2}$$

$$= \frac{i}{2} - 1$$

Total → Integral along *ON*+ Integral along *NA*

$$= \frac{1}{2} + \frac{i}{3} + \frac{i}{2} - 1$$

$$= \frac{1}{2} + i\frac{5}{6}$$

*Hence function is not analytic if function is analytic, we get some result, other not.

Example 5. Evaluate $\int_L \frac{dz}{z-a}$ where L represents a circle $|z-a| = r$.

Sol. The parametric equation L is

$$z = a + r\cos t + i\, r\sin t = a + r\, e^{it} \text{ where } 0 \le t \le 2\pi$$

$\Rightarrow$ $\qquad z - a = r\ e^{it}$

we have $\qquad z'(t) = \frac{dz}{dt} = rie^{it}$

Now $\displaystyle\int_L \frac{dz}{z-a} = \int_0^{2\pi} \frac{1}{z-a}\frac{dz}{dt}dt = \int_0^{2\pi} \frac{1}{re^{it}} r.ie^{it}\, dt = i\int_0^{2\pi} dt = 2\pi i.$

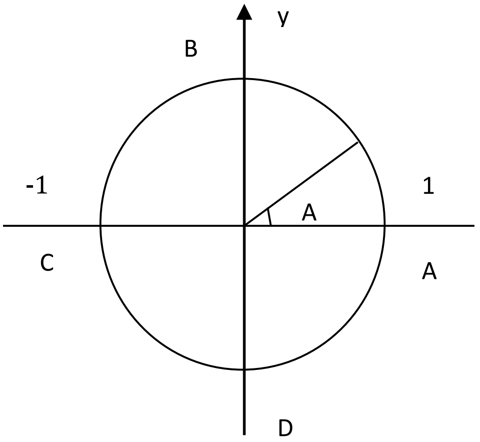
Example 6. Prove that the value of the integral of $\frac{1}{z}$ along a semi circular arc $|z| = 1$ $from - 1$ $to + 1$ $is - \pi i$ or πi according as the arc lies above or below the real axis.

Sol. The parametric equation of the circle $|z| = 1$ is $z = e^{it}$ so that $\frac{dz}{dt} = ie^{it}$ where $0 \le t \le 2\pi$.

As z moves from – 1 to 1 along the upper semi-circular arc, t varies from π to 0.

Hence in this case, we have

$$I_1 = \int_{CBA} \frac{dZ}{Z} = \int_{\pi}^{0} \frac{ie^{it}\ dt}{e^{it}} = i\int_{\pi}^{0} dt = -\pi i \qquad \ldots(1)$$



Also if z moves along the lower semi-circle from – 1 to 1 varies from π to 2π and we have

$$I_2 = \int_{CDA} \frac{dZ}{Z} = \int_{\pi}^{2\pi} \frac{ie^{it}\ dt}{e^{it}}$$

$$= i\int_{\pi}^{2\pi} dt = -\pi i \qquad \ldots(2)$$

Note. Since $\int_{CBA} \frac{1}{z} dz = -\pi i$ We see that $\int_{ABC} \frac{1}{z} dz = \pi i$

Hence from equation (2) and (3), we have

$$\int_{ABCD} \frac{1}{z} dz = \int_{ABC} \frac{1}{z} dz + \int_{CDA} \frac{1}{2}\ dz = \pi i + \pi i = 2\pi i.$$

Thus the integral of $\frac{1}{z}$ round the entire circle $|z| = 1$ is $2\pi i$.

Example 7. Using the definition of an integral as the limit of a sun evaluate the following integrals

$$(\boldsymbol{i}) \int_{L} \boldsymbol{dz} (\boldsymbol{ii}) \int_{L} |\boldsymbol{dz}| (\boldsymbol{iii}) \int_{L} \boldsymbol{z\ dz}$$

where L is any rectifiable arc joining the points $z = \alpha$ and $z = \beta$.

Sol. We know that the integrals exists since in integral is continuous in each parts.

(i) By definition, we have

$$\int_L dz = \lim_{n\to\infty}\sum_{k=1}^{n}(z_k - z_{k-1}) = \lim_{n\to\infty}[z_1 - z_0 + z_2 - z_1 \dots + z_n - z_{n-1}]$$

$$= \lim_{n\to\infty}[z_n - z_0] = \beta - \alpha \quad since\ z_0 = \alpha, \quad z_n = \beta$$

In particular if L is closed then $\propto = \beta$ $\ and\ $ $\int_L dz = 0$

$$(ii)\int_L |dz| = \lim_{n\to\infty}\sum_{k=1}^{n}|z_k - z_{k-1}| = \lim_{n\to\infty}|z_1 - z_0| + |z_2 - z_1| + \dots + |z_n - z_{n-1}|$$

$$= \lim_{n\to\infty}[chord\ z_1z_0 + \ chord\ z_2z_1 + \dots \dots + chord\ z_nz_{n-1}]$$

$$= [arc\ z_1z_0 + \ arc\ z_2z_1 + \dots \dots + arc\ z_nz_{n-1}]$$

$$= Arc\ length\ of\ L.$$

$$(iii)\ We\ have\ \ I = \int_L z\, dz = \lim_{n\to\infty}\sum_{k=1}^{n}\xi_k\,(z_k - z_{k-1}) \qquad \dots(1)$$

where ξ_k is any point on the sub-arc joining z_{k-1} to z_k. Since ξ_k is a arbitrary. We set $\xi_k = z_k$ and $\xi_k = z_{k-1}$successively in equation (1) and get

$$I = \lim_{n\to\infty}\sum_{k=1}^{n} z_k\,(z_k - z_{k-1}) \text{ and } \quad \lim_{n\to\infty}\sum_{k=1}^{n} z_{k-1}\,(z_k - z_{k-1})$$

Adding these two, we get

$$2I = \lim_{n\to\infty}\sum_{k=1}^{n}(z_k - z_{k-1}).(z_k + z_{k-1}) = \lim_{n\to\infty}\sum_{k=1}^{n}(z_k^2 - z_{k-1}^2)$$

$$= \lim_{n\to\infty}[z_1^2 - z_0^2 + z_2^2 - z_1^2 + \dots \dots + z_n^2 - z_{n-1}^2] = \lim_{n\to\infty}(z_n^2 - z_0^2)$$

$$= \beta^2 - \alpha^2 \quad since\ z_0 = \alpha, \quad z_n = \beta$$

$$Hence \quad I = \frac{1}{2}(\beta^2 - \alpha^2)$$

In particular, if L is closed, then $\alpha = \beta$ and $\int_L z\,dz$=0.

6.12 Summary

A continuous curve have no multiple point. A continuous are without multiple point (or arc) is called a Jordan curve or arc.A contour *C* is said to be simple closed contour if its ending points coincide.For example,circle, bounder of a triangle or rectangle.

A contour is a continuous chain of a finite number of regular arcs. A contour is an arc consisting of a finite number of smooth arcs joining end to end. A region *D* is said to be a connected region if any two points of region *D* can be connected by a curve which lies entirely within the region. A connected region *D* is said to be simply connected if all the interior points of closed curve *C* draw in the region *D* are points of the region *D*.

A region in which every closed curve can be shrunk to a point without passing out of the region is called a simply connected region otherwise it is said to be multiconnected region.

6.13 Terminal Questions

Q.1 What do you mean by Complex integration.

Q.2. Explain the connected and multiconnected region.

Q.3. Write a short note on contour and simple closed contour.

Q.4. Find the value of the integral $\int_C \frac{dz}{z-a}$, where C is circle, $|z - a| = \rho$.

Q.5. Evaluate $\int_0^{1+i}(x^2 + iy)dz$ along the paths $y = x$ and $y = x^2$.

Answer

4. $2\pi i$.

5. $\frac{1}{6}(5i-1)$.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-7: Cauchy's Theorem

Structure

7.1 Introduction

In 1831, Cauchy discovered his famous formula, the Cauchy integral formula. Cauchy began developing his theorem, the Cauchy fundamental theorem, in 1825. This theorem essentially establishes a connection between line integrals of complex functions and antiderivatives, similar to the Fundamental Theorem of Calculus in real analysis. Cauchy's theorem is a foundational result in complex analysis that underpins many key concepts and results in the field. Its importance extends far beyond pure mathematics, with applications in physics, engineering, and other scientific disciplines.

Cauchy's theorem guarantees the existence of antiderivatives for complex functions that are analytic within a region. This property is crucial in the development of complex analysis and has applications in various areas of mathematics and physics. Cauchy's theorem is also a key component in the development of the residue theorem, which provides a powerful method for computing complex integrals, especially around singularities of a function. The residue theorem has numerous applications in mathematics, physics, and engineering.

Hence the Cauchy theorem is central to the study of complex functions and their properties. It provides a framework for understanding the behavior of analytic functions and has implications for the behavior of real-valued functions as well.

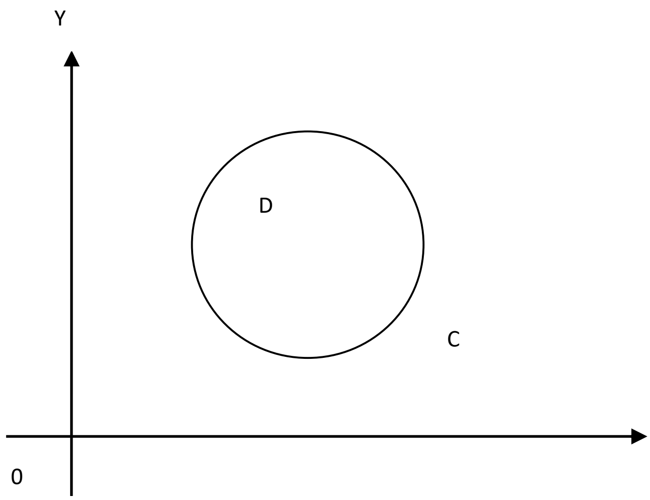
7.2 Objectives

After reading this unit the learner should be able to understand about:

- the Cauchy Fundamental theorem
- the Cauchy Goursat theorem
- the extension of Cauchy theorem to multiconnected region

7.3 Cauchy Fundamental Theorem

If $f(z)$ is an analytic function of z and if $f(z)$ is continuous at each point within and on a closed contour C, then



$$\int_C f(z)dz = 0. \qquad \ldots(1)$$

Proof. Let D be the region which consisting of all points within and on the contour.

If $P(x,y), Q(x,y), \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ are all continuous functions of x and y in the region D, then by Green's theorem

$$\int_C (Pdx + Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy \qquad \ldots(1)$$

Since $w = f(z) = u + iv$ is continuous on the curve C and $f'(z)$ exists and continuous in D therefore u, v, u_x, u_y, v_x, v_y are all continuous in the region D then these conditions by Green's theorem.

$$\int_C f(z)dz = \int_C (u + iv)(dx + i\, dy)$$

$$= \int_C (udx - vdy) + i\,(vdx + udy) \qquad \ldots.(2)$$

Now comparing equations (1) and (2), taking $P \to u,\ Q \to -v$

$$\int_C (udx - vdy) = \iint_D \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy$$

$$= -\iint_D \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\, dy$$

$= 0$ {by Cauchy-Reimann equation u_x=v_y, u_y=-v_x} …(3)

Now again comparing equations (1) and (2), taking $P \to v,\quad Q \to u$

$$i\int_C (vdx + udy) = i\iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy$$

$$= 0 \quad \{by\ C - R\ equation\ u_x = v_y,\ \ u_y = -v_x\} \qquad \ldots(4)$$

Adding equations (3) and (4), we get

$$\int_C (udx - vdy) + i(vdx + udy) = 0 \qquad \ldots(5)$$

By equation (2), we get

$$\int_C f(z)dz = 0.$$

7.4 Cauchy Goursat Theorem

Let $f(z)$ be analytic in a simple connected domain D and let C be any closed continuous rectifiable curve in D. Then $\int_C f(z)dz = 0$.

Proof. To prove the theorem we need two lemmas.

Lemma 1. Given $\varepsilon > 0$, it is possible to divided the region inside C into finite number of meshes (square or partial square) such that with in each mesh a point z_0 such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon \qquad \ldots(1)$$

for all value of z in the mesh. Where z$\neq$ z_o.

Suppose the lemma is false then it fails at least in one mesh i.e. in this mesh condition (1) is not satisfied.

In other words difference of $\frac{f(z)-f(z_0)}{z-z_0}$ and $f'(z_0)$ is not less then ε.

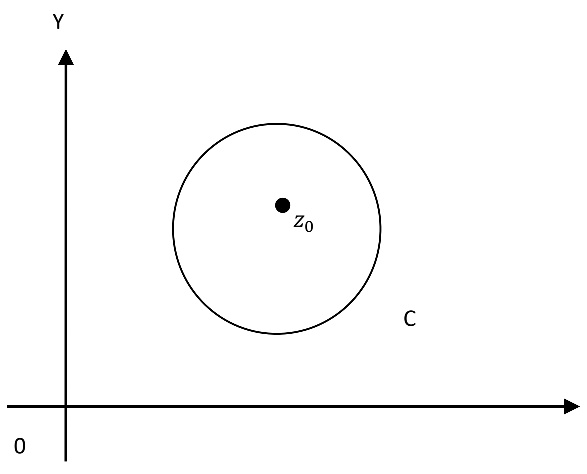
To reduce this difference we subdivided this mesh by lines joining middle points of opposite sides.

Now either the condition (1) is satisfied in each mesh or there is mesh in which it is not satisfied subdivided this part again the way described above this processes of either stop after finite number of step which condition is satisfied for every subdivision or the process may gone indefinitely in that case we reach at the point z_0 where

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \varepsilon \qquad \ldots(2)$$

Showing that function not differentiable at point z_0 this contradiction hypothesis that $f(z)$ is an analytic at all the points within and on C. Hence the lemma is true and hold for all point.

Lemma 2. On a closed contour C, we always have



$$\int_C dz = 0, \qquad \int_C z\ dz = 0$$

$$\int_{z_0}^{z_0} dz = [z]_{z_0}^{z_0} = 0, \qquad \int_C z\, dz = \left[\frac{z^2}{2}\right]_{z_0}^{z_0} = 0$$

Proof of the Main Theorem

$$Let \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) = \eta, \qquad \eta < \varepsilon$$

$$\Rightarrow f(z) - f(z_0) - (z - z_0)f'(z_0) = \eta(z - z_0)$$

$$\Rightarrow f(z) = f(z_0) + zf'(z_0) - z_0 f'(z_0) + \eta(z - z_0)$$

Let $C_1, C_2, \dots \dots . . C_m$ and so on be a complete squares and $P_1, P_2, \dots \dots . . P_n$ and so on partial square part of whose boundaries or part of C coincide

$$\sum_{r=1}^{m} \int_{c_r} f(z)dz + \sum_{s=1}^{n} \int_{p_s} f(z)dz$$

where integral along each continuous is taken in anticlockwise direction in the complete sum the integral along each straight side of each square {where the complete of partial cancel} and only those integral let which are taken along curved boundary of the partial squares then sum of the remaining integral

$$\int_C f(z)dz = \sum_{r=1}^{m} \int_{c_r} f(z)dz + \sum_{s=1}^{n} \int_{P_s} f(z)dz$$

$$= \int_C f(z)dz \qquad \dots(4)$$

$$\int_C f(z)dz = \int_{C_r} [f(z_0) + zf'(z_0) - z_0 f'(z_0) + \eta(z - z_0)]dz$$

$$= f(z_0)\int_{C_r} dz + f'(z_0)\int_{C_r} z\, dz - z_0 f'(z_0)\int_{C_r} dz + \int_{C_r} \eta(z-z_0)dz$$

Integral around C_r

$$\int_{C_r} f(z)dz = \int_{C_r} (z-z_0)\eta dz$$

Similarly

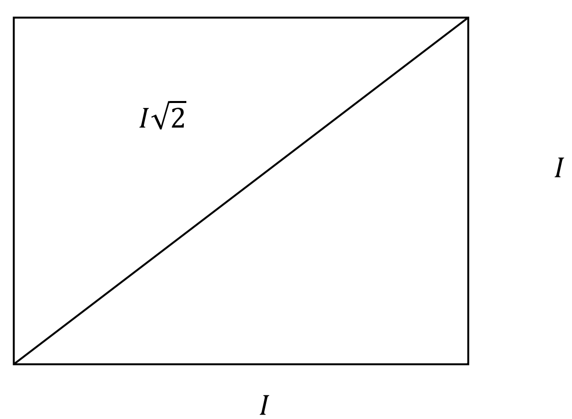
$$\int_{P_s} f(z)dz = \int_{P_s} (z-z_0)\eta dz \qquad \ldots(5)$$

Now by equations (4) and (5)

$$\int_C f(z)dz = \sum_{r=1}^{m}\int_{C_r} (z-z_0)\eta\, dz + \sum_{s=1}^{n}\int_{P_s} (z-z_0)\eta\, dz$$

Now we have

$$\left|\int_{C_r} (z-z_0)\eta\, dz\right| \le \int_{C_r} |z-z_0||\eta|\,|dz|$$

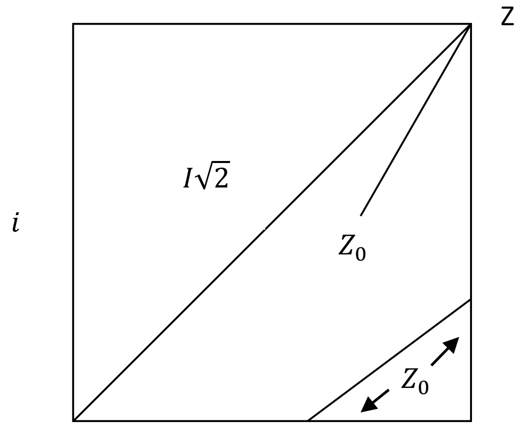


$$< I\sqrt{2}.\varepsilon\,.4I$$

$$\le 4\varepsilon\sqrt{2}I^2$$

$\le 4\varepsilon\sqrt{2}A_r$, where A_r is area of r complete square

$$And \left|\int_{P_s} (z - z_0)\eta \, dz\right| \leq \int_{P_s} |z - z_0||\eta| \, |dz|$$



$$\leq I\sqrt{2}\, \varepsilon(4i + S_n)$$

$$\leq \varepsilon\sqrt{2}[4i^2 + is_n]$$

$$\leq \varepsilon\sqrt{2}[4A_s + Is_n]$$

$$\therefore \int_C f(z)dz \leq \sum\left[\left\{4\varepsilon\sqrt{2}A_r\right\} + \varepsilon\sqrt{2}\{4A_s + Is_n\}\right]$$

$$\leq \varepsilon/2[4\{\Sigma(A_r + A_s)\} + \Sigma \mathrm{Is_n}]$$

$$\int_C f(z)dz \leq 4 \ \Sigma/2 \ A + La \, \Sigma/2$$

$$\leq \varepsilon/2[4A + aL]$$

$$\int_C f(z)dz \to 0$$

Since ε is arbitrary and so making $\varepsilon \to 0$.

Check your Progress

1. What do you mean by complex fundamental theorem?

2. State the Cauchy Goursat theorem.

Examples

Example.1. If C is the circle $|z-2|=5$, determine whether $\int_C \frac{dz}{z-3}$ is zero.

Sol. We have $z-2=5e^{i\theta} \Longrightarrow dz=5ie^{i\theta}d\theta$, we get

$$\int_c \frac{dz}{z-3}=\int_0^{2\pi}\frac{5ie^{i\theta}}{5ie^{i\theta}-1}d\theta$$

$$=i\int_0^{2\pi}\left\{\frac{5e^{i\theta}-1}{5e^{i\theta}}\right\}^{-1}d\theta$$

$$=i\int_0^{2\pi}\left\{1-\frac{1}{5e^{i\theta}}\right\}^{-1}d\theta$$

$$=i\int_0^{2\pi}\left[1+\frac{1}{5}e^{-i\theta}+\frac{1}{5^2}e^{-2i\theta}+\cdots\right]d\theta$$

$$=i\left[\theta-\frac{i}{5}e^{-i\theta}-\frac{2i}{5^2}e^{-2i\theta}\ldots\right]_0^{2\pi}$$

$$=i\left[2\pi-\frac{ie^{-2\pi i}}{5}-2i\frac{e^{-4\pi i}}{5^2}\ldots\right]$$

$$=2\pi i+\frac{1}{5}e^{-2\pi i}+\frac{2}{25}e^{-4\pi i}+\cdots$$

Now $\int_0^{2\pi} e^{-mi\theta}=-\left[\frac{e^{-mi\theta}}{mi}\right]_0^{2\pi}$

$$=-\frac{1}{mi}\left[e^{-2\pi i}-e^0\right]$$

$$= \frac{1}{mi}[1-1] = 0, \text{ when } m \neq 0.$$

$\therefore \qquad \int_C \frac{dz}{2-3} = i\int_0^{2\pi} d\theta = 2i \neq 0.$

The reason that the integral is not zero is that $\frac{1}{z-3}$ is not analytic at $z = 3$ which is an interior point of the circle $|z-2| - 5$.

7.5 Extension of Cauchy theorem to multiconnected region

If C is closed curve and $C_1, C_2, \ldots\ldots\ldots. C_n$ and so on are closed curves which lie inside C and if afunction $f(z)$ is analytic in the region between these curves and continuous on C then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots.$$

Proof. Drawing cuts $ABA_1B_1.CDC_1D_1$ ... ...etc.

Multiconnected region becomes simply connected.

We have, by Cauchy theorem

$$\int_C f(z)dz = \int_{C_1} f(z)dz - \int_{C_2} f(z)dz - \ldots\ldots. = 0$$

$$\Rightarrow \int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots\ldots.$$

If there is only one closed curve C_1 within C then

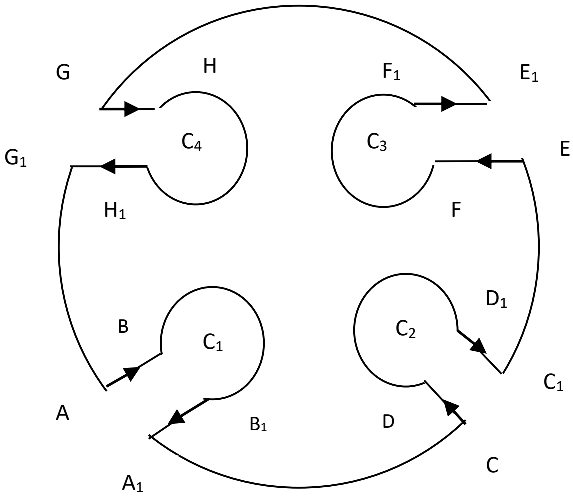
$$\int_C f(z)dz = \int_{C_1} f(z)dz$$

where integral along each curve is an anticlockwise direction.

Particular Case

Theorem.1. Let $f(z)$ be continuous on a contour L or length $i$and let $|f(z)| \leq M$ on i. Then

$$\left| \int_L f(z) dz \right| \leq Mi$$



Proof: We have $S_p = \sum_{k=1}^{n} (z_k - z_{k-1}) f(z_k)$

Since the modulus of the sum is less than or equal to the sum of the moduli, we have

$$|S_p| = \left| \sum (z_k - z_{k-1}) f(z_k) \right|$$

$$\leq \sum |(z_k - z_{k-1}) f(z_k)|$$

$$= \sum |(z_k - z_{k-1})||f(z_k)|$$

$$\leq M \sum |z_k - z_{k-1}|$$

$$\therefore \lim \left| \sum (z_k - z_{k-1}) f(z_k) \right| \leq \lim M \sum |z_k - z_{k-1}|$$

We know that

$$\int_L |dz| = \lim_{n\to\infty} \sum_{k=1}^{n} |z_k - z_{k-1}| = \text{arc length of L}$$

then $$\left|\int_L f(z)dz\right| \le M \int_L |dz| = Mi.$$

Theorem.2. $\int_C z\,dz$ **(initio** $\boldsymbol{ab}$**)** $= \frac{b^2-a^2}{2}$

Prof: We know that initio integral of $f(z)$ is defined as

$$\int_C f(z)dz = \sum_{r=1}^{n} f(\xi_r)(z_r - z_{r-1})$$

Where ξ_r is a point between z_r and z_{r-1}

Take $\xi_r = z_r$

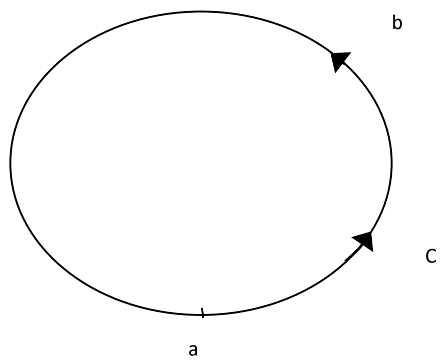
$$\int_C z\,dz = \sum_{r=1}^{n} \xi_r(z_r - z_{r-1}) = \sum_{r=1}^{n} z_r(z_r - z_{r-1}) \quad \ldots(1)$$

Again take $\xi_r = z_{r-1}$

$$\int_C z\,dz = \sum_{r=1}^{n} z_{r-1}(z_r - z_{r-1}) \quad \ldots(2)$$

Adding equations (1) and (2), we have

$$\int_C z\,dz = \sum_{r=1}^{n} (z_r - z_{r-1})(z_r + z_{r-1})$$



Closed contour C

$$= \sum_{r=1}^{n} (z_r^2 - z_{r-1}^2) = (z_1^2 - z_0^2) + (z_2^2 - z_1^2) + \cdots .. + (z_n^2 - z_{n-1}^2)$$

$$= z_n^2 - z_0^2$$

$$\int_C z\, dz = \frac{z_n^2 - z_0^2}{2} = \frac{b^2 - a^2}{2}$$

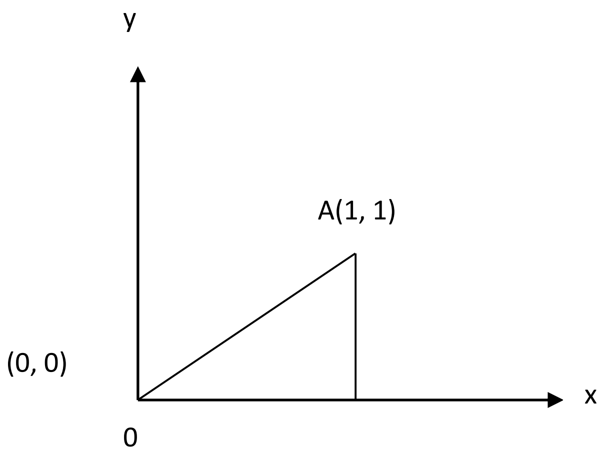
$$\int_C z\, dz \text{ (initio ab)} = \frac{b^2 - a^2}{2}.$$

Examples

Example.2.Evaluate the integral $\int_0^{1+i} z^2\, dz$.

Sol. $\int_0^{1+i} z^2\, dz$

Given $z = 0\ to\ z = 1 + i$ and we have $z = x + iy,\ z = 0 \Rightarrow = 0, y = 0, z = 1 + i \Rightarrow x = 1, y = 1, f(z) = z^2$, which is analytic for all finite values of z, therefore, its integral between two fixed points will be same is respective of the path joining the two fixed points integrating $f(z)$ along OA.



We have $y = x \quad this\ imples\ \ dy = dx$

$$\int_0^{1+i} z^2\, dz = \int_0^1 (x + ix)^2(dx + idx)$$

$$= \int_0^1 (1 + ix)^2 x^2\ (1 + i)\mathrm{d}x$$

$$= (1 + i)^3 \int_0^1 x^2\, dx$$

$$= (1 + i)\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{3}(1 + i)^3.$$

7.6 Summary

If $f(z)$ is an analytic function of z and if $f(z)$ is continuous at each point within and on a closed contour C, then $\int_C\ f(z)dz = 0.$

Let $f(z)$ be analytic in a simple connected domain D and let C be any closed continuous rectifiable curve in D then $\int_C\ f(z)dz = 0.$

If C is closed curve and $C_1, C_2, \dots \dots \dots .. C_n$ and so on are closed curves which lie inside C and if afunction $f(z)$ is analytic in the region between these curves and continuous on C then

$$\int_C\ f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \dots \dots .$$

7.7 Terminal Questions

Q.1. What do you mean by Cauchy fundamental theorem.

Q.2. State the Cauchy Goursat theorem..

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.

2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.

3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.

4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.

5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.

6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)

7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-8: Cauchy's Integral Formula and its Applications

Structure

8.1 Introduction

The Cauchy's Integral Formula is of paramount importance in complex analysis, providing a powerful tool for evaluating complex integrals and understanding the behavior of holomorphic functions. Complex integration is essential in fluid dynamics, where it helps analyze potential flow, irrotational flow, and complex fluid motion. It aids in calculating fluid velocities, pressure distributions, and flow patterns. In control theory, complex integration is crucial for analyzing the stability and performance of control systems. It assists in designing controllers and predicting the behavior of dynamic systems. Cauchy theorem is the foundation for the Cauchy integral formula, which allows for the calculation of complex contour integrals in terms of function values on the boundary of the region. This formula is essential in the study of complex analysis and has wide-ranging applications.

Cauchy's Integral Formula is a cornerstone of complex analysis, playing a crucial role in understanding the behavior of holomorphic functions and their applications in various areas of mathematics and physics.

8.2 Objectives

After reading this unit the learner should be able to understand about:

- Cauchy's Integral Formula
- Extension of Cauchy Integral Formula for Multiconnected Region
- Cauchy Integral Formula For Derivative
- The n^{th} Derivative of Cauchy Integral Formula
- Cauchy Inequality

8.3 Cauchy's Integral Formula

If $f(z)$ is analytic within and on a closed contour C and a is any point within C then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \text{ about the point } z = a.$$

Proof. Let a small circle λ of radius r lying entirely within C consider the function $\frac{f(z)}{z-a}$.

By Cauchy theorem to multiconnected Region we have

$$\int_C \frac{f(z)}{z-a}dz = \int_\gamma \frac{f(z)}{z-a}\, dz$$

$$\int_C \frac{f(z)}{z-a}dz - \int_\gamma \frac{f(a)}{z-a}\, dz = \int_\gamma \frac{f(z)}{z-a}dz - \int_\gamma \frac{f(a)}{z-a}\, dz$$

$$\int_C \frac{f(z)}{z-a}dz - f(a)\int_\gamma \frac{dz}{z-a} = \int_\gamma \frac{f(z)-f(a)}{z-a}dz$$

Let $\quad z - a = re^{i\theta} \Rightarrow |z-a| = r$

$$dz = re^{i\theta}\ d\theta \qquad \text{and} \qquad \gamma \text{ is a circle}$$

$$\int_C \frac{f(z)}{z-a}dz - f(a)\int_\gamma^{2\pi} \frac{rie^{i\theta}\, d\theta}{re^{i\theta}} = \int_\gamma \frac{f(z)-f(a)}{z-a}$$

$$\int_C \frac{f(z)}{z-a}dz - f(a)(2\pi i) = \int_\gamma \frac{f(z)-f(a)}{z-a}$$

$$\left|\int_C \frac{f(z)}{z-a}dz - 2\pi i\, f(a)\right| = \left|\int_\gamma \frac{f(z)-f(a)}{z-a}dz\right|$$

$$\leq \int_\gamma \frac{|f(z)-f(a)|}{|z-a|}|\, dz|$$

because $f(z)$ is analytic i.e. $f(z)$ is continuous $|f(z)-f(a)| < \varepsilon$ where $|z-a| < \delta$

$$\left\{\int_{\gamma} dz = 2\pi r \; for\ a\ circle\right\}$$

thus $$\left|\int_{C} \frac{f(z)}{z-a}dz - 2\pi i\, f(a)\right| \le \int_{\gamma} \frac{\varepsilon}{|z-a|}|dz| = \frac{\varepsilon}{r}\int_{r} |dz|$$

$$\left|\int_{C} \frac{f(z)}{z-a}dz - 2\pi i\, f(a)\right| \le \frac{\varepsilon}{r}2\pi r = 2\pi\varepsilon \to 0$$ {as ε→0 (arbitrary)}

$$\int_{C} \frac{f(z)}{z-a}dz = 2\pi i\, f(a)$$

⇒ $$f(a) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a}\, dz.$$

8.4 Extension of Cauchy Integral Formula for Multiconnected Region

If $f(z)$ is analytic in the region bounded by two closed contour C and C' and a is a point in the region, then

$$\boldsymbol{f(a) = \frac{1}{2\pi i}\int_{C} \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C^i} \frac{f(z)}{z-a}\, dz}$$ **where C is out contour.**

Proof. Let a small circle γ with centre a point a consider $\frac{f(z)}{z-a}$.

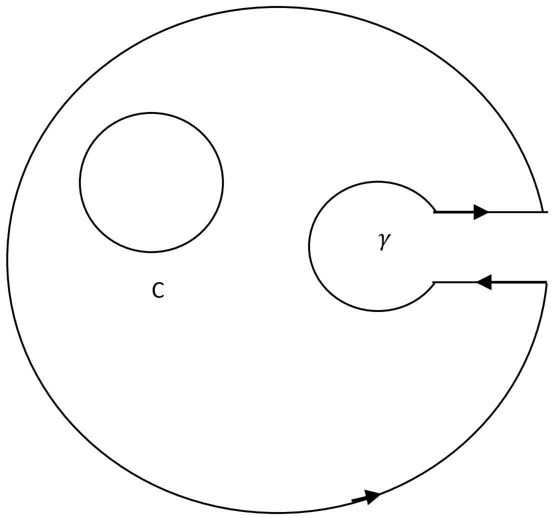
This is analytic in the region bounded by three contours C. C' and γ by Cauchy theorem for multiconnected region, we have

$$\int_{C} \frac{f(z)}{z-a}dz = \int_{C'} \frac{f(z)}{z-a}dz + \int_{\gamma} \frac{f(z)}{z-a}dz \qquad \ldots(1)$$

Now we solve this part

$$\int_{\gamma} \frac{f(z)}{z-a}dz = \int_{\gamma} \frac{f(z)-f(a)+f(a)}{z-a}dz$$

$$= \int_{\gamma} \frac{f(z) - f(a)}{z - a} dz + f(a) \int_{\gamma} \frac{dz}{z - a} \qquad \ldots(2)$$



Let $\quad I = \int_{\gamma} \frac{dz}{z - a} = \int_{0}^{2\pi} \frac{i\rho e^{i\theta}}{\rho e^{i\theta}} d\theta \qquad \left| \begin{array}{l} put\ |z - a| = \rho \\ z - a = \rho e^{i\theta} \\ dz = i\rho e^{i\theta}\, d\theta \end{array} \right.$

And $\quad \left| \int_{\gamma} \frac{f(z) - f(a)}{z - a} dz \right| \leq \int_{\gamma} \frac{|f(z) - f(a)|}{|z - a|} |dz|$

Because $f(z)$ is analytic i.e. $f(z)$ is continuous $\Rightarrow |f(z) - f(a)| < \varepsilon$ where $|z - a| < \delta$

$$\leq \varepsilon \int_{\gamma} \frac{|dz|}{r} \left| \begin{array}{l} z - a = \rho e^{i\theta} \ \ dz = i\rho e^{i\theta}\ \ d\theta \\ |z - a| = \rho \\ |dz| = |i||\rho|\left|e^{i\theta}\right||d\theta| = r\ \ d\theta \end{array} \right.$$

$$\leq \varepsilon \int_{\gamma} \frac{r d\theta}{r}$$

$$\leq \varepsilon \int_{0}^{2\pi} d\theta = 2\pi\theta \to 0$$

Since ε is arbitrary and so making $\varepsilon \to 0$

From equation (2), $\int_{\gamma} \frac{f(z)}{z - a}\, dz = f(a) 2\pi i$

From equation (1) $\int_C \frac{f(z)}{z-a}\,dz = \int_{C'} \frac{f(z)}{z-a}\,dz + 2\pi i\, f(a)$

$$\Rightarrow \qquad f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{C'} \frac{f(z)}{z-a}dz$$

8.5 Cauchy Integral Formula for Derivative

Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let a be any point within C. Then $f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz$.

Proof. Let $a+h$ be any point neighbourhood a point a. We know that by Cauchy integral formula

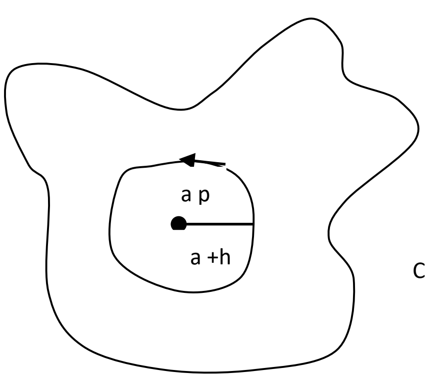
$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Also $$f(a+h) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a-h)}\,dz$$

Now $$f(a+h) - f(a) = \frac{1}{2\pi i}\int_C \left[\frac{1}{z-a-h} - \frac{1}{z-a}\right] f(z)\,dz$$

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{1}{2\pi i h}\int_C \frac{h}{(z-a-h)(z-a)} f(z)\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz$$



Check your Progress

1. Write the Cauchy integral formula.

2. State the Cauchy integral formula for derivative.

8.6 The n^{th} Derivative of Cauchy Integral Formula

Let $f(z)$ be analytic within and on the boundary C of simply connected region D and let a be any point within C. Then $f(z)$ possesses derivatives of all orders and these derivatives are themselves all analytic at a, their values being given by

$$\boldsymbol{f^m(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz.}$$

Proof. First we prove for one derivative as above theorem we solve by mathematical induction Let us assume the proof is true for $n = m$

$$f^m(a) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{m+1}}dz.$$

Also
$$f^m(a+h) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-a-h)^{m+1}}dz.$$

Now
$$f^m(a+h) - f^m(a)$$

$$= \frac{m!}{2\pi i}\int_C \left\{\frac{1}{(z-a-h)^{m+1}} - \frac{1}{(z-a)^{m+1}}\right\} f(z)dz$$

$$\lim_{h\to 0}\frac{f^m(a+h) - f^m(a)}{h} \lim_{h\to 0}\frac{m!}{2\pi i\, h}\int_C \frac{1}{(z-a)^{m+1}}\left\{\frac{1}{\left(1-\frac{h}{z-a}\right)^{m+1}} - 1\right\} f(z)dz$$

$$f^{m+1}(a) = \frac{m!}{2\pi i}\lim_{h\to 0}\frac{1}{h}\int_C \frac{1}{(z-a)^{m+1}}\left\{\left(1-\frac{h}{z-a}\right)^{-m-1} - 1\right\} f(z)dz$$

$$= \frac{m!}{2\pi i}\lim_{h\to 0}\frac{1}{h}\int_C \frac{1}{(z-a)^{m+1}}\left\{1 + \frac{h(m+1)}{z-a} + \frac{h^2(m+1)^2}{(z-a)^2} + \ldots\ldots\ldots -1\right\} f(z)dz$$

$$= \frac{m!\,(m+1)!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{m+2}} \text{ that is true for } m+1$$

Hence $\quad f^n(a) = \int_C \frac{f(z)dz}{(z-a)^{n+1}}.$

8.7 Cauchy Inequality

If $f(z)$ is analytic within a circle C given by $|z-a| = R$ and if $|f(z)| \le M\ on\ C\ then\ |f^n(a)| \le \frac{Mn!}{R^n}$.

Proof. We know that

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$$|f^n(a)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi i}\int_C \frac{|f(z)|}{|z-a|^{n+1}}|dz|$$

$$\le \frac{Mn!}{2\pi}\int_C \frac{R\,d\theta}{R^{n+1}}$$

$$\le \frac{Mn!}{2\pi R^n}\int_C d\theta = \frac{Mn!}{2\pi R^n}.2\pi$$

$$|f^n(a)| \le \frac{Mn!}{R^n}.$$

Theorem: *If C* if a closed contour origin prove that

$$(i)\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{n!\ z^{n+1}}dz \qquad (ii)\sum\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_0^{2\pi} e^{2a\cos\theta}d\theta.$$

Proof: (i) We take $f(z) = e^{az}$ then $f^n(z) = a^n e^{az}\ put\ z = 0$

$$\Rightarrow f^n(0) = a^n \qquad \ldots(1)$$

We know that by Cauchy integral formula for derivative

$$f^n(0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz$$

Put $a = 0$

$$f^n(0) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}dz \qquad \ldots(2)$$

From equations (1), and (2), we get

$$a^n = \frac{n!}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}dz$$ Now multiply by $\frac{a^n}{n!}$ both sides

$[f(z) = e^{az}]$

$$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n f(z)}{n!\, z^{n+1}}dz$$

$$\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C \frac{a^n e^{az}}{n!\, z^{n+1}}dz$$

(ii) Take summation on each sides, we get

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \sum_{n=0}^{\infty}\frac{1}{2\pi i\, n!}\int_C \frac{a^n e^{az}}{z^{n+1}}dz$$

$$= \frac{1}{2\pi i}\int_C e^{az}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{a}{z}\right)^n\frac{dz}{z}$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{az}e^{a/z}}{z}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{e^{a(z+\frac{1}{z})}}{z}dz$$

$$e^x = \sum_{n=0}^{\infty}\frac{a^n}{n!}$$

$$e^{a/z} = \sum_{n=0}^{\infty}\frac{a^n}{z^n n!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n\frac{1}{n!}$$

C is a unit circle

$z = e^{i\theta}$

$$= \frac{1}{2\pi i}\int_C^{2\pi} \frac{e^{2a\cos\theta} ie^{i\theta}}{e^{i\theta}} d\theta$$

$$\sum_{n=0}^{\infty}\left(\frac{a^n}{n!}\right)^2 = \frac{1}{2\pi i}\int_C^{2\pi} e^{2a\cos\theta} d\theta$$

$dz = i\, e^{i\theta}\, d\theta$

$z + \frac{1}{z} = e^{i\theta} + e^{-i\theta}$

$= 2\cos\theta$

Examples

Example 1. ***Using Cauchy Integral Formula***

$(\boldsymbol{i})$ ***Evaluate*** $\boldsymbol{\int_C \frac{dz}{z-2}}$***, where C is*** $\boldsymbol{|z| = 3}$

$(\boldsymbol{ii})$ ***Evaluate*** $\boldsymbol{\int_C \frac{z\,dz}{(9-z^2)(z+i)}}$***, where C is the circle***

$\boldsymbol{|z| = 2}$ ***described in positive sense.***

$(\boldsymbol{iii})$ ***Evaluate*** $\boldsymbol{\frac{1}{2\pi i}\int_C \frac{z^{tz}}{z^2+1} dz, t} > 0$, *where C is the circle* $\boldsymbol{|z| = 3}$.

Solution. The Cauchy integral formula is

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}$$

$$or \qquad \int_C \frac{f(z)dz}{z-a} = 2\pi i\, f(a) \qquad \ldots(1)$$

where $z = a$ is a point inside C and $f(z)$ is analytic function within and upon C.

(i) Given that $\int_C \frac{dz}{z-2}$, where C is $|z| = 3$.

$$\text{We have} \qquad \int_C \frac{f(z)dz}{z-a} = \int_C \frac{dz}{z-2} \qquad \ldots..(2)$$

Then $a = 2, f(z) = 1, C$ is a circle $|z| = 3$ whose radius is 3 and centre is at $z = 0$.

Therefore $a = 2$ lies inside C.

Using equations (1) and (2), we have

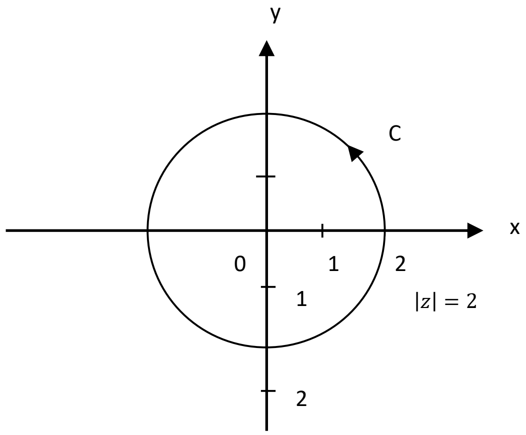
$$\int_C \frac{dz}{z-2} = 2\pi i\, f(a) \qquad \begin{Bmatrix} \text{for } f(z) = 1 \\ \Rightarrow f(2) = 1 \end{Bmatrix}$$

$$= 2\pi i\, f(2)$$

$$= 2\pi i\ (1)$$

$$= 2\pi i.$$

$(ii)\ Given\ that\ \int_C \frac{z\,dz}{(9-z^2)(z+i)}$, where C is the circle , $|z| = 2$ described in positive sense.



We have $\quad I = \int_C \frac{z\,dz}{(9-z^2)(z+i)}$

$$= \int_C \frac{f(z)\,dz}{[z-(-i)]} \ldots (3) \qquad \left\{\because \quad f(z) = \frac{z}{(9-z^2)}\right.$$

Using equations (1) and (3), we get

$$I = 2\pi i\, f(-i)$$

$$= 2\pi i \left[\frac{-1}{9-(-i)^2}\right] \qquad \left[\because \quad f(z) = \frac{z}{(9-z^2)}\right]$$

$$= \frac{2\pi}{9+1} = \frac{\pi}{5}$$

Here $f(z)$ is analytic function within and upon C such that $|z| = 2$ and $z = -i$ lies inside C.

(iii) $Given\ that$

$$I = \frac{1}{2\pi i}\int_C \frac{e^{tz}}{z^2+1}\,dz, \quad t > 0 \qquad \ldots(4)$$

where C is the circle $|z| = 3$.

We have $I = \frac{1}{2\pi i}\int_C \frac{e^{tz}}{z^2+1}\,dz$

Let $f(z) = e^{tz}$, then

$$I = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z+1)(z-i)}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{2i}\left\{\frac{1}{z-1} - \frac{1}{z+i}\right\}dz$$

$$= \frac{1}{4\pi i^2}\int_C \frac{f(z)}{z-i}dz - \frac{1}{4\pi i}\int_C \frac{f(z)}{z-i}dz$$

Using Cauchy integral formula, we get

$$= \frac{1}{2i}\{f(i)\} - \frac{1}{2i}\{f(-i)\}$$

$$= \frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}$$

$$= \frac{1}{2i}(e^{it} - e^{it})$$

$$= \sin t.$$

Example 2. Evaluate the following integral by Cauchy's integral formula $\int_C \frac{\sin xz^2 + \cos xz^2}{(z-I)(z-2)}$

where C is the circle $|z| = 3$.

Solution. The Cauchy integral formula is

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-a)}$$

or $$\int_C \frac{f(z)}{(z-a)} dz = 2\pi i\, f\,(a)$$

where $z = a$ is a point inside C and $f(z)$ is analytic function within and upon C.

Given that $$I = \int_C \frac{\sin xz^2 + \cos xz^2}{(z-I)(z-2)}\, dz$$

We have $f(z) = \sin xz^2 + \cos xz^2$, C is the circle $|z| = 3$ whose radius is 3 and centre at $z = 0$. Therefore 1 and 2 lies inside C.

Using equation (1) and (2), we get

$$I = \int_C (\sin xz^2 + \cos xz^2)\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$=\int_C \sin xz^2 \left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz + \int_C \cos xz^2 \left(\frac{1}{z-2} - \frac{1}{z-1}\right) dz$$

$$= \int_C \frac{\sin xz^2}{(z-2)} dz - \int \frac{\sin xz^2}{(z-1)} dz + \int_C \frac{\cos xz^2}{(z-2)} dz - \int_C \frac{\cos xz^2}{(z-1)} dz$$

$$= 2\pi i\{(\sin xz^2)_{z=2} - (\sin xz^2)_{z=1} + (\cos xz^2)_{z=2} - (\cos xz^2)_{z=1}\}$$

$$= 2\pi i[\sin 4\pi - \sin \pi + \cos 4\pi - \cos \pi]$$

$$= 2\pi i[0 - 0 + 1 - (1)]$$

$$= 4\pi i.$$

$\boldsymbol{Example. 3}$ *Evaluate* $\displaystyle\int_C \frac{\sin z\, dz}{\left(z - \frac{\pi}{4}\right)^3}$, *where* C *is* $\left|z - \frac{\pi}{4}\right| = \frac{1}{2}$.

Sol. We know that Cauchy integral formula for higher derivative is

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \qquad \ldots(1)$$

where $z = a$ lies inside C and $f(z)$ is analytic function within and upon C. Given that

$$I = \int_C \frac{\sin z\, dz}{\left(z-\frac{\pi}{4}\right)^3} \qquad \ldots(2)$$

Let $f(z) = \sin z$ then $f'(z) = \cos z,$ $f''(z) = -\sin z$

and $$f''(\pi/4) = -sing\,\frac{\pi}{4} = -\frac{1}{\sqrt{2}}.$$

Using equation (1) and (2), we have

$$\text{I}=\int_C \frac{f(z)}{\left(z-\frac{\pi}{4}\right)^3}dz = \frac{2\pi i}{2!}f''\left(\frac{\pi}{4}\right)$$

$$= \pi i\left(-\frac{1}{\sqrt{2}}\right)$$

$$= -\frac{\pi i}{\sqrt{2}}.$$

Example.4. Evaluate $\int_C \frac{dz}{z^2+2z+2}$, where C is the square having vertices at $(0,0), (-2,0), (-2,-2), (0,-2)$ oriented in anticlockwise direction.

$$\boldsymbol{Sol.} We\ have \int_c \frac{dz}{z^2+2z+2} \qquad \ldots(1)$$

Where C is square OPQR

Now we have $x^2 + 2z + 2 = (z+1)^2 + 1 = 0$

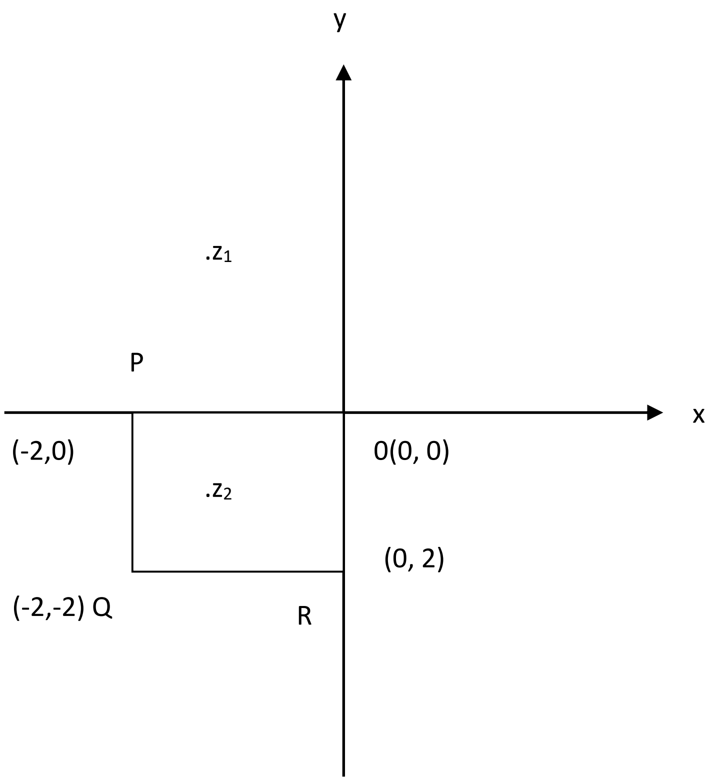
or $$(z+1)^2 = -1 = i^2$$

or z+1=±i

or $z = -1 \pm i(z_1 = -1 + i, z_2 = -1 - i,$ assuming

Here z=z_2 *lies inside C*.

Using equation (1), we have



$$I = \int_C \frac{dz}{(z-z_1)(z-z_2)} = \frac{f(z)}{(z-z_2)}, \text{ where } f(z) = \frac{1}{z-z_1}$$

$$= 2\pi i\, f(z_2) \qquad \text{\{Using Cauchy Integral formula\}}$$

$$= \frac{2\pi i}{z_2 - z_1}$$

$$= \frac{2\pi i}{(-1-i)-(-1+i)} = \frac{2\pi i}{-2i}$$

$$= -\pi.$$

Example.5. Using Cauchy integral formula

$(i)\ Evaluate \int_C \frac{\cosh(\pi z)dz}{z(z^2+I)}, where\ C\ is\ circle\ |z| = 2$

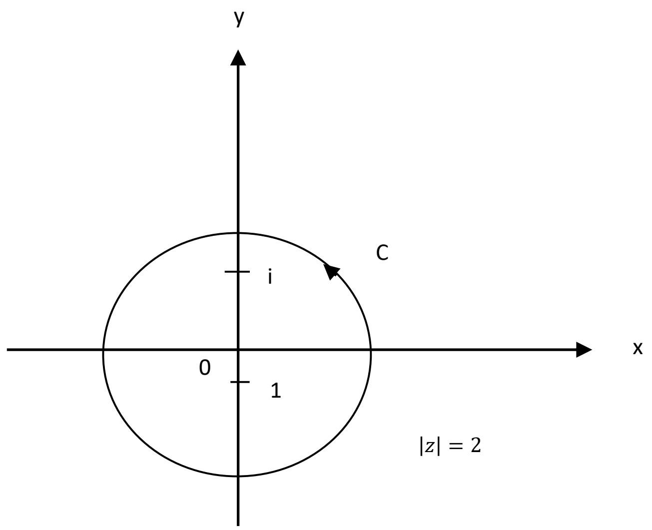
$(ii)\ Evaluate \int_C \frac{dz}{z(z+\pi i)}, where\ C\ is\ |z+3i| = 1.$

Sol. The Cauchy integral formula is

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)}$$

or
$$= \int_C \frac{f(z)dz}{(z-a)} 2\pi i\ f\ (a)$$

where $z = a$ is a point inside C and $f(z)$ is analytic function within and upon C.



$(i) Given\ that \int_C \frac{\cosh(\pi z)dz}{z(z^2+I)}, where\ C\ is\ circle\ |z| = 2$

We have

$$I = \int_C \frac{\cosh(\pi z)dz}{z(z^2+I)}. \qquad \ldots(2)$$

$$Let \quad f(z) = \cosh(\pi z) = \cos(i\pi z)$$

and $$C\ is\ |z| = 2.$$

Then $$I = \int_C \frac{f(z)}{z(z^2 + I)}\, dz$$

or $$I = \int_C \left[\frac{f(z)}{z(z+i)(z-i)}\right] dz$$

or $$I = \int_C \left[\frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}\right] f(z) dz. \qquad \ldots(3)$$

Now, we have

$$\frac{1}{z(z+i)(z-i)} = \frac{A}{z} + \frac{B}{z+i} + \frac{C}{z-i}$$

$$A = \frac{1}{(z-)(z+i)} = 1 \qquad \text{at } z = 0$$

$$B = \frac{1}{z(z-i)} = -\frac{1}{2} \quad at\ z = -i$$

$$C = \frac{1}{z(z+i)} = -\frac{1}{2} \quad at\ z = i$$

Here $z = 0, -i, i$ are points inside C.

Using equation (1) and (3), we get

$$I = 2\pi i[Af(0) + Bf(-1) + Cf(i)]$$

$$= 2\pi i \left[f(0) - \frac{1}{2} f(-i) - \frac{1}{2} f(i)\right]$$

$$= 2\pi i \left[\cos 0 - \frac{1}{2}\cos(i^2\pi) - \frac{1}{2}\cos(-i^2\pi)\right]$$

$$= 2\pi i\left[1 + \frac{1}{2} + \frac{1}{2}\right]$$

$$= 4\pi i.$$

$$(ii) Given\ that \int_C \frac{dz}{z(z+\pi i)}, where\ C\ is\ |z+3i| = 1$$

We have $$\text{I=}\int_C \frac{dz}{z(z+\pi i)}$$

or $$\text{I=}\int_C \frac{f(z)dz}{[z-(-\pi i)]} \qquad \dots(4)$$

$$\left\{\because \qquad f(z) = \frac{1}{z}\right\}$$

Using equations (1) and (2), we have

$$I = 2\pi i\, f(-\pi i)$$

$$= 2\pi i\left(-\frac{1}{\pi i}\right)$$

$$= 2.$$

Here $z = \pi i$ lies inside $Cf(z)$ is analytic within C.

Example.6. Evaluate $\int_C \frac{e^{2z}dz}{(z+1)^4} where\ C\ is\ |z| = 3.$

Sol. We know that Cauchy integral formula for higher derivative is

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

Put $a = -1\ and\ n = 3$ in equation (1), we get

$$f^{(n)}(-1) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z+1)^4}$$

Now take $$f(z) = e^{2z}$$

then $$f^n(z) = 2^n e^{2z}$$

$$f^3(-1) = 2^3 e^{-2} = \frac{8}{e^2}$$

Using equations (2) and (3), we get

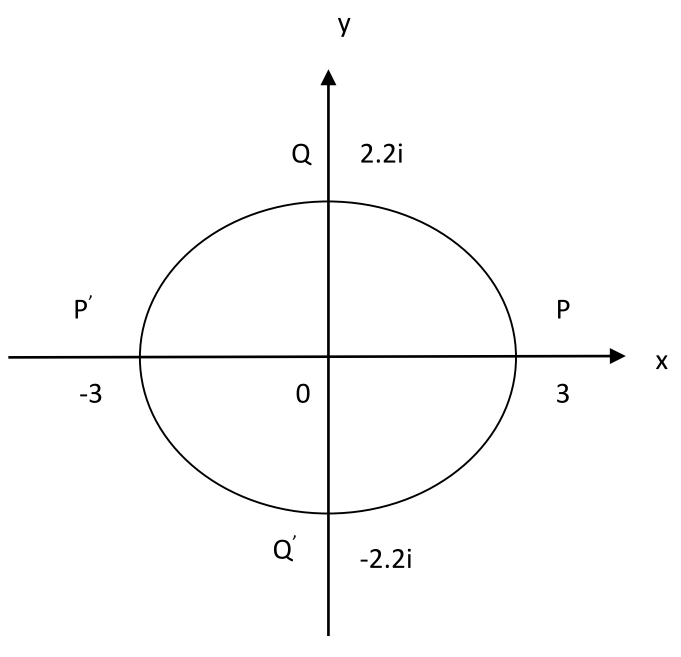
$$\frac{8}{e^2} = \frac{3!}{2\pi i}\int_C \frac{e^{2z}\,dz}{(z+1)^4}$$

$$\frac{8\pi i}{3e^2} = \int_C \frac{e^{2z}\,dz}{(z+1)^4}$$

Example. 7. Using Cauchy integral formula, evaluate $\int_C \frac{e^{2z}\,dz}{(z-\pi i)}$,

where C is the ellipse $|z-2| + |z+2| = 6$.

Sol. The Cauchy integral formula is



$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)}dz$$

or $$\int_C \frac{f(z)}{(z-a)}dz = 2\pi i\, f(a) \qquad \ldots(1)$$

We have $$I = \int_C \frac{e^{az}\,dz}{(z-\pi i)} \qquad \ldots(2)$$

Where C is the ellipse

$$|z-2| + |z+2| = 6$$

or $$|z-2| = 6 - |z+2|$$

or $$|(x-2)^2 + y^2|^{1/2} = 6 - |(x+2)^2 + y^2|^{1/2}$$

Squaring both sides, we get

$$x^2 + y^2 + 4 - 4x = 36 + (x^2 + y^2 + 4x + 4) - 12[(x+2)^2 + y^2]^{1/2}$$

or $$12(x^2 + y^2 + 4 + 4x)^{1/2} = 36 + 8x$$

or $$3(x^2 + y^2 + 4 + 4x)^{1/2} = 9 + 2x$$

again squaring both sides, we get

$$9(x^2 + y^2 + 4 + 4x) = 81 + 4x^2 + 36x$$

or $$5x^2 + 9y^2 = 45$$

or $$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

Comparing equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we get

$$a^2 = 9 \Rightarrow a = 3$$

and $b^2 = 5 \Rightarrow b = 2.2$ approximate

Given $z = \pi i = 3.14i$ lies outside C

Therefore $\frac{e^{az}}{z-\pi i}$ is an analytic function within and upon C.

$\Rightarrow \qquad I = 0$, by Cauchy's theorem.

8.8 Summary

If $f(z)$ is analytic within and on a closed contour C and a is any point within C then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \text{ about the point } z = a.$$

If $f(z)$ is analytic in the region bounded by two closed contour C and C' and a is a point in the region, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_{Ci} \frac{f(z)}{z-a} dz \text{ where C is out contour.}$$

Let $f(z)$ be analytic within and on the boundary C of a simply connected region D and let a be any point within C. Then

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz.$$

Let $f(z)$ be analytic within and on the boundary C of simply connected region D and let a be any point within C. Then $f(z)$ possesses derivatives of all orders and these derivatives are themselves all analytic at a, their values being given by

$$f^m(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz.$$

If $f(z)$ is analytic within a circle C given by $|z-a| = R$ and if $|f(z)| \leq M \; on \; C \; then \; |f^n(a)| \leq \frac{Mn!}{R^n}$.

8.9 Terminal Questions

Q.1. Explain State the Cauchy's Integral Formula.

Q.2. Write a short note on Extension of Cauchy Integral Formula for Multiconnected Region.

Q.3. What do you means by Cauchy Integral Formula For Derivative

Q.4. Explain the Cauchy Inequality.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-9: Important Theorems in Complex Integration

Structure

9.1 Introduction

Complex integration also plays a significant role in theoretical physics, especially in studying electromagnetic fields, quantum mechanics, and fluid dynamics. Overall, complex integration provides a rich framework for understanding and analyzing complex-valued functions, with applications across various fields of mathematics and science. Morera’s theorem is an important result in complex analysis that establishes a key condition for a complex-valued function to be holomorphic (analytic) in a region. Morera's theorem is important because it provides a simple and useful criterion for determining whether a function is holomorphic. It is often used in complex analysis to prove the holomorphicity of functions and to establish the existence of complex integrals.

Liouville's theorem provides a clear and concise characterization of entire functions. It states that if an entire function is bounded (i.e., its values do not become arbitrarily large), then it must be constant. This is a strong and useful result that helps classify and understand entire functions. Maximum Modulus Principle is a fundamental result that plays a key role in the theory of complex analysis and has many important applications.

9.2 Objectives

After reading this unit the learner should be able to understand about the:

- Morera’s theorem and its applications
- Liouville’s theorem and its applications
- Fundamental theorem of Integral Calculus for Complex Function
- Maximum modulus principle

9.3 Morera’s Theorem

Morera's theorem is an important result in complex analysis that establishes a key condition for a complex-valued function to be holomorphic (analytic) in a region. Morera's theorem is a key step in the proof of the fundamental theorem of calculus for complex integrals, which states that if *f*(*z*) is holomorphic on a simply connected domain D, then the integral of *f*(*z*) along any path in D between two points is independent of the path and depends only on the endpoints. This result has important consequences in complex analysis and its applications.

Theorem: If $f(z)$ is a continuous in a region D and if the integral $\int_c f(z)dz$ taken around on a simply closed contour in D is zero then $f(z)$ is analytic function inside D i.e., $\int_c f(z)dz = 0$.

Proof: Let Z_0 be a fixed point and z any variable point in the region D then the value integral

$\int_{z_0}^{z} f(z)dz$ is independent of the curve joining z_0 to z.

Hence taking ζ as the variable of integration

We write

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta \qquad \ldots\ldots. (1)$$

Also $$F(z+h) = \int_{z_0}^{z+h} f(\zeta)d\zeta \qquad \ldots\ldots. (2)$$

From equation (2) – (1), we get

$$F(z+h) - F(z) = \int_{z_0}^{z+h} f(\zeta)d\zeta - \int_{z_0}^{z} f(\zeta)d\zeta$$

$$\lim_{h\to 0} F(z+h) - F(z) = \lim_{h\to 0} \int_{z}^{z+h} f(\zeta)d\zeta \qquad \ldots\ldots. (3)$$

Where $h \to 0$, z and z + h as same point

$\Rightarrow \int_c f(\zeta)d\zeta$ i.e., integral along a closed curve = 0

$$\lim_{h\to 0} F(z+h)-F(z)=0$$

or $$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=0$$

Now subtract $f(z)$ both sides we get

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}-f(z)=\lim_{h\to 0}\left[-f(\zeta)\right] \qquad \ldots\ldots. (4)$$

R.H.S. of (4), we have

$$-f(z)=-\frac{1}{h}\left[h\,f(\zeta)\right]$$

$$=-\frac{1}{h}\left[(\zeta+h)-\zeta\right]f(\zeta)$$

$$=-\frac{1}{h}f(\zeta)\left[\zeta\right]_{z}^{z+h}$$

$$=-\frac{1}{h}f(\zeta)\int_{z}^{z+h}1.d\zeta$$

Now

$$\lim_{h\to 0}\left[\frac{F(z+h)-F(z)}{h}-f(z)\right]=\lim_{h\to 0}(-1/h)\,f(\zeta)\int_{z}^{z+h}d\zeta$$

But as $h\to 0$ $\int\limits_{z}^{z+h}d\zeta \to 0$

$$\lim_{h\to 0}\left[\frac{F(z+h)-F(z)}{h}-f(z)\right]=0$$

or $$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z)$$

$$F'(z)=f(z) \qquad \ldots\ldots. (5)$$

Hence $F'(z)$ exists because $F'(z) = f(z)$.

The inequality of equation (5) holds for every $z \in D$.

Hence $f(z)$ is analytic in D.

Or

From (3), we have

$$F(z+h) - F(z) = \int_z^{z+h} f(\zeta)\, d\zeta$$

or $$\frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(\zeta)\, d\zeta - \frac{1}{h} f(\zeta)$$

$$\Rightarrow \frac{F(z+h)-F(z)}{h} - f(z) = \frac{1}{h}\int_z^{z+h} f(\zeta)\, d\zeta - \frac{1}{h} f(\zeta)\int_z^{z+h} d\zeta$$

Now

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_z^{z+h}\{f(\zeta) - f(z)\}\, d\zeta\right|$$

$$\leq \frac{1}{|h|}\int_z^{z+h}\{f(\zeta) - f(z)\}(dz)$$

Since $f(\zeta)$ is continuous is region D the $\left|f(\zeta) - f(x)\right| < \in$ ….. (6)

$$\leq \frac{1}{|h|}\int_z^{z+h} \in |\mathrm{d}\zeta|$$

$$\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| \leq \frac{\in}{|h|}|h| = \in \to 0$$

$$\lim_{h\to 0}\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = 0$$

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h}=f(z) \quad \text{or} \quad F'(z)=f(z) \qquad \text{...... (7)}$$

Hence F'(z) exist because $F'(z)=f(z)$.

Hence $f(z)$ is analytic in equation (1).

Indefinite Integrals or Primitives

Let $f(z)$ be a single valued analytic function on a domain D. Then a function $F(z)$ is said to be indefinite integral (synonymously a primitive or anti-derivate -x) of $f(z)$ is single valued and analytic in D and

$$F'(z)=f(z) \qquad (z\in D)$$

Theorem. 1: A necessary and sufficient condition for a function $f(z)$ to possess on indefinite integral in a simple connected domain D is that the function f(z) is analytic in D. Further any two indefinite integrals differ by a constant.

Proof: Necessary:-Let $f(z)$ possess an indefinite integral $F(z)$ then

$$F'(z)=f(z) \qquad \text{....... (1)}$$

Thus $F(z)$ possesses a derivative $f(z)$ at every point $z\in D$ and so $F(z)$ is analytic in D. But the derivative of an analytic function is analytic it follows from (1) that $f(z)$ is analytic in D.

Sufficient:- Let $f(z)$ be analytic in D. Let Zo be any fixed point and z any variable point in D.

Then by {extension of Cauchy theorem to multi-connected Region} the integral $f(z)$ along every curve in D joining Zo to Z in the same Hence we may write

$$F(z)=\int_{zo}^{z} f(z)\,dz$$

Where z is the variable of Integration

By Morera's theorem, we can show that $F'(z) = f(z)$

Thus $f(z)$ possesses an indefinite integral $F(z)$ defined by the equation (1).

Hence every analytic function in a simply connected domain possesses an indefinite integral

We now show that any two indefinite integrals differ by a constant.

Let $F(z)$ and $\psi(z)$ be two indefinite integrals of $f(z)$

Then $$F'(z) = \psi'(z) \qquad z \in D$$

Or $$\psi'(z) - F'(z) = 0$$

$$[\psi(z) - F(z)]' = 0$$

Hence $\psi(z) - F(z) = c$, where c is an arbitrary constant.

Thus $\psi(z) = F(z) + c$.

Hence the general indefinite integral of an analytic function $f(z)$ is given by $F(z) + c$ where $F(z) = \int_{zo}^{z} f(z)\,dz$.

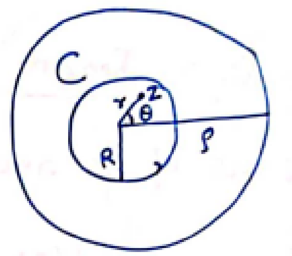
Poisson integral formula for a circle

Let $f(z)$ be analytic in a region according a circle $|z| < \rho$ then prove that for $0 < r < R$

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f\left(\mathrm{Re}^{i\phi}\right) d\phi}{R^2 - 2Rr\cos(\theta - \phi) + r^2}$$

Where $z = re^{i\theta}$ is any point of the domain $|z| = R$

Proof :- For circle $|z| = R$ such that $0 < r < R < \rho$ given $z = re^{i\theta}$



By Cauchy integral formula $f(a) = \dfrac{1}{2\pi i}\displaystyle\int_c \dfrac{f(z)}{z-a}dz$

Also $f(z) = \dfrac{1}{2\pi i}\displaystyle\int_c \dfrac{f(w)}{w-z}dw$ ...(1)

The inverse of the point z with respect to c is $\dfrac{R^2}{z}$ and lies outside C so that the function

$\dfrac{f(w)}{w - \dfrac{R^2}{z}}$ is analytic on and within C.

Therefore $\displaystyle\int_c \dfrac{f(w)}{w - \dfrac{R^2}{z}} dw = 0$

or $$\frac{1}{2\pi i}\int_c \frac{f(w)}{w - \dfrac{R^2}{z}} dw = 0 \qquad ....(2)$$

From equation (1) – (2), we have

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_c \frac{f(w)}{w - \dfrac{R^2}{z}} dw$$

$$= \frac{1}{2\pi i}\int_c \left\{ \frac{1}{w-z} - \frac{1}{w - \dfrac{R^2}{z}} \right\} f(w)\, dw$$

Put $z = re^{-\theta}\overline{z} = re^{-i\theta} w = \mathrm{Re}^{i\theta}, dw = Rie^{i\phi}d\phi$

$$f\left(re^{i\theta}\right) = \frac{1}{2\pi i}\int_c \frac{\left(re^{i\theta} - \dfrac{R^2}{r}e^{i\theta}\right) f\left(\mathrm{Re}^{i\phi}\right)}{\left(\mathrm{Re}^{i\phi} - re^{i\phi}\right)\left(\mathrm{Re}^{i\phi} - \dfrac{R^2}{r}e^{i\theta}\right)} .Rie^{i\theta}d\phi$$

$$=\frac{1}{2\pi}\int_{o}^{2\pi}\frac{e^{i\theta}\left(r^2-R^2\right)f\left(\mathrm{Re}^{i\phi}\right)\mathrm{Re}^{i\phi}\,d\phi}{r\left(\mathrm{Re}^{i\phi}-re^{i\theta}\right)e^{i\phi}e^{i\phi}R\left(e^{-i\theta}-\frac{R}{r}e^{-i\phi}\right)}$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{\left(r^2-R^2\right)f\left(\mathrm{Re}^{i\phi}\right)d\phi}{\left(\mathrm{Re}^{i\phi}-re^{i\theta}\right)\left(re^{-i\theta}-\mathrm{Re}^{-i\phi}\right)}$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{\left(r^2-R^2\right)f\left(\mathrm{Re}^{i\phi}\right)d\phi}{-R^2+\mathrm{Rre}^{i(\theta-\phi)}+\mathrm{Rre}^{-i(\theta-\phi)}-r^2}$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{\left(R^2-r^2\right)f\left(\mathrm{Re}^{i\phi}\right)d\phi}{R^2+\mathrm{Rr}\left(e^{i(\theta-\phi)}+e^{-i(\theta-\phi)}\right)+r^2}$$

$$=\frac{1}{2\pi}\int_{0}^{2\pi}\frac{\left(R^2-r^2\right)f\left(\mathrm{Re}^{i\phi}\right)d\phi}{R^2+2\,\mathrm{Rr}\,\cos(\theta-\phi)+r^2}\qquad\left\{\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}\right\}$$

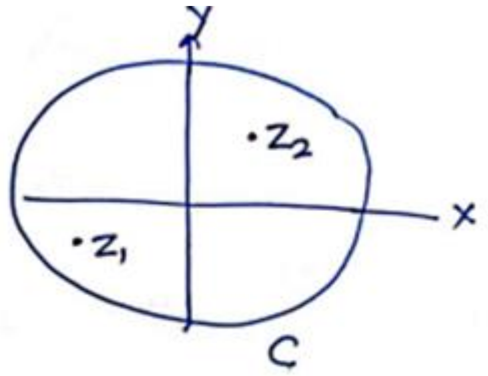
9.4 Liouville’s theorem

Liouville's theorem is a fundamental result in complex analysis that places significant constraints on the behavior of entire functions, which are functions that are holomorphic (analytic) on the entire complex plane. The theorem states that every bounded entire function must be constant. Liouville's theorem has important applications in complex analysis. For example, it is used to prove the fundamental theorem of algebra, which states that every non-constant polynomial with complex coefficients has at least one complex root.

Theorem: If a function If a function $f(z)$ is analytic for all finite value of z and is bounded then it is constant.

Proof:- Since $f(z)$ is bounded so there exist a positive integer M such that

$$|f(z)|\le M \qquad \ldots(1)$$



Let z_1 and z_2 be two point of z plane take the contour C to be a large circle with it can be origin and including points z_1 and z_2 that R be the radius of circle then $R > |z_1|, R > |z_2|$

By Cauchy integral formula, we have

$$f(z_1) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_1}dz,$$
$$f(z_2) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z - z_2}dz$$

Now we have

$$f(z_2) - f(z_1) = \frac{1}{2\pi i}\int_c \left\{\frac{1}{z - z_2} - \frac{1}{z - z_1}\right\} f(z)\, dz$$

$$|f(z_2) - f(z_1)| = \left|\frac{1}{2\pi i}\int_c \frac{(z_2 - z_1)}{(z - z_2)(z - z_1)} f(z) dz\right|$$

$$\leq \frac{1}{2\pi}\int_c \frac{|z_2 - z_1|}{|z - z_2||z - z_1|}|f(z)||dz|$$

$$\leq \frac{M}{2\pi}|z_2 - z_1|\int_c \frac{|dz|}{(R - |z_2|)(R - |z_1|)} \qquad \left(\int_c |dz| = 2\pi R\right)$$

$$\leq \frac{M|z_2 - z_1|}{2\pi R^2}\int_c |dz| \left\{1 - \frac{|z_2|}{R}\right\}\left\{1 - \frac{|z_1|}{R}\right\}$$

$$\leq \frac{M(z_2 - z_1)}{2\pi R^2} . 2\pi R \left\{1 - \frac{|z_2|}{R}\right\}\left\{1 - \frac{|z_1|}{R}\right\}$$

This implies for $R \to \infty$ then $|f(z_2) - f(z_1)| \to 0$

Consequently $f(z_1) = f(z_2)$.

Hence $f(z)$ is constant.

Check your Progress

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1. State the Morera's theorem.

2. State the Liouville's theorem.

9.5 Fundamental Theorem of Integral Calculus for Complex Function

The Fundamental Theorem of Integral Calculus for complex functions is a result that relates the integral of a function over a curve in the complex plane to its antiderivative (or primitive) along that curve. It is analogous to the Fundamental Theorem of Calculus for real functions, but it applies to complex-valued functions of a complex variable.

Theorem: Let $f(z)$ be single valued analytic function in a simply connected domain D if $a, b \in D$. Then we have

$$\int_a^b f(z)\,dz = F(b) - F(a), \quad \text{where } F(z) \text{ is any indefinite integral of } f(z).$$

Proof:- Let $\phi(z) = \int_a^b f(w)\,dw$ …….(1)

Then by theorem (1) above the indefinite integral $F(z)$ of $f(z)$ is given by

$$F(z) = \phi(z) + c$$

$\therefore \quad F(b) = \phi(b) + c$

and $\quad F(a) = \phi(a) + c$

Thus we have

$$F(b) - F(a) = \phi(b) - \phi(a) \qquad \ldots\ldots.(2)$$

Now from equation (1) we have

$$\phi(b) = \int_a^b f(w)dw$$

and $$\phi(a) = \int_a^b f(w)dw = \int_a^b f(z)dz \qquad \ldots\ldots.(3)$$

Since $F(z)$ is an indefinite integral of $f(z)$.

Now by the definition, we have $F'(z) = f(z)$

Hence the equation (3) may be written in the form

$$\int_a^b F'(z)dz = F(b) - F(a).$$

9.6 Maximum Modulus Principle

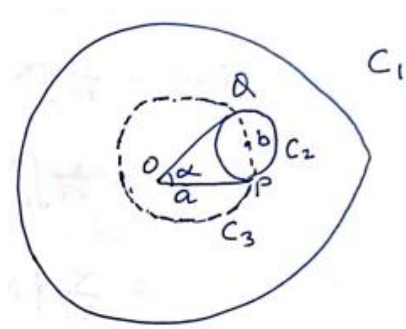
The Maximum Modulus Principle is a fundamental result in complex analysis that describes the behavior of holomorphic (analytic) functions on bounded domains. he principle can be used to prove properties of harmonic functions, which are real or complex valued functions that satisfy Laplace's equation. For example, it can be used to show that the maximum (or minimum) of a harmonic function in a domain occurs on the boundary.

Theorem. Let $f(z)$ be analytic within and on a simple closed contour c. Then $|f(z)|$ reaches its maximum value on c unless $f(z)$ is a constant in other words if M is the maximum value of $|f(z)|$ on and within c then unless f is constant. $|f(z)| < M$ for every within C.

Proof:- Since $f(z)$ is analytic then $f(z)$ continuous inside and on C. It follows that $|f(z)|$ does

have a maximum value M for at least one value of z inside or on C. suppose this maximum value is not attained on the boundary of c but it attained at an interior point i.e.

$$|f(a)| \leq M$$



Let c_1 be a circle inside C with centre at a if we exclude $f(z)$ from being constant inside c_1 then there must be a Point inside c_1 say b, such that

$$|f(b)| < M$$

Let $|f(b)| = M - \in$, where $\in > 0$

Since $f(z)$ is continuity at b for any $\in > 0$ we can find $\delta > 0$ st

$$\left||f(z)| - |f(b)|\right| < \frac{\in}{2} \quad \text{wherever } |z - b| < \delta \qquad \ldots(1)$$

i.e. $$|f(z)| < |f(b)| + \frac{\varepsilon}{2} = \mathrm{M} - \in + \frac{\in}{2} = \mathrm{M} - \frac{\in}{2} \qquad \ldots(2)$$

for all points interior to a circle c_2 with centre at b and radius b. Construct a circle c_3 with centre at a which passes through b. on part of this circle c_2 included PQ in it.

By equation (2) $|f(z)| < M - \frac{\in}{2}$ on the remaining part of circle c_3 we have

$$\left|f(z)\right| \leq M \ .$$

Radius of circle $c_3 \left|b-a\right| = r$, say By Cauchy integral formula

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}dz$$

Putting $z-a = re^{i\theta}$, we have

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f\left(a+re^{i\theta}\right)}{re^{i\theta}} rie^{i\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} f\left(a+re^{i\theta}\right) d\theta$$

If we measure a counterclockwise from OP and Let $< PoQ = \propto$ we have

$$f(a) = \frac{1}{2\pi}\int_0^{\propto} f\left(a+re^{i\theta}\right) d\theta + \frac{1}{2\pi}\int_{\propto}^{2\pi} f\left(a+re^{i\theta}\right) d\theta$$

$$\left|f(a)\right| \leq \frac{1}{2\pi}\int_0^{\propto} \left|f\left(a+re^{i\theta}\right)\right| d\theta + \frac{1}{2\pi}\int_{\propto}^{2\pi} \left|f\left(a+re^{i\theta}\right)\right| d\theta$$

$$< \frac{1}{2\pi}\int_0^{\propto} \left(M - \frac{\in}{2}\right) d\theta + \frac{1}{2\pi}\int_{\propto}^{2\pi} M \, d\theta$$

$$= \frac{\propto}{2\pi}\left\{M - \frac{\in}{2}\right\} + \frac{M}{2\pi}\left\{2\pi - \propto\right\}$$

$M = \left|f(a)\right| < M - \dfrac{\propto \in}{2\pi}$ which is contradiction

Hence $\left|f(z)\right|$ cannot attain its maximum at any interior point of C and so must attain its maximum on C.

Minimum Modulus Theorem

Let $f(z)$ be analytic inside and on a closed contour c and let $f(z) \neq 0$ inside C Then $\left|f(z)\right|$ must reach its maximum value on C.

Proof:- Since $f(z)$ is analytic inside and on c and since $f(z) \neq 0$ inside c, it follows that $1/f(z)$ is analytic inside C.

By the maximum modulus theorem it follows that $1/f(z)$ cannot assume its maximum value inside c and so $|f(z)|$ cannot assume its maximum value inside c.

Then since $|f(z)|$ has a minimum this minimum must be attained on C.

9.7 Summary

If $f(z)$ is a continuous in a region D and if the integral $\int_c f(z)\,dz$ taken around on a simply closed contour in D is zero then $f(z)$ is analytic function inside D i.e., $\int_c f(z)\,dz = 0$.

If a function If a function $f(z)$ is analytic for all finite value of z and is bounded then it is constant.

Let $f(z)$ be single valued analytic function in a simply connected domain D if $a, b \in D$. Then we have $\int_a^b f(z)\,dz = F(b) - F(a)$, where $F(z)$ is any indefinite integral of $f(z)$.

Let $f(z)$ be analytic within and on a simple closed contour c. Then $|f(z)|$ reaches its maximum value on c unless $f(z)$ is a constant in other words if M is the maximum value of $|f(z)|$ on and within c then unless f is constant. $|f(z)| < M$ for every within C.

9.8 Terminal Questions

Q.1. State and prove Morera's theorem.

Q.2. State and prove Liouville's theorem.

Q.3. Explain the Maximum Modulus Principle.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-10: Series

Structure

10.1 Introduction

In this unit we discuss the two types of series. Taylor's series and Laurent's series, which are very important series in complex analysis. This unit is devoted mainly to series representations of analytic functions. T Power series are important in complex analysis because they can represent many analytic functions, meaning functions that are locally given by convergent power series. Analytic functions play a fundamental role in complex analysis, and power series provide a powerful tool for studying them.he direct generalization of the Taylor's series of real function is complex Taylor's series. Laurent series is useful in evaluation of real and complex integrals and summations of series. A series of the type

$\sum_{n=0}^{\infty} a_n (z-z_0)^n = a_0 (z-z_0) + ...+ a_n (z-z_0)^n + ...$ is called a power series.

10.2 Objectives

After reading this unit the learner should be able to understand about:

- the entire function
- the Taylor's theorem and its applications
- the Laurent's theorem and its applications

10.3 Entire Function

A function which has no singularities in the finite part of the plane is called an entire function.In entire function in complex analysis is a function that is holomorphic (complex differentiable) at every point z in the complex plane C. In other words, an entire function is a function *f*: C→C that is complex differentiable at every complex number.

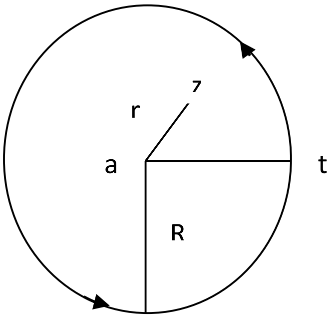
An entire function can be represented by its Taylor series expansion around any point in C. Since an entire function is holomorphic everywhere, its Taylor series converges to the function itself for all z in C. This property distinguishes entire functions from more general functions that may only be holomorphic in some region of the complex plane.

10.4 Taylor's Theorem

The Taylor series provides a way to approximate a function by a polynomial near a point. If the function is analytic (holomorphic) in a neighborhood of z_0, then its Taylor series converges to the function in that neighborhood.

If a function f(z) is analytic within a circle C with its centre z = a and radius R then at every point z inside C

$$f(z) = \sum_{n=1}^{\infty} f^{(n)}(a)\frac{(z-a)^n}{n!}$$



or $$f(z) = \sum_{n=1}^{\infty} a_n (z-a)^n$$

where $a_n = \dfrac{f^{(n)}(a)}{n!}$.

Proof. Let *f(z)* be analytic within a circle C whose equation in circle

$$|t-a| = R.$$

Let *z* be any point with in C such that

$|t-a| = r < R$

By Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-z}dt = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-a)-(z-a)} \qquad \ldots(1)$$

We have $$\frac{1}{t-z} = \frac{1}{(t-a)-(z-a)}$$

$$=\frac{1}{(t-a)}\left[1-\left(\frac{z-a}{t-a}\right)\right]^{-1}$$

$$=\frac{1}{(t-a)}\left[1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^{2}+.....+\left(\frac{z-a}{t-a}\right)^{n}+......\right]$$

$$\frac{f(t)}{t-z}=\frac{f(t)}{(t-a)}+\frac{(z-a)}{(t-a)^{2}}f(t)+.................+\frac{(z-a)^{n}}{(t-a)^{n+1}}\left\{1-\frac{z-a}{t-a}\right\}^{-1}$$

Expansion expresse we find series up to infinite n to n +1….∞

Now integrate each term around C and divide by 2πi

$$\frac{1}{2\pi i}\int_{C}\frac{f(t)dt}{t-z}=\frac{1}{2\pi i}\int_{C}\frac{f(t)}{(t-a)}f(t)+\frac{1}{2\pi i}\int_{C}\frac{f(t)(z-a)}{(t-a)^{2}}dt+....$$

$$+\frac{1}{2\pi i}\int_{C}\frac{(z-a)^{n}.\ f(t)}{(t-a)^{n+1}}\left\{1-\frac{z-a}{t-a}\right\}^{-1}dt$$

Now by equation (1), we have

$$f(z)=f(a)+\frac{(z-a)}{1!}f'(a)+\frac{(z-a)^{2}}{2!}f''(a)+.....$$

$$+\frac{(z-a)^{n}}{n!}f^{(n)}(a)+u_{n+1}\qquad ....(2)$$

where

$$\left.\begin{aligned}f(a)&=\frac{1}{2\pi i}\int_{C}\frac{f(t)dt}{t-a}\\ f'(a)&=\frac{1}{2\pi i}\int_{C}\frac{f(t)dt}{(t-a)^{2}}\\ &.\\ &.\\ f^{n+1}(a)&=\frac{(n+1)!}{2\pi i}\int_{C}\frac{f(t)dt}{(t-a)^{n+2}}\end{aligned}\right\}$$

and

$$U_{n+1}=\frac{(z-a)^{n+1}}{(n+1)!}f^{n+1}(a)\left[1-\frac{z-a}{t-a}\right]^{-1}\qquad .....(3)$$

$$= \frac{(z-a)^{n+1}}{(n+1)!} \quad f^{n+1}(a)\left(\frac{t-a}{t-z}\right) \qquad \text{(Using equation (3))}$$

$$= \frac{(z-1)^{n+1}}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}\left(\frac{t-a}{t-z}\right)$$

$$|U_{n+1}| \le \frac{|z-a|}{2\pi\,|t-a|}\int_C \frac{|f(t)||dt|}{|t-z|} \qquad \left| \begin{array}{l} M = \text{max, value of } |\mathrm{f}(\mathrm{x})| \, on\ C \\ \int_C |dt| = 2\pi R \end{array} \right.$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\int_C \frac{|dt|}{(|t-a|-|z-a|)}$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{1}{R-r}\int_C |dt|$$

$$\le \frac{M}{2\pi}\left(\frac{r}{R}\right)^{n+1}\frac{1}{R\left(1-\frac{r}{R}\right)}.2\pi R = M\left(\frac{r}{R}\right)^{n+1}\frac{1}{1-\frac{r}{R}}$$

For n $\to \infty$ $U_{n+1} \to 0$ or $\left\{ for \ \lim_{n\to\infty}\left(\frac{r}{R}\right)^{n+1} = 0 \ as \ \frac{r}{R} < 1 \right.$

Now by equation (2), we have

$$f(z) = \lim_{n\to\infty}\left[f(a) + \frac{(z-a)}{1!} f'(a) + \ldots + \frac{(z-a)^n}{n!} f^n(a)\right]$$

$$= \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!} f^n(a)$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n, \ where \ a_n = \frac{f_n(a)}{n!}$$

This is known as Taylor's series.

If we put a = 0 in the equation (3), we get $f(z) = \sum_{n=0}^{\infty} a_n z^n$

This known as Maclarin's series.

Check your Progress

1. What do you mean by entire function.

2. State the Taylor's theorem.

11.5 Laurent's Theorem

The Laurent series allows us to analyze the behavior of functions near singularities, such as poles and essential singularities. It provides a powerful tool for understanding complex functions and their properties in the complex plane.

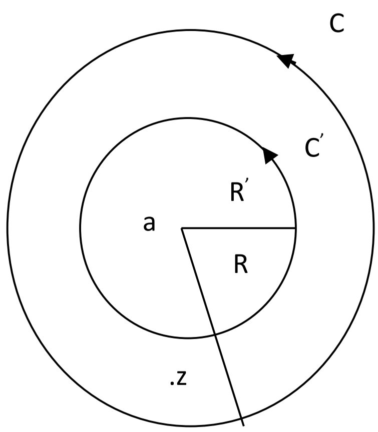
Suppose a function *f(z)* is analytic in the closed ring bounded by two concentric circles C and C' of centre *a* and radii *R* and *R'(R' < R).* If *z* is any point of the annulus. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{n+1}}, \quad b_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^{-n+1}}$$

Proof. Let *f(z)* be analytic in the closed ring bounded by two concentric circles and C and C' of centre *a* and radii *R and R' (R' < R).*

Then if *z* is any point within the ring space, then



$R' < |z - a| = r < R$

By Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z)} - \frac{1}{2\pi i}\int_{C'} \frac{f(t)\,dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-a)-(z-a)} + \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(z-a)-(t-a)}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-a)}\left[1-\frac{z-1}{t-a}\right]^{-1} dt + \frac{1}{2\pi i}\int_C \frac{f(t)}{(z-a)}\left[1-\frac{t-1}{z-a}\right]^{-1} dt$$

$$= \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{t-a}\left[1+\left(\frac{z-a}{t-a}\right)+\left(\frac{z-a}{t-a}\right)^2 +\ldots+\left(\frac{z-a}{t-a}\right)^n+\left(\frac{z-a}{t-a}\right)^{n+1}\left[1-\frac{z-a}{t-a}\right]\right]^{-1} dt$$

$$+ \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left[1+\left(\frac{t-a}{z-a}\right)+\ldots+\left(\frac{t-a}{z-a}\right)^n+\left(\frac{t-a}{z-a}\right)^{n+1}\left\{1-\frac{z-a}{t-a}\right\}^{-1}\right]dt \quad \ldots(1)$$

We let $a_n = \int_C \frac{f(t)dt}{(t-a)^{n+1}} \;.b_n = \frac{1}{2\pi i}\frac{f(t)}{(t-a)^{-n+1}}\Big|\,dt = a_{-n}$

Then by equation (1), we have

$$f(z) = [a_0 + (z-a)a_1 + \ldots + a_n(z-a)^n + U_{n+1}]$$

$$+ \left[\frac{b_1}{(z-a)} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_n}{(z-a)^n} + V_{n+1}\right] \quad \ldots(2)$$

where

$$U_{n+1} = \frac{1}{2\pi i}\int_C \frac{f(t)}{t-a}\left(\frac{z-a}{t-a}\right)^{n+1}\left\{1-\frac{z-a}{t-a}\right\}^{-1} dt \qquad \left[\because \left(1-\frac{z-a}{t-a}\right)^{-1} = \left(\frac{t-z}{t-a}\right)^{-1} = \left(\frac{t-a}{t-z}\right)\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)(z-a)^{n+1}}{(t-a)(t-a)^{n+1}}\left(\frac{t-a}{z-a}\right)dt$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)(z-a)^{n+1}}{(t-z)(t-a)^{n+1}}\, dt$$

and $V_{n+1} = \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\left(\frac{t-a}{z-a}\right)^{n+1}\left\{1-\frac{t-a}{z-a}\right\}^{-1} dt$

$$= \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{z-a}\,\frac{(t-a)^{n+1}}{(z-a)^{n+1}}\left(\frac{z-a}{z-t}\right)dt$$

$$= \frac{1}{2\pi i}\int_{C'} \frac{f(t)}{(z-1)}\,\frac{(t-a)^{n+1}}{(z-a)^{n+1}}\quad dt \qquad \ldots(4)$$

Let *M = max, value of |f(t)| on C*

M' = max, value of |f(t)| on C'

From equation (3), we have

$$|z-a| = r$$
$$|z-a| = R$$
$$|t-z| = (t-1)-(z-a) = R-r$$

$$\left|\int_C |dt| = 2\pi R\right.$$

$$U_{n+1} = \frac{1}{2\pi i}\int_C \frac{f(t)(z-a)^{n+1}}{(t-z)(t-a)^{n+1}}\, dt$$

$$|U_{n+1}| = \frac{1}{2\pi}\int_C \frac{f(t)|z-a|^{n+1}}{|t-z||t-a|^{n+1}}\, dt$$

$$= \frac{M}{2\pi}\int_C \frac{r^{n+1}}{(R-r)R^{n+1}}\,|dt|$$

$$= \frac{M}{2\pi}\int_C \frac{r^{n+1}}{R^{n+1}}\,\frac{1}{R-r}\,.2\pi R$$

$$= M\left(\frac{r}{R}\right)^{n+1}\frac{1}{1-\frac{r}{R}}$$

$$\lim_{n=\infty}|U_{n+1}|=\lim_{n+\infty} M\left(\frac{r}{R}\right)^{n+1}\frac{1}{1-\frac{r}{R}}\to 0\, as\frac{r}{R}<\,1 \qquad \ldots(5)$$

Now from equation (4), we have

$$|V_{n+1}|\leq\frac{1}{2\pi}\int_{C'}\frac{|f(t)|}{|z-t|}\frac{|t-a|^{n+1}}{|z-a|^{n+1}}|\,dt\,| \qquad \Big|\; |z-1|=|(z-a)-(t-a)|=r-R'$$

Solving above

$$\lim_{n=\infty}|V_{n+1}|=\lim_{n+\infty} M\left(\frac{R}{r}\right)^{n+1}\frac{1}{\frac{r}{R'}-1}\to 0 \quad as\ \frac{R}{r'}<\,1\ or\ \frac{r}{R'}>1 \qquad \ldots(6)$$

Now by equation (2), using equations (5) and (6), we have

$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

where $$a_n=\frac{1}{2\pi i}\int_C\frac{f(t)dt}{(t-a)^{n+1}},\ b_n\ \frac{1}{2\pi i}\int_C\frac{f(t)dt}{(t-a)^{-n+1}},$$

Examples

Example.1. Expand the following functions in a Taylor's series about $z = 0$ and determine the region of convergence in each case.

Sol. *(i)* Let $f(z) = e^z$. Then $f^{(n)}(z) = e^z$

Hence $f(0) = e^o = 1, f^{(n)}(z) = e^z$

Since e^z is analytic for every value of z, we have

$$f(z)=e^z=f(0)+\frac{z}{1!}f'(0)+\frac{z^2}{2!}f''(0)+\ldots\ldots$$

$$= 1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots\ldots\ldots = \sum_{n=0}^{\infty} \frac{z^n}{n!},\ where\ |z| = \infty$$

(ii) Let $f(z) = \sin z$

Then $f'(z) = \cos z, f''(z) = -\sin z, f'''(z) = -\cos z$ etc

$$f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(z) = -1$$

In general $f^{(2n-1)}(0) = (-1)^{n+1}$

Since sin z is analytic for all value of z, we have

$$f(z) = \sin z = f(0) + \frac{z}{1!} f'(0) = \frac{z^2}{2!} f''(0) + \ldots\ldots\ldots$$

$$= 0 + \frac{z}{1} + 0 - \frac{z^3}{3!} \ldots. + \frac{z^{2n-1}}{(2n-1)!}(-1)^{n+1} + \ldots\ldots.$$

$$= \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \ldots. + \frac{z^{2n-1}}{(2n-1)!}(-1)^{n+1} + \ldots\ldots.$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-1}}{(2n-1)},\ when\ |z| < \infty.$$

(iii) Let $f(z) = \cos z$

We can show that above part *(ii)*

$$\cos z = 1 + \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{2n!}\ when\ |z| < \infty$$

(iv) Let

$$\begin{aligned} f(z) &= \sinh z & f(0) &= 0 \\ f'(z) &= \cosh z & f'(0) &= 1 \\ f''(z) &= \sinh z & f''(0) &= 0 \\ f'''(z) &= \cosh z & f'''(0) &= 1 \\ f^{iv}(z) &= \sinh z & f^{iv}(0) &= 0 \end{aligned}$$

$$\cosh z = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + ...$$

$$= 1 + 0 + \frac{z^2}{2!} + 0 + \frac{z^4}{4!} + ...$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + ... = \sum_{n=0}^{\infty} \frac{z^{2n}}{2n!}, \ when\ |z| < \infty$$

Example 2. *Show that* $e^z = e + e\sum_{n=0}^{\infty} \frac{(z-1)}{2n!}$ *when* $|z| < \infty$.

Sol. $f(z) = e^z$.

We know that $e^z = 1 + \sum_{n=1}^{\infty} \frac{z}{n!}, \ |z| < \infty$

Replacing $z = z - 1$, *we get*

$$e^{z-1} = 1 + \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$$

Multiply by e both sides, we get

$$e^z = e + e\sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}.$$

Example 3. $(a) \frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n, \ when |z-2| < 1.$

$(b) \frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n, \ when\ |z-2| < 1.$

(c) $\frac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n, \ when |z| < 1$

Sol.(a) *Let* $f(z) = \frac{1}{z^2} = \frac{1}{[(z+1)-1]^2} = \frac{1}{[1-(z+1)]^2} = [1-(z+1)]^{-2}$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + ...$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n$$

(b) $\quad Let \quad f(z) = \dfrac{1}{z^2} = \dfrac{1}{[2+z-2]^2} = \dfrac{1}{4\left[1+\dfrac{z-2}{2}\right]^2} = \dfrac{1}{4}\left[1+\dfrac{z-2}{2}\right]^{-1}$

$$= \frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2 - 4\left(\frac{z-2}{2}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{4}+\frac{1}{4}\left[-2\frac{(z-2)}{2}+3\left(\frac{z-2}{2}\right)^2 - \ldots\ldots.\right]$$

$$= \frac{1}{4}+\frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n, \; when |\, z-2\,| < 1.$$

(c) $\qquad f(z) = \dfrac{1}{1+z}, \; f(0) = 1$

$$f'(z) = -\frac{1}{(1+z)^2}\;, \; f'(0) = -1$$

$$f''(z) = \frac{2}{(1+z)^3}, \; f''(0) = 2$$

$$f'''(z) = -\frac{6}{(1+z)^4}, \; f'''(0) = -6$$

$$f^{iv}(z) = \frac{24}{(1+z)^5}, \; f^{iv}(0) = 24$$

$$\frac{1}{1+z} = f(0)+\frac{z}{1!}f'(0)+\frac{z^2}{2!}f''(0)+\frac{z^3}{3!}f'''(0)+\ldots\ldots.$$

$$= 1-\frac{z}{1!}+\frac{2z^2}{2!}-\frac{6z^3}{3!}+24\frac{z^4}{4!}-\ldots\ldots..$$

$$= 1-z+z^2-z^3+z^4-\ldots\ldots..$$

$$\frac{1}{1+z} = \sum_{n=1}^{\infty} (-1)^n z^n \quad when\ |z| < 1.$$

Example 4. Explain cos z into a Taylor series about the point $z_0 = \frac{\pi}{2}$.

Sol. Let $f(z) = \cos z$

We know that the Taylor series

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \ldots\ldots\ldots$$

$$\begin{array}{lll} f(z) = \cos z & f(\pi/2) = 0 & f(0) = 1 \\ f'(z) = -\sin z & f'(\pi/2) = -1 & f'(0) = 0 \\ f''(z) = -\cos z & f''(\pi/2) = 0 & f''(0) = -1 \\ f'''(z) = \sin z & f'''(\pi/2) = 1 & f'''(0) = 0 \\ f^{iv}(z) = \cos z & f^{iv}(\pi/2) = 0 & f^{iv}(0) = 1 \\ f^{v}(z) = -\sin z & f^{v}(\pi/2) = -1 & f^{v}(0) = 0 \end{array}$$

$$\cos z = f(\pi/2) + \frac{\left(z - \frac{\pi}{2}\right)}{1!} f'(\pi/2) + \frac{\left(z - \frac{\pi}{2}\right)^2}{2!} f''(\pi/2) + \ldots..$$

$$= \quad 0 - \frac{\left(z - \frac{\pi}{2}\right)}{1!} + 0 + \frac{\left(z - \frac{\pi}{2}\right)^3}{3!} + \ldots\ldots$$

$$= -\left(z - \frac{\pi}{2}\right) + \frac{\left(z - \frac{\pi}{2}\right)^3}{3!} - \frac{\left(z - \frac{\pi}{2}\right)^5}{5!} + \ldots\ldots.$$

$$\cos z = \sum_{n=1}^{\infty} (-1)^n \frac{\left(z - \frac{\pi}{2}\right)^{2n-1}}{(2n-1)!}.$$

Example 5. Prove that when $0<|z|<4, \dfrac{1}{4z-z^2} = \sum_{n=0}^{\infty}\dfrac{z^{n-1}}{4^{n+1}}$

Sol. $f(z) = \dfrac{1}{4z-1} = \dfrac{1}{4z-z^2} = \dfrac{1}{4z\left(1-\frac{z}{4}\right)} = \dfrac{1}{4z}\left[1-\dfrac{z}{4}\right]^{-1}$

$$=\frac{1}{4z}\left[1+\frac{z}{4}+\frac{z^2}{4^2}+\frac{z^3}{4^3}+.......\right]$$

$$= \frac{1}{4z}+\frac{1}{4^2}+\frac{z}{4^3}+\frac{z^2}{4^4}+.......$$

$$= \frac{1}{4z}+\frac{1}{4^2}+\frac{z}{4^3}+\frac{z^2}{4^4}+.......$$

$$\frac{1}{4z-z^2} = \sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}.$$

Example 6. Represent the function $f(z)=\dfrac{z}{(z-1)(z-3)}$ by a series involving positive and negative powers of *z-1* which converges to *f(z)* when *0 < 1 |z – 1| < 2.*

Sol. $f(z) = \dfrac{z}{(z-1)(z-3)},\ putting\ z-1=u\ in\ f(z)$

$\Rightarrow z=u+1$

$f(u)=\dfrac{u+1}{u(u-2)} = -\dfrac{1}{2u}+\dfrac{3}{2(u-2)}$ $0<|u|<2$

$$= -\frac{1}{2u}+\frac{3}{2(-2)\left(1-\frac{u}{2}\right)} = -\frac{1}{2u}-\frac{3}{4}\left[1-\frac{u}{2}\right]^{-1}$$

$$= -\frac{1}{2u} - \frac{3}{4} + \left[1 + \frac{u}{2} + \frac{u^2}{2^2} + ...\right]$$

$$= \frac{u^{-1}}{2} - \frac{3}{2^2} - \frac{3u}{2^3} - \frac{3u^2}{2^4} - ....$$

$$= -\frac{1}{2}(z-1)^{-1} - 3\left[\frac{1}{2^2} - \frac{u}{2^3} - \frac{u^2}{2^4} - ....\right]$$

$$f(z) = -\frac{1}{2}(z-1)^{-1} - 3\sum_{n=1}^{\infty}\frac{(z-1)^n}{2^{n+2}}, \; where \; z = u + 1.$$

Example 7. Prove that $z \neq 0$.

$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + ...$$

Sol. Clearly the given function is analytic at $z \neq 0$.

$$f(z) = \frac{\sin(z^2)}{z^4} \qquad ...(1)$$

Replacing $z^2 = u$ in equation (1), we get

$$f(u) = \frac{\sin u}{u^2}$$ Now choosing $f(u_1) = sin\ u$

We know that

$$f(u_1) = \sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - .....$$

Now $$f(u) = \frac{\sin u}{u^2} = \frac{1}{u^2}\left[u - \frac{u^2}{3!} + \frac{u^5}{5!} - ....\right]$$

$$= \left[\frac{1}{4} - \frac{u}{3!} + \frac{u^3}{5!} - ....\right]$$

Again replacing $u = z^2$

$$\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \ldots\ldots..$$

Example 8. *Expand log (1 + z) in a Taylor's series about z = 0 and determine the region of convergence for the resulting series.*

Sol. $f(z) = \log(1+z)$ $\qquad f(0) = 0$

Then $f'(z) = \dfrac{1}{1+z}$ $\qquad f'(0) = 1$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z) = \frac{2}{(1+z)^3} \qquad f'''(0) = 2$$

$$f^{iv}(z) = -\frac{6}{(1+z)^4} \qquad f^{iv}(0) = -6$$

Therefore $f(z) = \log(1+z) = f(0) + \dfrac{z}{1!} f'(0) + \dfrac{z^2}{2!} f''(0) + \ldots..$

$$= 0 + \frac{z}{1!} - \frac{z^2}{2!} + 2.\frac{z^3}{3!} - 6\frac{z^4}{4!} + \ldots\ldots.$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots + (-1)^{n-1}\frac{z^n}{n} + \ldots\ldots$$

If U_n denotes that *nth* term of the series then

$$U_n = (-1)^{n-1}\frac{z^n}{n},\ U_{n+1} = \frac{(-1)^{n-1} z^{n+1}}{n+1}$$

$$\therefore \lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{nz}\right| = \frac{1}{|z|}$$

Hence by D' Alembert's ratio test, the series converge for $|z| < 1$.

The series can be shown to converge for $|z| = 1$ except for $z = -1$.

Note that $z = -1$ is the singularity of log (1 + z) nearest the point z = 0. Thus the series converges for all values of z within the circle |z| = 0.

Example 9. *Expand* $\dfrac{1}{z^2 - 3z + 2}$ *for*

$(i)\ 0 < |z| < 1,$ $\qquad$ $(ii)\ 1 < |z| < 2,$ $\qquad$ $(iii)\ |z| > 2$

Sol.

Let $\qquad f(z) = \dfrac{1}{z^2 - 3z + 2} = \dfrac{1}{(z-1)(z-2)}$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

(i) When $0 < |z| < 1$

By equation (1), we have

$$f(z) = -\frac{1}{2}\frac{1}{\left(1 - \frac{z}{2}\right)} + \frac{1}{(1-z)}$$

$$= (1-z)^{-1} - \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

$$= \sum_{n=0}^{\infty} z^n \left(1 - \frac{1}{2^{n+1}}\right).$$

which is Maclaurin's series

(ii) When $\qquad 1 < |z| < 2$

Then $\qquad \dfrac{1}{|z|}<1, \dfrac{|z|}{2}<1$

By equation (1), we have

$$f(z)=-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$
$$=-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^{n}-\frac{1}{z}\sum_{n=0}^{\infty}z^{-n}$$

which is Laurent's series

(iii) When $|z| > 2$.

Then $\dfrac{2}{|z|} < 1 \Rightarrow \dfrac{1}{|z|} < \dfrac{1}{2} < 1$

By equation (1), we have $f(z)=\dfrac{1}{z}\left(1-\dfrac{2}{z}\right)^{-1}-\dfrac{1}{z}\left(1-\dfrac{1}{z}\right)^{-1}$

$$=\frac{1}{z}\sum_{0}^{\infty}\left(\frac{2}{z}\right)^{n}-\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}$$

$$=\sum_{n=0}^{\infty}\frac{2^{n}}{2^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\left(2^{n}-1\right)$$

which is Laurent's series

Example 10. *Expand* $\dfrac{1}{z\left(z^{2}-2z+2\right)}$ *for the regions*

$(i)\ 0<|z|<1 \qquad (ii)\ 1<|z|<2 \qquad (iii)\ |z|>3$

Sol.

(i) When $0 < |z| < 1$, *then*

$$f(z)=\frac{1}{z(z^2-2z+1)}=\frac{1}{z(z-1)(z-2)}=\left[\frac{1}{2z}-\frac{1}{z-1}+\frac{1}{2(z-2)}\right]$$

$$=\frac{1}{2z}+[1+z+z^2+.....]-\frac{1}{4}\left[1+\frac{z}{2}+\frac{z^2}{2^2}+......\right]$$

$$=\frac{1}{2z}+\frac{3}{4}+\frac{7}{8}z+\frac{15}{16}z^2+......$$

(ii) Where $1 < |z| < 2$, Then

$$f(z)=\frac{1}{2z}+\frac{1}{z-1}-\frac{1}{4\left[1-\frac{z}{2}\right]}=\frac{1}{2z}-\frac{1}{z}\left[1-\frac{1}{z}\right]^{-1}-\frac{1}{4}\left[1-\frac{z}{2}\right]^{-1}$$

$$=\frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}-\frac{1}{6}\left[1+\frac{z}{3}\right]^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+......\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+.........\right]$$

$$=\left[\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+.........\right]-\left[\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+.....\right]$$

(iii) For$|z| > 3$, we have

$$f(z)=\frac{1}{2z}\left[1+\frac{1}{z}\right]^{-1}-\frac{1}{2z}\left[1+\frac{3}{z}\right]^{-1}$$

$$=\frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-........\right]-\frac{1}{2z}\left[1-\frac{3}{z}+\frac{3^2}{z^2}-........\right]$$

$$=\left[\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-.......\right]+\left[-\frac{1}{2z}+\frac{3}{2z^2}-\frac{9}{2z^3}+.........\right]$$

$$=\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-........$$

(iv) For $1 < |z+1| < 2$, we have

Let $z + 1 = u$, then $1 < |u| < 2$ and we have

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)} - = \frac{1}{2u} = \frac{1}{2(u+2)} = \frac{1}{2u} - \frac{1}{4}\left[1+\frac{u}{2}\right]^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left[1-\frac{u}{2}+\frac{u^2}{2^2}-........\right]$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + ...$$

$$= \frac{1}{2u} - \frac{1}{4} + \frac{(z+1)}{8} - \frac{(z+1)^2}{16} + ...$$

Example 11. Find the Taylor's and Laurent's series which represent the function $\frac{z^2-1}{(z+2)((z+3))}$

(i) When $|z| < 2$ *(ii) when* $2 < |z| < 3$ *(iii) when* $|z| > 1$

Sol. We have $f(z) = \frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{(z+2)} - \frac{8}{(z+3)}$

Now we find the expansion of $f(z)$

(i) When $|z| < 2$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left\{1-\frac{z}{2}+\frac{z^2}{4}-.......\right\} - \frac{8}{3}\left\{1-\frac{z}{3}+\frac{z^2}{9}-......\right\}$$

$$= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

(ii) When $2 < |z| < 3$

$$f(z)=1+\frac{3}{z+2}-\frac{8}{z+3}$$
$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$
$$=1+\frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n-\frac{8}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n.$$

Example 12. *Prove that cosh*

$$\left(z+\frac{1}{z}\right)=a_0+\sum_{l}^{\infty}a_n\left(z^n+\frac{1}{z^n}\right) \text{ where } a_n=\frac{1}{2\pi}\int^{2\pi}\cos n\theta\cosh(2\cos\theta)d\theta.$$

Sol. The function *f(z)* = cosh $\left(z+\frac{1}{z}\right)$ is analytic everywhere in the finite part of the plane except at z = 0.

i.e.., it is analytic in the annulus $r\le|z||\le R$ where r is small and R is large. Hence f(z) can be expanded in Laurent's series.

$$\cosh\left(z+\frac{1}{z}\right)=\sum_{n=0}^{\infty}a_u z^n+\sum_{n=0}^{\infty}b_n z^{-n}$$

where C is any circle with origin as center.

$$\left\{\text{where } a_n=\frac{1}{2\pi i}\int_C f(z)\frac{dz}{z^{n+1}},\ b_n=\frac{1}{2\pi i}\int_C f(z)z^{n-1}dz\right.$$

Taking *C*to be the circle |*z*| = *1*. So that $z=e^{i\theta}$, dz = $ie^{\theta}d\theta$

$$a_n=\frac{1}{2\pi i}\int_0^{2\pi}\cosh\left(e^{i\theta}+e^{-i\theta}\right)\frac{i^{e\theta}\,d\theta}{e^{i(n+1)\theta}}$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)e^{-ni\theta}d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\{\cos n\theta-i\sin n\theta\}\,d\theta$$
$$=\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta\,d\theta-\frac{i}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\sin n\theta\,d\theta$$
$$=\frac{1}{2\pi}\int_0^{2\pi}\cosh(2\cos\theta)\cos n\theta\,d\theta-0$$

Since by the integral property

$$\int_0^{2a} f(\theta)d\theta = 0 \; if \; f(2a-\theta) = -f(\theta)$$

Also $$\cosh\left(z+\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$$

$$= a_0 + \sum_{n=1}^{\infty} a_n \left(z^n + \frac{2}{z^n}\right)$$

where $$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh(2\cos\theta) \; \cos n\theta \; d\theta.$$

Example 13. Prove that

$$e^{\frac{u}{z}+vz} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $$a_n = \frac{1}{2\pi}\int_0^{2\pi} \exp\{(u+v)\cos\theta\} \cos\{(v-u)\sin\theta - n\theta\} \; d\theta$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \exp\{(u+v)\cos\theta\} \cos\{(u-v)\sin\theta - n\theta\} \; d\theta$$

Sol. *Let* $f(z) = e^{\frac{u}{z}+vz}$

Then *f(z)* is analytic everywhere except at $z = 0$. So, it can be expanded in the region $r < |z| < R$ in a Laurent's series where r is small and R is large but finite.

$$\therefore \qquad f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

where $$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}dz, \; b_n = \frac{1}{2\pi i}\int_C f(z)\, z^{n-1} dz$$

and C is any circle with centre at origin

Let C be the unit circle defined by $|z| = 1$. *Then* $z = e^{i\theta}$, $dz = ie^{i\theta}$, $dz = ie^{i\theta}\, d\theta$. Now we have

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\exp\{ue^{-i\theta} + ve^{i\vartheta}\}}{e^{i(n+1)\theta}}\, ie^{i\theta}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\, e^{(u+v)\cos\theta} e^{i[(v-u)\sin\theta - n\theta]} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\, \cos\{(v-u)\sin\theta - n\theta\} + i\sin\{(v-u\sin\theta - n\theta)\} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\, \cos\{(v-u)\sin\theta - n\theta\}\, d\theta + 0 \quad \{by\ \text{int}\ egral\ property\}$$

Also

$$b_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{(ue^{-\theta}+ve^{\theta})}\, e^{i(n-1)\theta} ie^{i\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{(u+v)\cos\theta}\, e^{i[(u-v1)\sin\theta - n\theta]} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta}\left[\cos\{(u-v)\sin\theta - n\theta\} - i\sin\{(u-v)\sin\theta - n\theta\}\right] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{(u+v)\cos\theta} \cos\{(u-v)\sin\theta - n\theta\} d\theta \quad [\text{the second integral becomes zero}]$$

Example 14. Show that $\sin\left\{C\left(z+\frac{1}{z}\right)\right\}$ can be expanded in a series of the type $\sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ where the coefficient of both z^n and z^{-n} are $\frac{1}{2\pi}\int_0^{2\pi} \sin(2C\cos\theta)\cos n\theta\, d\theta.$

Sol. *Let* $\quad f(z) = \sin\left\{C\left(z+\frac{1}{z}\right)\right\}$

Then *f(z)* is analysis everywhere except at z = 0. So it is analytic in the annulus $r < |z| < R$ where r is small and R is large. Therefore *f(z)* can be expressed in a Laurent's series in the region $r < |z| < R$.

Thus $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^{-n}$

where $a_n = \frac{1}{2\pi i}\int_C f(z)\frac{dz}{z^{n+1}}$, $b_n = \frac{1}{2\pi i}\int_C f(z) z^{n-1} dz$

and C is any circle with centre at origin.

Let C be the unit circle defined by |z| = 1

Then $z = e^{i\theta}, \; dz = ie^{i\theta}\, d\theta$

Now $$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \sin\left\{C\left(e^{i\theta} + e^{-i\theta}\right)\frac{ie^{i\theta} d\theta}{e^{i(n+1)\theta}}\right.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \sin\left(2C\cos\theta\right)e^{-in\theta} d\theta$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \sin\left(2C\cos\theta\right)\left(\cos n\theta - i\sin n\theta\right) d\theta$$

|Using property of odd integral II part is zero

$$= \frac{1}{2\pi i}\int_0^{2\pi} \sin\left(2C\cos\theta\right)\cos n\theta \; d\theta - 0$$

Since the given function remains unchanged if we put $\frac{1}{z}$ for z therefore we have

$$b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \sin\left(2\cos\theta\right)\cos\left(-n\theta\right) d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin\left(2\cos\theta\right)\cos n\theta \, d\theta = a_n$$

Hence $$\sin\left\{C\left(z + \frac{1}{z}\right)\right\} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

$$= \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$ [where $a_n = b_n$]

$$= a_0 + \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} a_n z^{-n}$$

$$\sin\left\{C\left(z+\frac{1}{z}\right)\right\}=a_0+\sum_{n=1}^{\infty}a_n\left(z_n+z^{n-1}\right)$$

where $$a_n=b_n=\frac{1}{2\pi}\int_0^{2\pi}\sin\left(2C\cos\theta\right)\cos\theta n\theta\ d\theta$$

Example 15. Show that $e^{\frac{1}{2}C\left(z-\frac{1}{2}\right)}=\sum_{m=-\infty}^{\infty}$ *where* $a_n=\dfrac{1}{2\pi}\int_0^{2\pi}\cos(n\theta-C\sin\theta)d\theta.$

Sol. Letting $f(z)=e^{\frac{1}{2}C\left(z-\frac{1}{2}\right)}$

The given function is analytic at every point the z-plane except at z = 0 so it is analytic in the annulus $r<|z|<R$, where r is small and R is large. Therefore it can be expanded in a Laurent's series in the region r < |z| < R.

$$f(z)=e^{\frac{1}{2}C\left(z-\frac{1}{z}\right)}=\sum_{n=0}^{\infty}a_nz^n+\sum_{n=1}^{\infty}b_nz^{-n}$$

Where $$a_n=\frac{1}{2\pi}\int_C\frac{f(z)}{z^{n+1}}dz,\ b_n=\frac{1}{2\pi}\int_C f(z)z^{n-1}dz$$

and C is any circle with the centre at origin.

Let C be the unit circle defined by |z| = 0. Then $z=e^{i\theta},\ dz=e^{i\theta}d\theta$

$$\therefore\qquad a_n=\frac{1}{2\pi i}\int_0^{2\pi}e^{\frac{1}{2}C\left(e^{i\theta}-e^{i\theta}\right)}.\ ie^{i\theta}\ d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}e^{\frac{1}{2}C(2i\sin\theta)}\ e^{-ni\theta}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}e^{i(C\sin\theta-n\theta)}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\left[\cos(C\sin\theta-n\theta)+i\sin\left(C\sin\theta-n\theta\right)\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\cos(C\sin\theta - n\theta)\,d\theta - i\sin(n\theta - C\sin\theta)d\theta\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - C\sin\theta)d\theta - 0 \qquad \left[\text{The second part of integral is zero}\right]$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos(n\theta - C\sin\theta)d\theta$$

Now the given function remains unchanged by replacing z by $\left(-\frac{1}{z}\right)$ therefore $b_n = (-1)^n a_n$.

Hence $e^{\frac{1}{2}C\left(z-\frac{1}{z}\right)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty}(-1)^n a_n z^{-n} = \sum_{-\infty}^{\infty} a_n z^n.$

Example 16. If the function f|z| is analytic and one valued in |z-a| < R. Prove that, when 0 < r < R.

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}\,d\theta \text{ where } P(\theta) \text{ is the real part of } f(a + re^{i\theta}).$$

Sol. Since f(z) is analytic in $|z| < R$ *and* $r < R$.

It follows that $f(z)$ is also analytic inside the circle C defined by $|z - a| = r$

Hence by Cauchy's formula for the derivative, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz = f'(a) \qquad \ldots(1)$$

Also $f(z)$ can be expanded as a Taylor's series about $z = a$ in the form

$$f(z) = \sum_{m=0}^{\infty} a_m (z-a)^m$$

Putting $z - a = re^{i\theta}$, we have {and $(z - a)^m = r^m e^{mi\theta}$

$$f(z) = f(a + re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{mi\theta}$$

Then $\overline{f(z)} = \sum_{m=0}^{\infty} \overline{a}_m \, r^m \, e^{-mi\theta}$

$$\frac{1}{2\pi i}\int \frac{\overline{f(z)}}{(z-a)^2}\; dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \overline{a}_m \, r^m e^{-mi\theta}}{(z-a)^2}\; dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \overline{a}_m \, r^m e^{-mi\theta} \, rie^{-mi\theta} d\theta}{r^2 \, e^{2i\theta}}$$

$$= \frac{1}{2\pi}\sum_{m=0}^{\infty} \overline{a}_m \, r^{m-1} \int_0^{2\pi} e^{-(m+1)\theta}\; d\theta$$

$$= 0 \qquad \left[\because \int_0^{2\pi} e^{-(m+1)\theta} d\theta = 0\right] \qquad \ldots(2)$$

Adding equations (1) and (2), we get

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z) + \overline{f(z)}}{(z-a)^2} dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } \mathrm{f}(z)}{(z-a)^2} dz$$

$$= \frac{1}{\pi i} \frac{\text{real part of } \left(\mathrm{a} + \mathrm{re}^{\mathrm{i}\theta}\right)}{r^2 e^{2i\theta}} \, rie^{i\theta} \; d\theta \qquad \left[\because z - a = re^{i\theta}\right]$$

$$= \frac{1}{r\pi}\int_0^{2\pi} P(\theta)\, d\theta \qquad \left[where\ P(\theta) \text{is the real part of } \mathrm{f}\left(\mathrm{a} + \mathrm{re}^{\mathrm{i}\theta}\right)\right]$$

10.6 Summary

The Taylor series is a powerful tool in complex analysis and is used to study the properties of complex functions, compute integrals, solve differential equations, and understand the behavior of functions near singularities. The Laurent series allows us to analyze the behavior of functions near singularities, such as poles and essential singularities. It provides a powerful tool for understanding complex functions and their properties in the complex plane.

In complex analysis, the Laurent series is a representation of a complex function that includes both positive and negative powers of the variable z. It is named after the French mathematician Pierre Alphonse Laurent. The Laurent series is particularly useful for functions with singularities, such as poles.

10.7 Terminal Questions

Q.1. Explain the entire function.

Q.2. State and prove Taylor's series.

Q.3. State and prove Laurent's series.

Q.4. Expand Taylor's series of

$$(i)\ f(z) = \frac{2z^3 + 1}{z^2 + z} \text{ about the point } z = i. \ \ (ii)\ f(z) = \frac{1}{(z+1)^2} \text{ about the point } z = i.$$

Q.5. Expand Taylor's series of

$$(i) f(z) = \sin z \text{ about the } z = \pi/4 \quad (ii) f(z) = \frac{\sin z}{z - \pi} \text{ about the } z = \pi.$$

Q.6. Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region

$(i)|z| > 1, (ii)1 < |z| < 2$

$(ii)|z| > 2\ (iv)0 < |z - 1| < 1.$

Q.7. Expand $f(z) = \frac{1-\cos z}{z^3}$ in Laurent's series about $z = 0$.

Q.8. Find the laurent's expansion of $f(z) = \frac{7z-2}{(z+1)(z-2)}$ in the region $1 < (z + 1) < 3$.

Q.9. Expand the Laurent's series for $f(z) = \frac{e^z}{(z-1)^2}\ bout\ z = 1.$

Q.10. Find the Laurent's series expansion of

$(i)\frac{z^2-6z-1}{(z-1)(z-3)(z+2)}$ *in the region* $3 < |z+2| < 5.$

$(ii)\frac{z^2-1}{z^2+5x+6}$ *aboutz* $= 0$*in the region* $3 < |z| < 3.$

Q.11. Expand the following function in Laurent's series:

(i) $\frac{z}{(z-1)(z-3)}$ *for* $|z-1| < 2$

(ii) $\frac{1}{(z+1)(z+2)}$ *in powers of* $(z+1)$*for the range* $0 < |z+1| < 2$

Q.12. Expand $\frac{1}{z(z^2-3z+2)}$ *for the regions*

$(i)\ 0 < |z| < 1 (ii) 1 < |z| < 2$

$(iii)\ |z| > 2.$

Answers

4. $(i)\left(\frac{i}{2}-\frac{3}{2}\right)+\left(3+\frac{i}{2}\right)(z-i)+\sum_{n=2}^{\infty}(-1)^n\left\{\frac{1}{i^{n+1}}+\frac{1}{(1+i)^{n+1}}\right\}(z-i)^n$

$(ii)\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right]$

5. $\sum_{n=0}^{\infty}\sin(2n+1)\frac{\pi}{4}\left(z-\frac{\pi}{4}\right)^n\ (ii) - 1 + \frac{1}{3!}(z-\pi)^2 - \frac{1}{5!}(z-\pi)^4 + \dots\dots..$

6. $(i)\frac{1}{2}+\frac{3}{4}z+\frac{7}{8}z^2+\frac{15}{16}z^3+\dots\dots.$which is a Taylor's series

$(ii) \dots\dots -\frac{1}{z^4}-\frac{1}{z^3}-\frac{1}{z^2}-\frac{1}{z}-\frac{1}{2}-\frac{1}{4}z-\frac{1}{8}z^2-\frac{1}{16}z^3-\dots$ which is a Laurent's series.

$(iii) \dots\dots\dots\dots\dots..+\frac{8}{z^4}+\frac{4}{z^3}+\frac{2}{z^2}-1-z-z^2-\dots\dots$

$(iv) -\frac{1}{(z-1)}-[1+(z-1)+(z-1)^2+(z-1)^3+\dots..]$

7. $\frac{1}{2z} - \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} z^{2n-1}}{(2n+2)!}$

8. $-\frac{2}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)^3} + \cdots \infty - \frac{2}{3}\left[1 + \frac{z+1}{3} + \frac{(z+1)^2}{3^2} + \frac{(z+1)^3}{3^3} + \cdots \infty\right].$

9. $\frac{e}{2}\left[2(z-1)^{-2} + 2(z-1)^{-1} + 1 + \frac{1}{3}(z-1) + \cdots\right]$

10. $(i)\ \frac{2}{z+2} + \frac{3}{(z+2)^2} + \frac{3^2}{(z+2)^3} + \cdots + \frac{1}{5}\left[1 + \frac{z+2}{5} + \frac{(z+2)^2}{5^2} + \frac{(z+2^3}{5^3} + \cdots\right]$

$(ii) 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right]$

11. $(i) - \frac{1}{2(z-1)} - 3\sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}}$

$(ii) \ldots\ldots + \frac{1}{(z+1)^4} - \frac{1}{(z+1)^3} + \frac{1}{(z+1)^2} + \frac{1}{(z+1)} - 1 + (z+1) - (z+1)^2 + \cdots \ldots$

12. $(i) \frac{1}{2z} + \sum_{n=0}^{\infty} z^n - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \ (ii) \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty} z^n + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

$(iii) \frac{1}{2z} - \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} + \frac{1}{2z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n.$

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

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Master of Science
PGMM -104N
Complex Analysis

Block

4

The Calculus of Residues

Block-4

The Calculus of Residues

Singularities are important in complex analysis because they provide information about the behavior of functions, especially near these points. The study of singularities is crucial for understanding the properties of complex functions and their applications in various areas of mathematics and physics. In complex analysis, the residue of a function at a singular point is a complex number that captures the behavior of the function around that point, particularly in the context of contour integration. Residues are important for evaluating complex integrals, especially those along paths that enclose singularities. Residues also play a crucial role in the study of meromorphic functions (functions that are analytic except for isolated singularities) and have applications in various areas of mathematics and physics.

Complex integration is a powerful tool with wide-ranging applications in mathematics, physics, engineering, and other fields. Complex integration is essential for calculating residues of functions at isolated singularities. Residues play a crucial role in evaluating complex integrals, solving differential equations, and analyzing the behavior of functions. Complex integration allows us to evaluate integrals of complex-valued functions along curves in the complex plane. This is useful for calculating quantities in physics, engineering, and mathematics that involve complex variables. In physics, complex integration is used in the study of electromagnetic fields and potentials. Techniques such as the method of images, which involves complex variables, are used to solve problems in electrostatics and magnetostatics. In engineering, complex integration is used in signal processing and control theory. The Fourier transform, which involves complex integration, is used to analyze and process signals in communication systems, audio processing, and image processing.In complex analysis, a meromorphic function is a function that is holomorphic (analytic) everywhere in its domain except at isolated points where it has poles. In other words, a meromorphic function is a function that is "mostly" analytic, except for a finite number of singularities, which are poles. Meromorphic functions play a fundamental role in complex analysis and provide a rich framework for understanding the behavior of complex functions with singularities.

UNIT-11: Singularities and Residues

Structure

11.1 Introduction

In this unit, we deal with the singularities, poles, residue at pole, Cauchy's residue theorem and their applications. Singularities are important in complex analysis because they provide information about the behavior of functions, especially near these points. The study of singularities is crucial for understanding the properties of complex functions and their applications in various areas of mathematics and physics.

In complex analysis, the residue of a function at a singular point is a complex number that captures the behavior of the function around that point, particularly in the context of contour integration. Residues are important for evaluating complex integrals, especially those along paths that enclose singularities.

11.2 Objectives

After reading this unit the learner should be able to understand about:

- The Zero of Analytic Function
- Singularities of an Analytic Function
- Principal part of $f(z)$
- Isolated Essential Singularity and Remove Singularity
- Pole, Polynomials and Residue at Pole
- Cauchy's Residue Theorem

11.3 The Zero of Analytic Function

A zero of an analytic function $f(z)$ is a value of z such that $f(z) = 0$.

If $f(z)$ is analytic in the neighbourhood of a point of a point $z = a$ then by Taylor's theorem. We have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \qquad where\ a_n = \frac{f^n(a)}{n!} \qquad \ldots(1)$$

i.e., $$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots\ldots + a_n(z-a)^n + \ldots..$$

If $a_0 = 0$, but $a_1 \neq 0$, then $f(z)$ is said to have a simple zero at $z = a$.

If $a_0 = a_1 = a_2 = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots = a_{m-1} = 0$, but $a_m \neq 0$, then $f(z)$ is said to have a zero of order m at $z = a$. This zero is said to be simple, double, triple in the cases in $n = 1, 2, 3\ldots..$

It may also be seen that, since $a_m = \dfrac{f^m(a)}{m!}$.

Hence, At a zero of order m, we have

$$f(a) = f'(0) = f''(a) = \ldots\ldots f^{m-1}(a) = 0$$

but $$f^m(a) \neq 0.$$

11.4 Singularities of an Analytic Function

If a function is analytic at all points of a bounded domain except at a finite number of points then these exceptional points are called singular points or singularities.

Thus the singularity of a function is a point at which the function, cases to be analytic.

For example. The function $f(z) = \dfrac{z^2+5}{z(z-3)(z^2+1)}$ is analytic at every point except $z = 0, 3, \pm i$.

There are isolated singular points of $f(z)$. Consider the another function $f(z) = \dfrac{1}{\tan(\pi/z)}$.

It has infinite number of isolated singularities which lie on the real axis from z = -1 to z = 1.

These isolated singularities are given by $z = \pm\frac{1}{n},\ n = 1, 2, 3, \ldots\ldots\ldots$ Then origin $(z = 0)$ is also a singular point but it is not isolated since in every neighbourhood of zero there are finite number of other singularities.

11.5 Principal part of *f*(z)

Let $f(z)$ be an analytic function in a domain D except at the point $z = z_0$. Then there exist a deleted neighbourhood $0 < |z - z_0| < R$ in which $f(z)$ is analytic.

In this annulus $0 < |z - z_0| < R$ the Laurent expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

The term $f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^{-n}$ of the Laurent series is called principal part of $f(z)$ at $z = z_0$.

Now there are three possibilities:

(i) The principal part contains infinite number of terms.

(ii) There are finite number of terms in the principal part.

(iii) All the b_n are zero i.e., there is no terms in the principal part.

The three possibilities defined above give three kinds of singularities.

11.6 Isolated Essential Singularity

If there are infinite of terms in the principal part of $f(z)$ at $z = z_0$, then z_0 is called an isolated essential singularity of $f(z)$.

For example. Sin $\frac{1}{z}$ has an isolated essential singularity at $z = 0$,

since $$\sin\frac{1}{z} = \frac{1}{z} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} + \ldots\infty$$

has infinite number of terms in negative powers of z and since $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \ldots.$

$z = 0$ is an essential singularity

Note. $\lim_{z\to a} f(z)$ does not exist if $z = a$ is an essential singularity.For example $\sin\frac{1}{z-a}, z = a$.

Check your Progress

1. Define the zero of an analytic function.

2. What do you mean by singularities?

11.7 Removable Singularity

If the principal part of $f(z)$ at $z = z_0$ consists of no term, i.e. if all the coefficients b_n are zero then z_0 is said to be a ‘removable singularity of $f(z)$.

For example. $f(z) = \frac{\sin z}{z}$ has removable singularity at $z = 0$

since $$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!}$$

Thus $\frac{\sin z}{z}$ contains no negative power of z. If we get $f(z) = 1$ at $z = 0$, then $f(z)$ becomes analytic at 0.

or

If a single value function *f(z)* is not defined at $z = a$ but $\lim_{z \to 0} f(z)$ exists, then $z = a$ is called a removable singularities.

For example. $f(z) = \dfrac{z^2 - a^2}{z - a}$ has removable singularit ies at $z = a$, because $\lim_{z \to a} \dfrac{z^2 - a^2}{z - a} = 2a$.

11.8 Pole

If the principal part of function *f(z)* at $z = z_0$ consists of a finite number of terms say *m*, then z_0 is said to pole of order *m* of *f(z)*.

A pole of order one is called a simple pole.

If z_0 is a pole of order *m*, then *f(z)* has an expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots\ldots + \frac{b_m}{(z - z_0)^m}$$

If *f(z)* has a pole at $z = 0$ then $\lim_{z \to a} \dfrac{z^2 - a^2}{z - a} = \infty$

For example, $f(z) = \dfrac{z^2 + a^2}{z - a}$ has a pole at $z = a$ because $\lim_{z \to 0} \dfrac{z^2 + a^2}{z - a} = \infty$

For example,

$$\frac{e^{(z-a)}}{(z-a)^2} = \frac{1}{(z-a)^2} \left\{ 1 + \frac{(z-a)}{1!} + \frac{(z-a)^2}{2!} + \ldots \right\}$$

$$= \frac{1}{(z-a)^2} + \frac{1}{(z-a)} + \frac{1}{2!} + \frac{(z-a)}{3!} + \frac{(z-a)^2}{4!} + \ldots$$

Here we have only two terms in the negative power of *(z-a) i.e.* there are only a finite number of terms in principal part of the function.

Hence the function has a pole at $z = a$ of order 2.

11.9 Polynomials

An expression of the form

$$P_n(z) = a_0 + a_1 z + a_2 z^2 + \ldots\ldots\ldots + a_n z^n$$

where $a_0, a_1, \ldots\ldots a_n$ are complex numbers and $a_n \neq 0$ is said to be a polynomial of degree n.

In particular every constant is a polynomial of degree.

The degree of the constant polynomial 0 remain undefined

For example, z^n is a polynomial of degree n.

4 $+ 4z^2 + 2z^3$ is a polynomial of degree 3.

11.10 Residue at Pole

Let z_0 be a pole of order m of the function $f(z)$. Then we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{(z - z_0)} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_m}{(z - z_0)^m}, \qquad b_m \neq 0$$

If $m = 1$, then z_0 is called a simple pole.

The coefficient b's which may also be zero is called the residue of $f(z)$ at $z = z_0$.

If $z = z_0$is simple pole, then we have

$$b_1 = \lim_{z \to z_0} (z - z_0) f(z)$$

Case I. The coefficient b_1 of $\dfrac{1}{z - z_0}$ in the Laurent's series is called residue whatever be the order of pole, then

$$b_1 = \lim_{z \to z_0} (z - z_0)\, f(z).$$

Case II. Let $z = z_0$ is a pole of order m, then *f(z)* is of the form $\dfrac{\varphi(z)}{(z-z_0)^m}$, where $\varphi(z)$ is analytic.

Residue of *f(z)* at $z = z_0$ is

$$\begin{aligned} b_1 &= \frac{1}{2\pi i}\int_\gamma f(z)\,dz \\ &= \frac{1}{2\pi i}\int_\gamma \frac{\varphi(z)}{(z-z_0)^m} \\ &= \frac{\varphi^{m-1}(z_0)}{(m-1)!} \qquad\qquad (\text{By Cauchy's integral formula}) \end{aligned}$$

Case III Residue at infinity

(i) $\lim\limits_{z\to\infty}(-z)f(z)$, if it has a definite value

(ii) Let f(z) has a pole of order m at infinity t hen $f(1/w)$ has a pole or order m at w = 0

The residue at infinity = $-\dfrac{1}{2\pi i}\int_C f(z)dz$

= is the *(-iv)* of the coefficient of $\dfrac{1}{z}$ in the expansion of *f(z)* in the neighbourhood of $z = \infty$.

Note. If degree of Numerator < degree of Denominator then we will apply (i) method.

11.11 Cauchy's Residue Theorem

Let *f(z)* be a single valued and analytic within and on a closed contour C except at a finite of singularities z_1, z_2, ...z_n and let R_1, R_2,….., R_n be respectively the residue to *f(z)* at these points then prove that $\int_C f(z)dz = 2\pi i(R_1 + R_2 + \ldots\ldots R_n)$.

Proof. *Let* $\gamma_1, \gamma_2, \ldots\ldots, \gamma_n$ be the circles with centres at z_1, z_2,....., z_n and radius which is small, then these circles lie completely in the closed curve C.

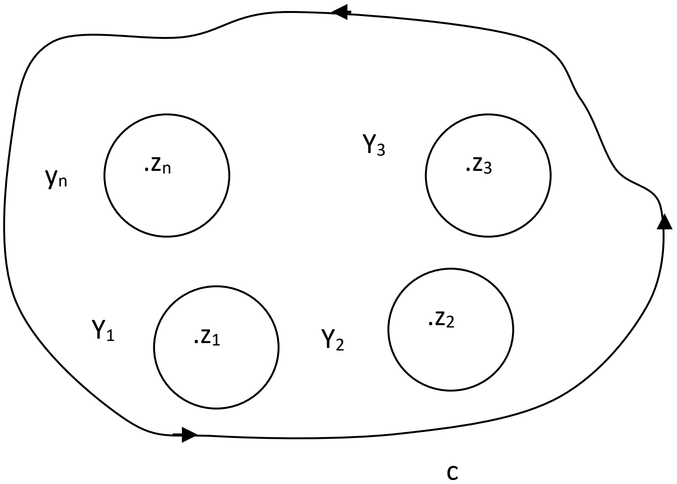
Let R_1, R_2, …., R_n *be the residue to f(z)* at z_1, z_2, …., z_n respectively, we have

$$R_1 = \frac{1}{2\pi i}\int_{\gamma_1} f(z)dz \qquad ...(1)$$

$$R_2 = \frac{1}{2\pi i}\int_{\gamma_2} f(z)dz \qquad ...(2)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$R_n = \frac{1}{2\pi i}\int_{\gamma_n} f(z)dz \qquad ...(n)$$



Now by Cauchy Goursat theorem for multiconnected regions, we have

$$\int_C f(z)dz - \int_{\gamma_1} f(z)dz - \int_{\gamma_2} f(z)dz - .... - \int_{\gamma_n} f(z)dz = 0$$

$$\int_C f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz$$

$$= 2\pi i R_1 + 2\pi i R_2 + ... + 2\pi i R_n$$
$$= 2\pi i\,(R_1 + R_2 + ... + R_n)$$
$$= 2\pi i \times (\text{Sum of residue of f}(z)\text{at its poles within C})$$

Check your Progress

1. Define the residue at pole.

2. State the Cauchy's Residue theorem.

Examples

Example 1. Show that the function $e^{1/z}$ actually takes every value except zero, an infinite number of times in the neighbourhood of $z = 0$.

Sol. $f(z) = e^{1/z}$

Now

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \ldots\ldots$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^n}.$$

Thus the principal part of $f(z)$ is $\sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{z^n}$

which contains an infinite number of terms. Hence $z = 0$ is the isolated essential singularity of $f(z)$ as desired.

Example 2. Show that the function e^z has an isolated essential singularity at $z = \infty$.

Sol. Let $f(z) = e^z$. *Then* $f\left(\frac{1}{w}\right) = e^{1\backslash w}$

The nature of singularity of the function $f(z)$ at $z = \infty$ will be the same as that of the function

$$f\left(\frac{1}{w}\right) \text{ at } w = 0$$

Now $$f\left(\frac{1}{w}\right) = e^{1/w} = 1 + \frac{1}{w} + \frac{1}{2!w^2} + \ldots\ldots$$

Hence the principal part of $f\left(\frac{1}{w}\right)$ *is* $\frac{1}{w} + \frac{1}{2!w^2} + \ldots\ldots$ which consists of an infinite number of terms. Hence $w = 0$ is an isolated essential singularity of $e^{1/w}$ and consequently $z = \infty$ is an isolated essential singularity of e^z.

Example 3. Discuss the nature of singularities of the following functions:

$(i)\ \tan z$ $\qquad$ $(ii)\dfrac{1}{z(1-z^2)}$

$(iii)\dfrac{z}{1+z^4}$ $\qquad$ $(iv)\dfrac{\sin z}{(z-\pi)^2}$

Sol. *(i)* Let $\qquad f(z)=\tan\ z=\dfrac{\sin z}{\cos z}$

Poles of *f(z)* are given by

$$\cos z = 0$$

or $\qquad z=(2n+1)\dfrac{\pi}{2}, n=(0,\pm 1,\pm 2,\ldots\ldots..)$

Hence $z=(2n+1)\dfrac{\pi}{2}$ are the simple poles of $f(z)$.

$(ii)\ f(z)=\dfrac{1}{z(1-z^2)}$

Let $\qquad f(z)=\dfrac{1}{z(1-z^2)}$ singularit y of $f(z)$ is given by

$$z(1-z^2)=0 \ or\ z=0,\pm 1$$

which are simple poles

$(iii)\ f(z)=\dfrac{z}{1+z^4}$

singularity of f(z) are $1 + z^4 = 0 \Rightarrow z = (-1)^{1/4}$

$$z=\{\cos(2n+1)\pi+i\sin\ (2n+1)\pi\}^{\ 1/4}$$

$$=\cos(2n+1)\frac{\pi}{4}+i\sin(2n+1)\frac{\pi}{4}=e^{i(2n+1)\frac{\pi}{4}}$$

Putting n = 0, 1, 2, 3…..

$z=e^{i\pi/4},\ e^{3\pi u/4}, e^{5\pi i/4}$

which are the simple poles of *f(z)*.

$(iv) = \frac{\sin z}{(z-\pi)^2}$

Let $\quad f(z) = \frac{\sin z}{(\pi - z)^2}$, singularity of f(z) are given by $(z - \pi)^2 = 0$

or $\quad z = \pi$ is a double pole

or $\quad z = \pi$ is a pole of order two of *f(z)*

Example 4. Find the kind of the singularities of the following functions:

$(i)\ \frac{\cot \pi z}{(z-a)^2}$ *at* $z=0$ *and* $z=\infty$

$(ii)\ \sin \frac{1}{1-z}$ *at* $z=1$

$(iii)\ \sin z - \cos z$ *and* $z = \infty$

$(iv)\ \operatorname{cosec} \frac{1}{z}$ *at* $z=0$

Sol. *(i)* Let $\quad f(z) = \frac{\cot \pi z}{(z-a)^2} = \frac{\cos \pi z}{\sin \pi z} \cdot \frac{1}{(z-a)^2}$

Poles of *f(z)* are given by putting the determinator equal to zero.

i.e $\{\sin \pi z . (z-a)^2\}$ and $\{(z-a)^2 = 0 \Rightarrow z = a$ Hence $z = a$ is a double pole.$\}$

i.e. $\quad \sin \pi z = 0$ *or* $\pi z = n\pi$

or $\quad z = n \ (n = 0, \pm 1, \pm 2, \ldots)$

and $z = 0, \pm 1, \pm 2, \ldots\ldots$ are simple pole.

$Z = \infty$ is the limit point of the sequence of these poles hence $z = \infty$ is a non-isolated

$(ii) \quad \sin \frac{1}{1-z}$ *at* $z = 1$

Let $\quad f(z) = \sin \frac{1}{1-z}$

zeros of *f(z)* are given by

$$\sin\frac{1}{1-z}=0 \qquad i.e. \qquad \frac{1}{1-z}=n\pi$$

or $$1-z=\frac{1}{n\pi} \quad \Rightarrow \quad z=1-\frac{1}{n\pi}$$

$$\Rightarrow \qquad z=1-\frac{1}{n\pi}\left(n=0,\ \pm 1, \pm 2,........\right)$$

z =1 is a limit point of these zeros. Obviously *z = 1* is an isolated essential singularity

(iii) *sin* z - cos z at z = ∞

Let *f(z) = sinz - cos z*

zeros of *f(z)* are given by

$$\sin z - \cos z = 0$$

$$\sin z = \cos z$$

$$\tan z = 1 \text{ or } z = n\pi\frac{\pi}{4}$$

$\therefore z=\infty$ is a limit point of these zeros which is there an isolated essential singularity.

$(iv)\ \operatorname{cosec}\frac{1}{z}\ at\, z=0$

Let $$\operatorname{cosec}\frac{1}{z}\ at\ z=\frac{1}{\sin\frac{1}{z}}$$

Poles of *f(z)* are given by

$$\sin\frac{1}{z}=0 \qquad or \qquad \frac{1}{z}=n\pi$$

i.e., $$z=\frac{1}{n\pi}\left(n=0.\pm 1, \pm 2,......\right)$$

Obviously $z = 0$ is the limit point of the sequence of these poles.

Hence $z = 0$ is a non-isolated singularity.

$(v)\ \tan\frac{1}{z}\ at\ z=0$

$$f(z) = \tan\frac{1}{z} = \frac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$$

Poles of $f(z)$ are given by $\cos\frac{1}{z} = 0$

$$\Rightarrow \qquad \frac{1}{z} = 2n\pi + \frac{\pi}{2}$$

or $$z = \frac{1}{\left(2n+\frac{1}{2}\right)\pi}\ (n = 0, \pm 1, \pm 2)$$

Since $z = 0$ is a limit point of these sequence therefore $z=0$ is a non-isolated essential singularity.

Example 5. Show that $z = 0$ is an isolated essential singularity of the function $\dfrac{e^{c/(z-a)}}{e^{z/a}-1}$.

Sol. We have $f(z) = \dfrac{e^{c/(z-a)}}{e^{z/a}-1} = \dfrac{e^{c/z-a}}{e^{1+\frac{(z-a)}{a}}-1}$

$$= \frac{1+\frac{c}{z-a}+\frac{c^2}{2!(z-a)^2}+....}{e\left\{1+\frac{z-a}{a}+\left(\frac{z-a}{a}\right)^2\frac{1}{2!}+.....\right\}-1}$$

$$= =-\left\{1+\frac{c}{z-a}+\frac{c^2}{2!(z-a)^2}+...\right\}\left[1-e\left\{1+\left(\frac{z-a}{a}\right)+\frac{1}{2!}\left(\frac{z-a}{a}\right)^2+...\right\}\right]^{-1}$$

$$=\left\{1+\frac{c}{z-a}+\frac{c^2}{2!(z-a)^2}+...\right\}\times\left[1+e\left\{1+\left(\frac{z-a}{a}\right)+\frac{1}{2!}\left(\frac{z-a}{a}\right)^2+....\right\}\right.$$

$$\left.+e^2\left\{1+\frac{z-a}{a}+\frac{(z-a)^2}{2!a^2}+....\right\}^2\right]$$

Since in the expansion of *f(z)* there are infinite numbers of terms containing negative powers at $\frac{1}{(z-a)}$.

Hence z = $a$is an isolated essential singularity

Example 6. Find the kind of singularities of the following functions:

$(i)\ \frac{1-e^z}{1+e^z}\ at\ z=\infty$ $\qquad (ii)\ \frac{1}{\sin z-\cos z}\ at\, z=\pi/4$

$(iii)\sin\frac{1}{z}\, at\, z=0$ $\qquad (iv)\, z\,\text{cosec}\, z\, at\, z=\infty$

Sol. (i) *Let* $\qquad f(z)=\frac{1-e^z}{1+e^z}\ at\, z=\infty$

Poles of *f(z)* are given by putting the denominator equal to zero.

$$\left[\begin{array}{l}\because\ -1=\cos\pi+i\sin\pi\ \Rightarrow\ e^{\pi i}=\cos\pi+i\sin\pi\\ \Rightarrow e^{2n\pi i}=\cos 2n\pi+i\sin 2n\pi\end{array}\right]$$

$$\left|\begin{array}{l}-1=\cos\pi+i\sin\pi\\ e^{\pi i}=\cos\pi+i\sin\pi\ \ i\sin 2\pi n\\ e^{2n\pi i}=\cos 2n\pi+\end{array}\right.$$

$$1+e^z=0\quad\Rightarrow e^z=-1=e^{\pi i}.\ 1$$

$$e^z=e^{2n\pi i+\pi i}$$

$$\therefore\qquad z=(2n+1)\pi i\quad (n=0,\pm1,\pm2,.....)$$

Hence $z = (2n+1)\pi i$ are the simple poles of $f(z)$. Obviously $z = \infty$ is a limit point of these poles therefore $z = \infty$ is a non-isolated singularities.

(ii) $\dfrac{1}{\sin z - \cos z}$ $at\ \ z = \dfrac{\pi}{4}$

Let $$f(z) = \frac{1}{\sin z - \cos z}$$

Poles of $f(z)$ are given by

$$\sin z - \cos z = 0$$

i.e. $$\tan z = 1$$

$$z = n\pi + \frac{\pi}{4} \quad (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

At n = 0, $z = \dfrac{\pi}{4}$ is a simple pole.

(iii) $\sin \dfrac{1}{z}$ $at\ z = 0$

Let $$f(z) = \sin\frac{1}{z}$$

Poles of $f(z)$ are given by $\sin\dfrac{1}{z} = 0$ or $\dfrac{1}{z} = n\pi$

$$z = \frac{1}{n\pi} \; (n = 0, \pm 1, \pm 2, \ldots\ldots)$$

Since $z = 0$ is a limit point of zeros of $f(z)$ *therefore* $z = 0$ is an isolated essential singularity of $f(z)$.

(iv) *z cosec z at* $z = \infty$

Let $$f(z) = z \operatorname{cosec} z = \frac{z}{\sin z}$$

Poles of $f(z)$ are given by

$$\sin z = 0 \text{ or } z = n\pi \qquad (n = 0, \pm 1. \pm 2, \ldots.)$$

$z = \infty$ is an non-isolated essential singularity.

Example 7. Show that the function e^{-1/z^2} has no singularities.

Sol. We have $f(z) = e^{-1/z^2}$ ...(i)

The zeros of $f(z)$ are given by

$$e^{-1/z^2} = 0 \qquad \{e^{-\infty} = 0\}$$

or $\qquad z^2 = 0$

$\therefore \qquad z = 0$ is a zero of order 2.

Since zeros have no limit point therefore, there is no singularity of $f(z)$.

Poles of $f(z)$ are given by

$e^{1/z^2} = 0$, which is not possible for any value of z, real or complex

Hence e^{-1/z^2} has no singularities.

Example 8. Find the residues of $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$ at all its poles in the finite plane.

Sol. Let $f(z) = \dfrac{z^2 - 2z}{(z+1)^2(z^2+4)}$

Poles are given by $(z + 1)^2 (z^2 + 4) = 4$

or $\qquad z = -1, -1, \pm 2i$

So $f(z)$ has a double pole at $z = -1$ and two simple poles at $z = \pm 2i$.

Now for double pole at $z = -1$, we have

$$f(z)=\frac{z^2-2z}{(z+1)^2(z^2+4)}$$

$$=\frac{\phi(z)}{(z+1)^2},\text{ where }\phi(z)=\frac{z^2-2z}{z^2+4}$$

Because $z = -1$ is a pole of order 2, then residue at $z = -1$ is $\frac{\phi'(-1)}{1!}$

Now $\phi(z)=\frac{z^2-2z}{z^2+4}=1-\frac{2z-4}{z^2+4}$

$$\phi'(z)=\frac{2z^2-8z-8}{(z^2-4)^2}$$

$$\phi'(-1)=\frac{2}{25}$$

Residue at z = 2i is

$$=\lim_{z\to 2i}(z-2i)\,f(z)$$

$$=\lim_{z\to 2i}(z-2i)\frac{z^2-2z}{(z+1)^2(z^2+4)}$$

$$=\lim_{z\to 2i}(z-2i)\frac{z^2-2z}{(z+1)^2(z+2i)}$$

$$=\frac{-4-4i}{(2i+1)^2(2i+2i)}$$

$$=\frac{-4(1+i)}{(-4+1+4i).\,4i}$$

$$=\frac{-(1+i)(-4+3i)}{i(-3+4i)}=\frac{-(1+i)}{(-4-3i)}$$

$$=\frac{-(1+i)(-4+3i)}{25}=\frac{-(-4-4i+3i-3)}{25}$$

$$= -\frac{(-7-i)}{25} = \frac{7+i}{25}$$

Similarity, residue at $z = -2i$ *is*

$$= \lim_{z\to -2i} (z+2i)\, f(z)$$

$$= \frac{7-i}{25}$$

Example 9. Evaluate $\int_C z^2 e^{1/z}\; dz \; around \; |z| = 1.$

Sol. *Let* $\quad f(z) = z^2 e^{1/z} = \frac{z^2}{e^{-1/z}}$

Poles of $f(z)$ are given by $e^{-1/z} = 0$

i.e., $\quad z = 0$

Here $z = 0$ is a simple pole

Now we have $f(z) = z^2 e^{1/z}$

$$= z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + ...\right]$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!z} + ... \qquad ...(1)$$

The residue at z = 0 is the coefficient of z^{-1} in the series expansion (1). Thus we have Residue at $z = 0$ *is* $\frac{1}{6}$

.

Using Cauchy's Residue theorem, we have

Therefore $\quad \int_C z^2 e^{1/z}\, dz = 2\pi i \,.\, \frac{1}{6} = \frac{\pi i}{3}.$

Example 10. Find residue of $f(z)=\dfrac{z^3}{(z-1)(z-2)(z-3)} or\ z=1,2,3$. Also find residue at infinity. Then prove that the sum of all these residue is zero.

Sol. Let $$f(z)=\frac{z^3}{(z-1)(z-2)(z-3)}$$

Here z = 1, 2, 3 are the simple poles of f(z)

$$R_1=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\ \frac{z^3}{(z-1)(z-2)(z-3)}$$

$$=\frac{1}{(1-2)(1-3)}=\frac{1}{2}\backslash$$

Residue at z = 2 is

$$R_1=\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}(z-2)\ \frac{z^3}{(z-1)(z-2)(z-3)}$$

$$=\frac{8}{(2-1)(2-3)}=-8$$

Residue at z = 3 is

$$R_1=\lim_{z\to 3}(z-3)f(z)=\lim_{z\to 3}(z-3)\ \frac{z^3}{(z-1)(z-2)(z-3)}$$

$$=\frac{27}{(3-1)(3-2)}=\frac{27}{2}$$

Now we have

$$f(z)=\frac{z_3}{(z-1)(z-2)(z-3)}=\frac{1}{\left(1-\frac{1}{z}\right)\left(1-\frac{2}{z}\right)\left(1-\frac{3}{z}\right)}$$

$$= \left(1-\frac{1}{z}\right)^{-1}\left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)$$

$$= \left(1+\frac{1}{z}+.....\right)\left(1+\frac{2}{z}+....\right)\left(1+\frac{3}{z}+....\right)$$

Coefficient of $\frac{1}{z} is = 1 + 2 + 3 = 6$

Residue at infinity = $-\left\{coeff\,.of\ \frac{1}{z} in\ f(z)\right\} = -6$

Hence the sum of residue *f(z)* at z = 1, 2, 3, ∞

$$= \frac{1}{2} - 8 + \frac{27}{2} - 6$$

$$= \frac{1-16+27-12}{2}.$$

$$= 0.$$

Example 11. Find residue of $\frac{1}{\left(z^2+a^2\right)^2}$ at $z = ai$ *and* $z = -ai$.

Sol. Let $f(z) = \frac{1}{\left(z^2+a^2\right)^2}$

Poles are given by

$$\left(z^2+a^2\right)^2 = 0 \ or \ z = ai, ai, -ai, -ai$$

So *f(z)* has pole of two order at $z = ai$ and $z = -ai$

Here $f(z) = \frac{1}{\left(z^2+a^2\right)^2} = \frac{1}{(z+ia)^2(z-ia)^2} = \frac{\varphi(z)}{(z-ia)^2}$

where $\phi(z) = \frac{1}{(z+ia)^2}$

The residue of function *f(z)* at $z = ia$ *is a* $\varphi'(ia)$

We have $\varphi'(z) = -\dfrac{2}{(z+ia)^3}$

Hence the residue at z = *ia is φ'(ia)*

$$= -\frac{2}{(2ia)^3} = -\frac{2}{8i^3a^2} = +\frac{1}{4ia^3} = \frac{-i}{4a^3}$$

and residue at *z* = *-ia is φ'*(ia)

Here $\varphi(z) = \dfrac{1}{(z-ia)^2}$

$\because$ $\phi'(z) = -\dfrac{2}{(z-ia)^3}$

So $\phi'(-ia) = -\dfrac{2}{(-2ia)^3} = \dfrac{-2}{-8a^3i^3} = -\dfrac{1}{4ia^3} = \dfrac{i}{4a^3}$

Example 12. Find the residue of $\dfrac{z^3}{(z-1)^4(z-2)(z-3)}$

Sol. We have $f(z) = \dfrac{z^3}{(z-1)^4(z-2)(z-3)}$

Here $z = 1$ is a pole of order 4

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)} = \frac{\phi(z)}{(z-1)^4}, where\,\phi(z) = \frac{z^3}{(z-2)(z-3)}$$

Because $z = 1$is a pole of order 4, then residue at $z = 1$ *is* $\dfrac{\phi'''(1)}{3!}$

Now we have

$$\phi(z)=\frac{z^3}{(z-2)(z-3)}=z+5-\frac{8}{z-2}+\frac{27}{z-3}$$
$$\phi'(z)=1+\frac{8}{(z-2)^2}-\frac{27}{(z-3)^3}$$
$$\phi''(z)=-\frac{16}{(z-2)^4}+\frac{54}{(z-3)^2}$$
$$\phi'''(z)=\frac{48}{(z-2)^4}-\frac{162}{(1-3)^4}=48-\frac{162}{16}=\frac{303}{8}$$

Residue at z = 1 $\frac{\phi'''(1)}{3!}=\frac{303/8}{3!}=\frac{101}{16}$.

Example 13. Find the residue of $\frac{z^2}{(z-a)\,(z-b)(z-c)}$

The residue of *f(z)* at infinity = $\lim_{z\to\infty}\{-z\,f(z)\}$

$$=\lim_{z\to\infty}(-z)\frac{z^2}{(z-a)(z-b)(z-c)}$$
$$=\lim_{z\to\infty}(-1)\left(1-\frac{a}{z}\right)^{-1}\left(1-\frac{b}{z}\right)^{-1}\left(1-\frac{c}{z}\right)^{-1}$$
$$=\lim_{z\to\infty}(-1)\left\{1+\frac{abc}{z}+higher\ power\ of\ \frac{1}{z}\right\}$$
$$=-1.$$

Example 14. *Evaluate* $\int_C \frac{z^2}{(z-1)^2(z-2)}$ dz over the contour C enclosed by $|z|=\frac{5}{2}$.

Sol. Let $$f(z)=\frac{z^2}{(z-1)^2\,(z-2)}$$

Poles are given by $(z-1)^2\,(z-2)=0$ *or* $z=1,1,2$

So *f(z)* has a double pole at $z = 1$ and $z = 2$ is a simple pole. Hence both poles lie inside $|z|=\frac{5}{2}$.

Now for double pole at $z = 1$, we have

$$f(z) = \frac{z^2}{(z-1)^2\,(z-2)}$$
$$= \frac{\phi(z)}{(z-1)^2},\ where\, \phi(z) = \frac{z^2}{z-2}$$

Because $z = 1$ is a pole of order 2, then residue at $z = 1$ *is* $\frac{\phi'(1)}{1!}$

Now

$$\phi(z) = \frac{z^2}{z-2} = (z+2) + \frac{4}{z-2}$$
$$\phi'(z) = 1 - \frac{4}{(z-2)^2}$$
$$\phi'(z) = 1 - \frac{4}{(z-2)^2} = 1 - 4 = -3$$

Residue at $z = 2$ is

$$= \lim_{z\to 2} (z-2)\, f(z)$$
$$= \lim_{z\to 2} (z-2)\frac{z^2}{(z-1)^2(z-2)}$$
$$= \lim_{z\to 2} \frac{z^2}{(z-1)^2}$$
$$= \frac{2^2}{(2-1)^2} = 4$$

By Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of f}(z)\,\text{at its poles with C})$$

We have

$$\int_C \frac{z^2}{(z-1)^2}\, dz = 2\pi i(-3+4) = 2\pi i$$

Example15. *Evaluate* $\int \frac{z^2 e^{zt}}{z^2+1}\, dz$, where C is the circle $|z| = 2$ and t is independent of z.

Sol. Let $$f(z) = \frac{z^2 e^{zt}}{z^2+1}$$

Poles are given by $z^2 + 1 = 0$ or $z = \pm i$

So *f(z)* has two simple pole $z = \pm i$ and both of them lie inside $|z| = 2$.

Residue at $(z = i)$ is

$$= \lim_{z \to i} (z-i) f(z)$$
$$= \lim_{z \to i} (z-i) \frac{z^2 e^{zt}}{(z^2+1)}$$
$$= \lim_{z \to i} \frac{z^2 e^{tz}}{(z+i)} = \frac{i^2 e^{tz}}{2i}$$
$$= \frac{i}{2} e^{it}$$

Residue at $(z = -i)$ is

$$= \lim_{z \to i} (z+i) f(z)$$
$$= \lim_{z \to i} (z+i) \frac{z^2 e^{zt}}{(z^2+1)}$$
$$= \lim_{z \to i} \frac{z^2 e^{tz}}{(z-i)}$$
$$= \frac{(-1)^2 e^{-it}}{(-i-i)}$$
$$= \frac{-e^{-it}}{-2i} = \frac{e^{-it}}{2i}$$
$$= -\frac{i}{2} e^{-it}$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)\,dz = 2\pi i \times (\text{Sum of reside of } f(z) \text{ at its poles within C})$$

Now we have

$$\int_C \frac{z^2 e^{zt}}{z^2+1}\,dz = 2\pi i \left(\frac{i}{2} e^{it} - \frac{i}{2} e^{it} \right)$$

$$= -2\pi\left(\frac{e^{it} - e^{-it}}{2}\right)$$

$$= -2\pi i\left(\frac{e^{it} - e^{-it}}{2i}\right)$$

$$= -2\pi i \sin t.$$

11.12 Summary

A zero of an analytic function *f(z)* is a value of *z* such that *f(z)* = 0.

If a function is analytic at all points of a bounded domain except at a finite number of points then these exceptional points are called singular points or singularities.

Let *f(z)* be an analytic function in a domain D except at the point $z = z_0$. Then there exist a deleted neighborhood $0 < |z - z_0| < R$ in which *f(z)* is analytic.

If there are infinite of terms in the principal part of *f(z)* at $z = z_0$, then z_0 is called an isolated essential singularity of *f(z).*

If the principal part of *f(z)* at $z = z_0$ consists of no term, i.e. if all the coefficients b_n are zero then z_0 is said to be a ‘removable singularity of *f(z).*

If the principal part of function *f(z)* at $z = z_0$ consists of a finite number of terms say *m*, then z_0 is said to pole of order *m* of *f(z).* A pole of order one is called a simple pole.

Let *f(z)* be a single valued and analytic within and on a closed contour C except at a finite of singularities $z_1, z_2, \ldots z_n$ and let $R_1, R_2, \ldots, R_n$ be respectively the residue to *f(z)* at these points then prove that

$$\int_C f(z)dz = 2\pi i\left(R_1 + R_2 + \ldots\ldots R_n\right).$$

11.13 Terminal Questions

1. Find the residue at the pole of the following function:

$(i)\ f(z) = \dfrac{z}{\sin z}$

$(ii)\ f(z) = \dfrac{1}{1+z^4}$

$(iii)\ f(z) = \dfrac{e^2}{z(z+1)^2}$

$(iv)\ f(z) = \dfrac{z+2}{(z+1)^2(z-2)}.$

2. find the residue at pole $f(z) = \dfrac{z\cos z}{\left(z-\dfrac{\pi}{2}\right)^2}.$

3. find the residue of $f(z) = \dfrac{\cot z \coth z}{z^3}\ \ at\ z = 0.$

4. Find the pole and their residues at these points for the function $f(z) = \dfrac{1+e^z}{\sin z + z\cos z}.$

5. Evaluate $\int_C \cos ech\ z$ around $|z| = 4.$

6. Evaluate $\int_C \dfrac{\tan z}{z} dz$ around $|z| = 2.$

7. Find the residues at the poles of the following functions.

$(i)\ \cos ec^2\ at\ \mathrm{z} = 0$

$(ii)\ \dfrac{e^2 \cos \pi z}{z(z-1)^2}$ at $z = 0$

$(iii)\ \dfrac{z^4}{(z^2+a^2)}$ at $z = -ai$

8. Evaluate $\int_C \frac{z^2}{(z-1)^2(z-2)}\,dz$ around $C:|z|=\frac{3}{2}$.

9. Evaluate $\int \frac{3z^2-2}{(z-1)(z^2+9)}\,dz$ around $C:|z+2|=4$.

10. Evaluate $\int_C \frac{z-1}{(z-1)^2(2z+3)}\,dz$ around $C:|z+i|=\sqrt{3}$.

11. Evaluate $\int_C \frac{e^{2z}}{z(z^2+1)}\,dz$ around enclosed by the square with vertices $2+2i,-2+2i,-2-2i,2-2i$.

Answers

1. $(i)\ (-1)^n n\pi$ $\qquad$ $(ii)\ -\frac{1+i}{4\sqrt{2}}, \frac{1-i}{4\sqrt{2}}, \frac{1+i}{4\sqrt{2}}, \frac{-1+i}{4\sqrt{2}}$

$(iii)\ 1, \frac{-2}{e}$ $\qquad$ $(iv)\ -\frac{4}{9}, \frac{4}{9}$.

2. $-\frac{\pi}{2}$

3. $-\frac{14}{90}$

4. 1.

5. $-2\pi i$ $\qquad$ 6. 0

7. $(i)\ 0$ $\qquad$ $(ii)\ 1$ $\qquad$ $(iii)\ \frac{ia^3}{2}$

8. $2\pi i$ $\qquad$ 9. $6\pi i$ $\qquad$ 10. $4\pi i$

11. $2\pi i(1-\cos 2)$.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-12: Applications of Complex Integration-I

Structure

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12.5 Evaluation of the integral $\int_{-\infty}^{\infty} f(x)dx$

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12.1 Introduction

Complex integration can be used to evaluate real integrals, particularly those involving trigonometric, rational, or exponential functions. By using techniques such as contour integration and the residue theorem, real integrals can sometimes be transformed into complex integrals that are easier to evaluate. Complex integration allows us to evaluate integrals of complex-valued functions along curves in the complex plane. This is useful for calculating quantities in physics, engineering, and mathematics that involve complex variables. Complex integration allows for the analytic continuation of functions from their domain of definition to larger domains. This is important in understanding the global behavior of functions defined on complex domains.

Complex integration has numerous applications in mathematics, physics, engineering, and other fields. Complex integration is used to solve differential equations, particularly in the context of complex analysis. Techniques such as the Laplace transform, which involves complex integration, are used to solve ordinary and partial differential equations in engineering and physics.

12.2 Objectives

After reading this unit the learner should be able to understand about the:

- Evaluation of real definite integrals
- Integral round the unit circle
- Evaluation of the integral $\int_{-\infty}^{\infty} f(x)\,dx$
- Jordan’s inequality
- Jordan’s Lemma

12.3 Evaluation of Real Definite Integrals

Some problems of real definite integrals whose evaluation by usual methods sometimes becomes very tedious, can be easily evaluated by Cauchy's theorem of residues. In this method we take a curve part of which consists of real axis and the remaining part consists of straight lines or arc of circles. We evaluate the real definite integrals using the method of contour integration. The contour may be a semi circle, circle and quadrant of circle.

The process of integrating along a countour is called contour integration. Before proceeding to the evaluation of definite integrals, we discuss two useful identities:

(i) If $\lim_{z\to a}(z-a)f(z)=A$, and if C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a|=r$, then

$$\lim_{z\to a}\int_C f(z)dz = iA(\theta_2-\theta_1).$$

(ii) If C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a|=R$, then if $\lim_{R\to\infty} z\ f(z)=A$, then

$$\lim_{R\to\infty}\int_C f(z)dz = iA(\theta_2-\theta_1).$$

Note. Cauchy's residue theorem apply only those pole or singularities that lie inside the contour otherwise, the residue at pole lying outside the contour is zero.

12.4 Integral round the unit circle

Consider the integral of the type

$$I=\int_0^{2\pi} f(\cos\theta,\sin\theta)d\theta \qquad ...(1)$$

where $f(\cos\theta,\sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

Put $z=e^{i\theta}$, then $dz = ie^{i\theta}d\theta = iz\ d\theta$

$$\Rightarrow \qquad \frac{dz}{iz} = d\theta$$

We know that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

Then we have

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta \quad = \frac{1}{\mathrm{i}} f\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{z}$$

$$= \int_C g(z)dz$$

where *g(z)* is a rational function of *z* and contour *C* is the unit circle $|z| = 1$.

Therefore we have

$$I = \int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$$

$$= \int_C g(z)dz$$

Using Cauchy's residue theorem, we get

$I = 2\pi i \times$(sum of residues of $g(z)$ at its poles within C).

Check your Progress

1. How to evaluate the real definite integrals.

2. What do you mean by integral round the unit circle?

Examples

Example 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a > b > 0.$

Sol. Let $I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}, a > b > 0$ ...(1)

Let C be the circle $|z| = 1$.

Putting $z = e^{i\theta} \quad \Rightarrow dz = ie^{i\theta} d\theta$

or $$dz = iz\ d\theta$$

or $$\frac{dz}{iz} = d\theta$$

Using equation (1), we get

$$I = \int_C \frac{dz}{iz\left\{a + \frac{b}{2}(z+z^{-1})\right\}}$$

$$= \frac{2}{i}\int_C \frac{dz}{[2az + bz^2 + b]}$$

$$= \frac{2}{ib}\int_C f(z)dz \qquad ...(2)$$

where C is the unit circle $|z| = 1$.

Poles of *f(z)* are given by

$$z^2 + \frac{2a}{b}z + 1 = 0$$

or $$bz^2 + 2az + b = 0$$

or $$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $$z_1 = \frac{-a+\sqrt{a^2-b^2}}{b}, z_2 = \frac{-a-\sqrt{a^2-b^2}}{b}$$

Since *a>b>0*, we have $|z_2| > 1$. But $|z_1 z_2| = 1$. so that $|z_1| < 1$. Hence $z = z_1$ is only simple pole inside *C*.

Residue at $(z = z_1) = \lim_{z \to z_1}(z - z_1) f(z)$

$$= \lim_{z \to z_1}(z - z_1)\frac{1}{(z - z_1)(z - z_2)}$$

$$= \lim_{z \to z_1}\frac{1}{(z - z_2)}$$

$$= \frac{1}{z_1 - z_2}$$

$$= \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{b}{2\sqrt{a^2-b^2}} \qquad \ldots(3)$$

Using Cauchy Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C) \qquad \ldots(4)$$

By equations (2), (3) and (4), we have

$$I = \frac{2}{ib}\int_C f(z)dz$$

$$= \frac{2}{ib}.2\pi i.\frac{b}{2\sqrt{a^2-b^2}}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}.$$

Example 2. Show that $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, a^2 < 1.$

Sol. Let $$I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta}, a^2 < 1. \qquad ...(1)$$

Let C be the circle $|z| = 1.$

Putting $z = e^{i\theta} \quad \Rightarrow dz = ie^{i\theta}d\theta = iz\, d\theta$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

Using equation (1), we get

$$I = \int_C \frac{dz}{iz\left\{a + \frac{b}{2}(z+z^{-1})\right\}}$$

$$= \frac{2}{i}\int_C \frac{dz}{[2z + az^2 + a]}$$

$$= \frac{2}{ib}\int_C f(z)dz \qquad ...(2)$$

where C is the unit circle $|z| = 1.$

Poles of *f(z)* are given by

$$z^2 + \frac{2}{a}z + 1 = 0$$

or $$az^2 + 2z + a = 0$$

or $$z = \frac{-2 \pm \sqrt{4-4b^2}}{2a} = \frac{-1 \pm \sqrt{1-a^2}}{a}$$

Let $$z_1 = \frac{-a+\sqrt{1-a^2}}{a}, z_2 = \frac{-1-\sqrt{1-a^2}}{a}$$

Since $z_2 > 1$, we have $|z_2| > 1$. But $|z_1 z_2| = 1$. so that $|z_1| < 1$. Hence $z = z_1$ is only simple pole inside C.

Residue at $(z = z_1) = \lim_{z \to z_1} (z - z_1) f(z)$

$$= \lim_{z \to z_1} (z - z_1) \frac{1}{(z - z_1)(z - z_2)}$$

$$= \lim_{z \to z_1} \frac{1}{(z_1 - z_2)}$$

$$= \frac{1}{z_1 - z_2}$$

$$= \frac{1}{\frac{2\sqrt{1-a^2}}{a}}$$

$$= \frac{a}{2\sqrt{1-a^2}} \qquad \text{...(3)}$$

Using Cauchy Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C) \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = \frac{2}{ia} \int_C f(z)dz$$

$$= \frac{2}{ia} .2\pi i. \frac{a}{2\sqrt{1-a^2}}$$

$$= \frac{2\pi}{\sqrt{1-a^2}}.$$

Example 3. Show that $\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = a > b > 0.$

Sol. Let $$I = \int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}, a > b > 0. \qquad \text{...(1)}$$

Let C be the circle $|z|=1$.

Putting $z=e^{i\theta} \quad \Rightarrow dz = ie^{i\theta}d\theta = iz\,d\theta$

$$\Rightarrow \frac{dz}{iz} = d\theta$$

Using equation (1), we get

$$I = \int_C \frac{dz}{iz\left[a+\frac{b}{2}(z+z^{-1})\right]^2}$$

$$= \frac{4}{i}\int_C \frac{z\,dz}{[2az+bz^2+b]^2}$$

$$= \frac{-4i}{b^2}\int_C f(z)dz \qquad ...(2)$$

where C is the unit circle $|z|=1$.

Poles of $f(z)$ are given by

$$z^2+\frac{2a}{b}z+1=0$$

or $$bz^2+2az+b=0$$

or $$z=\frac{-2a\pm\sqrt{4a^2-4b^2}}{2b}=\frac{-a\pm\sqrt{a^2-b^2}}{b}$$

Let $$z_1=\frac{-a+\sqrt{a^2-b^2}}{b},\, z_2=\frac{-a-\sqrt{a^2-b^2}}{b}$$

Since $a>b>0$, we have $|z_2|>1$. But $|z_1 z_2|=1$. so that $|z_1|<1$. Hence $z=z_1$ is only of order two inside C.

Now we have

$$f(z) = \frac{z}{(z-z_1)^2(z-z_2)^2}$$

$$= \frac{1}{(z-z_1)^2}\Phi(z), \text{ where } \Phi(z) = \frac{z}{(z-z_1)^2}$$

We have $$\Phi'(z) = \left[\frac{(z-z_2)^2 - z.(z-z_2)}{(z-z_2)^4}\right]$$

$$= \frac{z - z_2 - 2z}{(z-z_2)^3}$$

$$= \frac{(z-z_2)}{(z-z_2)^3}$$

$$\Phi'(z_1) = -\frac{(z_1-z_2)}{(z_1-z_2)^3}$$

$$= -\frac{-\frac{2a}{b}}{\frac{8(a^2-b^2)^{3/2}}{b^3}}$$

$$= \frac{ab^2}{4(a^2-b^2)^{3/2}} \qquad \text{...(3)}$$

Using Cauchy Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C) \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = -\frac{4i}{b^2}\int_C f(z)dz$$

$$= -\frac{4i}{b^2}.2\pi i.\frac{ab^2}{4(a^2-b^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2 - b^2)^{3/2}}.$$

Example 4. Show that $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta = \dfrac{2\pi}{n!}.$

Sol. Let $\quad \mathrm{I} = \int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta.$ ...(1)

Let C be the circle $|z| = 1.$

By equation (1), we have

$$= \int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta) d\theta$$

$$= \text{Real part of } \int_0^{2\pi} e^{\cos\theta} e^{i(\sin\theta - n\theta)} d\theta$$

$$= \text{R.P.of } \int_0^{2\pi} e^{\cos\theta + i\sin\theta - in\theta} d\theta$$

$$= \text{R.P.of } \int_0^{2\pi} e^{e^{i\theta}} e^{-in\theta} d\theta$$

Putting $z = e^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta} d\theta \quad \Rightarrow \quad \dfrac{dz}{iz} = d\theta$

$$= \text{R.P.of } \int_C e^z z^{-n} \frac{dz}{iz}$$

$$= \text{R.P.of } \frac{1}{\mathrm{i}} \int_C \frac{e^z}{z^{n+1}} dz$$

$$= \text{R.P.of } \frac{1}{\mathrm{i}} \int_C f(z) dz \qquad \text{...(2)}$$

Pole of *f(z)* are given by

$$z^{n+1} = 0$$

This has pole at $z = 0$ of order $n + 1.$

Residue at $(z=0)=\frac{1}{n!}\frac{d^n}{dz^n}\left[e^z\right]_{z=0}$

$$=\frac{1}{n!}.e^0$$

$$=\frac{1}{n!} \qquad \text{...(3)}$$

Using Cauchy Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C) \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I=\frac{1}{i}\int_C f(z)dz$$

$$=\frac{1}{i}.2\pi i.\frac{1}{n!}$$

$$=\frac{2\pi}{n!}.$$

Example 5. Show that $\int_0^\pi \frac{a d\theta}{a^2+\sin^2\theta}=\frac{\pi}{\sqrt{I+a^2}}, a>0.$

Sol. Let $I=\int_0^\pi \frac{a d\theta}{a^2+\sin^2\theta}, a>0.$

$$=\int_0^\pi \frac{2a\, d\theta}{2a^2+2\sin^2\theta}$$

$$=\int_0^\pi \frac{2a\, d\theta}{2a^2+1-\cos 2\theta}$$

Putting $2\theta=\varphi$, we get

$$= \int_0^{\pi} \frac{a\,d\phi}{2a^2+1-\cos\phi} \qquad \text{...(1)}$$

Again putting $z = e^{i\phi} \quad \Rightarrow dz = ie^{i\phi} d\phi$

$\Rightarrow \quad dz = iz\ d\varphi$

$$\Rightarrow \quad \frac{dz}{iz} = d\phi$$

Using equation (1), we get

$$I = \int_C \frac{adz}{iz\left[2a^2+1-\frac{1}{2}(z+z^{-1})\right]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{4a^2z+2z-z^2-1}$$

$$= -\frac{2a}{i}\int_C \frac{dz}{z^2-2z(2a^2+1)+1}$$

$$= 2ai\int_C f(z)dz \qquad \text{...(2)}$$

Where C is the unit circle $|z| = 1$.

Pole of f(z) are given by

$$z^2 - 2z(2a^2+1) = 0$$

or $$z = \frac{2(a^2+1) \pm \sqrt{4(2a^2+1)^2-4}}{2}$$

or $$z = 2(a^2+1) \pm 2a\sqrt{a^2+1}$$

Let $\quad z_1 = (2a^2+1) + 2a\sqrt{a^2+1}$

and $z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$

Since $|z_2 z_2| = 1$, we have $|z_2| < 1$ then clearly $|z_2| > 1$.

Hence $z = z_2$ is only pole inside C.

Residue at $(z = z_2) = \lim_{z \to z_2} (z - z_2) \dfrac{1}{(z - z_1)(z - z_2)}$

$$= \lim_{z \to z_2} \frac{1}{z - z_1}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{-4a\sqrt{a^2 + 1}} \qquad \text{...(3)}$$

Using Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C\text{)} \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = 2ai \int_C f(z)dz$$

$$= 2ai.2\pi i.\frac{1}{-4a\sqrt{a^2 + 1}}$$

$$= \frac{\pi}{\sqrt{a^2 + 1}}.$$

Example 6. Show that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2} = \frac{\pi(1 - p + p^2)}{1 - p}, 0 < p < 1.$

Sol. Let $\qquad I = \displaystyle\int_0^{2\pi}, \frac{\cos^2 3\theta \, d\theta}{1 - 2p\cos 2\theta + p^2}, 0 < p < 1$

Let C be the unit circle $|z| = 1$

$$= \int_0^{2\pi}, \frac{1+\cos^2 6\theta}{2(1-2p\cos\theta+p^2)} d\theta$$

$$= \frac{1}{2} \text{ Real part of} \int_0^{2\pi} \frac{1+e^{i6\theta}}{1-2p\cos\theta+p^2} d\theta \qquad \text{...(1)}$$

Putting $z = e^{i\theta} \quad \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow dz = iz\, d\theta$

$$\Rightarrow \quad \frac{dz}{iz} = d\theta$$

Using equation (1), we have

$$I_1 = \frac{1}{2} \int_C \frac{(1+z^6)}{iz\left[1-p(z^2+z^{-2})+p^2\right]} dz$$

$$= \frac{1}{2i} \int_C \frac{z(1+z^6)}{z^2-pz^4+p^2z^2-p} dz$$

$$= \frac{1}{2i} \int_C \frac{z(1+z^6)}{(z^2-p)(1-pz^2)} dz$$

$$= \frac{1}{2i} \int_C f(z) dz \qquad \text{...(2)}$$

Where C is the unit circle $|z| = 1$.

Poles of *f(z)* are given by

$$(z^2-p)(1-pz^2) = 0$$

or $\quad z = \pm\sqrt{p}, \pm\sqrt{\frac{1}{p}}$

Here only $z = \pm\sqrt{p}$ lie within C because $0<p<1$.

Residue at $(z = \sqrt{p}) = \lim_{z\to\sqrt{p}} (z-\sqrt{p})p(z)$

$$= \lim_{z \to \sqrt{p}} (z - \sqrt{p}) \frac{z(1+z^6)}{(z^2 - p)(1 - pz^2)}$$

$$= \lim_{z \to \sqrt{p}} \frac{z(1+z^6)}{(z+\sqrt{p})(1-pz^2)}$$

$$= \frac{\sqrt{p}(1+p^3)}{(2\sqrt{p})(1-p^2)}$$

$$= \frac{1+p^3}{2(1-p^2)}$$

Similarly, residue at $(z = -\sqrt{p})$

$$= \frac{\sqrt{p}(1+p^3)}{(-2\sqrt{p})(1-p^2)}$$

$$= \frac{1+p^3}{2(1-p^2)}.$$

Sum of the residues at $z = \pm\sqrt{p}$ is $\dfrac{1+p^3}{(1-p^2)}$ ...(3)

Using Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C) \qquad \ldots(4)$$

By equations (1), (2), (3) and (4), we have

$$I = \text{Real part of } \frac{1}{2i}\int_C f(z)dz$$

$$= \text{Real part of } \frac{1}{2i}.2\pi i.\frac{1+p^3}{1-p^2}$$

$$= \frac{\pi(1+p^3)}{(1-p^2)}$$

$$= \frac{\pi(1-p+p^2)}{1-p}.$$

Equating real part on both sides we get

$$\int_0^{2\pi} \frac{\cos^3 3\theta}{1-2p\cos\theta+p^2} = \frac{\pi(1-p+p^2)}{1-p}.$$

Example 7. Show that $\int_0^{2\pi} \frac{\sin^2\theta d\theta}{a+b\cos\theta} = \frac{2\pi}{b^2}\left[a-\sqrt{a^2-b^2}\right], a>b>0.$

Sol. Let $\quad I = \int_0^{2\pi} \frac{\sin^2\theta d\theta}{a+b\cos\theta}$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2(a+b\cos\theta)} d\theta$$

Then $I = R.P.\text{ of } \int_0^{2\pi} \frac{(1-e^{2i\theta})}{2a+b\cos\theta} d\theta$

Putting $z = e^{i\theta} \qquad \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow dz = iz\, d\theta$

$$\Rightarrow \quad \frac{dz}{iz} = d\theta$$

Using equation (1), we have

$$I = \text{R.P.of } \int_C \frac{(1-z^2)}{iz[2a+b(z+z^{-1})]} dz$$

$$= \text{R.P.of } \frac{1}{i}\int_C \frac{(1-z^2)dz}{[2a+bz^2+b]} dz$$

$$= \text{R.P.of } \frac{1}{bi}\int_C \frac{(1-z^2)}{\left[z^2+\frac{2a}{b}z+1\right]} dz$$

$$= \text{R.P.of } \frac{1}{bi}\int_C f(z)dz \qquad \ldots(2)$$

Where C is the unit circle $|z|=1$.

Poles of *f(z)* are given by

$$z^2 + \frac{2az}{b} + 1 = 0$$

or $$bz^2 + 2az + b = 0$$

or $$z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b} = \frac{-a \pm \sqrt{a^2 - b^2}}{b}$$

Let $$z_1 = \frac{-a + \sqrt{a^2 - b^2}}{b}, \frac{-a - \sqrt{a^2 - b^2}}{b}$$

Since *a>b>0*, we have

$|z_2| > 1$. But $|z_1 z_2| = 1$, so that $|z_1| < 1$. Hence $z - z_1$ is only simple pole inside C.

Residue at $(z = z_1) = \lim_{z \to z_1} (z - z_1) f(z)$

$$= \lim_{z \to z_1} (z - z_1) \frac{(1 - z^2)}{(z - z_1)(z - z_2)}$$

$$= \lim_{z \to z_1} \frac{(1 - z^2)}{(z - z_2)}$$

$$= \frac{1 - z_1^2}{z_1 - z_2}$$

$$= \frac{z_1\left[\frac{1}{z_1} - z_1\right]}{z_1 - z_2}$$

$$= \frac{z_1[z_2 - z_1]}{z_1 - z_2}$$

$$= -z_1 = \frac{-a + \sqrt{a^2 - b^2}}{b} \qquad \text{...(3)}$$

Using Cauchy's Residue theorem

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C\text{)} \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = \text{R.P.of } \frac{1}{bi} \int_C f(z)dz$$

$$= \text{R.P.of } \frac{1}{bi}.2\pi i \times \left[\frac{-a + \sqrt{a^2 - b^2}}{b} \right]$$

$$= \text{R.P. of } \frac{2\pi}{b^2} \left[a - \sqrt{a^2 - b^2} \right]$$

$$= -\frac{2\pi}{b^2} \left[a - \sqrt{a^2 - b^2} \right].$$

Example 8. Show that $\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta = \frac{\pi a^2}{1 - a^2}, 1 < a < 1.$

Sol. Let $I = \int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$

$$= \int_0^{\pi} \frac{\cos 2\theta}{(1 + a^2)^2 - 4a^2 \cos^2} d\theta$$

$$= \int_0^{\pi} \frac{\cos 2\theta (1 + a^2)}{(1 + a^2)^2 - 2a^2(1 + \cos 2\theta)} d\theta + 2a$$

$$= \int_0^{\pi} \frac{\cos 2\theta \cos\theta}{(1 + a^2)^2 - 2a^2(1 + \cos 2\theta)} d\theta$$

$$= \int_0^{\pi} \frac{(1+a^2)\cos 2\theta}{(1+a^2)^2 - 2a^2(1+\cos 2\theta)} d\theta + 0 \qquad \{\text{Using definite integrals property}\}$$

Putting $2\theta = \phi$, we have

$$= \int_0^{2\pi} \frac{(1+a^2)\cos\phi}{2[(1+a^2)^2 - 2a^2(1+\cos\phi)]} d\phi$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos\phi}{[1+a^4 - 2a^2\cos\phi]} d\phi \qquad \ldots(1)$$

Now putting $z = e^{i\phi}$ $\quad \Rightarrow dz = ie^{i\phi} d\phi \Rightarrow dz = iz\, d\phi$

$$\Rightarrow \quad \frac{dz}{iz} = d\phi$$

Using equation (1), we have

$$I = \frac{1}{2}\int_C \frac{(1+a^2)\left(\frac{z+z^{-1}}{2}\right)}{\left[1+a^4 - a^2(z+z^{-1})\right]} \frac{dz}{iz}$$

$$= \frac{(1+a^2)}{4i}\int_C \frac{(z^2+1)}{z\left[(1+a^4)z - a^2z^2 - a^2\right]} dz$$

$$= -\frac{(1+a^2)}{4a^2 i}\int_C \frac{(z^2+1)}{z\left[z^2 - \left(a^2 + \frac{1}{a^2}\right)z + 1\right]} dz$$

$$= \frac{(1+a^2)i}{4a^2}\int_C f(z)dz \qquad \ldots(2)$$

Where C is the unit circle $|z| = 1$.

Poles of $f(z)$ are given by

$$z\left[z^2 - \left(a^2 + \frac{1}{a^2}\right)z + 1\right] = 0$$

or $$z[a^2z^2-(a^4+1)z+a^2]=0$$

or $$z(z-a^2)\left(z-\frac{1}{a^2}\right)=0$$

or $$z=0,a^2,\frac{1}{a^2}.$$

This implies f(z) has simple poles at $z=0,a^2,\dfrac{1}{a^2}$.

Here only $z=0,\ a^2$ lie inside the circle since $a^2<1$.

Residue at $(z=0)=\lim\limits_{z\to 0} zf(z)$

$$=\lim_{z\to 0} z.\frac{(z^2+1)}{z\left[z^2-\left(a^2+\frac{1}{a^2}\right)z+1\right]}$$

$$=\lim_{z\to 0}\frac{(z^2+1)}{\left[z^2-\left(a^2+\frac{1}{a^2}\right)z+1\right]}$$

$$=1.$$

Residue at $(z=a^2)=\lim\limits_{z\to a^2}(z-a^2)\,f(z)$

$$=\lim_{z\to a^2}\frac{(z-a^2).(z^2+1)}{z(z-a^2)\left(z-\frac{1}{z^2}\right)}$$

$$=\lim_{z\to a^2}\frac{(z^2+1)}{z\left(z-\frac{1}{a^2}\right)}$$

$$=\frac{a^4+1}{a^2\left(a^2-\frac{1}{a^2}\right)}$$

$$= \frac{a^4+1}{a^4-1}.$$

Sum of residues at poles *z* = *0* and *z* = a^2, we have

$$= 1 + \frac{a^4+1}{a^4-1}$$

$$= \frac{a^4-1+a^4+1}{a^4-1}$$

$$= \frac{a^4+a^4}{a^4-1}$$

$$= \frac{2a^4}{a^4-1} \qquad \text{...(3)}$$

Using Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C) \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = \frac{(1+a^2)i}{4a^2}\int_C f(z)dz$$

$$= \frac{(1+a^2)i}{4a^2}.2\pi i.\frac{2.a^4}{a^4-1}$$

$$= \frac{-(1+a^2)\pi a^2}{a^4-a}$$

$$= -\frac{\pi a^2}{a^2-1}$$

$$= \frac{\pi a^2}{1-a^2}.$$

Example 9. Show that $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = -\frac{16}{5}(\pi+2i).$

Sol. Let $I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$ ...(1)

Putting $z = e^{i\theta}$ $\Rightarrow dz = ie^{i\theta} d\theta \Rightarrow dz = iz\, d\theta$

$\Rightarrow \frac{dz}{iz} = d\theta$

Using equation (1), we have

$$I = \int_C \frac{dz}{iz\left[3 - (z + z^{-1}) + \frac{(z - z^{-1})}{2i}\right]}$$

$$= 2\int_C \frac{dz}{[6iz - 2iz^2 - 2i + z^2 - 1]}$$

$$= 2\int_C \frac{dz}{[z^2 - 2iz^2 + 6iz - 1 - 2i]}$$

$$= 2\int_C \frac{dz}{z^2(1 - 2i) + 6iz - (1 + 2i)}$$

$$= 2\int_C f(z)dz \qquad ...(2)$$

Where C is the unit circle $|z| = 1$.

Poles of *f(z)* are given by

$$z^2(1 - 2i) + 6iz - (1 + 2i) = 0$$

or $$z = \frac{-6i \pm \sqrt{(6i)^2 + 4(1 - 2i)(1 \_ 2i)}}{2(1 - 2i)}$$

$$= \frac{-6i \pm \sqrt{-36 + 20}}{2(1 - 2i)}$$

$$= \frac{-3i \pm 2i}{(1-2i)}.$$

Let $\qquad z_1 = \frac{-3i+2i}{(1-2i)} = \frac{-i}{(1-2i)} = \frac{-i(1+2i)}{(1-4i^2)} = \frac{-i+2}{5} = \frac{2-i}{5}$

and $\qquad z_1 = \frac{-3i-2i}{(1-2i)} = \frac{-5i}{(1-2i)} = \frac{-5i(1+2i)}{(1-4i^2)} = \frac{-5(1+2i)}{5} = 2-i.$

Since $|z_1 z_2| = 1$, we have $|z_2| > 1$, then $|z_1| < 1$.

Hence the pole $z = z_1$ lies inside C.

Residue at $(z = z_1) = \lim_{z \to z_1} (z - z_1) f(z)$

$$= \lim_{z \to z_1} (z - z_1). \frac{1}{(z-z_1)(z-z_2)}$$

$$= \lim_{z \to z_1} \frac{1}{(z-z_2)}$$

$$= \frac{1}{(z_1 - z_2)}$$

$$= \frac{1}{\frac{2-i}{5} - (2-i)}$$

$$= \frac{-8+4i}{5} \qquad \text{...(3)}$$

Using Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C) \qquad \text{...(4)}$$

By equations (2), (3) and (4), we have

$$I = 2\int_C f(z)dz$$

$$= 2.2\pi i.\left(\frac{-8+4i}{5}\right)$$

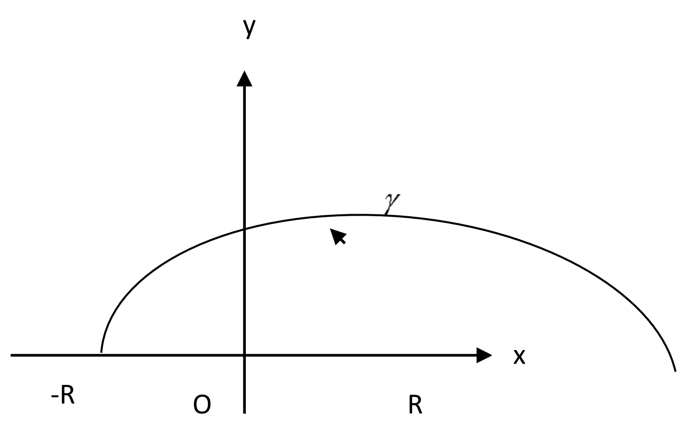
$$= -\frac{16\pi}{5}(1+2i).$$

12.5 Evaluation of the Integral $\int_{-\infty}^{\infty} f(x)dx$

Suppose the function *f(z)* is such that it has no poles lies on the real line and some poles lie in upper half of *z*-plane. Then we evaluate the integral

$$\int_{-\infty}^{\infty} f(x)dx$$

by considering the integral of *f(z)* taken around a closed contour *C*. Consisting of the upper half of the circle $|z| = R$ and part of the real axis from – R to R, provided that the integral round the semi-circle γ tends to a limit as $R \to \infty$.



Thus by Cauchy Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C)$$

or $$\int_\gamma f(z)dz + \int_{-R}^{R} f(x)dx = \int_C f(z)dz.$$

12.6 Jordan Inequality

We know that as θ increases from 0 to $\dfrac{\pi}{2}$, $\cos\theta$ decreases steadily and the mean ordinate of the graph of $y = \cos\theta$ over the range $0 \le x \le \theta$ also decreases steadily. But this mean ordinate is

$$\frac{1}{\theta}\int_0^{\theta} \cos x \, dx = \frac{\sin\theta}{\theta}$$

Therefore then $\qquad 0 \le \theta \le \dfrac{\pi}{2}$

We have $\qquad \dfrac{2}{\pi} \le \dfrac{\sin\theta}{\theta} \le 1$

or $\qquad \dfrac{2\theta}{\pi} \le \sin\theta \le \theta$

This is known as Jordan's inequality.

12.7 Jordan's Lemma

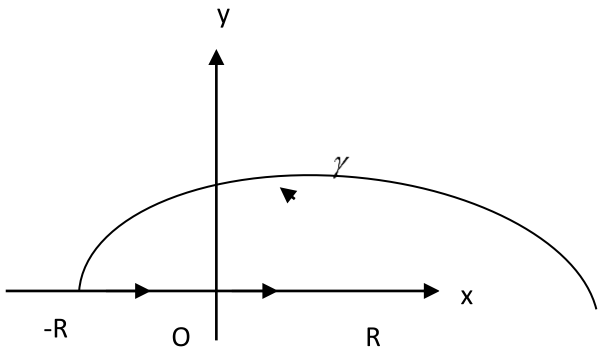
If *f(z)* is analytic except at finite number of singularities and if $f(z) \to 0$ uniformly as $z \to \infty$, then

$$\lim_{R\to\infty} \int_{\gamma} e^{imz} f(z) dz = 0, m > 0$$

Where γ denotes the semi-circle $|z| = R$, and R is taken so large that all the singularities of *f(z)* lie within the semi circle γ not lies on the boundary of semi-circle.

Examples

Example 10. Show that $\displaystyle\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}$.



Sol. Let $I = \int_0^\infty \frac{dx}{1+x^2}$ ...(1)

Suppose the integral $\int_C f(z)dz$

Where $f(z) = \frac{1}{1+z^2}$

We have $\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-R}^{R} f(x)dx$ ...(2)

Where C is the closed contour consisting of γ the upper half of the large circle $|z| = R$ and the real axis from $-R$ to R.

Poles of $f(z)$ are given by

$$1 + z^2 = 0$$

$$\Rightarrow \qquad z = \pm i$$

Here only one simple pole $z = i$ lie inside C

Residue at $(z = i) = \lim_{z \to i} (z-i) f(z)$

$$= \lim_{z \to i} (z-i) \frac{1}{(z^2+1)}$$

$$= \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \frac{1}{2i}$$

$$= \pi \qquad \ldots(3)$$

Now we have

$$\lim_{|z|\to\infty} af(z) = \lim_{|z|\to\infty} \frac{z}{1+z^2} = 0$$

Then $\lim_{R\to\infty} \int_\gamma f(z)dz = i(\pi - 0).(0) = 0 \qquad \ldots(4)$

Using theorem, if $\lim_{R\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_\gamma f(z)dz = i(\theta_2 - \theta_1)A$$

Where C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z| = R$.

Making $R \to \infty$ in equation (2) and using equation (3) and (4), we get

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx$$

$$\pi = 0 + \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

or $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$$

$\Rightarrow$ $$2\int_0^{\infty} \frac{dx}{1+x^2} = \pi$$

or $$\int_0^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2}.$$

Example 11. Show that $\int_{-\infty}^{\infty}\frac{dx}{x^4+1}=\frac{\pi}{\sqrt{2}}$.

Sol. Let $$I=\int_{-\infty}^{\infty}\frac{dx}{x^4+1} \qquad ...(1)$$

Suppose the integral $\int_C f(z)dz$

Where $$f(z)=\frac{1}{z^4+1}$$

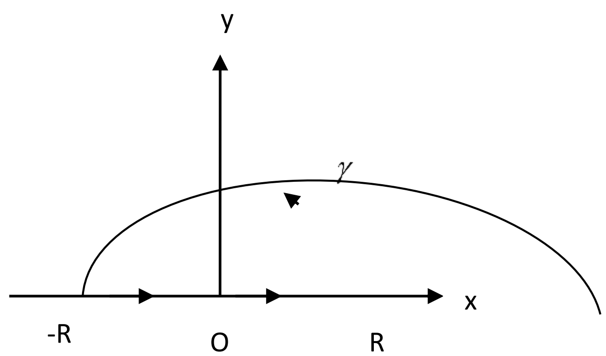
We have

$$\int_C f(z)dz=\int_\lambda f(z)dz+\int_{-R}^{R} f(x)dx \qquad ...(2)$$

Where C is the closed contour consisting of γ, the upper half of large circle $|z|=R$ and real axis from – R to R.

Poles of f(z) are given by

$$z^4+1=0$$



or $$z^4=-1$$

or $$z^4=e^{(2n+1)\pi i}$$

or $$z=e^{i(2n+1)\pi/4}, \text{ where } n=0,1,2,3.$$

Here only two simple pole $z=e^{i\pi/4}, e^{3i\pi/4}$ lie within C.

Let $z_1 = e^{i\pi/4}$ and $z_2 = e^{3i\pi/4}$.

Residue at $(z = z_1) = \lim_{z \to z_1} (z - z_1) f(z)$

$$= \lim_{z \to z_1} (z - z_1) . \frac{1}{z^4 + 1} \qquad \left\{ from \left(\frac{0}{0} \right) \right\} \Big]$$

$$= \lim_{z \to z_1} \frac{1}{4z^3}$$

$$= \lim_{z \to z_1} \frac{z}{4z^3}$$

$$= \lim_{z \to z_1} \frac{z}{4(-1)} \qquad [z^4 + 1 = 0]$$

$$= -\frac{z_1}{4}$$

Similarly, residue at $(z = z_2)$ is $= -\frac{z_2}{4}$.

Sum of residue at $z = z_1$ and $z = z_2$ is

$$= -\frac{z_1}{4} - \frac{z_2}{4}$$

$$= -\frac{1}{4} \left[e^{i\pi/4} + e^{-i\pi/4} \right]$$

$$= -\frac{1}{4} \left[e^{i\pi/4} - e^{-i\pi/4} \right]$$

$$= -\frac{1}{4} \left[2i \sin \frac{\pi}{4} \right]$$

$$= -\frac{1}{4} . 2i \frac{1}{\sqrt{2}}$$

$$= -\frac{i}{2\sqrt{2}}.$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \frac{-1}{2\sqrt{2}} \qquad \text{...(3)}$$

$$= \frac{\pi}{\sqrt{2}}$$

Now we have $\lim_{|z|\to\infty} zf(z) = \lim_{|z|\to\infty} \frac{z}{z^4+1} = 0,$ then

$$\lim_{R\to\infty} \int_\gamma f(z)dz = i(\pi - 0).(0) = 0 \qquad \text{...(4)}$$

Making $R \to \infty$ in equation (2) and using equations (3) and (4), we get

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx$$

or $$\frac{\pi}{\sqrt{2}} = 0 + \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$

or $$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}}$$

Example 12. Show that $\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}.$

Sol. Let $I = \int_0^{\infty} \frac{dx}{(1+x^2)^3}$ ...(1)

Suppose the integral $\int_C f(z)dz$, where $f(z) = \frac{1}{(1+z^2)^3}$

We have $\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-R}^{R} f(x)dx$ ...(2)

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R.

Poles of *f(z)* are given by

$$(z^2+1)^3=0$$

or $$z=\pm i, \pm i, \pm i$$

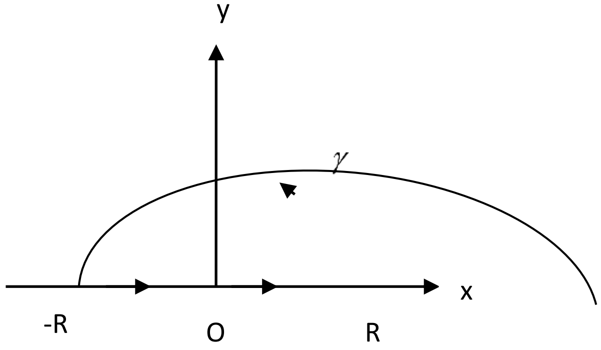
Here *f(z)* has poles of three order at $z=\pm i$ and only $z=i$ lies in the upper half plane

$$f(z)=\frac{1}{(z^2+1)^3}=\frac{\phi}{(z-i)^3}\text{, where } \phi(z)=\frac{1}{(z-i)^3}$$

Residue at *(z=i)* is $\dfrac{\phi^{''}(i)}{2!}$

Now $$\phi(z)=\frac{1}{(z+i)^3}$$

$$\phi^{'}(z)=-\frac{3}{(z+i)^4}$$



$$\phi^{''}(z)=\frac{12}{(z+i)^5}$$

$$\phi^{''}(i)=\frac{12}{(2i)^5}=\frac{12}{32i}=\frac{3}{8i}$$

Residue at $(z=i)=\dfrac{\theta^{''}(i)}{2i}$

$$=\frac{1}{2!}.\frac{3}{8i}$$

$$= \frac{3}{16i}$$

Now we have $\lim_{|z|\to\infty} z\, f(z) = \lim_{|z|\to\infty} \frac{z}{(1+z^2)^3} = 0$ then

$$\lim_{|z|\to\infty} \int_\gamma f(z)dz = i(\pi - 0).(0) = 0 \qquad ...(3)$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \frac{3}{16i}$$

$$= \frac{3\pi}{8} \qquad ...(4)$$

Making $R \to \infty$ in equation (2) and using equations (3) and (4), we get

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx$$

or $$\frac{3\pi}{8} = 0 + \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx$$

or $$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^3} dx = \frac{3\pi}{8}$$

or $$2\int_0^{\infty} \frac{1}{(1+x^2)^3} dx = \frac{3\pi}{8}$$

or $$\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}$$

Example 13. If $a > 0$, prove that $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi}{2} e^{-a}$

Sol. Let $$I = \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx \qquad ...(1)$$

Suppose the integral $\int_C f(z)dz$, where $f(z) = \frac{ze^{iz}}{z^2 + a^2}$

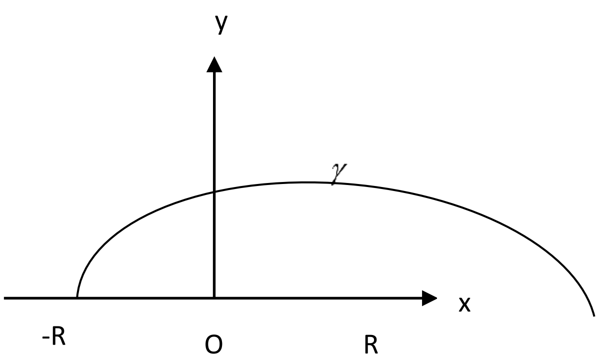
We have $\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-R}^{R} f(x)dx$ ...(2)

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R.

Poles of *f(z)* are given by

$$z^2 + a^2 = 0$$

$$\Rightarrow \qquad z = \pm ia$$



Here only $z = ia$ is simple pole lie inside *C*.

Residue at $(z = ia)$ is $\lim_{z \to ia} (z - ia). f(z)$

$$= \lim_{z \to ia} (z - ia) \frac{ze^{iz}}{z^2 + a^2}$$

$$= \lim_{z \to ia} \frac{ze^{iz}}{z^2 + ia}$$

$$= \frac{iae^{-a}}{2ia}$$

$$= \frac{e^{-a}}{2}.$$

By Cauchy’s Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C\text{)}$$

$$= 2\pi i \times \frac{e^{-a}}{2}$$

$$= \pi i e^{-a} \qquad ...(3)$$

Now we have $\lim_{|z|\to\infty} z\, f(z) = \lim_{|z|\to\infty} z.\frac{1}{z^2+a^2} = 0$

Then by Jordan Lemma

$$\lim_{R\to\infty}\int_\gamma f(z)dz = 0 \qquad ...(4)$$

Making $R \to \infty$ in equation (2) and using equations (3) and (4), we get

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx$$

$$\pi i e^{-a} = 0 + \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx$$

or
$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2}dx = \pi i e^{-a}$$

Equation imaginary part from both sides, we get

$$\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+a^2}dx = \pi e^{-a}$$

or
$$2\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \pi e^{-a}$$

or
$$\int_0^{\infty} \frac{x\sin x}{x^2+a^2}dx = \frac{\pi}{2}e^{-a}$$

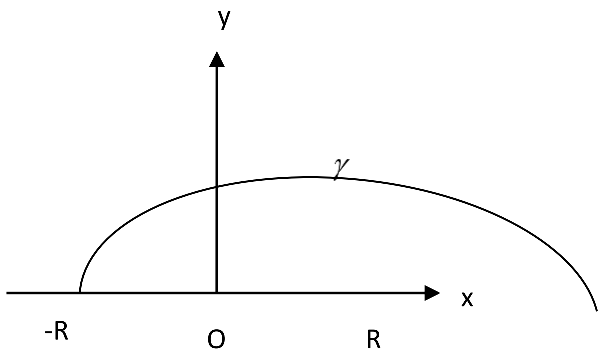
Example 14. Show that $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+2)} = \frac{2\pi}{5}$.

Sol. Let $$I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+2)} \qquad ...(1)$$

Consider the integral

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-R}^{R} f(x)dx \qquad ...(2)$$

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R.



Poles of f(z) are given by

$$(z^2+1)(z^2+2z^2+2) = 0$$

or $$(z^2+1) = 0$$

$\Rightarrow$ $$z = \pm i$$

or $$(z^2+2z+2) = 0$$

$\Rightarrow$ $$(z+1)^2 = -1$$

$\Rightarrow$ $$z = -1 \pm i$$

i.e., $$z = \pm i, -1 \pm i.$$

Here the pole $z = -1+i$ lie only inside C.

Residue at $(z = i)$ is $= \lim_{z \to i} (z-i) f(z)$

$$= \lim_{z \to i} (z-i).\frac{1}{(z^2+1)(z^2+2z+2)}$$

$$= \lim_{z \to i} \frac{1}{(z+i)(z^2+2z+2)}$$

$$= \frac{1}{2i(i^2+2i+2)}$$

$$= \frac{1}{2i(1+2i)}.$$

Now Residue at $(z=-1+i)$ is $\lim_{z \to 1+i} (z+1-i)\, f(z)$

$$= \lim_{z \to 1+i} (z+1-i)\frac{1}{(z^2+1)(z+1-i)(z+1+i)}$$

$$= \lim_{z \to 1+i} \frac{1}{(z^2+1)(z+1-i)}$$

$$= \frac{1}{[(-1+i)^2+1][-1+i+1+i]}$$

$$= \frac{1}{2i(1+2i)}$$

Sum of residue at $z=i$ and $z=-1+i$ is

$$= \frac{1}{2i(1+2i)} + \frac{1}{2i(1-2i)}$$

$$= \frac{1-2i+1+2i}{2i(1-4i^2)}$$

$$= \frac{2}{2i(5)}$$

$$= \frac{1}{5i}.$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \frac{1}{5i}$$

$$= \frac{2\pi}{5}. \qquad \text{...(3)}$$

Now we have

$$\lim_{|z|\to\infty} z\ f(z) = \lim_{|z|\to\infty} z \frac{1}{(z^2+1)(z^2+2z+2)} = 0$$

then $\displaystyle \lim_{|z|\to\infty} \int_\gamma f(z)dz = 0$ ...(4)

Making $R \to \infty$ in equation (2) and using equations (3) and (4), we get

$$\int_C f(z)dz = \int_\gamma f(z)dz + \int_{-\infty}^{\infty} f(x)dx$$

$$\frac{2\pi}{5} = 0 + \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+2)}$$

or $$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+2x+2)} = \frac{2\pi}{5}.$$

12.8 Summary

The process of integrating along a contour is called contour integration. Before proceeding to the evaluation of definite integrals, we discuss two useful identities:

(i) If $\lim_{z\to a}(z-a)f(z) = A$, and if C is the arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a| = r$, then

$$\lim_{z\to a} \int_C f(z)dz = iA(\theta_2 - \theta_1).$$

(ii) If C is an arc $\theta_1 \le \theta \le \theta_2$ of the circle $|z-a| = R$, then if $\lim_{R\to\infty} z\ f(z) = A$, then

$$\lim_{R\to\infty}\int_C f(z)dz = iA(\theta_2-\theta_1).$$

Cauchy's residue theorem apply only those pole or singularities that lie inside the contour otherwise, the residue at pole lying outside the contour is zero.

If *f(z)* is analytic except at finite number of singularities and if $f(z)\to 0$ uniformly as $z\to\infty$, then

$$\lim_{R\to\infty}\int_\gamma e^{imz} f(z)dz = 0, m>0$$

Where γ denotes the semi-circle $|z|=R,$ and R is taken so large that all the singularities of *f*(*z*) lie within the semi circle γ not lies on the boundary of semi-circle.

12.9 Terminal Questions

Q.1. What do you mean by evaluation of real definite integrals?

Q.2. Explain the Jordan inequality.

Q.3. State the Jordan Lemma.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-13: Applications of Complex Integration-II

Structure

13.1 Introduction

In complex analysis, the evaluation of integrals involving complex functions can be approached using several techniques, including contour integration, residues, and the Cauchy integral formulas. These methods are particularly powerful because they allow us to evaluate integrals that might be challenging or impossible to compute using real-variable calculus. Contour integration involves integrating a complex function along a curve (contour) in the complex plane. The key result used in contour integration is the Cauchy-Goursat theorem, which states that if a function is holomorphic inside a simply connected region and along its boundary, then its integral over any closed contour in that region is zero. The residue theorem is another important tool for evaluating complex integrals.

The evaluation of integrals is important in several areas of mathematics, science, and engineering. In the context of complex analysis, the evaluation of complex integrals plays a crucial role in understanding the behavior of complex functions and has various applications. These techniques, along with others in complex analysis, provide powerful tools for evaluating a wide range of complex integrals, leading to insights into the behavior of complex functions and their applications. In complex analysis, integrals along different paths between two points can be equal, provided the function being integrated is analytic in the region enclosed by the paths. This property is used to simplify complex integrals by choosing convenient paths of integration.

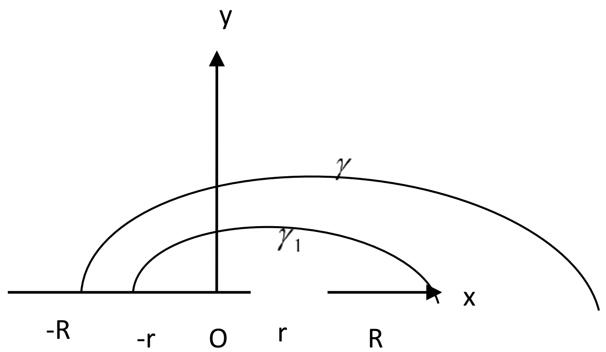
13.2 Objectives

After reading this unit the learner should be able to understand about the:

- Evaluation of Integrals in which poles lie on the real axis

13.3 Evaluation of Integrals in which poles lie on the real axis

In a special case when the poles of *f(z)* lie on the real axis and also within the semi-circular region, then those poles which lie on the real axis can be avoided by drawing semi-circles, γ_1, γ_2 etc., about those poles as centres and small radius r_1 and r_2 etc., in the upper half of the plane. This procedure known as “indenting at a point”.



We have

$$\int_C f(z)dz = \int_{-R}^{-r} f(x)dx + \int_{\gamma_1} f(z)dz + \int_r^R f(x)dx + \int_{\gamma} f(z)dz.$$

Check your Progress

1. How to evaluate the integrals when poles lie on the real axis.

Examples

Example.1. Apply the calculus of residue to prove that

$$\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

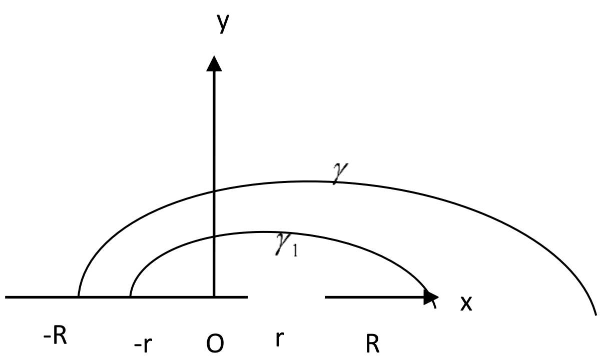
Sol. Let $$I = \int_0^\infty \frac{\sin mx}{x} dx \qquad ...(1)$$

Consider the integral $\int_C f(z)dz$, where $f(z) = \frac{e^{imz}}{z}$.

We have $$\int_C f(z)dz = \int_{-R}^{-r} f(x)dx + \int_{\gamma_1} f(z)dz + \int_r^R f(x)dx + \int_\gamma f(z)dz \qquad ...(2)$$

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R idented at $z = 0$. Let γ be the radius of small circle γ_1. Pole of f(z) are given by $z = 0$.

Here no pole of *f(z)* lie inside contour *C*.



By Cauchy's Residue theorem

$\int_C f(z)dz = 2\pi i \times$ (Sum of residue of *f(z)* at its poles within *C*)

$$= 2\pi i \times 0 = 0 \qquad ...(3)$$

For small circle γ_1, we have

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} z\frac{e^{imz}}{z} = 1$$

Then $$\lim_{z\to 0}\int_{\gamma_1} f(z)dz = -i(\pi - 0).1$$

$$= -i\pi \qquad ...(4)$$

Here negative sign is taken because the semi-circle γ_1 is negatively oriented.

Since $\frac{1}{z} \to 0$ as $z \to \infty$ and by Jordan's Lemma

$$\lim_{R\to\infty} \int_\gamma \frac{e^{imz}}{z} dz = \lim_{R\to\infty} \int_\gamma f(z)dz = 0 \qquad ...(5)$$

Making $r \to 0, R \to \infty$ in equation (2) and using equations (3), (4) and (5), we get

$$\int_C f(z)dz = \int_{-R}^{-r} f(x)dx + \int_{\gamma_1} f(z)dz + \int_r^R f(x)dx + \int_\gamma f(z)dz$$

or $$0 = \int_{-\infty}^{0} f(x)dx + \int_{\gamma_1} f(z)dz + \int_0^\infty f(x)dx + \int_\gamma f(z)dz$$

or $$0 = \int_{-\infty}^{0} f(x)dx - i\pi + \int_0^\infty f(x)dx + 0$$

or $$i\pi = \int_{-\infty}^{\infty} f(x)dx$$

or $$\int_{-\infty}^{\infty} \frac{e^{imx}}{x} dx = i\pi.$$

Equating imaginary parts both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi \quad \text{or} \quad \int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}.$$

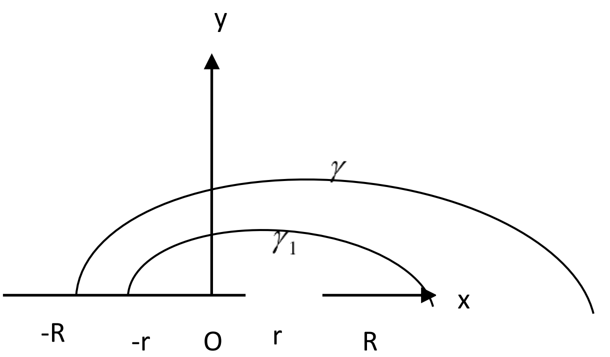
Example.2. Prove that $\int_0^\infty \frac{\sin x}{x(x^2+a^2)} dx = \frac{\pi}{2a^2}(1-e^{-a}), a > 0.$

Sol. Let $$I = \int_0^\infty \frac{\sin x}{x(x^2+a^2)} dx \qquad ...(1)$$

Consider the integral $\int_C f(z)dz$, where $f(z) = \frac{e^{iz}}{z(z^2+a^2)}$

We have $$\int_C f(z)dz = \int_{-R}^{-r} f(x)dx + \int_{\gamma_1} f(z)dz + \int_r^R f(x)dx + \int_{\gamma_1} f(x)dx \qquad ...(2)$$

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R idented at $z = 0$. Let z be the radius of small circle γ_1.



Poles of f(z) are given by

$$z(z^2 + a^2) = 0$$

or $$z = 0, \pm ia$$

Here only one pole $z - ia$ lie within C.

Residue at $(z - ia) = \lim_{z \to ia} (z - ia) f(z)$

$$= \lim_{z \to ia} (z - ia) \frac{e^{iz}}{z(z^2 + a^2)}$$

$$= \lim_{z \to ia} \frac{e^{iz}}{z(z + ia)}$$

$$= -\frac{e^{-a}}{2a^2}$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times (\text{Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \left(-\frac{e^{-a}}{2a^2} \right)$$

$$= -\frac{i\pi e^{-a}}{a^2} \qquad \text{...(3)}$$

For small circle γ_1, we have

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{ze^{iz}}{z(z^2+a^2)}$$

$$= \lim_{z\to 0} \frac{e^{iz}}{z^2+a^2}$$

$$= \frac{1}{a^2}$$

Then we have

$$\lim_{z\to 0} \int_{\gamma_1} f(z)dz = -i(\pi - 0).\frac{1}{a^2}$$

$$= -\frac{i\pi}{a^2} \qquad \text{...(4)}$$

Here negative sign is taken because the semi-circle γ_1 is negatively oriented.

Since $\lim_{z\to\infty} \frac{1}{z(z^2+a^2)} = 0$

Then by Jordan's Lemma, we have

$$\lim_{z\to\infty} \int_{\gamma} \frac{e^{iz}}{z(z^2+a^2)} dz = \lim_{z\to\infty} \int_{\gamma} f(z)dz = 0 \qquad \text{...(5)}$$

Making $r \to 0, R \to \infty$ in equation (2) and using equations (3), (4) and (5), we get

$$\int_C f(z)dz = \int_{-\infty}^{0} f(x)dx + \int_{\gamma_1} f(z)dz + \int_0^{\infty} f(x)dx + \int_{\gamma} f(z)dz$$

$$\frac{i\pi e^{-a}}{a^2} = \int_{-\infty}^{0} \frac{e^{ix}dx}{x(x^2+a^2)} - \frac{i\pi}{a^2} + \int_0^{\infty} \frac{e^{ix}dx}{x(x^2+a^2)} dx + 0$$

or $$\int_{-\infty}^{\infty} \frac{e^{ix}dx}{x(x^2+a^2)} = \frac{i\pi}{a^2}[1-e^{-a}]$$

Equating imaginary parts from both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+a^2)} dx = \frac{\pi}{a^2}[1-e^{-a}]$$

or $$\int_{0}^{\infty} \frac{\sin x}{x(x^2+a^2)} dx = \frac{\pi}{2a^2}[1-e^{-a}].$$

Example.3. Prove that $\int_0^{\infty} \frac{\log x\, dx}{(1+x^2)^2} = -\frac{\pi}{4}.$

Sol. Let $$I = \int_0^{\infty} \frac{\log x}{(1+x^2)^2} dx \qquad ...(1)$$

Consider the integral $\int_C f(z)dz,$ where $f(z) = \frac{\log z}{(1+z^2)^2}$

We have $$\int_C f(z)dz = \int_{-R}^{--r} f(x)dx + \int_{\gamma_1} f(z)dz + \int_r^R f(x)dx + \int_{\gamma} f(d)dz \qquad ...(2)$$

Where C is the closed contour consisting of γ, the upper half of large circle $|z| = R$ and real axis from – R to R idented at $z = 0$. Let r be the radius of small circle γ_1.

Poles of *f(z)* are given by

$$(1+z^2)^2 = 0$$

or $$z = \pm i, \pm i.$$

Here *f(z)* has only one pole of order two at $z = i$ within C.

Residue at $(z = i) = \frac{\phi'(i)}{1!}$, where $\phi(z) = \frac{\log z}{(z+i)^2}$

And $f(z) = \frac{\phi(z)}{(z-i)^2}$

$$\phi'(z) = \frac{(z+i)^2 \frac{1}{z} - \log z.2(z+i)}{(z+i)^4} = \frac{z^{-1}(z+i) - 2\log z}{(z+i)^3}$$

$$\varphi'(i) = \frac{\frac{1}{z}(2i) - 2\log i}{(z+i)^3}$$

$$= \frac{2 - 2\log i}{-8i}$$

$$= \frac{1}{4i}\left[1 - \log e^{i\pi/2}\right]$$

$$= \frac{i}{4}\left[1 - i\frac{\pi}{2}\right]$$

$$= \frac{1}{4}\left[i + \frac{\pi}{2}\right].$$

By Cauchy's Residue theorem, we have

$$\int_C f(z)dz = 2\pi i \times \text{(Sum of residue of } f(z) \text{ at its poles within } C)$$

$$= 2\pi i \times \frac{1}{4}\left(i + \frac{\pi}{2}\right)$$

$$= \frac{\pi}{2}\left(\mathrm{i}\frac{\pi}{2} - 1\right) \qquad \text{...(3)}$$

For small circle γ_1, we have

$$\lim_{z \to 0} z\, f(z) = \lim_{z \to 0} \frac{z \log z}{(1+z^2)^2} \qquad \left[\text{let } z = \frac{1}{t}\right]$$

$$= \lim_{t \to \infty} \frac{(1/t)\log(1/t)}{\left[1 + (1/t)^2\right]^2}$$

$$= \lim_{t \to \infty} \frac{t^3(\log 1 - \log t)}{(t^2+1)^2}$$

$$= \lim_{t \to \infty} \left(\frac{t^4}{(t^2+1)^2}\right)\left(-\frac{\log t}{t}\right)$$

$$=1.0=0$$

Then $\lim_{r\to 0}\int_{\gamma_1} f(z)dz = -i(\pi-0).(0)=0$ ...(4)

Here negative sign is taken because the semi-circle y_1 is negatively oriented.

Now we have

$$\lim_{z\to\infty} zf(z) = \lim_{z\to\infty}\frac{z\log z}{(1+z^2)^2}$$

$$=\lim_{z\to\infty}\left(\frac{z^2}{(1+z^2)^2}\right)\left(\frac{\log z}{z}\right)=(0).(0)=0$$

Then $\lim_{R\to\infty}\int_{\gamma} f(z)dz = -i(\pi-0).(0)=0$ ...(5)

Making $r\to 0, R\to\infty$ in equation (2) and using equations (3), (4) and (5), we get

$$\int_C f(z)dz = \int_{-\infty}^{0} f(x)dx + \int_{\gamma_1} f(z)dz + \int_0^{\infty} f(x)dx + \int_{\gamma} f(z)dz$$

or $$\frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)=\int_{-\infty}^{0} f(x)dx + 0 + \int_0^{0} f(x)dx + 0$$

or $$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dz + \int_{-\infty}^{0}\frac{\log x}{(x+x^2)^2}dx = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)$$

or $$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dz + \int_{-\infty}^{0}\frac{\log(-x)(-dx)}{\left[1+(-x)^2\right]^2} = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)$$

or $$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dz + \int_{-\infty}^{0}\frac{\log(-x)}{(1+-x^2)^2}dx = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)$$

$$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dz + \int_0^{\infty}\frac{\log(xe^{i\pi})}{(1+-x^2)^2}dx = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)$$

$$\int_0^{\infty}\frac{\log x}{(1+x^2)^2}dx + \int_0^{\infty}\frac{(\log x + i\pi)}{(1+-x^2)^2}dx = \frac{\pi}{2}\left(i\frac{\pi}{2}-1\right)$$

Equating real parts from both sides, we get

$$2\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{2}$$

or
$$\int_0^{\infty} \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4}.$$

13.4 Summary

The assessment of integrals holds significance across mathematics, science, and engineering fields. Particularly in complex analysis, the appraisal of complex integrals is pivotal for grasping the characteristics of complex functions and finding practical applications. These methodologies, coupled with others in complex analysis, furnish potent mechanisms for assessing a diverse array of complex integrals, unveiling deeper understandings of complex function behavior and their practical utilities.

In complex analysis, integrals taken along distinct paths between two points can be equivalent, provided the integrated function is analytically defined within the enclosed region. This feature is leveraged to simplify complex integrals by selecting advantageous paths for integration.

13.5 Terminal Questions

Q.1. Explain the concept of Evaluation of Integrals in which poles lie on the real axis.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.

2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.

3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.

4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.

5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.

6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)

7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.

UNIT-14: Meromorphic Functions

Structure

14.1 Introduction

In complex analysis, a meromorphic function is a function that is holomorphic (analytic) everywhere in its domain except at isolated points where it has poles. In other words, a meromorphic function is a function that is "mostly" analytic, except for a finite number of singularities, which are poles.Meromorphic functions are important in complex analysis because they behave nicely in many ways such as poles, residues, partial fraction decomposition and analytic continuation etc. Meromorphic functions arise naturally in many areas of mathematics and physics, including number theory, algebraic geometry, and quantum field theory. Meromorphic functions can often be analytically continued from their domain of definition to larger domains, allowing for the study of their behavior over a wider range of complex numbers.

The singularities of a meromorphic function are isolated poles, which are relatively simple to understand and analyze compared to other types of singularities. Hence the meromorphic functions play a fundamental role in complex analysis and provide a rich framework for understanding the behavior of complex functions with singularities.

14.2 Objectives

After reading this unit the learner should be able to understand about the:

- Meromorphic functions
- Mittag Leffler's expansion theorem
- Number of poles and zeroes of a meromorphic function
- Principle of argument
- Rouche's theorem

14.3 Meromorphic Functions

A function which has poles at its only singularities in the finite part of the plane is said to be a meromorphic function.

or

A function is said to be meromorphic in a region D if it is analytic in D except at a finite number of poles.

14.4 Mittag Leffler's expansion Theorem

Theorem: Suppose that the only singularities of $f(z)$ in the finite part of the z-plane are simple poles at $a_1, a_2, ...a_n$ arranged in the order of increasing absolute values.

Also suppose that

(i) residues of $f(z)$ at $a_1, a_2, ..., a_n$ be $b_1, b_2, ..., b_n$

(ii) $< C_n >$ is a sequence of circles (or rectangles or squares) of radii R_n or R_n is the

minimum distance of C_n from the origin) C_n encloses $a_1, a_2, ..., a_n$ and no other poles.

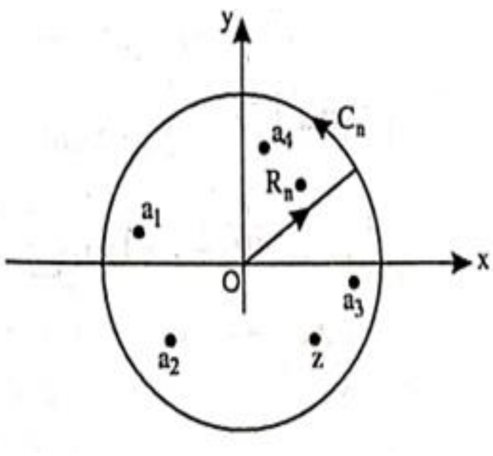
On the circle $C_n, |f(z)| < M$, where M is independent of n and $R_n \to \infty$ as $n \to \infty$.

Then for all values of z except poles,

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

Proof: Consider the integral

$$I = \int_{C_n} \frac{f(t)\,dt}{t - z}$$



z being any point (except pole) inside the circle C_n. The function $\dfrac{f(t)}{t-z}$ has simple pole at

$t = z.$

Evidently any pole of $f(t)$ is a pole of $f(t)/(t-z)$. But $f(t)$ has simple poles at

$t = a_1, a_2, ..., a_n$. Hence, $\dfrac{f(t)}{t-z}$ has simple poles at z, $a_m \ (m = 1, 2, ..., n)$.

Residue of $\dfrac{f(t)}{t-z}$ at z is $\lim\limits_{t\to z}(t-z)\dfrac{f(t)}{(t-z)} = f(z)$.

Residue of $\dfrac{f(t)}{t-z}$ at a_m is

$$\lim_{t\to a_m}(t-a_m)\frac{f(t)}{(t-z)} = \left[\lim_{t\to a_m}(t-a_m)\,f(t)\right].\left[\lim_{t\to a_m}\frac{1}{t-z}\right]$$

$$= \frac{b_m}{a_m - z}.$$

For residue of $f(t)$ at a_m is b_m.

By Cauchy's residue theorem, we have

$$\int_{C_n} \frac{f(t)\,dt}{t-z} = 2\pi i \text{ (Sum of residues of } \frac{f(t)}{t-z} \text{ within } C_n\text{)}$$

$$= 2\pi i \text{ [Residues at } z, a_1, a_2, ..., a_n\text{]}$$

$$= 2\pi i \left[f(z) + \sum_{m=1}^{n} b_m / (a_m - z) \right]$$

or $$\frac{1}{2\pi i} \int_{C_n} \frac{f(t)\,dt}{t-z} = f(z) + \sum_{m=1}^{n} \frac{b_m}{a_m - z} \qquad .....(1)$$

Suppose $f(t)$ is analytic at $t = 0$. Putting $z = 0$ in (1),

$$\frac{1}{2\pi i} \int_{C_n} \frac{f(t)}{t}\,dt = f(0) + \sum_{m=1}^{n} \frac{b_m}{a_m} \qquad ......(2)$$

Subtracting equation (2) from (1),

$$\frac{1}{2\pi i} \int_{C_n} \frac{z\,f(t)\,dt}{(t-z)t} = f(z) - f(0) + \sum_{m=1}^{n} \frac{zb_m}{a_m(a_m - z)} \qquad ......(3)$$

But $$\left| \frac{1}{2\pi i} \int_{C_n} \frac{z\,f(t)\,dt}{t(t-z)} \right| \le \left| \frac{1}{2\pi i} \right| \int_{C_n} \frac{|z|.|f(t)|.|dt|}{|t|(|t| - |z|)}$$

$$\le \frac{1}{2\pi} \frac{M|z|}{R_n(R_n - |z|)} \int_{C_n} |dt| = \frac{M|z|}{R_n - |z|}$$

For $\lim_{n\to\infty} R_n = 0 \qquad \therefore \lim_{n\to\infty} \int_{C_n} \frac{zf(t)\,dt}{(t-z)t} = 0$

Making $n \to \infty$ in (3) and noting this,

$$f(z) - f(0) + \sum_{m=1}^{\infty} \frac{zb_m}{a_m(a_m - z)} = 0$$

or $$f(z) = f(0) + \sum_{n=1}^{\infty} \left(\frac{b_n}{z - a_n} + \frac{b_n}{a_n} \right)$$

Finally $f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z-a_n} + \frac{1}{a_n} \right)$.

Check your Progress

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1. What do you mean by meromorphic function?

2. State the Mittag Leffler's expansion theorem.

Examples

Example.1 Prove that

$$\cot z = \frac{1}{z} + 2z \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2\pi^2}.$$

Solution. Let $f(z) = \cot z - \frac{1}{z} = \frac{z\cos z - \sin z}{z\sin z}$ ......(1)

Poles of $f(z)$ are given by $\sin z = 0,$ or $z = n\pi$ where $n \pm 1, \pm 2, ...$

$z = 0$ is not a pole of $f(z)$.

For $\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{z\cos z - \sin z}{z\sin z} = \left[\text{form } \frac{0}{0} \right]$

$$= \lim_{z\to 0} \frac{-z\sin z + \cos z - \cos z}{\sin z + z\cos z}, \left[\text{form } \frac{0}{0} \right]$$

$$= \lim_{z\to 0} \frac{-\sin z - z\cos z}{\cos z + \cos z - z\sin z} = \frac{0}{2} = 0$$

or $\lim_{z\to 0} f(z) = 0,$ so that $f(z)$ has a removable singularity at $z = 0$.

So we can define

$f(0)=0.$ Res $(z=n\pi)=\lim_{z\to n\pi}(z-n\pi)f(z)$

$$=\lim_{z\to n\pi}\frac{(z-n\pi)(z\cos z-\sin z)}{z\sin z}, \text{from } \frac{0}{0}.$$

$$=\lim_{z\to n\pi}\frac{(z-n\pi)(-z\sin z+\cos z-\cos z)+1.(z\cos z-\sin z)}{\sin z+z\cos z}$$

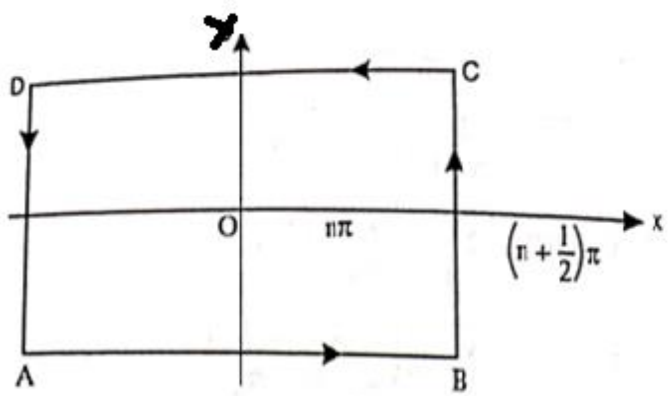
$$=\frac{n\pi\cos n\pi}{n\pi\cos n\pi}=1.$$

Let Res $(z=n\pi)=b_n$ and $z=n\pi=a_n$.

Then Res $(z=a_n)=b_n=1.$ $(z=a_n)=b_n=1.$

Let the contour C_n be the square $ABCD$ with centre at the origin and each side of length

$(2n+1)\pi.$ R_n = minimum distance of C_n from $z=0$ is $\left(n+\frac{1}{2}\right)\pi\to\infty$ as $n\to\infty$.



Also C_n encloses all the poles $a_1, a_2, ...a_n$. Along $AB, z=x-i\left(n+\frac{1}{2}\right)\pi$ so that

$2iz=2ix+(2n+1)\pi.$ Here x varies from $-\left(n+\frac{1}{2}\pi\right)$ to $\left(n+\frac{1}{2}\right)\pi.$

Hence $\left|e^{2iz}\right|=\left|e^{2ix}.e^{(2n+1)\pi}\right|=1.e^{(2n+1)\pi}=e^{(2n+1)\pi}$

Also $\frac{1}{|z_1 - z_2|} \le \frac{1}{|z_1| - |z_2|}$ and $|i| = 1$.

$$\cot z = i\frac{e^{iz} + e^{iz}}{e^{iz} - e^{-iz}} = i.\frac{e^{2iz} + 1}{e^{2iz} - 1}.$$

$$\therefore \qquad |\cot z| \le |i|.\frac{|e^{i2z}| + 1}{|e^{2iz}| - 1} = \frac{e^{(2n+1)\pi} + 1}{e^{(2n+1)\pi} - 1}$$

$$= \frac{1 + e^{-(2n+1)\pi}}{1 - \exp\left[-(2n+1)\pi\right]} \to 1 \text{ as } n \to \infty.$$

$\therefore \qquad |\cot z| \le 1$ as $n \to \infty$ along AB.

The same can be proved for other sides also.

Therefore $|\cot z| \le 1$ as $n \to \infty$ on C_n. Evidently $\frac{1}{z} \to 0$ as $n \to \infty$. It means that $f(z)$ is bounded. Now applying Mittag Leffler's theorem, we get

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{a_n} + \frac{1}{z - a_n}\right) + \sum_{n=-1}^{-\infty} b_n \left(\frac{1}{a_n} + \frac{1}{z - a_n}\right)$$

$$= 0 + \sum_{n=1}^{\infty} 1.\left(\frac{1}{n\pi} + \frac{1}{z - n\pi}\right) + \sum_{n=-1}^{-\infty} 1.\left(\frac{1}{n\pi} + \frac{1}{z - n\pi}\right)$$

$$= \sum_{n=1}^{\infty} \left[\left(\frac{1}{n\pi} + \frac{1}{z - n\pi}\right) + \left(\frac{1}{-n\pi} + \frac{1}{z + n\pi}\right)\right.$$

$$= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

or $$\cot z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2\pi^2}$$

Example.2. If C is a circle $|z - a| = r$, then

$$\frac{1}{2\pi i}\int_C \frac{dz}{z-a} = 1.$$

Solution. On circle $|z-a| = r$, we can take $z - a = re^{i\theta}$ so that

$$dz = re^{i\theta} i \; d\theta.$$

$$= \frac{1}{2\pi i}\int_C \frac{dz}{z-a}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{re^{i\theta} i \, d\theta}{re^{i\theta}}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} i \, d\theta$$

$$= 1.$$

14.5 The number of poles and zeroes of a meromorphic function

Theorem. If $f(z)$ is meromorphic inside a closed contour C and has no zero on C, then $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)} dz = N - P$

Where N is the no. of zeros and P the number of poles inside C, (a pole of zero of order m must be counted m times).

Proof: Let $z = a_i \, (i = 1, 2, ...m)$ be the zeros of $f(z)$ which lie inside C and r_i be the order of a_i .

Again suppose $b_i \; i = (1, 2, ...n)$ be the poles of $f(z)$ inside C and S_i be the order of b_i . Then we have to show that

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)} dz = \sum_{r=1}^{m} r_i - \sum_{i=1}^{n} S_i$$

Enclosing each zero and pole by non- overlapping circles $A_1, A_2, ..., A_m$ and $B_1, B_2, ..., B_n$

respectively each of radii ρ. Since poles and zeros are isolated. We can always find such ρ. Therefore

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{m}\int_{A_i}\frac{1}{2\pi i}\frac{f'(z)}{f(z)}dz + \sum_{i=1}^{n}\frac{1}{2\pi i}\int_{B_i}\frac{f'(z)}{f(z)}dz \text{ \_\_\_\_\_\_\_\_\_\_ } (1)$$

Since a_i is a zero of order r_i of $f(z)$. We may write $f(z) = (z-a_i)^{r_i}\phi_i(z)$

Where ϕ_i is analytic and non zero at a_i Taking log of both sides. We get

$$\log f(z) = r_i \log(z-a_i) + \log \phi_i(z)$$

Differentiating both sides w.r. to z we get

$$\frac{f'(z)}{f(z)} = \frac{r_i}{z-a_i} + \frac{\phi'_i(z)}{\phi_i(z)}$$

We have $\int_{A_i}\frac{\phi'_i(z)}{\phi_i(z)}dz = 0$. Since $\frac{\phi'_i(z)}{\phi_i(z)}$ is analytic at $z = a_i$ and $\int_{A_i}\frac{r_i}{z-a_i}dz = r_i\int_0^{2\pi}\frac{\rho i e^{i\theta}}{\rho e^{i\theta}}d\theta = 2\pi i\ r_i$

$$\therefore \sum_{i=1}^{m}\frac{1}{2\pi i}\int_{A_i}\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{m}\frac{1}{2\pi i}2\pi i r_i = \sum_{i=1}^{m} r_i$$

Since b_i is a pole of order S_i of $f(z)$. We have $f(z) = \psi_i(z)/(z-b_i)^{S_i}$

Where ψ_i is analytic and nθn zero at b_i Taking log of both sides and diff. w. r .t. z. We get

$$\frac{f'(z)}{f(z)} = \frac{\psi'_i(z)}{\psi_i(z)} - \frac{S_i}{(z-b_i)}$$

As above similarly

$$\sum_{i=1}^{n}\frac{1}{2\pi i}\int_{B_i}\frac{f'(z)}{f(z)}dz = \sum_{i=1}^{n}\frac{1}{2\pi i}(-2\pi i S_i) = -\sum_{i=1}^{n} S_i \text{ \_\_\_\_\_\_\_\_\_\_ } (3)$$

Using (2) and (3) we get by (1)

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = \sum_{i=1}^{m} r_i - \sum_{r=i}^{n} S_i = N - P \_\_\_\_\_\_ (4)$$

14.6 Principle of Argument

Theorem: $N - P = \frac{1}{2\pi}\Delta C \arg f(z)$

Where ΔC denotes the variation in $\arg\ f(z)$ as z moves once round C .

Proof: Using above theorem

$$N - P = \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz$$

Let $f(z) = \mathrm{Re}^{i\phi}$. Then $R = |f(z)|, \phi = \arg f(z)$ and

Let $f'(z)dz = \frac{d}{dz}(f(z))dz = d\left(\mathrm{Re}^{i\phi}\right) = e^{i\phi}(dR + iRd\phi)$

We have $$N - P = \frac{1}{2\pi i}\int_C \left(\frac{dR}{R} + id\phi\right) = \frac{1}{2\pi i}\int_C \frac{dR}{R} + \frac{1}{2\pi}\int_C d\phi$$

Now $\int_C \frac{dR}{R} = [\mathrm{logR}]_C = 0$

Since $\log R$ retains its original value if z moves once round C. Also

$$\frac{1}{2\pi}\int_C d\phi = \frac{1}{2\pi}[\phi]_c = \frac{1}{2\pi}\Delta c \arg f(z)$$

Hence we have

$$N - P = \frac{1}{2\pi}\Delta c \arg f(z) \_\_\_\_\_\_\_\_\_ (5)$$

Thus the excess of the no. of zeros over the no. of poles of a meromorphic function equals $(1/2\pi)$ times the increase in arg $f(z)$ in z goes once round C.

This is known as argument principle.

Note: When $f(z)$ is analytic. We have $P=0$ and in this case

$$N=\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz=\frac{1}{2\pi}\Delta c \arg f(z)$$

i.e. the number of zeros of an analytic function $f(z)$ within C is $(1/2\pi)$ times the increase in $\arg f(z)$ as z goes once round c. This is known as the argument principle for an analytic function.

14.7 Rouche's Theorem

Theorem. Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $\log|g(z)|<|f(z)|$ on C. Then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

Proof: Give that $|g(z)|<|f(z)|$ on C __________(1)

First we shall prove that neither $f(z)$ nor $f(z)+g(z)$ has a zero on c if possible let

$f(z)=0$ or C then $|f(z)|=0$ on C ________(2)

By (1) and (2) $\Rightarrow |g(z)|<0$ on C. which is contradiction Again let $g(z)+\phi(z)=0$ on C

then $g(z)=-f(z)$ on C $\Rightarrow |g(z)|=|f(z)|$ on c which contradiction by relation (1)

Hence neither $f(z)$ nor $f(z)+g(z)$ has a zero on C. Now by argument principle zeros of $f(z)$ and $f(z)+g(z)$ are respectively given by $(P=0)$. Let $F(z)=g(z)/f(z)$ then $g(z)=F(z)f(z) \Rightarrow g'=f'F+fF'$

$$M = \frac{1}{2\pi i}\int_C \frac{f'}{f}dz \text{ and } N = \frac{1}{2\pi i}\int_C \frac{f'+g'}{f+g}dz$$

Now $N - M = \frac{1}{2\pi i}\int_C \frac{f'+f'F+fF'}{f+g}dz - \frac{1}{2\pi i}\int_c \frac{f'}{f}dz$

$$= \frac{1}{2\pi i}\int_c \frac{f'(1+F)}{f+fF}dz + \frac{1}{2\pi i}\int_C \frac{fF'}{f+fF}dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\int_c \frac{f'}{f}dz + \frac{1}{2\pi i}\int_C F'(1+F)^{-1}dz - \frac{1}{2\pi i}\int_C \frac{f'}{f}dz$$

$$= \frac{1}{2\pi i}\int_C F'(1+F)^{-1}dz$$

$$= \frac{1}{2\pi i}\int_C F'(1-F+F^2 - - - - - -)dz \quad \{\because |F(z)<1|\}$$

$$= \frac{1}{2\pi i}\int_C F'dz + \frac{1}{2\pi i}\int_C F'Fdz + ... = 0$$

Since $f(z)$ and $g(z)$ are analytic and $g(z) \neq 0$ at any point on C. So F and F' are also analytic within and on C. Consequently each integral is separately zero.

Hence $N = M$ i.e., $f(z)$ and $f(z)+g(z)$ have the same of zeros inside C.

Schwarz lemma: Theorem. If $f(z)$ is analytic for $|z|<1$ and satisfies the conditions $|f(z)| \leq 1$ $f(0)=0$ then $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ equality sign holds only if $f(z)$ is a linear transformation $w = e^{i\propto}z$, where $\propto$ is a real constant.

Proof: Since $f(z)$ is analytic in disc $|z|<1$ expanding $f(z)$ in taylor's series about the origin.

We get $f(z) = a_0 + a_1 z + a_2 z^2 + ...$

By hypothesis $f(0)=0$. So we have $a_0 = 0 \therefore f(z) = a_1 z + a_2 z^2 + ...$ Let

$$g(z)=\frac{f(z)}{z}=a_1+a_2z+a_3z^2+........$$

In the annulus $|z|<1$ $g(z)$ $g(z)$ is analytic (if $g(0)=a_1=f'(0)$) suppose $z=a$ is any arbitrary point of this region choosen r s.t. $|a|<r<1$. Since $|f(z)|\le 1$. We have

$$|g(z)|=\frac{|f(z)|}{|z|}\le\frac{1}{r} \text{\_\_\_\_\_\_\_\_\_\_\_\_} (2)$$

By the maximum modulus principle (2) also holds in the region $|z|\le r$ so $|g(a)|=\left|\frac{f(a)}{a}\right|\le\frac{1}{r}$

When $r\to 1$. We have $|g(a)|\le\left|\frac{f(a)}{a}\right|\le 1$ or $|f(a)|\le|a|$ in particular $|g(0)|=|f'(0)|\le 1$

Since a is arbitrary. We have $|f(z)|\le|z|$ for all z satisfying $|z|\le 1$. If this result holds at a single point then $|g(z)|$ will attain its maximum value and hence $g(z)$ must reduce to a constant by (maximum modulus theorem). Thus $g(a)=1$ holds only when $g(z)=\frac{f(z)}{z}we^{i\propto}$. Where $\propto$ is a real constant

Hence $f(z)=ze^{i\propto}$. Where $\propto$ is a real constant.

Examples

Example.3 Use Rouche's theorem to show that the equation $z^7-5z^3+12=0$ lies between the circle $|z|=1$ and $|z|=2$.

Solution. Let c_1 represent the circle $|z|=1$ and c_2 represent the circle $|z|=2$. Suppose $f(z)=12$ and $g(z)=z^{7-5z^3}$ we observe that both $f(z)$ and $g(z)$ are analytic within and on c_1. Now

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{z^7-5z^3}{12}\right| \le \frac{|z|^7+|-5z^3|}{12} = \frac{|z|^7+5|z^3|}{12} \le \frac{1+5}{12} = \frac{1}{2}.$$

Since $|z|=1$ on C $\therefore \left|\frac{g(z)}{f(z)}\right|<1$

or

$|g(z)|<|f(z)|$ on c_1

Hence by Rouche's theorem $f(z)+g(z)=z^7-5z^3+12$ has the same no. of zero inside c_1 as $f(z)=12$.

Since $f(z)=12$ has no zeros inside c_1 theorem $f(z)+g(z)=z^7-5z^3+12$ has no zeros inside C.

Now consider the circle C_2. Take $F(z)=z^7\phi(z)=12-5z^3$. We observe that both $F(z)$

and $\phi(z)$ are analytic within and on c_2. We have

$$\left|\frac{\phi(z)}{f(z)}\right| = \frac{|12-5z^3|}{|z|^7} \le \frac{|12|+5|z|^3}{|z|^7} = \frac{12+5.2^3}{2^7} = \frac{52}{128} < 1$$

since $|z|$=2 on C_2

Thus on C_2, $|\phi(z)|<|F(z)|$

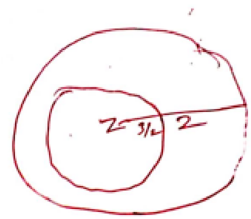
Hence by Rouche's theorem $F(z)+\phi(z)=z^7-5z^3+12$ has the same no. of zeros as $F(z)=z^7$ inside C_2. Since $F(z)=z^7$ has all the locked at the origin therefore ll the seven zero of z^7-5z^3+12 lie inside the circle C_2.

Hence all the roots of the equation $z^7-5z^3+12=0$ lie between the circle $|z|=1$ and $|z|=2$

Example.4: Use Rouche's theorem to show that the equation $z^5+15z+1=0$ has one root in the disc $|z|<\frac{3}{2}$ and fur roots in the annulus $\frac{3}{2}<|z|<2$.

Solution: Let $f(z) = z^5$ and $g(z) = |5z + 1$ Now on the circle $|z| = 2$. we have $|f(z)| = |z|^5 = 2^5 = 32$

$|g(z)| = |15z+1| \le 15|z| + 1 = 31$ $|g(z)| \le |f(z)|$ on the circle $|z| = 2$



Hence by Rouche's theorem the function $f(z) = z^5$. Since the function $f(z)$ has a zero of order 5 at $z = 0$ therefore all the five roots of $z^5 + 15z + 1 = 0$ must lie inside the disc $|z| < 2$ Again for $|z| < \frac{3}{2}$. We have

$$|z^5 + 1| \le |z|^5 + 1\frac{243}{32} + 1\frac{45}{2} = |15z| i \left(z = \frac{3}{2} \right)$$

The function $z^5 + 15z + 1$ has an many zeros in $|z| < \frac{3}{2}$ as the function 15z. Since 15z has exactly one zero in this region. So does $z^5 + 15z + 1$. Hence four of the zeros of $z^5 + 15z + 1$ must lie in the ring $\frac{3}{2} < |z| < 2$.

Example.5: If $f(z) = z^5 - 3iz^2 + 2z + i - 1$, then evaluate $\int_C \frac{f'(z)}{f(z)} dz$, where C encloses zero of $f(z)$?

Solution. Given $f(z)$ has 5 zeros. $\therefore N = 5$, it has no poles.

$\therefore P = 0.$ By Theorem $N - P = \frac{1}{2\pi} \Delta C \arg f(z)$

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P = 5 - 0 = 5$$

or $$\int_C \frac{f'(z)}{f(z)} dz = 10\pi i.$$

14.8 Summary

A function which has poles at its only singularities in the finite part of the plane is said to be a meromorphic function.

Suppose that the only singularities of $f(z)$ in the finite part of the z-plane are simple poles at $a_1, a_2, ...a_n$ arranged in the order of increasing absolute values. Also suppose that

(i) residues of $f(z)$ at $a_1, a_2, ..., a_n$ be $b_1, b_2, ..., b_{lv}$

(ii) $< C_n >$ is a sequence of circles (or rectangles or squares) of radii R_n or R_n is the

minimum distance of C_n from the origin) C_n encloses $a_1, a_2, ..., a_n$ and no other poles. On the circle $C_n, |f(z)| < M$, where M is independent of n and $R_n \to \infty$ as $n \to \infty$.

Then for all values of z except poles, $$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - a_n} + \frac{1}{a_n} \right)$$

If $f(z)$ is meromorphic inside a closed contour C and has no zero on C, then $\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = N - P$

Where N is the no. of zeros and P the number of poles inside C, (a pole of zero of order m must be counted m times).

$N - P = \frac{1}{2\pi} \Delta C \arg f(z)$, where ΔC denotes the variation in $\arg f(z)$ as z moves once round C .

Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let $\log|g(z)| < |f(z)|$ on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

14.9 Terminal Questions

Q.1. What do you mean by meromorphic function?

Q.2. Explain the Mittag-Leffler’s expansion theorem.

Q.3. What do you mean by principle of argument?

Q.4. State and Prove Rouche’s theorem.

Suggested Text Book Readings:

1. Ponnusamy, Foundations of Complex Analysis. 2nd Edition, Narosa Book Publication, 2008.
2. K.P. Gupta, Functions of complex variable, Sixteen Edition, Pragati Prakashan, 2002.
3. J. B. Conway, Functions of One Complex Variable, Narosa Publishing House, New Delhi, 2002.
4. Priestly, H. A.: Introduction to Complex Analysis, Oxford University Press, 2008.
5. Ahlfors, L.V.: Complex Analysis, McGraw Hill Education; 3rd Edition, 2017.
6. M. Spiegel, J. Schiller, S. Lipschutz, Schaum's Outline of Complex Variables, 2ed (Schaum's Outlines)
7. James W. Brown & R. V. Churchill: Complex variables and applications, McGraw-Hill, 2006.