MScSTAT – 404 (N)A/ MASTAT – 404(N)A

U. P. Rajarshi Tandon Open University, Prayagraj

Actuarial Statistics

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences
U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Ex. Head, Department of Statistics
University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Ex. Head, Department of Statistics
University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics
D. D. U. Gorakhpur University, Gorakhpur.

Prof. Shruti **Member-Secretary**
Professor, School of Sciences
U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Sandeep Mishra **Writer**
Department of Statistics (Unit 1, 2, 3, 4 & 5)
Shahid Rajguru College of Applied Sciences for Women
University of Delhi, New Delhi

Dr. Mahendra Saha **Writer**
Department of Statistics (Unit 6, 7, 8, 9, 10, 11, 12 & 13)
Faculty of Mathematical Sciences,
University of Delhi, New Delhi

Dr. Mahendra Saha **Editor**
Department of Statistics (Unit 1, 2, 3, 4, & 5)
Faculty of Mathematical Sciences
University of Delhi, New Delhi

Prof. Jitendra Kumar **Editor**
Department of Statistics (Unit 6, 7, 8, 9, 10, 11, 12 & 13)
Central University of Rajasthan, Ajmer, Rajasthan

Prof. Shruti **Reviewer & Course Coordinator**
School of Sciences
U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT– 404 NA/MASTAT– 404 NA **ACTUARIAL STATISTICS**

First Edition: ***July 2025***
ISBN:

Printed and Published by Col. Vinay Kumar, Registrar, Uttar Pradesh Rajarshi Tandon Open University, 2025.
Printed By:

Blocks & Units Introduction

The ***Block - 1 –Probability Models and Life Tables,*** is the first block of said self-learning material (SLM), which is divided into five units.

The ***Unit – 1: Basic Concepts - Probability Distributions*****:** is the first unit of present SLM, which serves as the introductory unit. In this unit the topics which have been discussed areas follows Introduction, Important Discrete Distributions, Important Continuous Distributions, Mixed Distributions, Insurance applications, Sum of Random variables.

The ***Unit – 2: Utility Theory:*** discussed about the utility criterions, Jensen's inequality, types of utility functions.

The ***Unit – 3: Principles of Premium Calculation:*** concentrates on the properties and examples of premium calculations.

The ***Unit – 4: The Collective Risk Model:*** explores about the model, compound Poisson distribution, the effect of reinsurance, recursive calculation of aggregate claims distribution, expansion of Panjer recursion formula, the application of recursion formula, approximate calculation of aggregate claims distributions.

The ***Unit – 5: Survival Distributions and Life Tables:*** discuss about the probability for age at death, survival function, time until death, curate future lifetime and force of mortality.

The ***Block - 2 –Reliability Estimations for Actuarial,*** is the second block of said SLM, which is divided into four units.

Unit – 6: Basic Concepts and Ageing: This unit lays the Reliability Theory with Applications in Insurance and Annuities, Failure and Failure Rates, Survival Function as a Reliability Measure in Insurance, Core Measures, Statistical Tools in Insurance and Annuity Modelling, Probability Distributions Used in Life Modelling, Force of Mortality Function, Importance in Insurance and Annuity Design.

This unit also focus on the Ageing across the Policy Lifecycle: Stages and Claim Behaviour, Infant Mortality Period, Normal Life, Wear-Out-Phase, Bathtub Curve, Ageing in Actuarial and Insurance Context, Models and Strategies for Managing Ageing in Insurance.

Unit – 7: Reliability Estimation and Repairable Systems: This unit is dedicated to the Lifetime Data Analysis, Key distributions, Actuarial Applications of Lifetime Distributions, Censored Data Analysis, Types of Censoring, Kaplan-Meier Estimator, Reliability Block Diagrams, Common Configurations, Actuarial Applications of RBDs and Fault Tree Analysis.

This unit also examines about the Maintenance Strategies, Preventive Maintenance, Corrective Maintenance, and Markov Models for Repairable Systems, Two- State Markov Model, Actuarial and Insurance Applications of Markov Models, Model Extensions, The Renewal Process, Availability Modeling, Instantaneous Availability, Steady State Availability, Steady State Unavailability, Mean Down Time, Impact of Repair Policies, and Maintenance Optimization.

Unit – 8: Growth Models, Accelerated Life Testing and Multiple Life Functions: This unit discuss about the, reliability growth models and accelerated life testing techniques are explored. The unit covers various reliability growth models, such as the Duane model and the AMSAA model. Accelerated life testing involves Arrhenius Model, Eyring Model, and Inverse Power Law Model.

In this unit, we also discussed about the Introduction, Joint Distribution of Future life time, joint life and last survivor status, insurance and annuity benefits through multiple life functions evaluation for special mortality law.

Unit – 9: Decrement Theory and Application of Multiple Decrement Theory: In this unit, we discussed about the multiple decrement models, deterministic and random survivorship groups, associated single decrement tables, central rates of multiple decrement, net single premiums and their numerical evaluations.

The ***Block - 3 –Insurance and Annuities,*** is the third block of said SLM, which is divided into four units.

Unit – 10: Fundamentals of Computation of Interest Rate: In this unit, we discussed about the Principles of compound interest. Nominal and effective rates of interest and discount, force of interest and discount, compound interest, accumulation factor, continuous compounding.

Unit – 11: Life Insurance & Life Annuities: In this unit, we discussed about the Insurance payable at the moment of death and at the end of the year of death-level benefit insurance, endowment insurance, deferred insurance and varying benefit insurance, recursions, commutation functions. Single payment, continuous life annuities, discrete life annuities, life annuities with monthly payments, commutation functions, varying annuities, recursions, complete annuities-immediate and apportion able annuities-due.

Unit – 12: Net Premiums & Net Premium Reserves: In this unit, we discussed about the ccontinuous and discrete premiums, true monthly payment premiums, apportion able premiums, commutation functions, and accumulation type benefits. Payment premiums, apportion able premiums, commutation functions, accumulation type benefits. Continuous and discrete net premium reserve, reserves on a semi continuous basis, reserves based on true monthly premiums, reserves on an apportion able or discounted continuous basis, reserves at fractional durations, allocations of loss to policy years, recursive formulas and differential equations for reserves, commutation functions.

Unit – 13: Some Practical Considerations: In this unit, we discussed about the practical considerations of premiums that include expenses-general expenses types of expenses, per policy expenses. Claim amount distributions, approximating the individual model, stop-loss insurance.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

U. P. Rajarshi Tandon Open
University, Prayagraj

MScSTAT – 404 (N)A/ MASTAT – 404(N)A

Actuarial Statistics

Block: 1 Probability Models and Life Tables

Course Design Committee

Dr. Ashutosh Gupta **Chairman**
Director, School of Sciences
U. P. Rajarshi Tandon Open University, Prayagraj

Prof. Anup Chaturvedi **Member**
Ex. Head, Department of Statistics
University of Allahabad, Prayagraj

Prof. S. Lalitha **Member**
Ex. Head, Department of Statistics
University of Allahabad, Prayagraj

Prof. Himanshu Pandey **Member**
Department of Statistics
D. D. U. Gorakhpur University, Gorakhpur.

Prof. Shruti **Member-Secretary**
Professor, School of Sciences
U.P. Rajarshi Tandon Open University, Prayagraj

Course Preparation Committee

Dr. Sandeep Mishra **Writer**
Department of Statistics
Shahid Rajguru College of Applied Sciences for Women
University of Delhi, New Delhi

Dr. Mahendra Saha **Editor**
Department of Statistics
Faculty of Mathematical Sciences
University of Delhi, New Delhi, India

Prof. Shruti **Reviewer & Course Coordinator**
School of Sciences
U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT–404 NA/MASTAT–404 NA **ACTUARIAL STATISTICS**
©UPRTOU
First Edition: ***July 2025***
ISBN:

Printed By:

Block & Units Introduction

The ***Block - 1 –Probability Models and Life Tables,*** is the first block of said self-learning material (SLM), which is divided into five units.

The ***Unit – 1: Basic Concepts - Probability Distributions*****:** is the first unit of present SLM, which serves as the introductory unit. In this unit the topics which have been discussed areas follows Introduction, Important Discrete Distributions, Important Continuous Distributions, Mixed Distributions, Insurance applications, Sum of Random variables.

The ***Unit – 2: Utility Theory:*** discussed about the utility criterions, Jensen's inequality, types of utility functions.

The ***Unit – 3: Principles of Premium Calculation:*** concentrates on the properties and examples of premium calculations.

The ***Unit – 4: The Collective Risk Model:*** explores about the model, compound Poisson distribution, the effect of reinsurance, recursive calculation of aggregate claims distribution, expansion of Panjer recursion formula, the application of recursion formula, approximate calculation of aggregate claims distributions.

The ***Unit – 5: Survival Distributions and Life Tables:*** discuss about the probability for age at death, survival function, time until death, curate future lifetime and force of mortality.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

UNIT:1 BASIC CONCEPTS-PROBABILITY DISTRIBUTIONS

Structure

1.1 Introduction

This unit delves into risk theory, with a special focus on two primary aspects: probability distributions and their applications in insurance. Risk theory forms the mathematical backbone for analysing and managing uncertainties in various insurance scenarios, from everyday health and car insurance to travel and home insurance.

To make this topic more accessible, consider a common situation: buying car insurance. When you purchase a policy, you pay a premium upfront to cover potential damages or losses over a specified period. From the insurer's perspective, this transaction involves a calculated gamble. They must estimate how many claims will be made and the cost of these claims throughout the policy period.

Imagine you're driving to work and have a minor accident. You file a claim for the repair costs. Now, multiply this scenario by thousands of policyholders. The insurer needs sophisticated models to predict the number and severity of such incidents. One model might predict the frequency of claims based on historical data, while another estimates the potential cost of each claim.

These predictive models are not just theoretical. They have real-world applications across various types of insurance policies. For instance, in health insurance, actuaries might use them to estimate the number of medical claims and their associated costs, considering factors like age, lifestyle, and medical history. Similarly, in travel insurance, models might predict the likelihood of trip cancellations or medical emergencies abroad.

To make these concepts more tangible, let's think about how insurance companies balance their books. They need to ensure that the premiums collected from policyholders are sufficient to cover the claims made, plus their operational costs and profit margins. This delicate balancing act relies heavily on accurate risk assessment and prediction.

In the upcoming units, we'll explore these probability distributions in greater detail, looking at individual claims and mixed distributions, which combine different types of claims. We'll also introduce two basic insurance arrangements and illustrate their mathematical underpinnings, providing a clearer picture of how these models work in practice.

1.2 Objectives

By the end of this lesson, learners will be able to:

1. Understand and differentiate between important discrete and continuous probability distributions.

2. Apply probability distributions to insurance problems, including reinsurance and policy excess.
3. Analyse sums of random variables using moment-generating functions, convolution, and recursive methods.
4. Interpret the role of mixed distributions in insurance and risk modelling.

1.3 Important Discrete Distributions

A discrete probability distribution is a probability distribution characterized by a probability mass function (PMF), which assigns probabilities to each possible value of a discrete random variable. Here are some key points about discrete probability distributions:

- **Discrete Random Variable**: The variable can only take on distinct, separate values. For example, the number of heads in three-coin flips (0, 1, 2, or 3) is a discrete random variable.
- **Probability Mass Function (PMF)**: This function gives the probability that the discrete random variable equals a particular value. For instance, if X is a discrete random variable,$P(X = x)$ represents the probability that X takes the value x.

Some of the important probability distributions are as follows:

1.3.1 The Poisson Distribution

The Poisson distribution is a probability distribution that describes the number of events occurring within a fixed interval of time or space, given a known average rate of occurrence and assuming that events happen independently of each other.

When a random variable N has a Poisson distribution with parameter λ> 0, its probability function is given by

$$P_r(N = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

For x= 0, 1, 2,… The moment generating function is

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx}.e^{-\lambda}\frac{\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^x}{2!} + - - - - - \right\} = e^{-\lambda}.e^{\lambda e^t}$$

$$= e^{\lambda}(e^{t} - 1) = \exp\{\lambda(e^{t} - 1)\} - - - - - - - - - - - - (1.1)$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^x . e^{-\lambda}\frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda}\frac{(\lambda r)^x}{x!} = e^{-\lambda}e^{\lambda r}$$

$$= e^{\lambda(r-1)} = \exp\{\lambda\,(r - 1)\}$$

The moments of N can be found from the moment generating function. For example,

$$\mu_r'' = \left[\frac{d^r}{dt^r} M_N(t)\right]$$

$$M_N'(t) = \frac{d}{dt}[exp\{\lambda(e^t - 1)\}]$$

$$= \exp\{\lambda(e^t - 1)\}.\lambda^{e^t} = \lambda^{e^t} M_N(t)$$

and

$$M_N''(t) = \frac{d^2}{dt^2}[exp\{\lambda(e^t - 1)\}] = \frac{d}{dt}(M_N'(t))$$

$$= \lambda^{e^t} M_N(t) + \lambda^{e^t}.\frac{d}{dt} M_N(t)[differentation\ by\ par]$$

$$= \lambda e^t M_N(t) + \lambda e^t.\lambda e^t M_N(t)$$

$$M_N''(t) = \lambda e^t M_N(t) + (\lambda e^t)^2 M_N(t)$$

From which it follows that

$$E[N] = \lambda\ and\ E\ [N^2] = \lambda + \lambda^2\ so\ that\ V[N] = \lambda.$$

We use $P(\lambda)$ the notation to denote a Poisson distribution with parameter λ.

1.3.2 The Binomial Distribution

The Binomial distribution is a discrete probability distribution that describes the number of successes (or failures) in a fixed number of independent Bernoulli trials, where each trial has two possible outcomes (often labeled as success and failure).

When a random variable N has a Binomial distribution with parameter n and q, where n is a positive integer and 0 < q <1. Its probability function is given by

$$P_r(N = r) = \binom{n}{r} q^r (1-q)^{n-x}$$

For x = 0,1, 2, …, n. The moment generating function is

$$M_N(t) = \sum_{x=0}^{n} e^{tx} \binom{n}{r} q^r (1-q)^{n-x} = \sum_{x=0}^{n} \binom{n}{r} (qe^t)^x (1-q)^{n-x}$$

$$= (qe^t + 1 - q)^n$$

and the probability generating function is

$$P_N(r) = \sum_{x=0}^{n} r^x \binom{n}{r} q^r (1-q)^{n-x} = \sum_{x=0}^{n} \binom{n}{r} (qr)^x (1-2)^{n-x} = (qr + (1-q))^n$$

As,

$$M_N'(t) = n(qe^t + 1 - q)^{n-1}.qe^t$$

and

$$M_N''(t) = n(n-1)(qe^t + 1 - q)^{n-2}.(qe^t)(qe^t) + n(qe^t + 1 - q)^{n-1}.qe^t$$

$$= n(n-1)(qe^t + 1 - q)^{n-2}.(qe^t)^2 + n(qe^t + 1 - q)^{n-1}.qe^t$$

It follows that $E[N] = nq, E\,[N^2] = n\,(n-1)\,q^2 + nq$ and $V[N] = nq(1-q)$.

We use the notation $B\ (n, q)$ to denote a binomial distribution with parameters *n* and *q*.

1.3.3 The Negative Binomial Distribution

The Negative Binomial distribution is a probability distribution that describes the number of trials needed to achieve a specified number of successes in a series of independent and identically distributed Bernoulli trials, where each trial can result in either success or failure.

When a random variable N has a negative Binomial distribution with parameters k > 0 and p. where, 0 < p < 1. Its probability function is given by

$$P_r(N = x) = \binom{x + k - 1}{x} p^k q^x$$

For 0, 1, 2…, where q =\ 1 − p. Where k is an integer calculation of the probability function is straight forward as the probability function can be expressed in terms of factorials. An alternative method of calculating the probability function, regardless of whether k is an integer, is recursively as

$$P_r(N = x + 1) = \frac{k + x}{x + 1} q P_r(N = x)$$

$$P_r(N = x) = \binom{x + k - 1}{x} p^k q^x, P_r(N = x + 1) = \binom{x + k}{x + 1} p^k q^{x+1}$$

$$\frac{P_r(N = x + 1)}{P_r(N = x)} = \frac{\binom{x + k}{x + 1} p^k q^{x+1}}{\binom{x + k - 1}{x} p^k q^x} = \frac{x + k}{x + 1} q$$

$$P_r(N = x + 1) = \frac{k + x}{x + 1} q P_r(N = x)$$

For x= 0,1,2----- with starting value $P_r(N = x) = p^k$. The moment generating function can be found by making use of identity.

$$\sum_{x=0}^{\infty} P_r(N = x) = 1 - - - - - - - - - - - - (1.2)$$

From this it follows that

$$\sum_{0}^{\infty} \binom{k + x - 1}{x} (1 - qe^t)^k (qe^t)^x = 1$$

Provided that 0 <qe^t<1. Hence x

$$M'_N(t) = \sum_{x=0}^{\infty} e^{tx} \binom{k + x - 1}{x} p^k q^x$$

$$= \frac{p^k}{(1 - qet)^k} \sum_{x=0}^{\infty} \binom{k + r - 1}{x} (1 - qet)^k (qe^t)^x$$

$$= \frac{p^k}{(1 - qet)^k} = \left(\frac{1}{1 - qe^t}\right)^k$$

Provided that 0 <qe^t< 1, or equivalently, t < -logq. Similarly, the probability generating function is

$$P_N(r) = \sum_{x=0}^{\infty} r^k \binom{k + x - 1}{x} p^k q^x = \left(\frac{p}{1 - qr}\right)^k$$

Moments of this distribution can be found by differentiating the moment generating function and the mean and variance are given by E{N}= kq/p and V[N]= kq/p^2.

$$M_N'(t) = \frac{d}{dt}\left(\frac{p}{1-qe^t}\right)^k$$

$$M_N''(t) = \frac{d^2}{dt^2}\left(\frac{p}{1-qe^t}\right)^k$$

Equality (1.2) trivially gives

$$\sum_0^{\infty}\binom{k+x-1}{x}p^k q^x = 1-p^k - - - - - - - - (1.3)$$

We use the notation $NB\ (k,p)$ to denote a negative binomial distribution with parameter k and p.

1.3.4 The Geometric Distribution

The Geometric distribution is a probability distribution that describes the number of trials X needed to achieve the first success in a sequence of independent and identically distributed Bernoulli trials, where each trial can result in either success or failure.

The geometric distribution is a special case of the Negative Binomial distribution. When the Negative Binomial parameter k is L, the distribution is called a geometric distribution with parameter p and the probability function is

$$P_r(N=x) = pq^x$$

For x= 0, 1, 2, …, From above, it follows that E [N] = q/p, V[N] = q/p^2 and

$$M_N(t) = \frac{p}{1-qe^t}f \quad or \quad t < \log q.$$

This distribution plays an important role in ruin theory.

1.4 Important Continuous Distributions

Continuous probability distributions are distributions where the random variable can take on any value within a specified range or interval. Unlike discrete distributions which deal with countable outcomes, continuous distributions deal with uncountable outcomes typically associated with measurements or observations that can take any value within the range.

Here are some key points about continuous probability distributions:

- For continuous random variables X, the probability is described by the probability density function f(x), which gives the density of probability at each point x in the range of X.
- Unlike discrete distributions where P (X = x) gives the probability at a specific point, for continuous distributions, f(x) gives the relative likelihood of X taking a value near x.

1.4.1 The Gamma Distribution

The Gamma distribution is a continuous probability distribution that generalizes the concept of the exponential distribution. It is characterized by two parameters: a shape parameter (also known as the shape or concentration parameter) and a scale parameter (also known as the rate parameter).

When a random variable X has a gamma distribution with parameter $\backslash alpha > 0\backslash$ and\ $\backslash lambda > 0$ its density is given by

$$f(x) = \frac{\lambda^{\alpha}}{\tau(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

For, $x > 0$, where $\tau(\alpha)$is the gamma function defined as

In the special case when, α is an integer the distribution is also known as an Erlang distribution, and repeated integration by parts gives the distribution function as

$$f(x) = 1 - \sum_{j=0}^{\alpha-1} e^{-\lambda x} \frac{(\lambda x)^j}{j!}$$

for, $x \geq 0$. The moments and moments generating function of the gamma distribution can be found by nothing that

$$\int_0^{\infty} f(x)dx = 1$$

yields

$$\int_0^{\infty} x^{\alpha-1} e^{-\lambda x} dx = \frac{\tau(\alpha)}{\lambda^{\alpha}} ------- (1.4)$$

The n^{th} moment is

$$E[X^n] = \int_0^{\infty} x^n \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\tau(\alpha)} dx = \frac{\tau(\alpha)}{\lambda^{\alpha}} \int_0^{\infty} x^{n+\alpha-1} e^{-\lambda x} dx$$

and form identity (1.4) it follows that

$$E[X^n] = \frac{\lambda^\alpha}{\tau(\alpha)} . \frac{\tau(\alpha+n)}{\lambda^{\alpha+n}} = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda^n} - - - - - - - - (1.5)$$

In particular,

n=1

$$E\,[X] = \frac{\tau(\alpha+n)}{\tau(\alpha)\lambda} = \frac{\alpha}{\lambda}$$

and, n=2

$$E\,[X^2] = \frac{\tau(\alpha+2)}{\tau(\alpha)\lambda^2} = \frac{\alpha(\alpha+1)}{\lambda^2},$$

So that

$$V[x] = E\,[X^2] - \{E[x]\}^2 = \frac{\alpha(\alpha+1)}{\lambda^2} - \frac{\alpha^2}{\lambda^2} = \frac{\alpha}{\lambda^2}$$

We can find the moment generating function in a similar fashion. As

$$M_x(t) = \int_0^\infty e^{tx} \frac{\lambda^\alpha x^{\alpha-1} \mathrm{e}^{-\lambda x}}{\tau(\alpha)} dx = \frac{\lambda^\alpha}{\tau(\alpha)} \int_0^\infty \mathrm{x}^{\alpha-1} \mathrm{e}^{-(\lambda-t)x} dx - - - - - - (1.6)$$

Application of identity (1.4), gives

$$M_x(t) = \frac{\lambda^\alpha}{\tau(\alpha)} . \frac{\tau(\alpha)}{(\lambda-t)^\alpha} = \left(\frac{\lambda}{\lambda-t}\right)^\alpha - - - - - - (1.7)$$

Note that in identity (1.4) $\lambda > 0$. Hence, in order to apply (1.4) to (1.6) we require that $\lambda - t > 0$, so that the m. g. f. exists when $t < \lambda$. The coefficient of skewness of X, which we denoted by $Sk[x]$, is $\frac{2}{\sqrt{\alpha}}$. This follows from the definition of the coefficient of skewness, namely third central moment divided by standard divided by standard division cubed and the fact that the third central moment is

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = E[X^3] - \frac{3\alpha}{\lambda}\, E[X^2] + 2\left(\frac{\alpha}{\lambda}\right)^3$$

$$E\left[\left(X - \frac{\alpha}{\lambda}\right)^3\right] = \frac{\alpha(\alpha+1)(\alpha+2)}{\lambda^3} - \frac{3r}{\lambda} . \frac{\alpha(\alpha+1)}{\lambda^2} + \frac{2\,\alpha^3}{\lambda^3}$$

$$= \frac{\alpha(\alpha+1)(\alpha+2) - 3\alpha^2(\alpha+1) + 2\,\alpha^3}{\lambda^3}$$

$$= \frac{2\alpha}{\lambda^3}$$

We use the notation $\gamma(\alpha, \lambda)$ to denote a gamma distribution with parameter α and λ.

1.4.2 The Exponential Distribution

The exponential distribution is a special case of gamma distribution. It is just a gamma distribution with parameter x=1. Hence, the exponential distribution with parameter λ> 0 has density function.

$$f(x) = \lambda e^{-\lambda x}$$

For x > 0 and has the distribution function

$$f(x) = 1 - e^{-\lambda x}$$

For $x \geq 0$ form equation (1.5), the nth moment of the distribution is

$$E[X^n] = \frac{n!}{\lambda^n}$$

And from equation (1.7), the moment generating function is

$$M_x(t) = \frac{\lambda}{\lambda - t}, \quad fort < \lambda$$

1.4.3 The Pareto Distribution

The Pareto distribution is a continuous probability distribution that is often used to model the distribution of wealth, income, and other quantities where a small number of observations (the "vital few") are significantly larger than the majority (the "trivial many")

When a random variable X has a Pareto distribution with parameters x > 0, its density function is given by

$$f(x) = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}.$$

For x>0 integrating this density, we find that the distribution functions is

$$F(x) = 1 - \left(\frac{\lambda}{(\lambda + x)}\right)^\alpha$$

for x ≥ 0. Whenever moments of the distribution exist, they can be found from.

$$E[X^n] = \int_0^\infty x^n f(x)dx$$

by integration by parts. However, they can above find individual using the following approach. Since the integral of the density function over $(0,\infty)$ equal 1, we have

$$\int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}dx = 1 \int_0^\infty \frac{dx}{(\lambda+x)^{\alpha+1}} = \frac{1}{\alpha\lambda^\alpha}$$

an identity which holds provided that x > 0. To find E[X], we can write

$$E[x] = \int_0^\infty xf(x)dx = \int_0^\infty (x+\lambda-\lambda)f(x)dx = \int_0^\infty (x+\lambda)f(x)dx - \lambda$$

and inserting for f, we have

$$E[x] = \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}.(x+\lambda)dx - \lambda$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha}}dx - \lambda$$

We can evaluate the integral by rewriting the integrand in terms of a α pareto density function with parameters $(\alpha-1)$ and λ. thus,

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)}\int_0^\infty \frac{(\alpha-1)\lambda^{\alpha-1}}{(\lambda+x)^\alpha}\,dx - \lambda \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.8)$$

and since the integral equals 1.

$$E[x] = \frac{\alpha\lambda}{(\alpha-1)} - \lambda = \frac{\lambda}{\alpha-1}$$

It is important to note that the integrand in equation (1.8) is a Pareto density function only if α>1, and hence E[x] exists only for α>1. Similarly, we can find E $[x^2]$ from

$$E[x] = \int_0^\infty ((x+\lambda)^2 - 2\lambda x - \lambda^2)f(x)dx$$

$$= \int_0^\infty (x+\lambda)^2\, f(x)dx - 2\lambda\, E[x] - \lambda^2$$

$$= \int_0^\infty (x+\lambda)^2 . \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}\,dx - 2\lambda\,\frac{\lambda}{\alpha-1} - \lambda^2$$

$$= \int_0^\infty \frac{\alpha\lambda^\alpha}{(\lambda + x)^{\alpha+1}}\, dx - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} \int_0^\infty \frac{(\alpha-2)\lambda^{\alpha-2}}{(\lambda + x)^{\alpha-1}}\, dx - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{\alpha\lambda^2}{(\alpha-2)} - \frac{2\lambda^2}{\alpha - 1} - \lambda^2$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)},\ \mathit{proved\ that}\ \alpha > 2.$$

and hence that

$$V[x] = E[x^2] - [E[x]^2]$$

$$= \frac{2\lambda^2}{(\alpha-1)(\alpha-2)} - \frac{\lambda^2}{(\alpha-1)^2}$$

$$V[x] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)}$$

We use the notation Pa (α, λ) to denote a Pareto distribution with parameters α and λ .

1.4.4 The Normal Distribution

The Normal distribution, also known as the Gaussian distribution, is one of the most important continuous probability distributions in statistics. It is characterized by a bell-shaped curve that is symmetrical around its mean (average) value.

When a random variable X has a normal distribution with parameter $\mu\ and\ \sigma^2$, its density function is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}\ for -\infty < x < \infty$$

We use notation N (μ, σ^2) to denote a normal distribution with parameter μ and σ^2. The standard normal distribution has parameter 0 and 1 and its distribution function is denoted by $\emptyset$, where

$$\emptyset(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\Lambda}} \exp\left\{\frac{-Z^2}{2}\right\} dz.$$

A key relationship is that if $X \sim \mu\ (\mu, \sigma^2)\ and\ if\ Z = \frac{(X-\mu)}{6}, then\ Z \sim N(0,1)$.

The moment generating function is

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)dx = \frac{1}{\sigma\sqrt{2\Lambda}} \int_{-\infty}^{\infty} e^{tx} \left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx$$

$$= \frac{1}{\sqrt{\alpha\Lambda}} \int_{-\infty}^{\infty} exp\,\{t(\mu + \sigma z)\} \exp\left(-\frac{t}{2}\right) dz, \left(z = \frac{x-\mu}{\sigma}\right)$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\, \left\{-\frac{1}{2}(z^2 - 2t\sigma z)\right\} dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\Lambda}} \int_{-\infty}^{\infty} exp\, \left[-\frac{1}{2}\{(z-\sigma t)^2 - \sigma^2 t^2\}\right] dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp \left\{-\frac{1}{2}(z-\sigma t)^2\right\} dz$$

$$= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \times \frac{1}{\sqrt{2\Lambda}} \int_{0}^{\infty} exp \left(-\frac{\mu^2}{2}\right) du$$

$$Hence, M_x(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\} - - - - - - - - - - - (1.9)$$

Twice differentiating $M_x(t)$ with respect to t, we get

$$M_X'(t) = (\mu + \sigma^2 t). M_X(t) and M_X''(t) = [(\mu + \sigma^2 t) + \sigma^2]. M_X(t)$$

$$So\ that, M_X'(0) = \mu\ and\ M_X''(0) = \mu^2 + \sigma^2. Thus,$$

$$E[X] = \mu\ and\ V(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2$$

1.4.5 The Log Normal Distribution

The log-Normal distribution is a continuous probability distribution of a random variable whose natural logarithm is normally distributed. It is commonly used to model quantities that are strictly positive and often skewed to the right (positively skewed).

When a random variable X has a log-normal distribution with parameter μ and σ where $-\infty < \mu < \infty\ and\ \sigma > 0$, its density function is given by

$$f(x) = \frac{1}{x\sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(logx-\mu)^2}{2\sigma^2}\right\}$$

For x>0. The distribution function can be obtained by integrating the density function as follows:

$$F(x) = \int_{-\infty}^{x} \frac{1}{y\sigma\sqrt{2\Lambda}}\ exp\left\{\frac{(logx-\mu)^2}{2\sigma^2}\right\} dy$$

and the substitution Z= logy yields

$$F(x) = \int_{-\infty}^{logx} \frac{1}{\sigma\sqrt{2\Lambda}}\ exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\} dz.$$

As the integrand is the N (μ, σ^2) density function

$$F(x) = \emptyset\left(\frac{logx-\mu}{\sigma}\right).$$

Thus, probabilities under a lognormal distribution can be calculated from the standard normal distribution function.

We use the notation LN (μ, σ^2) to denote a log-normal distribution with parameters and σ . From the preceding argument it follows that if X ~ LN(μ, σ^2) , then log X ~ LN(μ, σ^2)

This relationship between normal and log-normal distributions is extremely useful, particularly is deriving moments. If X ~ LN$(\mu, \sigma^2)$and $Y = logX$, then

$$E[X^n] = E[e^{ny}] = M_y(n) = \ exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\}$$

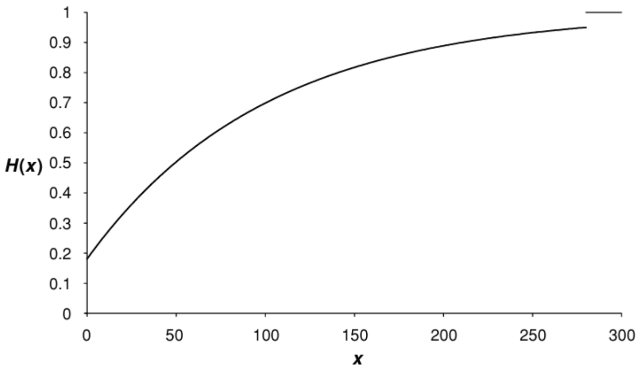
Where the final equality follows by equation (1.9).

1.5 Mixed Distribution

A mixed distribution, in statistics, refers to a probability distribution that is a mixture of two or more other distributions.

The illustrate the idea of a mixed distribution, Let X be exponentially distribution with mean 100, and let the random variable Y be defined by

$$Y = \begin{cases} 0 & if\ X < 20 \\ X - 20 & if\ 20 \le X < 300 \\ 280 & if\ X \ge 300 \end{cases}$$



Then,

$$P_r(Y = 0) = P_r(X < 20) = t - e^{-0.2} = 0.1813$$

and similarly, P_r (Y=280) =0.0498. Thus, Y has masses of probability at the point 0 and 280. However, in the interval (0,280), the distribution of Y is continuous with for example,

$$P_r(30 < Y \le 100) = P_r(50 < X \le 120) = 0.3053$$

Figure shows that the distribution function, H of Y notes that there are jumps at 0 and 280, corresponding to the masses of probability at these points. As the distribution is differentiable in the interval (0, 280), Y has a density function in this interval. Letting h denote the density function of Y, the moments of Y can be found from

$$E[Y^2] = \int_0^{180} x^r h(x)dx + 280^r P_r(Y = 280).$$

It will be convenient to use satisfies integral notation so that we do not have to specify with a distribution is discrete, continuous, or mixed. In the notation, we write the r^{th} moment of Y as

$$E[Y^r] = \int_0^{\infty} x^r dH(x).$$

More generally, if $k(x) = P_r(Z \le x)$ is a mixed distribution on [0,∞] and m is a function, then

$$E[m(Z)] = \int_0^{\infty} m(x)d\,k(x),$$

Where we interpret the integral as

$$\sum_{x_i} m(x_i)P_r(Z = x_i) + \int m(x)k(x)dx.$$

Where summation is over the points $\{x_i\}$ at which there is a mass of probability and integration is over the intervals in which K is continuous with density function k.

Exercise: Let the random variable X have the distribution function F given by:

$$F_{(x)}(x) = \begin{cases} 0 & X < 20 \\ \dfrac{x+20}{80} & 20 \le X < 40 \\ 1 & X \ge 40 \end{cases}$$

Check the nature of random variable X, Hence, calculate

$i)\ P\ (X < 30)$ $\qquad ii)\ P\ (X = 40)$

$iii)\ E(X)$ $\qquad iv)\ V\ (X)$

Calculation: - To check the nature of random variable x-

$$P(X = 20) = \frac{20+20}{80} = 0.5 > 0$$

$$P(X = 40) = 1 - \frac{40+20}{80} = 0.25 > 0$$

Points X=20 and X=40 are jump points. Thus, X has a mixed distribution.

$$i)\ P\ (X \le 30) = \frac{30+20}{80} = 0.625$$

$$ii)\ P\ (X = 40) = 1 - \frac{40+20}{80} = 0.25$$

The pdf of the distribution is given as:

$$f_x(x) = \begin{cases} \dfrac{1}{80}, & 20 \le x < 40 \\ 0, & otherwise \end{cases}$$

$$iii)\ E(X) = 20 \times 0.5 + \int_{20}^{40} x.\frac{1}{80}dx + 40 \times 0.25$$

$$= 20 + \left[\frac{x^2}{2 \times 80}\right]_{20}^{40} = 20 + \frac{40^2 - 20^2}{160} = 27.5$$

$$E(X^2) = 20^2 \times 0.5 + \int_{20}^{40} \frac{x^2}{80} dx + 40^2 \times 0.25$$

$$= 600 + \left[\frac{x^3}{3 \times 80}\right]_{20}^{40} = 600 + \frac{40^3 - 20^3}{240} = 833.33$$

$iv)\ V(X) = E[X^2] - \{E[X]\}^2 = 833.33 - (27.5)^2 = 77.083$

Result:

$i)\ P(X \leq 30) = 0.625$

$ii)\ P(X = 40) = 0.25$

$iii)\ E(X) = 27.5$

$iv)\ V(X) = 77.083$

1.6 Insurance Applications

In the context of reinsurance, several functions naturally come into play, particularly in managing risk and financial stability. Let us denote the amount of a claim by X and assume that X follows a distribution function F. Importantly, we assume all claim amounts are non-negative, which means F(x) = 0 for x < 0. Furthermore, we assume X is a continuous random variable with a corresponding density function f.

A reinsurance arrangement is a contractual agreement between an insurer and a reinsurer. Under this agreement, claims occurring within a specified period, such as one year, are divided between the insurer and the reinsurer according to pre-defined terms. Essentially, the insurer transfers a portion of its risk to the reinsurer, thereby sharing potential losses. In exchange for this coverage, the insurer pays a premium to the reinsurer. This arrangement is particularly beneficial for the insurer, as it helps to stabilize their financial outcomes by reducing the variability of claim payments.

Reinsurance serves multiple purposes beyond simply sharing risk. It allows insurers to underwrite policies with higher coverage limits than they could independently support, thereby enabling them to serve larger clients or handle more substantial risks. Additionally, reinsurance can provide protection against catastrophic events, which could otherwise threaten the insurer's solvency. By ceding a portion of their risk, insurers can manage their capital more efficiently, maintain reserve requirements, and enhance their capacity to write new business. Moreover, reinsurance can be structured in various ways to meet the specific needs of the insurer. For instance, proportional reinsurance involves sharing a fixed percentage of premiums and losses between the insurer and reinsurer, while non-proportional reinsurance, such as excess of loss reinsurance, involves the reinsurer covering losses that exceed a certain threshold. Each type of reinsurance agreement offers different benefits and considerations, depending on the insurer's risk profile and strategic objectives.

1.6.1 Proportional Reinsurance

In a proportional reinsurance arrangement, the insurer covers a fixed portion of each claim, denoted by a, during the reinsurance period. The reinsurer covers the remaining portion, 1−a.

Let Y represent the amount of a claim paid by the insurer and Z represent the amount paid by the reinsurer. These can be expressed as Y = aX and Z = (1−a)X respectively, where X is the total claim amount. It follows that Y+Z = X, indicating that the total claim is split between the insurer and reinsurer. Thus, both Y and Z are scaled versions of the original claim amount X.

Suppose X is continuous then calculate

(i) cdf of Y (ii) cdf of Z

(iii) pdf of y (iv) pdf of z

$$(i)\ cdf\ \ of\ \ y:\quad F_y(y) = P(aX \le y) = P\left(X \le \frac{y}{a}\right) G_y(y) = F_x(y/a)$$

$$(ii)\ cdf\ \ of\ \ Z: F_z(Z) = P(Z \le z) = P\big((1-q) \times \le z\big) = P\left(X \le \frac{z}{(1-a)}\right)$$

$$= H_z(Z)$$

$$(iii)\ pdf\ of\ \ y:\qquad g_y = \frac{d}{dy} G_y(y) = f_x(y/a) \,.\frac{1}{a}$$

$$(iv)\ pdf\ of\ \ Z:\qquad h_z = \frac{d}{dz} H_z(z) = f_x\left(\frac{Z}{1-a}\right) y \left(\frac{1}{1-a}\right)$$

Example 1: Let X~(α, λ) what is the distribution of ax?

Solution: As

$$f_{(x)} = \frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}$$

It follows that the density function of ax is

$$\frac{\lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}}{a^{\alpha}\Gamma(\alpha)}$$

Thus, the distribution of ax is $\Upsilon\left(\alpha, \frac{\lambda}{a}\right)$.

Example 2: Let $X \sim LN\ (\mu, \sigma)$. What is the distribution of ax?

Solution: As

$$f_{(x)} = \frac{1}{x\sigma\sqrt{2\Lambda}} exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}$$

It follows that the density function of ax is

$$\frac{1}{x\sigma\sqrt{2\Lambda}} exp\left\{-\frac{(\log x - loga - \mu)^2}{2\sigma^2}\right\}$$

Thus, the distribution of ax is LN $(\mu + \log a,\ \sigma)$.

1.6.2 Excess of Loss Reinsurance

In an excess of loss reinsurance arrangement, the insurer and reinsurer collaborate to handle claims only when they exceed a predefined amount known as the retention level, denoted by MMM. Below this threshold, the insurer independently covers the entire claim amount.

If a claim surpasses the retention level MMM, the insurer pays up to MMM, while the reinsurer assumes responsibility for the excess amount. This setup ensures that the insurer is protected from exceptionally large losses, as the reinsurer shares the financial burden for claims that exceed the retention level.

Mathematically, the amounts paid under this arrangement are represented as follows:

The insurer pays $y = \min(X, M)$ and the reinsurer pays $Z = \max(0, X - M)$, with $Y + Z = X$.

This strategic partnership allows insurers to manage risk effectively, ensuring financial stability and operational resilience in the face of unexpected or catastrophic events.

1.6.2.1 The Insurer's Position

Let F_y be the distribution of Y. Then is follows from the definition of Y that

$$F_y(x) = \begin{cases} F(x) for x < M \\ 1 \qquad for x \geq M \end{cases}$$

Thus, the distribution of y is mixed with a dentition function f(x) for 0 < x < M, and a mass of probability at M, with $P_r(Y = M) = 1 - F(M)$.

As Y is a function of X, the moment of Y can be calculated from.

$$E[Y^n] = \int_0^\infty (\min(x, m))^n f(x)dx.$$

and this integral can be split into two parts since min (x, m) equals x for x < M and equals M for x ≥ M. Hence

$$E[Y^n] = \int_0^M x^n f(x)dx + \int_0^\infty M^n f(x)dx$$

$$= \int_0^M x^n f(x)dx + M^n(1 - F(M)) ---------(1)$$

In particular,

$$E[Y] = \int_0^M x\, f(x)dx + M\,(1 - F(M)).$$

So that

$$\frac{d}{dm} E[Y] = 1 - F(M) > 0\,.$$

Thus, as a function of M, E[Y] increase from 0 when M = 0 to E [X] as M → ∞ .

Example 3: Let F (x) =1- $e^{-\lambda x},\ x \geq o.\ find\ E[y]$.

Solution: We have $E[Y] = \int_0^M x.\ \lambda e^{-\lambda x}\ dx +\ M_e{}^{-\lambda M}$

and integration by parts yields

$$= \lambda \int_0^M x_I.\ e^{-\lambda x}\ dx + M\, e^{-\lambda M}$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} \right]_0^M - \lambda \int_0^M 1. \frac{e^{-\lambda x}}{-\lambda} dx + M \quad e^{-\lambda x M}$$

$$= -M \quad e^{-\lambda x M} + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^M + M \quad e^{-\lambda x M}$$

$$= \left[\frac{-e^{-\lambda x}}{\lambda} + \frac{1}{\lambda}\right] = \frac{1}{\lambda}\left[1 - e^{-\lambda x}\right].$$

Example 4: Let $X \sim LN\ (\mu, \sigma). Find\ E[y^n]$.

Solution: Inserting the log-normal density function into the integral in equation (1) we get

$$E[y^n] = \int_0^M x^n . \frac{1}{x\, \sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\} dx.$$

To deal with an integral of this type there is a standard substitution namely y = log x. This gives

$$I = \int_{-\infty}^{\log M} \exp\{yn\} \frac{1}{x\, \sigma\sqrt{2\Lambda}} \exp\left\{-\frac{(\log y - \mu)^2}{2\sigma^2}\right\} dy.$$

The technique in evaluating this integral is to write the integral in terms of a normal density function (different to the N (μ, σ^2) density function). To achieve this we apply the technique of "completing the square" in the exponent, as follows:

$$y_n - \frac{(y-\mu)^2}{2\sigma^2} = -\frac{1}{2\sigma^2}\left[(y-\mu)^2 - 2\sigma^2 y_n\right]$$

$$= -\frac{1}{2\sigma^2}\left[(y^2 - 2\mu y + \mu^2) - 2\sigma^2 y_n\right]$$

$$= -\frac{1}{2\sigma^2}\left[y^2 - 2y\,(\mu + \sigma^2 n) + \mu^2\right]$$

Nothing that the terms inside the square brackets would give the square of $y - (\mu + \sigma^2 n)$ if the final term $(\mu + \sigma^2 n)^2$were instead of we can write the exponent as.

$$= -\frac{1}{2\sigma^2}\left[\left(y - (\mu + \sigma^2 n)\right)^2 - (\mu + \sigma^2 n)^2 + \mu^2\right]$$

$$= -\frac{1}{2\sigma^2}\left[\left(y - (\mu + \sigma^2 n)\right)^2 - 2\mu + \sigma^2 n - \sigma^4 n^2\right]$$

$$= \mu n + \frac{1}{2}\sigma^2 n^2 - \frac{1}{2\sigma^2}\left(y - (\mu + \sigma^2 n)\right)^2.$$

Hence, $\quad I = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \int_{-\infty}^{logM} \frac{1}{\sigma\sqrt{2\Lambda}}\, exp\left\{-\frac{1}{2\sigma^2}\left(y - (\mu + \sigma^2 n)\right)^2\right\} dy$

And as the integrand is the $N\ (\mu + \sigma^2 n, \sigma^2)$ density function.

$$I = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \phi\left(\frac{\log M - \mu}{\sigma}\right).$$

Finally, using the relationship between normal and log-normal distributions,

$$1 - F(M) = 1 - \phi\left(\frac{\log M - \mu}{\sigma}\right)$$

So that $\quad E[Y^n] = \exp\left\{\mu n + \frac{1}{2}\sigma^2 n^2\right\} \phi\left(\frac{\log M - \mu - \sigma^2 n}{\sigma}\right) + M^n\left(1 - \phi\left(\frac{\log M - \mu}{\sigma}\right)\right).$

1.6.2.1 The Reinsurer's Position

From the definition of Z it follows that Z takes the values Zero if $X \leq M,$ and takes the values Hence, if Fz denotes the distribution function of Z, then Fz (0)= F(M) and for x > 0 , Fz (x) = F (X+M). Thus, Fz is a mixed distribution with a mass of probability at 0.

The moments of Z can be found in a similar fashion to these of Y. We have

$$E[Z^n] = \int_0^\infty (0, x - M)^n\, f(x)dx,$$

and since max (0, x-M) is 0 for $0 \leq x \leq M$ we have

$$E[Z^n] = \int_0^\infty (\,x - M)^n\, f(x)dx - - - - - - - - - - (3)$$

Example 5: Let $F(x) = 1 - e^{-\lambda x},\ x \geq 0,$ Find $E\ [Z].$

Solution: Setting n =1 is equation (3) we have

$$E\ [Z] = \int_0^\infty (x - M)\lambda\, e^{-\lambda x} dx$$

$$= \int_0^\infty y\lambda\, e^{-\lambda(y+m)} dy$$

$$= e^{-\lambda M}\ E[X]\ = \frac{1}{\lambda} e^{-\lambda M}$$

Alternatively, the indent E [Z] = E [X] – E [y] yields the answer with E[X] = 1/λand E [y] given by the solution to example 3.

Example 6: $Let\ F(x) = 1 - e^{-\lambda x},\ x \geq 0, Find\ M_Z(t).$

Solution: By definition $M_Z(t) = E[e^{tz}]$ and as Z = max (0,X-M),

$$M_Z(t) = \int_0^{\infty} e^{t\max(0,2-M)} \lambda e^{-\lambda x} dx$$

$$= \int_0^{M} e^0\, \lambda\, e^{-\lambda x}\, dx + \int_0^{\infty} e^{t(x-M)}\, \lambda\, e^{-\lambda x}\, dx$$

$$= 1 - e^{-\lambda M}\ + \lambda \int_M^{\infty} e^{ty-\lambda\,(y+M)}\, dy \qquad [y = x - M)$$

$$=\ 1 - e^{-\lambda M}\ + \frac{\lambda\, e^{-\lambda M}}{\lambda - t}.$$

Provided that $t < \lambda$.

The above approach is a slightly artificial way of looking at the reinsurer's position since include zero as a possible "claim amount" for the reinsurer. Alternatively consider the distribution of the non-zero amounts paid by reinsurer. In practice the reinsurer is likely to have information only on these amounts as the insurer is unlikely to inform the reinsurer each time there is a claim whose amount is less than M.

Example 7: Let X have a discrete distribution as follows.

$P_r(x = 100) = 0.6$ $\qquad$ $P_r(x = 175) = 0.3$ $\qquad$ $P_r(x = 200) = 0.1$

If the insurer affects excess of loss reinsurance with retention level 150, what is the distribution of the non-zero payments made by the reinsurer?

Solution: First, we note that the distribution of Z is given by

$P_r(Z = 0) = 0.6$ $\qquad$ $P_r(Z = 25) = 0.3$ $\qquad$ $P_r(Z = 50) = 0.1$

Now, let W denote the amount of a non-zero payment made by the reinsurer. Then W can take one of the two values: 25 and 50. Since payments of amount 25 are three times as likely as payments of amount 50. We can write the distribution of W as

$$P_r(W = 25) = 0.75 \qquad\qquad P_r(W = 50) = 0.25$$

The argument in above example can be formalized as follows. Let W denote the amount of a non-zero payment by the reinsurer under an excess of loss reinsurance arrangement with retention level M. The distribution of W is identical to that of z | Z > 0. Hence.

$$P_r(W \leq x)P_r(Z < x \mid Z > 0) = P_r(X \leq x + M \mid x > M)$$

From which it follows that

$$P_r(W \leq x) = \frac{P_r(M < X < x + M)}{P_r(X > M)} = \frac{F(x + M) - F(M)}{1 - F(M)} - - - - -(4)$$

Differentiation gives the density function of was

$$\frac{F(x + M)}{1 - F(M)} - - - - - - - -(5)$$

Example 8: Let F (x) = 1-$e^{-\lambda x}, x \geq 0$. What is the distribution of the non-zero claim payments made by the reinsurer?

Solution: By formula (5), the density function is

$$\frac{\lambda e^{-\lambda(x+M)}}{e^{-\lambda m}} = e^{-\lambda x}$$

So that, the distribution of W is the same as that of X. (This rather surprising result is a consequence of the "memory less" property of the exponential distribution.

Example 9: What is the distribution of the non-zero claim payments made by the reinsurer?

Solution: By formula (5), the density function is

$$\frac{\alpha\lambda^{\alpha}}{(\lambda + M + x)^{\alpha+1}}\left(\frac{\lambda + M}{\lambda}\right)^{\lambda} = \frac{\alpha(\lambda + M)^{\alpha}}{(\lambda + M + x)^{\alpha+1}}$$

So that the distribution of W is $P_o(\alpha, \lambda + M)$

1.6.3 Policy Excess

An insurance policy excess, commonly referred to as a deductible, is the initial amount of any claim that the policyholder must bear themselves before the insurance coverage kicks in. It serves multiple purposes within insurance contracts:

1**. Risk Sharing:** The excess helps to align the interests of the insurer and the policyholder in managing small and frequent claims. By requiring policyholders to cover a portion of the claim, insurers can focus on larger, more significant risks.

2. **Cost Control:** Policy excesses help insurers manage costs by reducing the number of small claims processed, thereby lowering administrative expenses and potentially reducing premiums for policyholders.

3. **Risk Management:** For policyholders, selecting a higher excess can be a strategic choice to lower their premiums. It allows them to tailor their insurance coverage to their specific risk tolerance and financial situation.

4. **Claims Handling:** Excesses streamline the claims process by distinguishing between minor and major claims. This simplifies administrative tasks and ensures that insurers can efficiently handle claims that significantly impact their financial stability.

For example, in motor vehicle insurance, the policy excess acts as a deductible that policyholders must pay out of pocket before their insurance coverage applies. It helps manage costs by encouraging responsible driving behavior and reducing administrative overhead for minor claims.

Overall, insurance policy excesses play a crucial role in balancing risk, managing costs, and aligning incentives between insurers and policyholders, contributing to the sustainability and efficiency of insurance operations.

- **Notation:** If a policy is issued with an excess of other the insured party pays any loss of amount less than or equal to d in full and pays d on any loss in excess of d. thus if X represents the amount of a loss, when a loss occurs the insured party pays $\min(X, d)$ and the insurer pays $\max(0, X - d)$
- **Comparison with Excess of Loss Reinsurance:** The payment structure observed in an excess of loss reinsurance arrangement closely mirrors these quantities. Consequently, no new mathematical considerations are necessary beyond what is already

discussed. This similarity underscores the consistency in approach across different scenarios involving insurance claims and reinsurance agreements.

Remark: It is important however, to recognize that x represents the amount of a loss, and not the amount of claim.

1.7 Sum of Random Variables

In most insurance application we are interested in distribution of sum of identically and independently distribution random variables.

Example: Suppose that an insurer issues n policies and the claim amount from policy i= 1, 2,….,n, is a random variable Xi. Then the total amount the insurer pays in claim from these n policies is $S_n = \sum_{i=1}^{n} X_i$

An obvious question is the mind of the insurer is to know the distribution of S_n. Whenever the distribution exists is a closed form, we make use one of the following 3 methods:

(i) Moment Generating Function Method
(ii) Convolution
(iii) Recursive Calculation for Discrete random variables.

1.7.1 Moment Generating Function Method

(i) This is a very neat way of finding the distribution of S_n. Define M_s to be the moment generating function of X_L. Then

$$M_s(t) = E[e^{tS_n}] = E\left[e^{t(X_1+X_2+\cdots..X_n)}\right]$$

Using independence, it follows that

$$M_s(t) = E[e^{tX_1}] = E[e^{tX_2}] \ldots \ldots [e^{tX_n}]$$

And as the X_i's are identically distributed

$$M_s(t) = M_s(t)^n$$

Hence, if we can identify $M_s(t)^n$as the mgf of a distribution we know the distribution of S_n by the uniqueness property of m.g.f.

Example 10: Let X_1 have a Poisson distribution with parameter λ. what is the distribution of S_n?

Solution: As

$$M_x(t) = \exp\{\lambda(e^t - 1)\}$$

We have

$$M_s(t) = \exp\{\lambda n(e^t - 1)\}$$

and so, S_n has a Poisson distribution with parameter λn.

Example 11: Let X_1 have an exponential distribution with mean $1/\lambda$. What is the distribution of S_n?

Solution: As $\quad M_x(t) = \frac{\lambda}{\lambda - t} \; for \; t < \lambda, we \; have$

$$M_s(t) = [M_x(t)]^n = \left(\frac{\lambda}{\lambda - t}\right)^n$$

And so, S_n has a $\Upsilon\ (n, \lambda)$distribution

1.7.2 Direct Convolution of Distributions

Direct convolution is a more direct and less elegant method of finding the distribution of S_n. Let us first assume that $\{X_i\}_{i=1}$ is discrete random variable, distribution on the non-negative integers, so that S_n is also distributed on the non-negative integers. Let x be a non-negative integer and consider first the distribution of S_2. The convolution approach to finding Pr ($S_2 \leq$ x) consider how the event $\{S_z \leq x\}$ can occur. This event occurs when X_2 takes the value j, where j can be any values less than or equal to x-j, so that their sum is less than or equal to x. Summing over all possible values of j and using the fact that X_1 and x_2 are independent, we have

$$P_r(S_2 \leq x) = \sum_{j=0}^{x} P_r(X_1 \leq x - j)P_r(x_2 = j)$$

The same argument can be applied to find $P_r(S_3 \leq x)$by writing $S_3= S_2+X_3$ and by noting that S_2 and X_3 are independent (as $S_2= X_1 + X_2$). Thus,

$$P_r(S_3 \leq x) = \sum_{j=0}^{x} P_r(X_1 \leq x - j)P_r(x_3 = j). ------- (1)$$

The same reasoning gives

$$P_r(S_n = x) = \sum_{j=0}^{x} P_r(S_{n-1} \leq x - j)P_r(x_n = j)$$

Now, let F be the distribution function of X_1 and let $f_j = P_r\ (X_1=j)$. We define

$$F^{n*}(x) = P_r(S_n \leq x)$$

And call the n- fold convolution of the distribution F with itself. Then by equation (1)

$$F^{n*}(x) = \sum_{j=0}^{x} F^{(n-1)*}(x-j)f_j$$

Now that $F^{1*} = F$ and by convention, se define $F^{0*}(x) = 1$ for $x \geq 0$ with $F^{0*}(x) = 0$ for $x < 0$. Similarly, we define $F_x^{n*}(x) = P_r(S_n = x)$ so that

$$F_x^{n*} = \sum_{j=0}^{x} f_{x-j}^{(n-1)*} f_j$$

with $f^{1*} = f$

When F is a continuous distribution on $(0, \infty)$ with density function f, the analogues of the above result are

$$F^{n*}(x) = \int_0^x F^{(n-1)*}(x-y)f(y)dy$$

and

$$f^{n*}(x) = \int_0^x f^{(n-1)*}(x-y)f(y)dy ------(2)$$

This result can be used to find the distribution of S_n directly.

Example 12: What is the distribution of S_n when $\{Xi\}_{i=1}^{n}$ are independent exponentially distributed random variables, each with mean 1/λ?

Solution: Setting n = 2 is equation (2), we get

$$F^{2*}(x) = \int_0^x f(x-y)f(y)dy$$

$$= \int_0^x \lambda \quad e^{-\lambda(x-y)} \, \lambda \, e^{-\lambda y} dy$$

$$= \lambda^2 e^{-\lambda x} \int_0^x dy = \lambda^2 x \; e^{-\lambda x}$$

So that, S_2 has a $\Upsilon(2, \lambda)$ distribution. Next, setting n = 3 is equation (2), we get

$$f^{3*}(x) = \int_0^x f^{2*}(x-y)f(y)dy$$

$$= \int_0^x f^{2*}(y) f\,(x-y)dy$$

$$= \int_0^x \lambda^2 y\ e^{-\lambda y}\ \lambda\, e^{-\lambda(x-y)} dy$$

$$= \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}$$

So that, the distribution of S_3 is $\Upsilon(3, \lambda)$. An inductive argument can now be used to show that you a general value of n, S_n have $\Upsilon(n, \lambda)$a distribution.

In general, it is much easier to apply the moment generating function method of find the distribution of S_n.

1.7.3 Recursive Calculation for Discrete Random Variables

In the case when X_i is a discrete random variable, distributed on the non-negative integers, it is possible to calculate the probability function of S_n recursively. Define

$$f_j = P_r(X_i = j) and g_j = P_r(S_n = j)$$

Each for j = 0, 1, 2, … We denote the probability generating function of X_1 by P_x so that

$$P_r(x) = \sum_{k=0}^{\infty} {}^j f_j$$

and the probability generating function of S_n by P_s, so that

$$P_r(r) = \sum_{k=0}^{\infty} r^k g_k$$

Using argument that have previously been applied to moment generating functions, we have

$$P_s(r) = P_x(r)^n$$

and differentiation with respect to r gives.

$$P_s' = nP_x(r)^{n-1} P_s'(x).$$

When we multiply each side of the above identity by $rP_x(r)$ we get

$$P_x(r)r = P_s' = nP_s(r) r P_x'(x)$$

This can be expected as

$$\sum_{j=0}^{\infty} r^j f_j \sum_{k=1}^{\infty} kr^k g_k = n \sum_{k=0}^{\infty} r^k g_k \sum_{j=1}^{\infty} jr^j f_j \text{ -------- } (3)$$

To find an expression for g_x, we consider the coefficient of r^x on each side of equation (3), where x is a positive integer. On the left-hand side the coefficient of r^x can be found as follows. For j = 0,1, 2, ……, x-1, multiply together the coefficient of r^j in the first sum with the coefficient of r^xj is the second sum. Adding these products together gives the coefficient of r^x, namely

$$f_0 x g_x + f_1(x-1)g_{x-1} \pm -- \mp f_{x-1} g_1 = \sum_{j=0}^{\infty} (x-j) f_j g_{x-j}$$

Similarly, on the right-hand side of equal, we have

$$x\, g_x f_0 + \sum_{j=1}^{x-1} (x-j) f_j g_{x-j} = n \sum_{j=1}^{x} j\, f_j g_{x-j}$$

This gives (nothing that the sum on the left-hand side unaltered when the upper limit of summation is increased to x)

$$g_x = \frac{1}{f_0} \sum_{j=1}^{x} \left((n+1)\frac{j}{x} - 1 \right) f_j g_{x-j} \text{ ------ } (4)$$

The important point about this result is that it gives a recursive method of calculating the probability function $\{g_x\}_{x=0}^{\infty}$. Given the values $\{f_j\}_{j=0}^{\infty}$. We can use the value of go to calculate g1, then the values of g_0 and g_1 to calculate g_2, and so on. The starting value for the recursive calculation is g_0 which is given by f_0^n since S_n takes the value 0 if and only if each x_j, j= 1, 2,…, n1 takes the value 0.

This is very useful result as it permits much more efficient evaluation of the probability function on of S_n than the direct convolution approach of the previous section.

We conclude with three remarks about this result:

(i) Computer implementation of formula (4) is necessary especially when n is large. It is however, an easy task to program this formula.

(ii) It is straight forward to adapt this result to the situation when x_1 is distributed on m, m+1, m+2, -----, where m is a positive integer.

(iii) The recursion formula is unstable. That is, it may give numerical should be employed when sense. Thus, caution should be employed when applying their formula. However, for most practical purposes, numerical stability is not an issue.

Example 13: Let $\{X_i\}_{i=1}^{4}$ be independent and identically distribution random variable with common probabilities function $f_j = P_r(X_1 = j)$ given by

$$f_0 = 0.4, \qquad f_1 = 0.3, f_2 = 0.2, \qquad f_3 = 0.1$$

Let $S_4 = \sum_{i=1}^{4} X_i$ recursively calculate $P_r(S_4 = r)$ for r = 1, 2, 3 and 4.

Solution: The starting value for the recursive calculation is

$$g_0 = P_r(S_4 = 0) = f_0{}^4 = 0.4^4 = 0.0256$$

Now note that as $f_j = 0$ and for j = 4, 5, 6…. Equation

(4) Can be written with a different upper limit of summation as

$$g_x = \frac{1}{f_0} \sum_{j=1}^{\min(3,x)} \left(\frac{5j}{x} - L\right) f_j g_{x-j}$$

and so $$g_1 = \frac{1}{f_0} 4 f_i g_o = 0.0768.$$

$$g_2 = \frac{1}{f_0}\left(\frac{3}{2} f_1 g_1 + 4 f_1 g_o\right) = 0.1376$$

$$g_3 = \frac{1}{f_0}\left(\frac{2}{3} f_1 g_2 + \frac{7}{3} f_2 g_1 + 4 f_3 g_o\right) = 0.1840$$

$$g_4 = \frac{1}{f_0}\left(\frac{1}{4} f_1 g_3 + \frac{3}{2} f_2 g_2 + \frac{11}{4} f_3 g_{12}\right) = 0.1905$$

1.8 Summary

In actuarial science, basic probability distributions are essential for modeling and assessing risks associated with uncertain events. These distributions, such as the normal, binomial, and Poisson distributions, provide a mathematical framework for describing the likelihood of various outcomes and are used to estimate the frequency and severity of insurance claims, mortality rates, and other actuarial phenomena. Mixed distributions are particularly important in actuarial applications as they combine different probability distributions to better capture the complexity of real-world data. For example, a mixed distribution might be used to model claim sizes by combining a discrete distribution for small claims with a continuous distribution for large claims, providing a more accurate and comprehensive risk assessment. Understanding and applying these distributions enable actuaries to develop more precise pricing, reserving, and risk management strategies.

1.9 References

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.

1.10 Further Readings

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

UNIT : 2 UTILITY THEORY

Structure

2.1 Introduction

Utility theory encompasses a wide range of applications, particularly within the field of economics. In this chapter, however, we will focus exclusively on its application within the insurance sector. We will commence with a general discussion on the concept of utility, exploring its foundational principles and significance in various contexts. Following this, we will introduce the concept of decision making, which represents the primary application of utility theory. Decision making under uncertainty is particularly relevant to insurance, where assessing risks and benefits is crucial.

Furthermore, we will present several mathematical functions that may be utilized as utility functions. These functions help quantify preferences and make informed decisions. We will examine their respective uses and limitations, providing examples to illustrate their practical applications. By understanding these utility functions, we can gain insights into how individuals and organizations make choices that involve risk and uncertainty.

2.2 Objectives

By the end of this lesson, learners will be able to:

1. Understand the concept of utility and its role in insurance decision-making.
2. Explain different types of utility functions and their implications in risk assessment.
3. Apply the Expected Utility Criterion and Jensen's Inequality to insurance problems.
4. Use utility theory to analyze and solve decision-making problems under uncertainty.

2.3 Insurance and Utility

Consider a scenario where a decision maker owns a property that may be damaged or destroyed in the next accounting period. The potential amount of loss, which may be zero, is represented by a random variable denoted as x. It is assumed that the distribution of $x$is known. The expected value $E[x]$ represents the average loss over the long term if the property were exposed to damage under identical conditions numerous times. While it is clear that, this extensive set of trials cannot be conducted by an individual decision maker, this example illustrates the utility theory, providing insights into the economic role of insurance.

Suppose an insurance organization (the insurer) is established to mitigate the financial consequences of property damage or destruction. The insurer issues contracts (policies) that promise to pay the property owner a defined amount equal to or less than the financial loss if the property is damaged or destroyed during the policy period. This contingent payment, tied to the amount of the loss, is called a claim payment. In exchange for this promise, the property owner (the insured) pays a consideration (premium).

The premium payment amount is determined based on an economic decision principle adopted by both the insurer and the insured. There is an opportunity for a mutually beneficial insurance policy when the premium set by the insurer is less than the maximum amount the property owner is willing to pay for insurance.

Within the range of financial outcomes for an individual insurance policy, the insurer's utility function might be approximated by a straight line. In this scenario, the insurer would adopt the expected value principle in setting its premium, meaning the insurer would set its basic price for full insurance coverage as the expected loss, $E[x] = \mu$. In this context,

μ is referred to as the pure or net premium for the one-period insurance policy. To account for expenses, taxes, profit, and a margin for security against adverse loss experience, the insurer would set the premium by loading, or adding to, the pure premium. For insurance, the loaded premium, denoted by H, would be calculated accordingly as follows:

$$H = (1 + a)\,\mu + c. \quad a > 0, \qquad c > 0$$

In this expression, the term$a\mu$ represents expenses that vary with expected losses and the risk that actual claims experience may deviate from what is expected. The constant C accounts for expected expenses that do not vary with the number of losses.

2.4 Utility Functions

A utility function can be described as a function which measures the value or utility that an individual or institution attached to the monetary amount x. This function satisfies the following conditions:

$i)\ \mu'(x) > 0 \qquad\qquad ii)\ \mu''(x) < 0$

- **Interpretations of the two conditions**

Condition I: It says that u is an increasing function of x i.e. an individual whose utility function is u prefers amount y to amount z if, y>z, i.e. the individual prefers more money to less.

Condition II: It says that u is a concave function. This means that as the wealth of individual increase in wealth. For e.g. An increase in wealth to 1000 is worthless to the individual if his wealth is 20,00,000 compared to the case when the independent wealth is 10,00,000.

An individual whose utility function satisfies condition (i) & (ii) is said to be risk averse, this leads to definition of coefficient of

- **Coefficient of Risk aversion:** An individual whose utility function $\mu(x)$satisfies $\mu'(x) > 0$ &$\mu''(x) < 0$is risk average with coefficient of risk aversion being given as:

$$\gamma(x) = \frac{-\mu''(x)}{\mu'(x)}$$

2.5 Expected Utility Criterion (EUC)

Decision making using a utility function is based on EUC. This criterion says that a decision maker should calculate the expected utility of resulting wealth under each course of action & then select that course of action that gives that greatest value for expected utility of resulting wealth.

If two course of action, yield the same expected utility of resulting wealth, then, the decision maker has no preference between these two courses of action, so mathematically

Let us consider an investor with utility function u. Suppose, he is choosing between two investments which will lead to random net gains of x_1 and x_2 respectively.

Further, suppose that the investor has current wealth w so that result of investing in investment I is $w + x_i; i = 1,2$. Then, under the expected utility criterion, the investor would choose Investment 1 over Investment 2 if and only if

$$E[\mu(w + x_1)] > E[\mu(w + x_2)]$$

Further, the investor would be indifferent between the two investments if

$$E[\mu(w + x_1)] > E[\mu(w + x_2)]$$

Example: Suppose that an investor has a utility function given as

$$\mu\,(x) = -\exp[-0.002x]$$

i) Check whether the utility function satisfies the necessary conditions

ii) Is the individual risk averse. If yes calculate the coefficient of risk aversion.

iii) Suppose that the investor comes across two investments that lead to gains of x_1f and $x_2$respectively such that

$$X_1 \sim N(10^4, 500^2)$$

$$X_2 \sim N(1.1 \times 10^4, 2000^2)$$

Which of these two investments will the investor prefers (assuming hi wealth w).

Solution: -

$$\mu'(x) = 0.002 \exp[-0.002x]$$

$$\mu''(x) = -4 \times 10^6 \exp[-0.002x]$$

Coeff. of risk aversion = $\frac{-\mu''(x)}{\mu'(x)} = 2 \times 10^{-3} = 0.002.$

$$E[\mu(w+x_1)] = -E[\exp\{-0.002(w+x_1)\}]$$

$$= -exp\{-0.002w\}E[exp\{-0.002x_1\}]$$

$$= -exp\{-0.002w\}exp\left\{-0.002 \times 10^4 + \frac{1}{2}0.002^2 \times 500\right\}$$

$$= -exp\{-0.002w\}exp\{-19.5\}$$

Where the third line from the fact that the expectation in the second line is M_{x_1} . Similarly,

$$E[\mu(w+x_1)] = -exp\{-0.002w\}exp[u(w+x_1)]$$

Hence, the investor prefer investment 1 as $E[u(w+x_1)]$is greater than $E[u(w+x_2)]$.

EUC may lead to an outcome i.e., inconsistent with other criterion: for e.g.

$$E(x_1) = 10^4$$

$$E(x_2) = 1.1 \times 10^4$$

An investor can choose the investment that gives the greater net gain and this may be the faulty decision.

Suppose that an investor make decision based on utility function given as

$$\mu\,(x) = \beta\; logx \quad ;x > 0, \beta > 0$$

i) Does it satisfy the condition of a utility function?
ii) What is the risk condition of the investor? (Coeff. of risk aversion).
iii) Suppose he has a wealth B & has an option available in terms of buying shares of companies I, i =1, 2, 3, …, n.

Which course of action will the investor prefer based on EUC?

Solution:

i) $\mu'(x) = \beta/x$ $\qquad\qquad$ $\mu''(x) = \beta/x^2$

ii) Coefficient = $(\frac{\beta}{x^2})\,/(\frac{\beta}{x}) = \frac{1}{x}$ $\;$ $risk\ averse$

iii) $E[U(\beta\, x_1)] = E[\beta\; log(\beta x_i)]$

The investor will prefer the share of company i to that of company j, by EUC if,

$$E[U(\beta x_i)] > E[U(\beta x_j)]$$

$$= E[\beta\ log(\beta x_i)] > E[\beta log(\beta x_j)]$$

$$= E[\log \beta + log x_i] > E[\log \beta + log x_j]$$

$$= E[log x_i] > E[log x_j]$$

This choice of investment does not have to do anything with the initial wealth he possesses; this is because of choosing the logarithmic function as the utility function.

So, one must be very careful in making the right choice about the utility function.

An important result of utility function:

Suppose that a utility function v is defined is terms of a utility function by

$j(x) = a\ \mu(x) + b \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(1)$ for constant a & b, s.t. a > 0. Then decision made under EUC will be the same under j as they are u.

Proof: Suppose a per EUC we prefer an investment x_1 over x_2, so that

$$E[\partial(w + x_1)] > E[\partial(w + x_2)]$$

$$= E[a\mu(w + x_1) + b] > E[a\mu(w + x_2) + b]$$

$$= aE[\mu(w + x_1)] + b > aE\ [\mu(w + x_1)] + b$$

which is true if

$$E[U(w + x_1)] > E[U(w + x_2)]$$

Remarks:

i) If means necessary (N) sufficient (s) which holds.
ii) Necessary means you start with result = condition
iii) S means you start with condition = result.
iv) If only n then condition may hold but result may not hold.
v) If only S then condition may not hold but result may hold.

2.6 Jensen's Inequality

It is a well-known result is probability theory & it has though important applications in actuarial science. Jensen's Inequality starts that if u is a concave function, then.

$$E[\mu(x)] \leq \mu[E\{x\}] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (1)$$

Provided their quantities exists

<u>Remark</u>: if u is a convex function, then,

$$E[\mu(x)] \geq \mu[E\{x\}] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (2)$$

<u>Proof:</u> We prove Jensen's Inequality on the assumption that there is a <u>Taylor Series expansion</u> of u about the point a.

Thus, writing a Tayer series expansion with a reminder term as

$$\mu(x) = \mu(a) + u'(a)(x - a) + \frac{1}{2}u''(z)(x - a)^2 \ldots \ldots \ldots \ldots .. (3)$$

where, Z lies between a & x. Now if the function is concave, we know that

$$u''(z) < 0 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (4)$$

So, from (3), we have

$$\mu(x) \leq \mu(a) + u'(a)(x - a) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . (5)$$

Now replace x by X is (5) & put a = E[x] then we get

$$u(x) \leq u(E[x]) + u'(E[x])(x - E[x]) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (6)$$

Take expectation on the both sides, we have

$$E[u(x)] \leq U\ [E(x)]$$

(Hence proved)

2.6.1 Application of Jensen's Inequality to Insurance

Appropriate premium levels for insurance cover from the view point of both an individual & insurer maximum premium given by insure – $P \geq E(x)$ Minimum Premium charged by company (insurer) - $\Pi \geq E(x)$.

2.6.1.1 Result 1:

The maximum premium that an individual is prepared to pay is at least equal to the expected less, i.e.

Proof: Consider first an individual whose wealth is w. suppose that the individual can obtain completes pursuance protector against a random loss x. then the maximum premium that the individual is prepared it pay for this protection is P, where U(W-P) = E [U(W-X)] ----------(1)

Reason for (1) to be true

(i) Expected utility criterion (EUC)

(ii) We know $U'(x) > 0$, so for any premium$\bar{P} < p, u(w - \bar{p}) > U(w - P)$.

Now by Jensen's inequality

$$E\ [U(W - x)] \leq UE[(w - x)] - \ U\left[w - E[x]\right] - - - - - - - - -(2)$$

Now from (1) & (2)

$$U(W - P) \leq U(W - E\ [x]) - - - - - - - - - - - - - -(3)$$

As it is an increasing function

(3) = W-P $\leq W - E\ [x]$

=$P \geq E\ [x]$

2.6.1.2 Result 2:

The insurer requires a premium π that is at least equal to the expected loss i.e. $\pi \geq E[x]$.

Proof: Suppose that an insurer has a utility function 0. Let this wealth be w.

Now, he provides complete insurance protection an individual against a random loss x. Suppose the minimum acceptable premium for him is π.

Now according to EUC

$$\partial(w) = E[\partial(w + \pi - X)] - - - - - - - - (1)$$

Because the insurer is choosing between offering & net offering insurance.

Also, we can see that as 0 is an increasing function for any premium $\bar{\Pi} > \boldsymbol{\Pi}$

$$E[\pi(w+\bar{\pi}-x)] > E[\pi(w+\pi-X)] -------- (2)$$

Now, applying Jensen's Inequality is the RHS of equation (1)

We have

$$E[\partial(w+\bar{\pi}-x)] \leq \partial[E(w+\pi-X)] = \partial\,(w+\pi-E[x]) ----(3)$$

Using (1) & (3)

$$\partial(\mathrm{w}) \leq \partial(w+\pi-E[x]) -------- (4)$$

As Ə is an increasing function

$$= \mathrm{W} \leq w+\pi-\boldsymbol{E}[\boldsymbol{x}]$$

$$= \pi \geq \boldsymbol{E}[\boldsymbol{x}] ------(\mathbf{5})$$

2.7 Types of Utility Functions

It is possible to construct a utility function by assigning different value to different levels of wealth. e.g. An individual might set U (0) = 0, U (10) = 5; U (20) = 8 & so on …. (Clearly, it more practical to assign value through a suitable mathematical function now consider some mathematical function's which may be regarded as having suitable forms to be used as utility function:

(1) Exponential
(2) Quadratic
(3) Logarithmic
(4) Fractional Power

2.7.1 Exponential Utility Function

A utility function of the form

$$\mu(x) = -\exp\{-\beta x\} ------(1) \text{ where, } \beta > 0$$

Is called an exponential utility function.

Properties:

(1) Suppose that an individual has wealth w. suppose he has a choice between n course of action such that the i^th course of action will result is a wealth w+x_i; i= 1,2…,,n. Then by EUC the individual would calculate $E[\mu(w+w_i)] \forall i = 1,2,\ldots\ldots, n \&$ would choose say course of action j, if

$$E\left[\mu\left(w+x_j\right)\right] > E\left[u\left(w+x_j\right)\right]$$

$$= E\left[-exp\left\{-\beta\left(w+x_j\right)\right\}\right] > E\left[-exp\left\{-\beta\left(w+x_j\right)\right\}\right]$$

$$= E\left[exp\left\{-\beta x_j\right\}\right] < E\left[exp\left\{-\beta x_j\right\}\right]$$

Which is independent of initial wealth w. this is an important property of EUF.

(2) The maximum premium P that an individual with utility function.

$$\mu(x) = -\exp\{-\beta x\}$$

Would be prepared to pay for insurance against the random loss x.

$$P = \beta^{-1} \log M_x(\beta)$$

Proof: By EUC

$$\mu(w-p) = E\left[\mu(w-x)\right] \qquad \ldots\ldots\ldots \qquad (1)$$

$$= -\exp\{-\beta(w-p)\} = E[-\exp\{-\beta(w-x)\}]$$

$$= -\exp\{-\beta(w-p)\} = -E[\exp\{\beta(w-x)\}]$$

$$= -\exp\{-\beta w\}.\exp\{\beta p\} = -\exp\{-\beta w\}.E[\exp\{\beta x\}]$$

$$\exp\{\beta p\} = E[\exp\{\beta x\} \;\; = M_x(\beta)$$

$$\beta p = logM_x(\beta)$$

$$p = \beta^{-1} log\, M_x(\beta)$$

Example: Show that the maximum premium p, that an individual with utility function $\mu(x) = -\exp\{-\beta x\}$ is prepared to pay for complete insurance cover against a random loss x. where $x \sim N(\mu.\sigma^2)$ is an increasing function of β and explain this result.

Solution: $\mu(x) = -\exp(-\beta x)\, M_x(\beta) = exp\left\{\mu\beta + \frac{1}{2}\beta^2\right\}$.

$$\mu'(x) = \frac{1}{\beta}\exp(-\beta x)$$

$$\mu''(x) = -\frac{1}{\beta^2}\exp(-\beta x)$$

$$\gamma(x) = -\frac{\mu''(x)}{\mu'(x)} = \beta$$

$$P = \beta^{-1}loge[M_x(\beta)] = \beta^{-1}log\ \left[exp\left\{\beta\mu + \frac{1}{2}\beta^2\sigma^{-2}\right\}\right]$$

$$= \mu + \frac{1}{2}\sigma^2\beta(\beta = increasing\ function)$$

Coefficient of risk aversion is an increasing function of & it is independent of x. Thus, the higher the risk aversion of an investor, the higher would be the value of & this would mean that the investor would be ready to pay a maximum premium (p).

Example: An individual it facing a random loss, x, where $x \sim \gamma$ (2,0.01), and can obtain complete insurance cover against this loss fcl a premium of 208. The individual makes decision on the bases of an exponential utility function with parameter 0.001. is the individual prepared to insure for this premium.

Solution: $\mu(x) = -\exp(-0.001x)$

$$\mu'(x) = -\frac{1}{0.001}\exp(-0.001x)$$

$$\mu'(x) = -10^6\exp(-0.001x)$$

$$\gamma(x) = -\frac{\mu''(x)}{\mu'(x)} = 10^3$$

$$P = (0.001)^{-1}log\left[\left(1-\frac{\beta}{\gamma}\right)^{-\alpha}\right]$$

$$= \left[(0.001)^{-1}log\left[\left(1-\frac{0.001}{0.01}\right)^{-2}\right]\right]$$

$$= 210.72 = P = maximum\ premium\ he\ is\ willing\ to\ pay$$

$$Since \quad 208 < P = 208 < 210.72$$

2.7.2 Quadratic Utility Function

A utility function of the form

$\mu(x) = x - x - \beta x^2 \quad for\ x < \frac{1}{\alpha\beta}; \quad \beta > 0.$ is called a Quadratic utility function.

$$\mu'(x) = 1 - 2\beta x > 0 = x < \frac{1}{2\beta}$$

$$\mu''(x) = -2\beta < 0\,; \quad \beta > 0.$$

The biggest drawback of this utility function is that we can't use it for problems where random outcomes are distributed on $(-\infty, \infty)$.

Under EUC; we will prefer outcome x_j to outcome x_i if;

$$E\left[\mu\left(W + x_j\right)\right] > E[\mu(w + x_i)]$$

$$\mu\left(W + x_j\right) = W + x_j - \beta\left(W + x_j\right)^2$$

$$\mu(W + x_i) = W + x_i - \beta(W + x_i)^2$$

$$E\left[U\left(W + x_j\right)\right] > E[\mu(w + x_i)]$$

$$= E\left[x_j\right] - \beta E\left[x_j^2\right] - 2\beta w E\left[x_j\right] > E[x_i] - \beta E[x_i^2] - 2\beta w E[x_i]$$

$$= (1 - 2\beta w)E\left[x_j\right] - \beta E\left[x_j^2\right] > (1 - 2\beta w)\, E[x_i] - \beta E[x_i^2]$$

Thus, the result is based only on first 2 (raw) moments of x_i&x_j and also depends on w (i.e. the initial wealth)

Example: An individual whose wealth is w has choice between investment 1 and 2, which will result is wealth of w+x_1 and w+ x_2 respectively, where E[x_i]=10, v[x_1]=2 and E [x_2]=10.1 . The individual makes decision on the basis of a quadratic utility function with parameter β = 0.002. for what range of values for v[x_2] will the individual choose investment 1 when W=200? Assume that $P_r(w + x_i < 250) = 1\ for\ i = 1\ and\ 2.$

Solution: The individual will choose investment 1 if $E[\mu(W + x_1)] > E[\mu(w + x_2)]$ or equivalently

$E[200 + x_1 - \beta(200 + x_1^2)] > E[200 + x_2 - \beta(200 + x_2)^2]$. where = 0.002. After some straight forward algebra, this condition becomes

$$E[x_1](1 - 400\beta) - \beta E[x_1^2] > E[x_2](1 - 400\beta) - \beta E[x_2^2] \; or$$

$E[x_2^2] > (E[x_2] - E[x_1])(\beta^{-1} - 400\beta) + E[x_1^2] = 112$

Which is equivalent to $V[x_2] > 9.99$

Example: An insurer is considering offering complete insurance cover against a random loss x, where. E[x]= V[x]= 100 and Pr (x>0)=1. The insurer adopts the utility function $\mu(x) = x - 0.001x^2$ for decision making purposes. Calculate the minimum premium that the insurer would accept for this insurance cover when the insurer's wealth, w, is (a) 100, (b) 200 and (c0 300.

Solution: The minimum premium, is given by $\mu(w) = E[\mu(w + \pi - x)]$ so when W = 100, we have μ (100) = 90

$$= E[100 + \pi - x - 0.001(1000 + \pi^2) - 2(100 + \pi)x + x^2]$$

$$= 100 + \pi - E[x] - 0.001(100 + \pi)^2 - 2(100 + \pi)E[x] + E[x^2]$$

This simplifies to

$$\pi^2 - 1{,}000\pi + 90.100 = 0$$

which gives π = 100.13. Similarly, when W = 200, we find that π = 100.17 and when W=300, π = 100.25. We note the π increases as W increase and the this is an undesirable property as we would expect that as the insurer's wealth increases the insurer should be better placed to absorb random losses and hence should be able to reduce the minimum acceptable premium.

2.7 Logarithmic Utility Function

A utility function of the form $\mu(x) = \beta \log x \; for \; x > 0, \beta > 0$ is called a logarithmic utility function.

Constraints: As $\mu(x)$is defined only for positive value of x, the utility function is unsuitable for use in situations where outcomes could lead to negative wealth.

(1) Check that it is a properly defined utility function.

$$\mu(x) = \beta \log \quad x$$

$$\mu'(x) = \frac{\beta}{x}$$

$$\mu''(x) = \frac{\beta}{x^2}$$

$$\gamma(x) = \frac{\mu''(x)}{\mu'(x)} = \frac{1}{x} = coefficient\ of\ risk\ aversion.$$

$$\gamma'(x) = \frac{1}{x^2} decreasingfunction \frac{\theta}{x}\ (aswealthfunction)$$

This means that risk aversion is a decreasing j^n of wealth i.e. as the wealth increase, risk aversion would from other utility function. Therefore, one must be cautious about the logarithmic utility j^n.

An investor who makes decisions on the basis of a logarithmic utility function is considering investing is

Example: shares of one n company. The investor has wealth B, and investment in shares of company I will result of wealth Bx_i for {=1,2,……,n. show that the investment decision is independent of B.

Solution: The investor prefers the shares of company I to these of company I if,

$$E[\mu(Bx_i)] > E\left[\mu\left(Bx_j\right)\right]$$

Now $\quad E[\mu(Bx_i)] = E[\beta\ log(Bx_i)] = \beta E[logB]\beta E[logx_i]$

So, the investor prefers the share of company I to those of company j if and only if

$$E[logx_i] > E\left[logx_j\right] \text{ Independent of B.}$$

2.7.4 Fractional Power Utility Function

A utility function of the form $\mu(x) = x^{\beta},\ x > 0 \,\&\, 0 < \beta < 1.$ is called a fractional power utility function. As with a logarithmic utility function $\mu(x)$, is defined only for positive x and so its applications are limited in the same way as for a logarithmic utility function.

To check it is a properly utility function

$$\mu(x) = x^{\beta}$$

$$\mu'(x) = \beta x^{\beta-1} = \frac{\beta}{x^{1-\beta}} > 0$$

$$\mu''(x) = \beta(\beta-1)x^{\beta-2} = -\frac{\beta(1-\beta)}{x^{2-\beta}} < 0$$

$$\gamma(x) = \frac{-\mu''(x)}{\mu'(x)} = \frac{\beta(1-\beta)x^{\beta-2}}{\beta x^{\beta-1}} = (1-\beta)x^{-1} = \frac{1-\beta}{x} > 0$$

Example: An individual is facing a random loss x, that is uniformly distributed on (0,200). The individual can buy partial insurance cover against this loss under which the individual would pay y = min (x, 100) so that the individual would pay the loss in fall if the loss was less than 100 and would pay 100 otherwise. The individual makes decisions using the utility function $\mu(x) = x^{2/5}$. is the individual prepared to pay 80 for this partial insurance cover if the individual's wealth is 300?

Solution:$x \sim \mu(0,200),\ \ x = amount\ of\ loss\ \mu(x) = x^{2/5}$

$$y = amount\ paid\ by\ (insurer)\ investment$$

$$Y = \begin{cases} x, & x < 100, \quad y = \min(x,100) \\ 100, & x \geq 100, \end{cases}$$

$$f_x(y) = \begin{cases} fx^{(x)}, & x < 100 \\ 1 - fx^{(x)}, & x \geq 100 \end{cases}$$

Z= amount paid by reinsurer

$$Z = \begin{cases} 0, & x < 100, \\ x - 100, & x \geq 100, \end{cases}$$

$$\begin{cases} P = max.\ premium\ investor\ ready\ to\ pay \\ p = \ \beta^{-1} \log M_x(\beta) \end{cases}$$

(We can’t use this here before their p gives full insurance cover)

If he does not by partial insurance, then he would pay the amount x in full: he would leave with (300-x). if he buys partial insurance with 80← premium & will also pay y→ he would leave – (300-80-y).

Then by EUC

$$E[U(300-x)] > E[U(300-80-y)]$$

$$E[U(300-x)] = E\left[U(300-x)^{2/5}\right]$$

$$= \int_{300}^{200} (300-x)^{2/5} \frac{1}{200} dx$$

Let 30-x=y = dx= dy

$$= \int_{300}^{100} y^{2/5} \frac{1}{200} (-dy)$$

$$= \frac{1}{200} \left[\frac{y^{7/5}}{7/5}\right]_{100}^{200} = 8.237$$

$$E[U(300-80-y)] = E\left[(300-80-y)^{2/5}\right]$$

$$= \int_{0}^{200} (220-y)^{\frac{2}{5}} f_x(y) dy$$

$$\int_{0}^{100} (220-x)^{2/5} \frac{1}{200} dx + \int_{100}^{200} (120)^{215} dx$$

$$= \frac{1}{200} \left(\frac{-5}{7} (220-x)^{7/5}\right) \int_{0}^{140} +100 \times 120^{2/5} = 7.280$$

Hence, the individual is not prepared to pay 80 for this partial insurance cover.

2.8 Utility Theory and Decision Problems

Now turn to applying utility theory to the decision problem faced by the property owner subject to potential loss. The property owner has a utility function U(w), where (w) represents wealth measured in monetary terms. The owner faces the possibility of a loss due to random events that may damage the property. The distribution of the random loss (w) is assumed to be known. The owner will be indifferent between paying an amount G to the insurer, who will then assume the random financial loss, and bearing the risk themselves. This scenario can be expressed as follows:

$$\mu(w-G)=E[\mu(w-x)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The RHS of (1) represents the expected utility of not buying insurance when the owner's current wealth is w. The LHS of (1) represents the expected utility of paying G for incomplete financial protection.

If the owner has an increasing linear utility function that is, $\mu(w)=bw+d$ with b > 0. The owner will be adopting, the expected value principle. in this case the owner prefers, or is indifferent to, the insurance when

$$\mu(w-G)= b(w-G)+d \geq E[\mu(w-x)]=E[b(w-x)+d], b(w-G)+d \geq b(w-\mu)+d$$

$$G \leq \mu$$

That is, if the owner has an increasing linear utility function, the premium payments that will make the owner prefer, of be indifferent to, complete insurance absence of a subsidy, an insurer, over the long term must charge more than its expected losses. Therefore, in this case, there seems to be little opportunity for a mutually advantageous insurances contract. If an insurance contract it is result the insurer must charge a premium in excess of expected losses and expenses to avoid a bias toward insufficient income. The property owner then cannot us a linear utility function.

The preferences of a decisions maker must satisfy certain consistency requirements to ensure the existences of a utility function. Although these requirements were into listed they do not include any specifications that would force a utility function to be linear, quadratic, exponential, logarithmic, or any other particular form. In fact, each of these named functions might serve as a utility function for some decision maker or they might be spliced together to reflect some other decision maker's preferences.

Nevertheless, it seems natural to assume that u(w) is an increasing function. "More is better". In addition, it has been observed that for many decision makers, each additional equal increment of wealth results in a smaller increment of associated utility. This is the idea of decreasing marginal utility is economics. We know that $\Delta^2\mu(w) \leq 0$. if these ideas are extended to smoother functions the two properties suggested by observation are $\mu'(w) > 0$ and $\mu''(w) < 0$. The second inequality indicates that u(w) is a strictly concave downward function.

In discussing insurance decisions using strictly concave downward utility functions, we will make use of one form of Jensen's inequalities. There inequalities states that for a random variable x and functions u(w),

$$if\ \mu''(w) < 0, then\ E[\mu(x)] \leq \mu(E[x]), \dots \dots \dots \dots \dots (2)$$

$$if\ \mu''(w) > 0, then\ E[\mu(x)] \geq \mu(E[x]), \dots \dots \dots \dots \dots (3)$$

Jensen's inequalities require the existence of the two expected values.

Proof of equation (2)

If $E[x] = \mu$ exists, one considers the line tangent to $\mu(w)$,

$$y = \mu(\mu) + \mu'(\mu)(w - \mu),$$

At the point $\mu.\mu(\mu)$. Because of the strictly concave characteristic of the graph of u(w) will be below the tangent line, that is

$$\mu(w) \leq u(\mu)(w - \mu) - - - - - - - - - - - - - (4)$$

For all values of w. if w is replaced by the random variable x and the expectation is taken on each side of the inequality have several applications in actuarial mathematics. Let us apply Jensen's inequality (2) to the decision maker's insurance problem as formulated in (1). We will assume that the decision maker's preferences are such that $\mu'(w) > 0$ and $\mu''(w) < 0$. Applying Jensen's inequality to (1), we have.

$$\mu(w - G) = E[\mu(w - x)] \leq u(w - \mu) - - - - - - - - - - (5)$$

Because $\mu'(w) > 0, u(w)$is an increasing function. Therefore (5) implies that $w - G \leq w - \mu, or\ G \geq \mu\ with\ G > \mu$ unless x is a constant. In economic terms, we have found that $\mu'(w) > 0\ and\ \mu''(w) < 0$ if the decision maker will pay an amount greater than the expected loss for insurance. If G is at least equal to the premium set by the insurer, there is an opportunity for a mutually advantageous insurance policy.

Formally we say a decision maker with utility function is risk averse if and only if,

$$\mu''(w) < 0.$$

We now employ a general utility function for the insurer. We let u_i (w) denote the current wealth of the insurer measured in monetary terms. Then the minimum acceptable premium H for assuming random loss X, from the viewpoint of the insurer may be determined from (6);

$$\mu_i(w_i) = E[U_i(w_i + H - X)] ------(6)$$

The LHS of (6) is the utility attached to the insurer's current position. The r.h.s. is the expected utility associated with collecting premium H and paying random loss X. In other words, the insurer is indifferent between the current position and providing insurances for X at premium H. If the insurer's utility function is such that $\mu_i(w) > 0, u_I''(w) < 0$we can use Jenson's inequality (2) along with (6) to obtain.

$$\mu_i(w_i) = E[U_i(w_i + H - X)] \leq U_i(w_i + H - \mu)$$

Following the same line of reasoning displayed in connects with (5). We can conclude that $H \geq \mu$. If G as determined by the decision maker by solving (5) is such that $G \geq H \geq \mu$.an insurances contract is possible that is the expected utility of neither party to the contracts is decreased.

A utility function is based on the decision maker's preferences for various distractions of outcomes. An insurer need not be an individual. It may be a partnership, corporation, or government agency. In this situation the determination of $\mu_i(w)$ the insurer utility function may be a rather complicated matter. For example, if the insurer is a corporation, one of management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflating attitudes towards risk among the groups of stockholders.

Several elementary functions are used to illustrate properties of utility functions.

An exponential utility function is of the form $\mu(w) = -e^{-\alpha w}$ for all w and for a fixed x>0. And has several attractive features.

First $\mu'(w) = \alpha \ - e^{-\alpha w} \ > 0$

Second $\mu'^{\prime(w)} = -\alpha^2 \ - e^{-\alpha w} \ < 0$

Therefore, may serve as the utility function of a risk-averse individual third finding

$$E[-e^{-\alpha x}] = -E[-e^{-\alpha x}] = -M_x(-\alpha)$$

is essentially the same as finding the MGF of x in this expression.

$$M_x(\pm) = E[e^{tx}]$$

denotes the MGF of x. Fourth insurance premium do not depends on the wealth of the decision maker. This statement is verified for the insured by substituting the exponential utility function into (1) that is,

$$-e^{-\alpha(w-G)} = E\left[-e^{-\alpha(w-x)}\right]$$

$$e^{\alpha G} = M_x(\alpha)$$

$$G = \frac{log M_x(\alpha)}{\alpha}$$

and G does not depend on W.

The verification you the insurer is done by substituting the exponential utility function with parameter α_I into (6)

$$-e^{-\alpha_I W_I} = E\left[-e^{-\alpha_I (W_I + H - X}\right]$$

$$-e^{-\alpha_I W_I} = -e^{-\alpha_I (W_I + H)} M_x(\alpha_I)$$

$$H = \frac{\log M_x(\alpha_I)}{\alpha_I}.$$

Example: A decision maker's utility function is given by $(w) = e^{-5w}$. the decision has tow random economic prospects (gains) available. The outcome of the first denoted by x. has normal distribution with mean 5 and variance 2. Hence forth, a statement about a normal distribution with mean μ and variance σ^2 ill be abbreviated as $N(\mu, \sigma^2)$. The second prospect. Denoted by Y, is distributed as N ~ (6.25). Which prospect will be preferred?

Solution: We have

$$E[u(x)] = E[-e^{-5x}]$$

$$= -M_x(-5) = -e^{\left[-5(5) + \frac{(5^2)(2)}{2}\right]} = -L$$

and

$$E[u(Y)] = E[-e^{-sy}]$$

$$= -M_x(-5) = -e^{\left[-5(6)+\frac{(5^2)(2.5)}{2}\right]} = -e^{1.25}$$

Therefore

$$E[u(x)] = -L > E[u(x)] = -e^{1.25}$$

and the distribution X is preferred to the distribution of Y.

* The family of fractional power utility function is given by

$$u(w) = w^r, \qquad w > 0.0 < 1 < 1.$$

A member of this family might represent the preferences of a risk averse decision maker since $u'(w) = \gamma h^{r-1} > 0$ and $u''(w) = \gamma(r-1)^{r-2} < 0$

In this family premiums depend on the wealth of the decision maker in a manner that may be sufficiently realistic is many situations.

Example: A decision maker's utility function is given by $u(w) = \sqrt{w}$. The decision maker has wealth of W-10 and faces random loss X with a uniform distribution on (0,10). What is the maximum amount this decision maker will pay for complete insurance against the random loss?

Solution: Substituting into (1), we have

$$\sqrt{10-G} = E\left[\sqrt{10-X}\right] = \int_0^{10} \sqrt{10-X}10^{-1} = \frac{-2(10-x)^{\frac{3}{2}}}{3(10)} = \frac{2}{3}\sqrt{10}$$

$G = 5.556$

The decision maker is risk averse and has u'(w)>0. Following the discussion of (5), we would expect G>E[x]. and in this example G= 5.5556>E[x]=5.

*The family of quadratic utility function is given by

$$u(w) = w - \alpha w^2,\ \ w < (2\alpha)^{-1}, \alpha > 0$$

A member of this family represents the preferences of a risk- averse decision maker since $u''(w) = -2\alpha$. while a quadratic utility function is convenient because decision depends only on the first two moments of the distributions of outcomes under considerations there are certain consequences of its use that strike some people as being unreasonable Example given below illustrate one of their consequences.

Example: A decision maker's utility of wealth function is given by u(w) = w-0. 0/w^2, w < 50.

The decision maker will retain wealth of amount w with probability p and suffer a financial loss of amount C with probability 1-p. for the values of w, c, and p exhibited in the table the decision maker will pay for complete insurance assume $C \leq w < 50$.

Solution: For the facts stated. (1) becomes

$$u(w-G) = p\,u(w) + (1-p)u(w-c),$$

$$(w-g) - 0.01\,(w-G)^2 = p\left(w - \frac{0.0}{w^2}\right) + (1-\mathrm{p})[(\mathrm{w}-\mathrm{c}) - 0.01(\mathrm{w}-\mathrm{c})^2]$$

For given values of w, p and c this expression becomes a quadratic equation. Two solutions are shown.

Wealth	Loss	Probability	Insurance Premium
W	c	p	G
10	10	0.5	5.2p
20	10	0.5	5.37

2.9 Summary

Utility theory is a fundamental concept in economics and decision theory that describes how individuals and organizations make choices under conditions of uncertainty and varying preferences. It provides a mathematical framework for understanding decision-making processes, especially in situations involving risk and uncertainty.

Utility theory provides a powerful and flexible framework for analysing and understanding decision-making under uncertainty. By modelling preferences and incorporating risk attitudes, it helps explain and predict a wide range of economic behaviours. The concepts of utility functions, expected utility, and risk aversion are central to the theory, offering insights that are applicable across various fields such as economics, finance, and behavioural sciences.

2.10 References

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.

- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

2.11 Further Readings

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

UNIT: 3 PRINCIPLES OF PREMIUM CALCULATION

Structure

3.1 Introduction

In actuarial terms, **premium** refers to the amount of money that an insurance company charges to policyholders for providing coverage under an insurance policy. Premiums are a critical aspect of the insurance business, as they represent the primary source of revenue for insurance companies. Actuaries play a key role in determining the appropriate premium amounts by evaluating various factors, including:

1. Risk Assessment: Actuaries assess the risk associated with insuring individuals or entities. This involves analyzing historical data, statistical models, and trends to predict the likelihood and cost of future claims.

2. Cost of Claims: Estimating the expected cost of claims that the insurer will need to pay out over the policy term.

3. Administrative Costs: Including the costs associated with underwriting, issuing, and servicing the policy.

4. Profit Margin: Ensuring that the premium charged is sufficient to cover all costs and provide a profit to the insurance company.

5. Regulatory Requirements: Complying with legal and regulatory requirements that might influence premium calculations.

6. Market Competition: Considering the competitive landscape and pricing strategies of other insurers.

In summary, the premium in actuarial terms is the price charged by an insurance company to cover the expected costs of claims, administrative expenses, and provide a margin for profit, all while ensuring competitiveness and compliance with regulations.

Notation:

- We denoted π_x by the premium that an insurer changes to cover a risk x.
- When we refer to a risk X. what we mean is that the claim from this risk is distributed as the random variable X.
- The premium $\pi_x$is then some function of X.
- A rule that assigns a numerical value to $\pi_x$is referred to as a Premium Calculation Principle.
- Thus, a premium principle is of the form where is some function.
- We now consider some desirable properties of premium.

3.2 Objectives

By the end of this lesson, learners will be able to:

1. Understand the fundamental properties that a premium principle should satisfy.
2. Differentiate between various premium principles and their implications in insurance pricing.

3. Apply specific premium principles, such as the Expected Value, Variance, and Esscher principles, in risk assessment.
4. Evaluate and select the most appropriate premium principle based on different risk scenarios.

3.3 Properties of Premium Principles

There are many desirable properties for premium calculation principle. We list 5 most important of them.

(i) **Non- negative Loading:** This property requires that we should have $\pi_x \geq E[x]$ i.e. the premium should not be less than the expected claims.

(ii) **Additivity:** This property requires that if x_1& x_2 are independent risk, then the premium for the combined risk i.e $\pi_{x_1+x_2}$ should be $\pi_{x_1} + \pi_{x_2}$. if this property is satisfied, then these is no advantage to either an individual or an insurer, in combining risk or splitting them as the total premium does not alter under each course of action.

(iii) **Scale invariance:** This property requires that if Z= ax. Where a is (+ve), then $\pi_z = a\pi_{x.}$

For Example: Imagine that the currency of Great Britain changes from sterling to euro with 1 pound sterling being converted to a Euros. The, if a British insurer uses a scale invariant premium principle, a premium of 100 pounds sterling would change to 100a Euros.

(iv) **Consistency:** This property requires that if Y= X+C where C > 0, then, we should have $\pi_y = \pi_x + c$. Thus if the distribution of Y is the distribution of X shifted by C units, then the premium for risk Y should be that for risk X. increased by C.

(v) **No Rip-off Property:** (For upper limit of premium) This property requires that if there is finite max. claim amount of the risk say x_m. Then we should have $\pi_x \leq \pi_m$. if this condition is not satisfied then there is no incentive for an individual to effect insurance i.e.

$$E[x] \leq \pi_x \leq x_m$$

3.4 Examples of Premium Principles

Premium principles are essential guidelines that help actuaries determine appropriate premiums for insurance policies. These principles ensure that premiums are fair, adequate, and competitive while covering the expected costs of claims, administrative expenses, and providing a profit margin for the insurance company. Each premium principle has its approach to balancing these needs, often incorporating considerations of risk, uncertainty, and the time value of money. Below are some common premium principles:

3.4.1 The Pure Premium Principle

The pure premium principle sets $\pi_x = E(x)$i.e. the pure premium is equal to the insurer's expected claims under the risk. To check if all the properties are satisfied.

(i) $\pi_x \geq E[x] = \pi_x = E[x]$

(ii) $\pi_{x_1+x_2} = E[x_1 + x_2] = E(x_1) + (x_2) = \pi_{x_1} + \pi_{x_2}$

(iii) $if\ x = az.\quad \pi_x\ \ E[az] = aE[z] = \pi_x = a\pi_z$

(iv) $if\ Y = X + C\quad \pi_x = \pi_{y+c} = E[y + c] = E(y) + C = \pi_x = \pi_y + C$

(v) $if\ x \leq x_m$

$E[x] \leq x_m$

$\pi_x = E[x] \leq x_m = \pi_x \leq x_m$

3.4.2 The Expected Value Principle

The Expected Value Principal sets. $\pi_x = (1 + \theta)E[x]$ where $\theta > 0$ is referred to the premium loading factor (imp term). The loading in the premium is thus $\theta E[x]$.

Important Remark about Expected Value Principle:

Clearly this principle is a very simple one but it has a major drawback that it assigns the same premium to all risks with the same mean. However, risks with identical means but different variance should have different premium. Checking which properties the expected value principle satisfies:

(i) $\pi_x \geq E[x] = \pi_x = (1 + \theta)E[x] = E[x] + \theta E[x] \geq E[x]$

(ii) $\pi_{x_1+x_2} = (1 + \theta)E[x_1 + x_2] = E(x_1)(1 + \theta) + (x_2)(1 + \theta) = \pi_{x_1} + \pi_{x_2}$

(iii) $if\ x = az.$

$\pi_x(1 + \theta)E[az] = (1 + \theta)aE[z] = a\pi_z = \pi_z = a\pi_x.$

(iv) $if\ Y = X + C\quad \pi_y = (1 + \theta)E[y] = (1 + \theta)E[x + c]$

$= (1 + \theta)E[x] + C(1 + \theta) = \pi_x + C(1 + \theta) \neq \pi_x + C(until\ \theta = 0)$

(v) $x \leq x_m$

$$(1+\theta)x \leq (1+\theta)x_m$$

$$E((1+\theta)x) \leq (1+\theta)x_m$$

$$\pi_x \leq (1+\theta)x_m = \pi_x \& x_m$$

$$\therefore (1+\theta) > 0$$

Rip off Property: To check whether rip-off property holds we will consider example. Let x be a risk such that$P(x-b) = 1$; b>0

i.e. X is a degenerate random variable with its entire mass concentrated at point b. Now, we calculate $\pi_x = (1+\theta)\, b > b \; (\because \; \theta > 0\,)$

$$but\ x = b \;\; (or\ i.e. xm = b)$$

$\pi_x > \pi_m$. That is why no rip off is not being satisfied. This is a counter example; therefore, we can say that in general, the no rip off property may not be satisfied by expected value principle.

3.4.3 The Variance Principle

The variance principle sets:

$\pi_x = E[x] + \alpha v[X], where\ \alpha > 0.$

Thus, the loading in the premium is proportional to v[x]. In this way, this principle removes the drawback of the expected value principle which takes account only of the expected chain.

Checking which properties it satisfies:

(i) $\pi_x \geq E[x]\ (Non-negavtive\ loading)$

$$E[X] + \alpha v[X] \geq E[X]$$

(ii) $\pi_{x_1+x_2} = E[x_1 + x_2]$

$$E[x_1] + E[x_2] + \alpha\ [v(x_1) + v(x_2)]\ (b'oz\ of\ independpent\ of x_1\ \&\ x_2)$$

(iii) $\pi_z = \pi_{ax}$

$$= E[aX] + \alpha v[aX] = aE\ [X] + \alpha a^2 v[X]$$

$$= a[E(x) + \alpha a v(x)] \neq a(E(x) + \alpha v(x))$$

(iv) Y= X+C

$\pi_y = \pi_{x+c} \;\; = E(X+C) + \alpha v\,(X+C)$

$E(x) + C + \alpha v(x)$

$\pi_y = \pi_x + C.$

(v) $x \;\leq\; x_m$

$\pi_x \leq\; x_m$

$E(x) + \alpha v(x) \leq x_m$

Example: $P(x=8) = P(x=12) = 0.5$

E(x)= 4+6= 10

$v(x) = E\,(x^2) - (E(x))^2 = 4$

$\pi_x = 10 + \alpha 4 \quad \leq 12, \qquad x > 0$

If x =10 then π_x= 50 $\neq$ 12 or if > 0.5, π_x> xm.

It does not satisfy no rip off values (property) when α> 05. It satisfies no. rip off property when, $\alpha \leq 0.5$

3.4.4 The Standard Deviation Principle

Clearly the variance principle does not satisfy the scale in-variance property which is a very desirable property for premium setting & therefore motivated by this fact, we define the std. deviation principle.

Therefore, it sets

$\pi_x = E[x] + \alpha\sqrt{v(x)} = E[x] + \alpha S(x).$

<u>Checking</u>:

(i) $\pi_x \geq E[X]$

$= E[X] + \alpha\, S(x) \geq E\,(x).$

(ii) $\pi_{x_1+x_2} = E[x_1 + x_2] + \alpha\sqrt{v(x_1+x_2)}$

$= E[x_1]E\,[x_2] + \alpha\,\sqrt{v(x_1)v(x_2)}$

$$\neq \pi_{x_1} + \pi_{x_2}$$

$(iii)\quad \pi_Z = \pi_{ax}$

$$= E(ax) + \alpha\sqrt{v(ax)}$$

$$= aE(x) + a\alpha\sqrt{v(x)}$$

$$= a\pi_x$$

$(iv)\quad \pi_y = \pi_{x+c}$

$$= E(x+c) + \alpha\ \sqrt{v(x)} + c$$

$$= E(x) + \alpha\ \sqrt{v(x)} + c$$

$$= \pi_x + C$$

$x\ \leq x_m,\quad \pi_x = E(x) + \alpha\sqrt{v(x)}\ \leq\ x_m.$

$$= \pi_x \leq x_m$$

$if\ P\ (x = 8) =\ P\ (x = 12) =\ 0.5$

$$E(x) =\ 10$$

$$V(x) = 4\ = SD(x) = 2$$

$$\pi_x = 10 + \alpha\ \ 2 \leq 12$$

$when\ \alpha < 1,\ \ \pi_x \leq 12\ \rightarrow No. Rip\ off\ satisfies$

Exercise: An insurer thinks about an innovative way of setting up a premium such that. $\pi_x = V^{-1}\ (E[v(x)])$ where v is a f^ns.t. v^1 (x) > 0 & v" (x) < 0, for x > 0

(i) Calculate π_x → when V(x) = x^2& x ~ $\gamma(2, 2)$.

(ii) Check which properties does not this property satisfy.

(iii) In case if you feel a particular prop. Is not satisfy then give appropriate counter example.

Solution:

$$V(x) =\ x^2\ X{\sim}\gamma(\alpha, \beta)$$

$E[x] = \frac{\alpha}{\beta} = \frac{2}{2} = 1$

$v[x] = \frac{\alpha}{\beta^2} = \frac{2}{4} = \frac{1}{2}.$

$E[v(x)] = E(x^2)$

$= V(x) + (E(x))^2 = \frac{1}{2} + 1 = \frac{3}{2}$

$\pi_x = V(x) = x^2 = v^{-1}(x) = \sqrt{x} = v^{-1}(E(x)) = \sqrt{E(x)}$

$\pi_x = \sqrt{Ev(x)}\pi_x = \sqrt{\frac{3}{2}} = \sqrt{1.5} = 1.225$

Checking:

$(i)\pi_x = v^{-1}[E(v(x))] = \sqrt{Ev(x)} \geq E(x) = 1.225 \geq 1.$

$Since, \quad E(x^2) \geq (E(x))^2 = \sqrt{E(x^2)} \geq E(x)$

$(2)\pi_{(x_1+x_2)} = \sqrt{E[x_1 + x_2]^2} = \sqrt{E(x_1^2) + E(x_2^2) + 2E(x_1, x_2)}$

$\neq \sqrt{E(x_1^2)} + \sqrt{E(x_2^2)}$

(3) $\pi_z = \pi_{ax} = \sqrt{E((ax)^2)} = \sqrt{E(a^2x^2)} = a\sqrt{E(x^2)} = a\pi_x$

(4) $Y = X + C = \bar{\wedge}_y = \pi_{x+c} = \sqrt{E\ (x^2 + c^2 + 2xc)} \neq \sqrt{E(x^2)} + C$

(5) $X \leq x_m$

$= \pi_x = \sqrt{E(x^2)}$

$= \pi_x = \sqrt{6.25} = 7.905$

$x_m = -5 = \pi_m > x_m$

$P(x = -5) = P\ (x = -10) = 0.5$

$E(x^2) = 25 \times 0.5 + 100 \times 0.5$

$= 12.5 + 50 = 62.5$

3.4.5 Principle of Zero Utility

This principle sets the premium π_x to satisfy the equation

$$u(w) = E\ [u(W + \pi_x - x)].$$

where u(x)is the insurer′s utility function & W *is the insurer′s surplus*

Clearly, the utility function u(x) satisfies

$\mu'(x) > 0$ $\mu''(x) > 0$

In general, the premium will depend on the insurer's surplus (expect for the case when we have an exponentially utility function.

Checking which property does the principle of Zero utility satisfy.

$(i)\quad \pi_x \geq E[x]$

$E\ [\mu(W \neq \pi_x - x)] \leq \mu(w + \pi_x - E_x)\quad (By\ Jensen's\ Inequlity\ u\ is\ a\ convex\ f^n)$

$= \ \mu(w) \leq \mu(w + \pi_x - E(x))$

We know that, $\mu'(x) > 0$

$\therefore if\ a\ \leq b = \ \mu\ (a) \leq \ \mu\ (b)$

$= \ w < w + \pi_x - E[x]$

$\pi_x \geq aE[x]$

(iii) Now,

$\pi_Z = \ \pi_{ax} = a\pi_x$

$E\ \left(\mu(w + \pi_z - ax)\right) \leq \ \mu(w + \pi_z - aE(x))$

$= \ \mu\ (w) \leq \ \mu\ \left(w + \pi_z - aE(x)\right)$

$= \ \pi_x \geq aE(x)\ = a\pi_x \geq aE(x)$

$$\boxed{\begin{gathered}\pi_z \geq E(z)\\ \pi_z \geq E(ax)\\ \pi_z \geq aE(x)\end{gathered}}$$

For a counter example

$X \sim N\ (\mu, \sigma^2)$

$\mu(x) = \exp[-\beta x]$

$= -\exp[-\beta w] = E\left[-Exp\left(-\beta(w + \pi_x - X)\right)\right]$

$= \exp[-\beta w] = \exp(-\beta w) \exp(-\beta\pi_x) E(\exp(\beta x)) \exp(\beta\pi_x) = E\ \left(e^{\beta x}\right)$

$= \ \pi_x = \beta^{-1}\log\left[E\left(e^{\beta x}\right)\right] ---------(*)$

$= \pi_x = \dfrac{L}{\beta}\log M_x(\beta)$

$= \dfrac{1}{\beta}\ log\left[\exp\left(\beta\mu + \dfrac{1}{2}\beta^2\sigma^2\right)\right]$

$= \pi_x = \mu + \dfrac{1}{2}\beta\sigma^2 ------(A)$

$\pi_Z = \dfrac{1}{\beta}\log M_z(\beta)$

$\pi_Z = \frac{1}{\beta}\log\left(e^{\mu a\beta} + \frac{1}{2}\sigma^2 a^2\beta^2\right)\ =\ \mu a + \frac{1}{2}\sigma^2 a^2\beta^2 \neq a\pi_x.$

$(iii)\quad Y = X + C$

$X_Y = \pi_x + C$

$\mu\ (w) \leq \mu(w + \pi_x -\ E(x))$

$\mu(w) \leq \mu(w + \pi_Y - E(x) + C)$

$\mu(w) = E\left[\mu(w + \pi_y - Y)\right] = E[\mu(w + \pi_Y - x - c)] --- (1)$

Also by PZU for the claim X, we have

$\mu(w) = E[\mu(w + \pi_x - X)] -----------(2)$

From (1) & (2) on comparing

$\pi_Y - c = \pi_x$

$\pi_Y = \pi_x + c$

No. Rip off property

We know rip off property will hold if $x \leq x_m\ then\ \pi_x \leq x_m$

Now it is clear that if

$x \leq x_m$

$-x \geq x_m$

$= w + \pi_x - \ x \geq w + \pi_x - x_m - - - - - - (1) \begin{bmatrix} w \geq 0 \\ \pi_Y \geq 0 \end{bmatrix}$

For there by PZU

$\mu(w) = E[\mu(w + \pi_x - X)] - - - - - - - - - (2)$

Now, by the property of expectation if

$X \geq Y\ then, E(x) \geq E[Y] - - - - (3)$

From (1), (2) & (3)

$$\mu(w) = E[\mu(w + \pi_x - x)] \geq E[\mu(w + \pi_x - X_m)] > \mu(w + \pi_x - x_m)$$

$$(w, \pi_x \& x_m areallpureno.)$$

$= \ \mu(w) > \& \mu''(w) < 0$

$= \ \pi_x \leq w + \pi_x - x_m$

$= \pi_m \leq \ x_m$

In general the principle of utility is not additive but the exponential principle is.

From (*)

$\pi_{(x_1+x_2)} = \beta^{-1} \log E[exp\{\beta(x_1 + x_2)\}]$

$= \beta^{-1} \log E[exp\{\beta x_1\}] E[exp\{\beta x_2\}]$

$= \beta^{-1} \log E[exp\{\beta x_1\}] + \log \beta^{-1} E[exp\{\beta x_2\}]$

$= \pi_{x_1} + \pi_{x_2}$

Where, the second live followed by the independence of $x_1 and\ x_2$
The principle of zero utility is not scale invariant,

- **Particular Case of PZU**

Exponential Principle:

The exponential principle set:

$$\mu(w) = E[\mu(w + \pi_x - x)] - - - - - - - - - - - - - (*)$$

Where $\mu(w) = -\exp[-\beta w]$& w is the insurer's surplus. We already know that (*) yields

$$\pi_x = \frac{1}{\beta}\log\big(M_x(\beta)\big) - - - - - - - - - - (**)$$

Where $M_x(.)$ is the mgf of x.

1. $\pi_x \geq E(x)$

$$\frac{1}{\beta}log\big(M_x(\beta)\big) \geq E(x)$$

$$\frac{1}{\beta}\log E\left(e^{\beta x}\right) \geq E(x)$$

$$= \pi_x \geq E(x)$$

$$e^{\beta x} \geq \beta x$$

$$E\left[e^{\beta x}\right] \geq E(\beta x)$$

$$\log M_x(\beta) \geq \beta E(x)$$

$$= \frac{1}{\beta}logM_x(\beta) \geq E(x)$$

2. Consistency

Y=X+C

$$\pi_Y = \pi_x + C$$

$$e^{\beta Y} \geq \beta Y$$

$$= e^{\beta(x+c)} \geq \beta(x+c)$$

$$= \frac{1}{\beta}logM_x(\beta) + C$$

$$= \pi_x + C$$

$$\pi_Y = \frac{1}{\beta}log\left(M_y(\beta)\right)$$

$$= \frac{1}{\beta}logE\left(e^{\beta Y}\right)$$

$$= \frac{1}{\beta}logE\left(e^{\beta x+\beta c}\right)$$

$$= \frac{1}{\beta}\log\left(E\left(e^{\beta x}\right).e^{\beta c}\right)$$

$$= \frac{1}{\beta}\left(logE\left[e^{\beta x}\right] + loge^{\beta c}\right)$$

3. Additive

$$\pi_{x_1} = \frac{1}{\beta}logM_{x_1}(\beta)$$

$$\pi_{x_2} = \frac{1}{\beta}logM_{x_2}(\beta)$$

$$\pi_{x_1+x_2} = \frac{1}{\beta}logM_{(x_1+x_2)}(\beta)$$

$$= \frac{1}{\beta}log\left(e^{\beta(x_1)}.e^{\beta(x_2)}\right)$$

$$\pi_{(x_1+x_2)} = \frac{1}{\beta}logM_{x_1}(\beta) + \frac{1}{\beta}logM_{x_2}(\beta)$$

4. Scale is invariant

$$Z = ax \quad = \pi_Z = a\pi_x$$

$$\pi_Z = \frac{1}{\beta} log M_z(\beta)$$

$$= \frac{1}{\beta} log\left[E\left(e^{\beta ax}\right)\right]$$

$$= \frac{1}{\beta} log\left[E\left(e^{\beta xa}\right)\right] = \frac{1}{\beta} log M_x(a\beta)$$

$$\neq a \, \frac{1}{\beta} log M_x(\beta)$$

$$\neq a\pi_x$$

Remark: Although PZU in general does not satisfy the additive property, its particular case i.e. the exponential principle satisfies this property. This is because in the exponential the premium has a well define form & does not depend on the initial wealth W, while for other utility function like the quadratic utility on initial wealth W & does not have a compact form like for the exponential function.

3.4.6 The Esscher Premium Principle

The Esscher premium principle sets: -

$$\pi_x = \frac{E[Xe^{hx}]}{E[e^{hx}]}, where\ h > 0.$$

In fact, we can interpret the Esscher premium as being the pure premium for a risk, $\tilde{X}$ (where by pure premium, we mean premium based on the first principle). For this, we define $\tilde{X}$based on X as follows:

Suppose that X is a continuous random variable on (0,∞) with density function f & we define function g by:-

$$g(x) = \frac{e^{hx} f(x)}{\int_0^\infty e^{hy} f(y)\, dy}$$

Then, g is the density function of a random variable$\tilde{X}$which has distribution function G(x) given as:

$$G(x) = \int_0^x g(w) du \; = \int_0^x \frac{e^{hx} f(x) du}{\int_0^\infty e^{hy} f(y)\, dy}$$

$$G(x) = \frac{1}{\int_0^\infty e^{hy} f(y)\, dy} \int_0^x e^{hu} f(u)\, du.$$

The distribution function G is known as Esscher transform of F with parameter h, where F (.) is CDF of the random variable. To summarize we are performing the following transformation:

$$X \to \tilde{X} \; (using\ the\ pdf\ of\ X)$$

i.e. we transform f(x) to g(x) by weighing f(x) by e^{hx}& density by

$$\int_0^\infty e^{hy} f(y) dy$$

This mean that we also transform the CDF of x i.e. F(x) to G(x). since by def. of CDF

$$G(x) = \int_0^x g(u) du = \int_0^x \frac{e^{hu} f(u)\, du}{\int_0^\infty e^{hy} f(y)\, dy}$$

$$= \frac{1}{\int_0^\infty e^{hy} f(y)\, dy} \int_0^x e^{hy} f(y)\, dy$$

i.e.

$$G(x) = \frac{\int_0^x e^{hy} f(y)\, dy}{M_x(h)}$$

Then G is called the Esscher transform of F. Further now the MGF of the random variable. $\tilde{X}$is given below:

$$M_{\tilde{X}}(t) = E\left[e^{\tilde{X}t}\right]$$

$$M_{\tilde{X}}(t) = \int_0^\infty e^{zt}, g(z) dz - - - - - (1)$$

Then, using the definition of g(z) in (1), we have

$$M_{\tilde{X}}(t) = \int_0^\infty e^{zt} \left(\frac{e^{hz} f(z)}{\int_0^x e^{hy} f(y) dy} \right) dz$$

$$= \frac{1}{M_{\tilde{X}}(h)} \int_0^\infty e^{z(h+t)} f(z) dz$$

$$= \frac{M_x{}^{(t+h)}}{M_x(h)}$$

$$\text{i.e. } M_{\tilde{X}}(t) = \frac{M_x{}^{(t+h)}}{M_x(t)}$$

i.e., we have also transformed the MGF of X into MGF of $\tilde{X}$

Thus, we can see that the premium which is set up by the Esscher principle which is

$$\pi_x = \frac{E[xe^{hx}]}{E[e^{hx}]} isinfact \pi_x = E[\tilde{X}]$$

$$E[\tilde{X}] = \int_0^\infty \mu . g(u)\, du$$

$$= \int_0^\infty u \left(\frac{e^{hu} f(u)}{\int_0^\infty e^{hy}\, f(y)dy} \right) du$$

$$= E[\tilde{X}] = \frac{E[xe^{hx}]}{E[e^{hx}]}$$

Example 1: Suppose that X$\sim \exp(\lambda)$ calculate F(x) & the Esscher transform of x with parameter h s.t. h <λ.

Solution: X$\sim \exp(\lambda)$, $f(x) = \lambda e^{-\lambda x}$

$$F(x) = 1 - e^{-\lambda x}$$

Now, Esscher transform of x is then G (x) be distribution function PDF$\tilde{X}$& g(x) be its p.d.f.

$$g(x) = \frac{e^{hx} f(x)}{\int_0^\infty e^{hx} f(x)dx} = \frac{\lambda\, e^{x(h-\lambda)}}{\lambda \int_0^\infty e^{x(h-\lambda)} dx} = \frac{\lambda e^{-x(h-\lambda)}}{\frac{-\lambda}{(\lambda-h)} [e^{x(h-\lambda)}]_0^\infty} = (\lambda - h)e^{-x(h-\lambda)}$$

$\tilde{X} \sim \exp(\lambda - h)$.

$$G(x) = \frac{\int_0^\infty e^{hx} f(x)dx}{M_x(h)} = \frac{\lambda \int_0^\infty e^{x(h-\lambda)} dx}{\lambda/\lambda - h} = \frac{-1}{\lambda - h} \frac{\left[e^{-x(h-\lambda)} - 1\right]}{(\lambda - h)}$$
$$= (\lambda - h)^{-2}\left[1 - e^{-x(h-\lambda)}\right].$$

Example 2: For the same Question. Suppose λ =1 now calculate the premium using the Esscher principle. Also draw the graph depicting the values of f & g v/s the values of X.

Solution:

$$\pi_x = E[\tilde{X}] = \frac{1}{\lambda - h}$$

$\lambda - h,\ \pi_x = \frac{1}{\lambda - h}$

3.4.6.1 The Esscher Transform

Under Esscher transform, we have talked about transforming the CDF F into the CDF G.

Looking more closely at the density transformation, we see that the density g is in fact a weighted version of the density function, since we can write

$$g(x) = w(x)f(x) - - - - - - - - - - - - - - - - (1)$$

Where,

$$w(x) = \frac{e^{hx}}{M_x(h)} - - - - - - -(2)$$

Also, as $h > 0,\ w'(x) = \frac{he^{hx}}{M_x(h)} > 0 - - - - - - - - (3)$

i.e.

W(x) is an increasing f^n of x & so increasing weight attaches as the value of x increases. Now let's under a claim,

$$X \sim exp(x)$$

Then,

$$M_x(t) = \frac{\lambda}{(\lambda - t)} - - - - - - -(4)$$

$$M_{\tilde{X}}(t) = \frac{\lambda}{(\lambda - h - t)} - - - - - - -(5)$$

Esscher Transform of F is given by

$$G(x) = L - exp\{-(\lambda - h)x\}$$

Let's take

h= 0,2;λ =1

then

$$f(x) = e^{-x}\ ; \quad w(x) = 0.8\, e^{0.2x}$$

$$g(x) = 0.8\ e^{0.2x},\ \ x > 0$$

Example: Suppose that an insurer faces claim ~ Poisson (λ)calculate:

1) The Esscher transform of the CDF of the claim with parameter h.

2) Calculate the premium that the insurer will change based on the Esscher principle.

Solution:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad\qquad mgf - e^{u(e^t-1)}$$

$$g = \frac{e^{hx}f(x)}{\int_0^\infty e^{hx}f(x)dx} = \left(e^{\lambda}\lambda\right)^x$$

$$M_{\tilde{X}}(t) = \frac{e^{\mu\left(e^{t+h}-1\right)}}{e^{\mu(e^t-1)}} = \frac{e^{\mu\left(e^t.e^h-1\right)}}{e^{\mu(e^t-1)}} = e^{\mu}(e^te^h - 1 - e^t + 1)$$

$$= e^{\mu}(e^te^h - e^t) = e^{\mu}\Big(e^t(e^h - 1)\Big)$$

$$g(x) = \frac{e^{hx}e^{-\lambda}\lambda^x}{x!\,M_x(h)} = \frac{e^{-\lambda+hx}\lambda^x}{x!\ e^{\lambda(e^h-1)}}$$

$$= \frac{e^{\lambda+hx-\lambda e^h+\lambda}\lambda^x}{x!} = \frac{e^{hx-\lambda e^h}.\lambda^x}{x!} = \frac{e^{hx}e^{-\lambda e^h}\lambda^x}{x!} = \frac{\left(\lambda e^h\right)^x e^{-\lambda e^h}}{x!}$$

$$\therefore \tilde{X} \sim Pois(\lambda^{eh})$$

$$\pi_x = E[\tilde{X}] \qquad\qquad \pi_x = \lambda e^h$$

To check whether Esscher principle satisfies the five properties:

1. **Non-negativity:**

$$E[\tilde{X}] = \pi_x$$

$$M_x(0) = E[e^{tx0}] = 1.$$

Case 1: h = 0

$$M_{\tilde{X}}(t) = \frac{M_x(t+h)}{M_x(h)} = \frac{M_x(t)}{M_x(0)} = M_x(t) - - - - - (1)$$

$$= \pi_x = E[\tilde{X}] = E[X] - - - - - - - - - - - - - (2)$$

now

Case 2: h > 0

$$E[\tilde{X}^r] = \frac{d^r}{dt^r} M_{\tilde{X}}(t)\Big]_{t=0} \quad = \left|\frac{d^r}{dt^r}\left[\frac{M_x(t+h)}{M_x(h)}\right]\right|_{t=0} - - - - - - -(3)$$

$$\frac{1}{M_x(h)} = \frac{d^r}{dt^r} M_{\tilde{X}}(t)\Big]_{t=0} = \frac{1}{M_x(h)} {M_x}^{(r)}(h) - - - - - - - - -(4)$$

and so,

$$\frac{d}{dh}\pi_x = \frac{d}{dh}E[\tilde{X}] = \frac{d}{dh}\frac{M_x'(h)}{M_x(h)}(put r = 1)$$

$$= \frac{{M_x}^{(2)}(h)}{\left(M_x(h)\right)^2} - \left[\frac{M_x'(h)}{M_x(h)}\right]^2 = E[\tilde{X}^2] - \left(E[\tilde{X}^2]\right)^2 = V\left(\tilde{X}\right)$$

$$\therefore \frac{d}{dh}\pi_x \geq 0$$

$\pi_x$is a non – decreasing function of h.

$$= \pi_x \geq E[X] \qquad\qquad h \geq 0$$

2) **Consistency:**

(Take Y= X+C) Calculate π_Y

$$\pi_x = \frac{E[xe^{hx}]}{E[e^{hx}]}(by\, esscher\;\; principle)$$

$$\pi_y = \frac{E[ye^{hy}]}{E[e^{hy}]} = \frac{E(x+c)e^{h(x+c)}}{E[e^{h(x+c)}]}$$

$$\frac{E\left[xe^{h(x+c)} + ce^{h(x+c)}\right]}{E[e^{hx}.e^{hc}]}$$

$$\frac{e^{hc}E[e^{hx}]+c.E[e^{hx}.e^{hc}]}{e^{hc}E[e^{hx}]}=\frac{E[xe^{hx}]}{E[e^{hx}]}+c$$

$\pi_y = \pi_x + c$

3) **Additive Property:**

Take Z= $X_1 + X_2$, s.t. x_1& x_2 are independent.

$$\pi_{x_1+x_2} = \frac{E\left[(x_1+x_2)e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]} = \frac{E\left[x_1 e^{h(x_1+x_2)} + x_2 e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]}$$

$$=\frac{E\left[x_1 e^{h(x_1+x_2)}\right]E\left[x_2 e^{h(x_1+x_2)}\right]}{E[e^{h(x_1+x_2)}]}$$

$$=\frac{E[x_1 e^{hx_1}].E[e^{x_2 h}]+E[x_2 e^{hx_2}]\,E[e^{hx_1}]}{E[e^{hx_1}]\,E[e^{hx_2}]}$$

$$=\frac{E[x_1 e^{hx_1}]}{E[e^{hx_1}]}+\frac{E[x_2 e^{hx_2}]}{E[e^{hx_2}]}$$

$= \pi_{x_1} + \pi_{x_2} = \pi_{x_1+x_2}$

4) Scale Invariance:

Define u= ax

$$\pi_u = \frac{E[ue^{hu}]}{E[e^{hu}]} = \frac{E[axe^{hax}]}{E[e^{hax}]} (theequality\ holdswhen, a = 1)$$

$$a = \frac{E(xe^{hax})}{E(e^{hax})}$$

$$= \pi_u \neq a\pi_x \qquad (a \neq 1)$$

5) **No Rip off Property:**

$x \leq\ x_m$

$E[\tilde{X}] \leq x_m\ \ or\ \pi_x \leq\ x_m\ \leftarrow To\ proove$

$x \leq\ x_m$

$xe^{hx} \leq x_m e^{hx}$

$E[xe^{hx}] \leq x_m E[e^{hx}]$

$= \dfrac{E[xe^{hx}]}{E[e^{hx}]} \leq x_m [e^{hx} \geq 0 = E[e^{hx}] \geq 0]$

$\pi_x \leq x_m$

3.4.7 The Risk Adjusted Principle

Let X be a non-negative random variable with distribution function F. The risk adjusted premium principal sets:

$$\pi_x = \int_0^\infty [P(X > x)]^{1/f} dx = \int_0^\infty [1 - F(x)]^{1/f} dx$$

where, $f \geq 1$ is known as **Risk Index**.

The basis of this is very similar to that of the Esscher principle is based on a transform called Esscher transform, such that this Esscher transform weight the distribution of X giving increasing weight to right tail probabilities.

Similarly, the risk adjusted premium is also based on transform defined as follows-

Define the distribution function H of a non-negative random variable x* by

$1 - H_{x^*}(x) = [1 - F(x)]^{1/f}$

When, $f \geq 1$, is the risk- index.

Now, we are actually transforming

$$X \to X^*$$

By transforming F(.) → H()

Now, by definition of expectation

$E[X^*] = \int_0^\infty [1 - H(x)]\, dx - - - - - - - - - - - -(1)$

$= \int_0^\infty [1 - F(x)]^{1/f}\, dx - - - - - - - - - - - - - (2)$

$\pi_x - - - - - - - - - - - - - - - -(3)$

Example: Suppose an incurs claim by exponential (λ) calculate

1) The risk adjusted transform of the distribution of the claims,

2) The risk adjusted premium π_x.

Solution:

1) $X \sim \exp(\lambda)$ $\qquad\qquad F(x) = 1 - e^{-\lambda x}$

$$1 - H(x) = [1 - F(x)]^{\frac{1}{f}} = \left[e^{-\lambda x}\right]^{1/f}$$

$$H(x) = 1 - e^{-\lambda x/f} = X^* \sim \exp(\lambda/f)$$

2) $E(x^*) = f/\lambda$

$$\pi_x = f/\lambda$$

Example: Suppose that an insurer incurs claims according to pareto distribution with parameter α &λ.

1) Find the Risk adjusted transform of distribution of claims.
2) The Risk adjusted premium π_x.
3) Do you have any condition under which premium will lead do a valid answer. If yes, mention the condition.

Solution:

1) $F(x) = 1 - \left(\frac{\lambda}{\lambda+x}\right)^{\alpha}$

$$1 - H(x) = [1 - F(x)]^{\frac{1}{f}} = \left(\frac{\lambda}{\lambda+x}\right)^{\alpha/f}$$

$$H(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^{\alpha/f}$$

$$X^* \sim Pareto\left(\frac{\alpha}{f}, \lambda\right)$$

2) $E(x^*) = \int_0^{\infty} \left(\frac{\lambda}{\lambda+x}\right)^{\frac{\alpha}{f}} dx, \;\; \lambda + x = y = dx = dy$

$$= \lambda^{x/f} \int_0^{\infty} (\lambda + x)^{-\alpha/f} \text{ dx}$$

$$= \lambda^{\alpha/f} \int_{\lambda}^{\infty} y^{-\alpha/f} \;\; dy \;=\; \lambda^{\alpha/f} \left[\frac{y^{-\frac{\alpha}{f}+1}}{-\frac{\alpha}{f}+1}\right]_{\lambda}^{\infty} = \lambda^{\alpha/f} \left[\frac{-\lambda^{-\frac{\alpha}{f}+1}}{1-\alpha/f}\right] = \frac{-\lambda^{-\frac{\alpha}{f}} \lambda^{-\frac{\alpha}{f}} . \lambda}{1-\alpha/f}$$

$$= \frac{-\lambda}{1-\alpha/f} = \frac{-f\lambda}{f-\alpha} = \frac{f\lambda}{\alpha-f}$$

For $\pi_x$to be positive, $f < \alpha$ (provided) ← condition

3.4.7.1 General Risk Adjusted Principle

In particular, if the random variable X is a continuous with CDF $F_x(x)$, then the transformed rv X^* is also a continuous rv with CDF H_X^* (y) & pdf

$$h_{x^*}(y) = \frac{1}{f}[1 - F_x(y)]^{\frac{1}{f}-1} * f_x(y) ------(*)$$

This means that just like Esscher transform, the density function of rv x* is simply a weight version of the density function of x. The corresponding weights are

$$w(x) = \frac{1}{f}[1 - F(x)]^{\frac{1}{f}-2}$$

Now, we check w(x)as a f^n of X.

$$w'(x) = \frac{1}{f}\left(\frac{1}{f} - 1\right)[1 - F(x)]^{\frac{1}{f}-2}(-f_x(x))(f > 1)$$

∴W(x) is an increasing function of x.

Suppose that is pareto example, α =2, λ=1 & f= 1.5, depict the densities' function f & h on the same graph & check whether the statement w(x) is an increasing function of x is justified.

If X~ Pareto (α, λ)

$$f(x) = \frac{(\alpha\lambda)}{(\lambda + x)\alpha + 1}(for\ \alpha = 2, \lambda = 1\ \& f = 1.5)$$

$$f(x) = \frac{4}{(1+x)^3} = 4(1+x)^{-3}$$

and we see that x* ~ Pareto $\left(\frac{\alpha}{f}, \lambda\right)$

$$f_x(x) = \frac{\left(\frac{\alpha}{f}.\lambda\right)^{\alpha/f}}{(\lambda + x)^{\frac{\alpha}{f}+1}}$$

$$\frac{1.467}{(1+x)^{2.33}} = 1.467(1+x)^{-2.33}$$

To check which properties Risk Adjusted Principle Satisfies-

1) $\pi_x \geq E[x]$

We know that

$$E[x] = \int_0^\infty \left(1 - F(x)\right) dx - - - - - - - - - (1)$$

$$= \left(1 - F(x)\right).x]_0^\infty - \int_0^\infty -f(x).x\,dx = (0-0) + \int_0^\infty f(x)\,x\,dx = \int_0^\infty f(x)\,x\,dx$$

Now,

$$\int_0^\infty \left(1 - H(x)\right)dx = \int_0^\infty [1 - F(x)]^{1/\delta}dx - - - - - (2)$$

When δ =1

$\pi_x = \int_0^\infty \left(1 - F(x)\right)dx(\delta \geq 1) = E[x]$

$[1 - F(x)]^{1/\delta} \geq \left(1 - F(x)\right)\left(from\ (1)\ \&\ (2)\right)$

$\int_0^\infty [1 - F(x)]^{1/\delta} \geq \int_0^\infty \left(1 - F(x)\right)dx$

$\pi_x \geq E[x]$

2) Scale Invariance

Y= ax

$\pi_y = \int_0^\infty [1 - F(y)]^{1/\delta}dy = \int_0^\infty [1 - F(ax)]^{1/\delta}\,dx$

$= \int_0^\infty [P\ (Y < y)]^{1/\delta}\,dy = \int_0^\infty [P\ (ax < y)]^{1/P}\,dy$

$= \int_0^\infty [P\ (x < y/a)]^{1/\delta}\,dy = \int_0^\infty [1 - F_x(Y/a)]^{1/\delta}\,dy$

$F(y)$

$= P\ (Y < y)$

$= P\ (ax < y)$

$= P\ (x < y/a)$

$= F_x(Y/a)$

$Let\ \dfrac{y}{a} = x \quad = dy = adx$

$\pi_y = \int_0^\infty [1 - F_x(x)]^{1/\delta}adx = a\,\pi_x.$

3) Consistency

Y= x+c

$$\pi_y = \int_0^\infty [1 - F(y)]^{1/\delta}dy$$

$= \int_0^\infty [P\ (Y > y)]^{1/\delta}\,dy = \int_0^\infty \left(P\ (x + c > y)\right)^{1/\delta}\,dy = \int_0^\infty [P\ (x > y - c)]^{1/\delta}\,dy$

$y - c = x$

$dy = dx$

$$\pi_y = \int_{-c}^{\infty} \left(P - (x > x)\right)^{1/\delta} dx$$

$$= P(Y > x) = \begin{cases} 1, & x > c \\ 1 - F_x(y - c), & x \geq c \end{cases}$$

$$\pi_y = \int_0^c (1)^{1/\delta} dx + \int_c^{\infty} \left(1 - F_x(y - c)\right) dy \begin{cases} y - c = x \\ dy = dx \end{cases}$$

$$\pi_y = c + \int_c^{\infty} \left(1 - F_x(x)\right)^{1/\delta} dx$$

$\pi_y = c + \pi_x$

From (*)

$$\int_{-c}^{0} \left(P.(X > x)\right)^{1/\delta} dx + \int_0^{\infty} \left(P(X > x)\right)^{1/\delta} dx$$

we know that

$P(X > x) = 1 \qquad for \qquad x < c$

$1 - F_x(x) = 1\,, \qquad\qquad x < c$

$$= \pi_y = \int_{-c}^{0} 1 \quad dx + \pi_x$$

$\pi_y = C + \pi_x$

4) No Rip –off Property

If $x = x_m$

Then $\pi_y = x_m ----To\ prove$

Where x_m is the maximum value that X can assume.

$$\pi_y = \int_0^{\infty} [1 - F(ax)]^{\frac{1}{\delta}} dx$$

$x \geq x_m$

$E(x) \leq x_m$

$$= \int_0^{\infty} \left(1 - F_x(x)\right) d_x \leq x_m = \int_0^{x_m} \left(1 - F_x(x)\right) dx$$

$$\pi_x = \int_0^{x_m} \left(1 - F_x(x)\right)^{1/\delta} dx - - - - (1)$$

Further,

$1 - F_x(x) \leq 1$

$$\pi_x = \int_0^{x_m} \left(1 - F_x(x)\right)^{1/\delta} dx \leq \int_0^{x_m} 1 \; dx$$

$\pi_x = x_m$

5) Additive Property

RAP does not satisfy the additive property & to show this we take a simple counter example: Let x_1& x_2 be two identical discrete rv such that

$$P(x_1 = 0) = P\ (x_1 = 1) = 0.5$$

Let the risk index be $\delta = 2$.

Calculate $\pi_{x_1}, \pi_{x_2}\ and\ \pi_{x_1+x_2}$

$\pi_{x_1} = E[x_1^*]$

$1 - H(x) = \left(1 - F(x)\right)^{1/\delta}$

$$E[x] = \sum_{x=0}^{1} \left(1 - F(x)\right)^{\frac{1}{\delta}}$$

$$= \left(1 - F(0)\right)^{1/\delta} + \left(1 - F(1)\right)^{1/\delta} = (1 - 0.5)^{1/2} = (0.5)^{1/2} = 0.707$$

Similarly,

$\pi_{x_2} = (0.5)^{1/2} = 0.707$

$$P(Y \leq y) \ = \begin{cases} 0.25, & y < 0 \\ 0.75, & 0 \leq y < 1 \\ 1 & y < 1 \end{cases}$$

$y = x_1 + x_2 \rightarrow 0,1,2.$

$\pi_{x_1+x_2} = 0.5\left(1 + 3^{1/2}\right) = 1.366.$

$$\pi_{x_1+x_2} = E(X) = \sum_{x=0}^{2} \left(1 - F(x)\right)^{1/\delta} dx. = \pi_{x_1} + \pi_{x_2} > \pi_{x_1+x_2}$$

3.5 Selecting the Correct Premium Principle

When determining which premium principle an insurer should use, it is important to recognize that there is no universally correct answer. Each premium principle has its own set of desirable properties and suitability for different types of risks and market conditions. The decision on which principle to use should be guided by a combination of mathematical considerations and practical relevance to the specific risk being insured.

Key Considerations in Selecting a Premium Principle:

1. Risk Profile: The nature and variability of the risk play a crucial role in determining the appropriate premium principle. For example, high-variability risks such as natural disasters might benefit from principles that account for variance or standard deviation, while more predictable risks might be better suited to simpler principles like the Expected Value Principle.

2. Desirable Properties: Some principles possess properties that are particularly advantageous in certain contexts. For instance, principles that incorporate measures of dispersion (variance, standard deviation) are useful for high-risk scenarios, while the Exponential Principle is beneficial for scenarios with significant risk aversion.

3. Regulatory and Market Considerations: Insurers must consider regulatory requirements and market conditions. Some regions or industries may have specific guidelines that favor certain principles. Additionally, competitive dynamics in the market might influence the choice of principle to ensure premiums are competitive yet sufficient to cover risks.

4. Administrative and Practical Feasibility: The complexity of calculating premiums using a particular principle can influence its practicality. Insurers must balance the precision of a premium principle with the administrative costs and feasibility of its implementation.

- ### Examples of Contextual Suitability

Life Insurance and Annuities: The Equivalence Principle is often used because it ensures long-term balance between premiums collected and benefits paid out, which is crucial for products with long durations and predictable payouts.

General Insurance: The Expected Value Principle might be suitable for lines with high claim frequency and lower variability, providing simplicity and efficiency in premium calculation.

Catastrophe Insurance: The Variance Principle or Standard Deviation Principle may be appropriate due to their consideration of loss variability and uncertainty, essential for high-severity, low-frequency events.

High-Risk Insurance: The Exponential Principle can be useful where the insurer's risk aversion towards large losses is significant, emphasizing the protection against extreme events.

Limited Data Scenarios: The Credibility Principle is valuable when historical data is sparse, allowing insurers to blend their own data with broader industry experience to derive a more reliable premium.

3.6 Summary

Selecting a premium principle is a nuanced decision that involves evaluating the specific properties of each principle in the context of the risk being insured. Insurers should identify which properties are most relevant to the given risk, such as predictability, variability, and regulatory requirements, and choose a principle that best satisfies these criteria. By doing so, they can ensure that the premiums charged are both fair and sufficient to cover the expected costs, while also being competitive in the market.

3.7 References

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

3.8 Further Readings

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

UNIT: 4 THE COLLECTIVE RISK MODEL

Structure

4.1 Introduction

In this unit, the focus is on the total claims from a general insurance risk over a short period, typically one year. The term "risk" is used to describe a group of similar policies, although it can also apply to a single policy. It was indicated in Chapter 1 that, at the start of an insurance period, the insurer does not know how many claims will occur or the amounts of these claims. Therefore, it is necessary to construct a model that accounts for these two sources of variability. Claims over a one-year period are considered for ease of presentation, but any time unit can be used.

It is stated that aggregate claims are modeled as a random variable S in Section 4.2, with expressions derived for its distribution function and moments. There is also consideration of the significant case when S follows a compound Poisson distribution, and an important result concerning the sum of independent compound Poisson random variables is given. Section 4.4 discusses the effect of reinsurance on aggregate claims from both the insurer’s and reinsurer’s perspectives.

The unit continues with addressing the practical question of calculating an aggregate claims distribution. Section 4.5 introduces certain classes of counting distributions for the number of claims from a risk. The importance of these classes is highlighted because assuming individual claims under the risk are modeled as discrete random variables, it is then possible to calculate the probability function for aggregate claims recursively. The chapter concludes by describing two methods for approximating an aggregate claims distribution.

4.2 Objectives

By the end of this lesson, learners will be able to:

- Understand the structure and properties of the aggregate claims model.
- Analyze the distribution and moments of the total claim amount (S).
- Apply the Compound Poisson Distribution in modelling aggregate claims.
- Evaluate the impact of different reinsurance strategies on the aggregate claims distribution.
- Utilize recursive methods, including the Panjer recursion formula, for computing aggregate claims distributions.

4.3 The Model

The random variable S represents the total amount of claims arising from a risk over one year.

Let N be the random variable indicating the number of claims from the risk during the year, and let X_i be the random variable representing the amount of the i-th claim. The aggregate claim amount is simply the sum of the individual claim amounts, expressed as:

$$S = \sum_{i=1}^{N} X_i$$

with the understanding that $S = 0$ when $N = 0$ (if there are no claims, the aggregate claim amount is zero). Throughout this unit, individual claim amounts are modeled as non-negative random variables with a positive mean.

Two key assumptions are made:

- it is assumed that $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables.
- it is assumed that the random variable N is independent of $\{X_i\}_{i=1}^{\infty}$.

Typically, the risk in question is a portfolio of insurance policies, which is why the term "collective risk model" is used. This approach examines the risk as a whole, concentrating on the total number of claims from the entire portfolio instead of from individual policies.

4.3.1 The Distribution of S

Let us define the notations initially.

Let $G(x) = \Pr(S \leq x)$ denote the distribution function of aggregate claims,

$F(x) = \Pr(X_1 \leq x)$ denote the distribution function of individual claim amounts, and

Let $p_n = \Pr(N = n)$ so that $\{p_n\}$ n ∞ is the probability function for the number of claims.

We can derive the distribution function of S by noting that the event $\{S \leq x\}$ occurs if n claims occur, n = 0, 1, 2,... , and if the sum of these n claims is not more than x . Thus, we can represent the event $\{S \leq x\}$ as the union of the mutually exclusive events $\{S \leq x$ and $N = n\}$, so that

$$\{S \leq x\} = \cup_{n=0} \{S \leq x \text{ and} N = n\}$$

So that,

$$inf$$

$$\overline{\underline{inf}}$$

$$G(x) = Pr(S \leq x) = \sum_{n=0}^{\infty} Pr(S \leq x \text{ and} N = n)$$

Now,

$$Pr(S \leq x \text{ and } N = n) = \Pr(S \leq x | N = n) \Pr(N = n)$$

And

$$\Pr(S \leq x | N = n) = \Pr\left(\sum_{i=1}^{n} X_i \leq x\right) = F^{n*}(x)$$

Thus, when $x \geq 0$,

$$G(x) = \sum_{n=0}^{\infty} p_n F^{n*}(x) \quad ------------ (4.1)$$

We know that $F^{0*}(x)$ is defined to be 1 for $x \geq 0$, and zero otherwise.

The equation 4.1 gives us the means of calculating the aggregate claims distribution. However, the convolution F^{n*} may not exist in a closed form for all individual claim distributions of practical interest like Pareto and log-normal. Even in cases when a closed form does exist, the distribution function in equation (4.1) still has to be evaluated as an infinite sum.

By an analogous argument, in the case when individual claim amounts are distributed on the positive integers with probability function

$$f_j = F(j) - F(j-1)$$

For, j=1,2, 3…, the probability function $\{g_x\}_{x=0}^{\infty}$ of S is given by $g_0 = p_0$, and $x = 1,2,3, ...$,

$$g(x) = \sum_{n=0}^{\infty} p_n f_x^{n*} \quad ------------ (4.2)$$

Where $f_x^{n*} = \sum_{i=1}^{n} X_i = x$

The formula 4.2 is not more useful than 4.1. However for certain distributions N, $g(x)$ can be calculated recursively for x = 1, 2, 3,… using g0 as the initial value for the recursive calculation which is discussed in detail later.

4.3.2 The Moments of S

The moments and m.g.f. of S can be calculated using conditional expectation arguments. The key results are that for any two random variables Y and Z for which relevant moment exist,

$$\mathrm{E[Y] = E[E(Y|Z)]} ------- (4.3)$$

and

$$\mathrm{V[Y] = E[V(Y|Z) + V[E(Y|Z)]} ------- (4.4)$$

The immediate application of 4.3 we have,

$$\mathrm{E[S] = E[E(S|N)]}$$

Now let $\mathrm{m_k = E\left[X_1^k\right]}$ for $\mathrm{k = 1,2,3, ...}$ then

$$\mathrm{E[S|N = n] = E\left[\sum_{i=1}^{n} Xi\right] = \sum_{i=1}^{n} E[Xi] = nm_1}$$

And this holds for n=0, 1, 2,…., E[S|N] = Nm1 and hence

$$\mathrm{E[S] = E[Nm_1] = E[N]m_1} --------(4.5)$$

This is a very appealing result as it states that the expected aggregate claim amount is the product of the expected number of claims and the expected amount of each claim.

Similarly, using the fact that $\{X_i\}_{i=1}^{\infty}$are independent random variables,

$$V\,[S|N\ =\ n]\ =\ V\left[\sum_{i=1}^{n} Xi\right] = \sum_{i=1}^{n} V[Xi] = n(m_2 - m_1^2)$$

So that $V\,[S|N\,] = n(m_2 - m_1^2)$. Then by applying 4.4 we get,

$$V[S]\ =\ E\,[V\,(S|N)]\ +\ V\,[E\,(S|N)]$$

$$=\ E\,[N\,(m_2 - m_1^2)] +\ V\,[Nm_1]$$

$$=\ E\,[N](m_2 - m_1^2)\ \ +\ V[Nm_1^2] ------(4.6)$$

Formula (4.6) does not have the same type of natural interpretation as formula (4.5), but it does show that the variance of S is expressed in terms of the mean and variance of both the claim number distribution and the individual claim amount distribution. The same technique can be used to obtain the moment generating function of S.

We have,

$$M_s(t) = E[e^{tS}] = E\big[E(e^{tS}\big|N)\big],$$

and

$$E(e^{tS}\big|N = n) = E\left[\exp\left\{\sum_{t=1}^{n} X_i\right\}\right] = \prod_{i=1}^{n} E[\exp\{tX_i\}]$$

Where, we have again used the independence $\{X_i\}_{i=1}^{\infty}$.

Further as $\{X_i\}_{i=1}^{\infty}$ are identically distributed,

$$E(e^{tS}\big|N = n) = M_X(t)^n$$

Where $M_X(t)\ \ = E[\exp\{tX_1\}]$

This leads to,

$$M_s(t) = E[M_X(t)^N]\ = E[\exp\{\log M_X(t)^N] = E\big[\exp\{N\log M_X(t)\}\ \ \big] = M_N[\log M_x(t)]$$

Thus MS is expressed in terms of MN and MX.

Similarly, when X1 is a discrete random variable distributed on the non-negative integers with probability generating function PX, the above arguments lead to

PS(r) = PN [PX (r)]

Where, PS and PN are the probability generating functions of S and N respectively. In the above development of results, it has been tacitly assumed that all relevant quantities exist. However, we have seen before that moments and moment generating functions may exist only under certain conditions. Thus, for example, if the second moment of X1 does not exist, then the second moment of S does not exist either.

4.4 The Compound Poisson Distribution

When N has a Poisson distribution with parameter λ, we say that S has a compound Poisson distribution with parameters λ and F, and similar terminology applies in the case of other claim number distributions. Since the mean and variance of the P(λ) distribution are both λ, it follows from formulae (4.5) and (4.6) that when S has a compound Poisson distribution

$$E\,[S] \;=\; \lambda m_1$$

and

$$V[S] \;=\; \lambda m_2$$

Third Central Moment,

$$\mathrm{E}[(\mathrm{S}-\lambda \mathrm{m}_1)^3] = \lambda \mathrm{m}_3 \;------(4.8)$$

To derive formula (4.8), we can use the moment generating function of S which by equation (1.1) and formula (4.7) is

$$\mathrm{M_S(t)} = \exp\{\lambda(\mathrm{M_x(t)}-1)\}.$$

Differentiating with respect to t yields

$$\mathrm{M'_S(t)} = \lambda\big(\mathrm{M'_x(t)M_S(t)}\big)$$

$$M''_S(t) = \lambda\big(M''_x(t)M_S(t)\big)+\lambda\big(M'_x(t)M'_S(t)\big)$$

and

$$M'''_S(t) = \lambda\big(M'''_x(t)M_S(t)\big)+2\lambda\big(M''_x(t)M'_S(t)\big)+\lambda\big(M'_x(t)M''_S(t)\big) \qquad ----------\ (4.9)$$

Setting t=0 in (4.9) we get

$$E[S^3] = \lambda m_3 + 2\lambda m_2 E[S] + \lambda m_1 E[S^2]$$

$$= \lambda m_3 + 2V[S]E[S] + E[S]E[S^2]$$

$$= \lambda m_3 + 3E[S]E[S^2] - 2E[S]^3$$

Which, yields formula (4.8).

An important point about the compound Poisson distribution that will be relevant in the context of approximation methods discussed in Section 4.8 is that the coefficient of skewness is positive under our assumptions that $\Pr(X_1 < 0) = 0$ and $m_1 > 0$ (which gives $m_k > 0$ for k = 2, 3, 4,...), and hence

$$Sk[S] = \frac{E[(S - \lambda m_1)^3]}{V[S]^{3/2}} = \frac{\lambda m_3}{(\lambda m_2)^{3/2}}$$

Example 1: Let S have a compound Poisson distribution with Poisson parameter 100, and let the individual claim amount distribution be Pa(4, 1500). Calculate E[S], V[S] and Sk[S].

Solution:

When $X \sim Pa(\alpha,\beta)$,

$E[X] = \beta/(\alpha - 1)$ and $E[X^2] = 2\beta^2/(\alpha - 1)(\alpha - 2)$

And $E[X^3] = 6\beta^3/(\alpha - 1)(\alpha - 2)(\alpha - 3)$. Thus,

$$E[S] = 100 \times \frac{1500}{3} = 50000$$

$$V[S] = 100 \times \frac{2 \times 1500^2}{6} = 7.5 \times 10^7$$

And

$$E[(S - E[S])^3] = 100 \times 1500^3 = 1.5^3 \times 10^{11}$$

So that

$$Sk[S] = \frac{1.5^3 \times 10^{11}}{(7.5 \times 10^7)^{3/2}} = 0.5196$$

An important property of compound Poisson random variables is that the sum of independent, but not necessarily identically distributed, compound Poisson random variables is itself a compound Poisson random variable.

Formally, let $\{Si\}_{i=1}^{n}$ be independent compound Poisson random variables, with parameters λ_i and F_i, and let $S = \sum_{i=1}^{n} S_i$. Then S has a compound Poisson distribution with parameters Λ and F where $\Lambda = \sum_{i=1}^{n} \lambda_i$ and

$$F(x) = \frac{1}{\Lambda}\sum_{i=1}^{n} \lambda_i F_i(x) ------(4.10)$$

To prove this, we use the MGF of S,

$$E[\exp\{tS\}] = E[\exp\{t(S_1 + \cdots + S_n)\}] = \prod_{i=1}^{n} E[\exp\{tS_i\}]$$

Since $\{Si\}_{i=1}^{n}$ are independent. Now let M_i denote the moment generating function of a random variable whose distribution function is F_i ,then, as S_i has a compound Poisson distribution,

$$E[\exp\{tS_i\}] = \exp\{\lambda_i(M_i(t) - 1)\}$$

and hence

$$E[\exp\{tS\}] = \prod_{i=1}^{n} \exp\{\lambda_i(M_i(t) - 1)\}$$

$$= \exp\{\sum_{i=1}^{n} \lambda_i(M_i(t) - 1\ \ )\}$$

$$= \exp\{\Lambda\left(\sum_{i=1}^{n} \frac{\lambda_i M_i(t)}{\Lambda} - 1\right)\}$$

It then follows that S has a compound Poisson distribution by the uniqueness of a moment generating function and by the fact that a random variable whose distribution function is F has moment generating function

$$\int_0^\infty e^{tx} dF(x) = \frac{1}{\Lambda} \sum_{i=1}^{n} \lambda_i \int_0^\infty e^{tx} dF_i(x) = \sum_{i=1}^{n} \frac{\lambda_i M_i(t)}{\Lambda}$$

This is an important result that has applications not just in the collective risk model, but, also in individual risk model.

Example 2: Let S_1 have a compound Poisson distribution with Poisson parameter $\lambda_1 = 10$, and distribution function F_1 for individual claim amounts where $F_1(x) = 1 - e^{-x}$, $x \geq 0$. Let S_2 have a compound Poisson distribution with Poisson parameter $\lambda_2 = 15$, and distribution function F_2 for individual claim amounts where $F_2(x) = 1 - e^{-x}(1 + x)$, $x \geq 0$. What is the distribution of $S_1 + S_2$ assuming S_1 and S_2 are independent?

Solution: The distribution of $S_1 + S_2$ is compound Poisson by the above result, and the Poisson parameter is $\lambda_1 + \lambda_2 = 25$. From equation (4.10), the individual claim amount distribution is:

$$F(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} F_1(x) + \frac{\lambda_1}{\lambda_1 + \lambda_2} F_2(x)$$

$$= \frac{2}{5}(1 - e^{-x}) + \frac{3}{5}\left(1 - e^{-x}(1 + x)\right)$$

$$= (1 - e^{-x})(1 + \frac{3}{5}x)$$

4.5 The Effect of Reinsurance

In Unit 1, we discussed the concepts of both proportional and excess of loss reinsurance. In this section, we will delve deeper into the effects these reinsurance arrangements have on the distribution of aggregate claims. Reinsurance plays a crucial role in risk management by allowing insurers to share part of their risk with reinsurers, thus stabilizing their financial outcomes and reducing the impact of large claims.

Regardless of the type of reinsurance arrangement, the aggregate claim amount is divided between the insurer and the reinsurer. Therefore, we can express the total aggregate claims S as the sum of the insurer's share S_I and the reinsurer's share S_R. Specifically, S_I denotes the insurer’s aggregate claims after accounting for reinsurance, effectively representing the net amount of claims the insurer retains. On the other hand, S_R represents the reinsurer’s aggregate claim amount, which is the portion of the claims transferred to the reinsurer.

This division of claims affects how risks are distributed and managed between the insurer and the reinsurer. Proportional reinsurance involves sharing claims in a fixed proportion, while excess of loss reinsurance involves the reinsurer covering losses that exceed a specified threshold. Understanding these effects is essential for accurately modeling and managing aggregate claims distributions and for making informed decisions about reinsurance strategies.

4.5.1 Proportional Reinsurance

Under a proportional reinsurance arrangement with proportion retained ‘a’, the insurer pays proportion ‘a’ of each claim. Thus, the insurer’s net aggregate claim amount is:

$$S_I = \sum_{i=1}^{N} aX_i = aS$$

With $S_I = 0$ if S = 0. Similarly S_R = (1-a) S

Example: Aggregate claims from a risk have a compound Poisson distribution with Poisson parameter 100 and an individual claim amount distribution which is exponential with mean 1000. The insurer effects proportional reinsurance with proportion retained 0.8. Find the distribution of S_R

Solution: As the reinsurer pays 20% of each claim, S_R has a compound Poisson distribution with Poisson parameter 100 and an individual claim amount distribution that is exponential with mean 200.

Example: Aggregate claims from a risk are distributed with mean μ and standard deviation σ. The insurer of the risk has effected proportional reinsurance with proportion retained a.

Find Cov(S_I , S_R)

Solution: By definition,

$$Cov(S_I, S_R) \ = \ E[(S_1 - a\mu)(S_R - (1-a)\mu)]$$

Since $E[S_i] = a\mu$ and $E[S_R] = (1-a)\mu$. As S_I = aS and S_R = (1-a) S,

$$Cov(S_I, S_R) = E[a(S-\mu)(1-a)(S-\mu)]$$

$$= a(1-a)E[(S-\mu)^2] = \ a(1-a)\sigma^2$$

4.5.2 Excess of Loss Reinsurance

Let us assume that the insurer of a risk, S, has affected excess of loss reinsurance with retention level M. Then we can write

$$S_I = \sum_{i=1}^{N} min(X_i, M)$$

With $S_I = 0$ if N = 0, and

$$S_R = \sum_{i=1}^{N} max(0, X_i - M) \ -------- \ (4.11)$$

With $S_R = 0$ if N = 0

An important point to note is that S_R can equal 0 even if N is greater than zero. This situation would arise if n > 0 claims occurred and each of these n claims was for an amount less than M, so that the insurer would pay each of these claims in full. Thus, there are two ways of considering the reinsurer's aggregate claim amount. The first is as specified by equation (4.11), in which case the interpretation is that each time a claim occurs for the insurer, a claim also occurs for the reinsurer, and we allow 0 to be a possible claim amount. The second approach is to count only claims (for the insurer) whose amount exceeds M as these claims give rise to non-zero claim payments by the reinsurer. Thus, an alternative way of writing S_R is:

$$s_R = \sum_{i=1} \hat{X}_i$$

with S_R = 0 when N_R = 0. Here, N_R denotes the number of non-zero claims for the reinsurer, and $\hat{X}_i$denotes the amount of the ith claim payment by the reinsurer giving:

$$Pr(\hat{X}_i \leq x) = \frac{F(x+m) - F(m)}{1 - F(m)}$$

To find the distribution of N_R, we introduce the sequence of independent and identically distributed indicator random variables$\{I_j\}_{j=1}^{\infty}$, where I_j takes the value 1 if $X_j > M$, so that there is a non-zero claim payment by the reinsurer, and I_j takes the value 0 otherwise. Then,

$$Pr(I_j = 1) = Pr(X_j > M) = 1 - F(M) = \pi_M$$

And

$$N_R = \sum_{i=1}^{N} I_j$$

with $N_R = 0$ when $N = 0$. As N_R has a compound distribution, its probability generating function is

$$P_{NR}(r) = P_N[P_I(r)]$$

where PI is the probability generating function of each indicator random variable, and

$$P_I(r) = 1 - \pi_M + \pi_M r$$

Example: Let $N \sim P(\lambda)$. What is the distribution of N_R ?

Solution: As

$$P_N(r) = \exp\lambda(r - 1)$$

We have

$$P_{N_R}(r) = exp\lambda(1 - \pi_M + \pi_M r - 1)$$

$$= exp\{\lambda\pi_M(r - 1)\}$$

Hence $N_R \sim P(\lambda\pi_M)$ by the uniqueness property of probability generating function.

Example: Let $N \sim NB(k, p)$. What is the distribution of N_R?

Solution: As

$$P_N(r) = (\frac{p}{1 - qr})^k$$

Where, $q = 1-p$, we have

$$P_N(r) = (\frac{p}{1 - q(1 - \pi_M + \pi_M r)})^k$$

$$= (\frac{p}{1 + q\pi_M - q\pi_M r})^k \quad ---------- \quad (4.12)$$

Now let $p* = p/(p + q\pi_M)$ and $q* = q\pi_M/(p + q\pi_M) = 1 - p*$. Then division of both the numerator and the denominator inside the brackets in formula (4.12) by $p + q\pi_M$ yields

$$P_{N_R}(r) = (\frac{p*}{1 - q*r})^k$$

so that NR ~ N B(k, p∗)

4.6 Recursive Calculation of Aggregate Claims Distribution

In this unit, we derive the Panjer recursion formula, which allows for the recursive calculation of the aggregate claims distribution when individual claim amounts are non-negative integers and the number of claims distribution belongs to the (a, b, 0) class of distributions.

4.6.1 The (a, b, 0) class of distributions

A counting distribution is said to belong to the (a, b, 0) class of distributions if its probability function $\{p_n\}_{n=0}^{\infty}$, can be calculated recursively from the formula

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1} - - - - - - - - -(4.13)$$

For n= 1,2,3,…, where a and b are constants.

The starting value for the recursive calculation is p_0 which is assumed to be greater than 0, and the term '0' in (a, b, 0) is used to indicate this fact. There are exactly three non-trivial distributions in the (a, b, 0) class, namely, Poisson, Binomial and negative Binomial. To see this, we note that the recursion scheme given by formula (4.13) starts from

$$p_1 = (a + b)p_0$$

Hence, we require that a + b ≥ 0 since we would otherwise obtain a negative value for p_1. Members of the (a, b, 0) class can be identified by considering possible values for a and b, as follows. Suppose first that a + b = 0. Then $p_n = 0$ for n=1, 2, 3,… and as $\sum_{n=0}^{\infty} P_n \ = \ 1$, we see that p_0 must equal 1, so that the distribution is degenerate at 0. Secondly, let us consider the situation when a = 0. This gives $p_n = (b/n)\, p_{n-1}$ for n = 1, 2, 3,... so that

$$p_n = \frac{b}{n}\frac{b}{n-1} \cdots \frac{b}{2} b p_0 = \frac{b^n}{n!} p_0$$

And again, using the fact that $\sum_{n=0}^{\infty} P_n \ = \ 1$,

$$\sum_{n=0}^{\infty} p_n = \ p_0 \sum_{n=0}^{\infty} \frac{b^n}{n!} = p_0 e^b$$

giving $p_0 = e^{-b}$. Hence, when a = 0, we obtain the Poisson distribution with mean b. thirdly, we consider the situation when a > 0 and a ≠ −b, so that a + b > 0. Then by repeated application of formula (4.13) we have

$$p_n = (a+\frac{b}{n})(a+\frac{b}{n-1})\ldots(a+\frac{b}{2})(a+b)p_0$$

$$= \frac{(an+b)(a(n-1)+b)\ldots(2a+b)(a+b)}{n(n-1)\ldots2}p_0$$

$$= \frac{a^n}{n!}(n+\frac{b}{a})(n-1+\frac{b}{a})\ldots(2+\frac{b}{a})(1+\frac{b}{a})p_0$$

If we now write α for 1 + b/a, then we have

$$p_n = \frac{a^n}{n!}(n-1+\alpha)(n-2+\alpha)\ldots(1+\alpha)\alpha p_0$$

$$= \binom{\alpha+n-1}{n}a^n \mathrm{p}_0 \text{ ---------------------------(4.14)}$$

By the ratio test, we have absolute convergence if

$$\lim_{n\to\infty}\left|\frac{P_n}{p_{n-1}}\right| < 1$$

and as $p_n = (a + b/n)p_{n-1}$, we have absolute convergence if $|a| < 1$, and as we have assumed $a > 0$, this condition reduces to $a < 1$. Then

$$p_0 + p_0\sum_{n=1}^{\infty}\binom{\alpha+n-1}{n}a^n = 1$$

And for NB (k, p)

$$p^k\sum_{n=1}^{\infty}\binom{k+n-1}{n}q^n = 1-p^k$$

Where p + q = 1

Hence $p_0 = (1-a)^{\alpha}$ and the distribution of N is negative binomial with parameters s $1 - a$, where $0 < a < 1$, and $\alpha = 1 + b/a$.

The final case to consider is when $a + b > 0$ and $a < 0$. As $a < 0$, there must exist some positive integer k such that

$$a + \frac{b}{k+1} = 0$$

so that $p_n = 0$ for $n = \kappa + 1, \kappa + 2, \ldots$ If this were not true, then as $a < 0$ and $b > 0$, we would find that there would be a first value of n such that $a + b/n$ would be less than 0, generating a negative value for pn. Proceeding as in the third case above,

$$p_n = \frac{a^n}{n!}(n+\frac{b}{a})(n-1+\frac{b}{a})\ldots(2+\frac{b}{a})(1+\frac{b}{a})p_0$$

and as κ = −(1 + b/a), we can write this as,

$$p_n = \frac{a^n}{n!}(-k+n-1)(-k+n-2)\ldots(-k+1)(-k)p_0$$

$$= (-1)^n \frac{a^n}{n!}(k-n-1)(k-n+2)\ldots(k-1)(k)p_0$$

$$= (-1)^n \frac{a^n}{n!}(\frac{k!}{n!(k-n)!})p_0 = (-a)^n \binom{k}{n} p_0$$

We conclude our discussion of the (a, b, 0) class by considering the probability generating function of a distribution in this class, and deriving a result:

	a	**b**
$P(\lambda)$	0	λ
$B(n,q)$	$-\frac{q}{(1-q)}$	$\frac{(n+1)q}{(1-q)}$
$NB(k,p)$	$1-p$	$(1-p)(k+1)$

4.6.2 The Panjer Recursion Formula

The Panjer recursion formula is a significant result in risk theory. It is not only useful for calculating aggregate claims distributions but also has applications in ruin theory. The recursion formula enables us to compute the probability function of aggregate claims when the counting distribution belongs to the (a, b, 0) class and the individual claim amount distribution is discrete, with a probability function $\{f_j\}_{j=0}^{\infty}$,

Up to this point, we have implicitly assumed that individual claims follow continuous distributions, such as lognormal or Pareto. We have not yet considered discrete distributions for modeling individual claim amounts, and this topic will be discussed later. Additionally, it is worth noting that allowing $f_0 > 0$ is useful, even though a zero individual claim amount would not be considered a claim in practice.

The Panjer recursion formula is a powerful computational tool in actuarial science, primarily used for calculating the aggregate claims distribution when individual claim amounts follow a discrete distribution. This formula is particularly useful for insurers in various practical applications:

1. **Calculation of Aggregate Claims Distribution:** The Panjer recursion formula allows actuaries to efficiently compute the probability distribution of total claims over a fixed period, typically one year. By utilizing this method, actuaries can handle complex aggregate

claims models where individual claim amounts are discrete and the claim number distribution belongs to the (a, b, 0) class of distributions, such as the binomial, negative binomial, and Poisson distributions.

2. **Premium Setting and Risk Pricing:** Accurately determining the distribution of aggregate claims enables insurers to set premiums that are commensurate with the underlying risk. The Panjer recursion formula helps in deriving the expected value and variance of total claims, which are critical inputs for calculating risk-based premiums and ensuring that the insurer's pricing strategy is both competitive and sustainable.

3. **Reserving:** Insurance companies need to hold adequate reserves to cover future claims. The Panjer recursion formula assists in estimating the distribution of aggregate claims liabilities, which in turn informs the calculation of required reserves. By understanding the distribution and potential variability of total claims, insurers can better plan their capital allocation and maintain solvency.

4. **Reinsurance Strategy:** The formula is instrumental in evaluating the impact of different reinsurance arrangements on the aggregate claims' distribution. Insurers can use it to model the effect of proportional and excess-of-loss reinsurance treaties, allowing them to optimize their reinsurance purchases and manage their risk exposure more effectively.

5. **Risk Management and Solvency Analysis:** Regulatory frameworks often require insurers to demonstrate their ability to withstand adverse scenarios. The Panjer recursion formula helps actuaries perform stress testing and scenario analysis by providing a detailed understanding of the aggregate claims distribution under various conditions. This analysis supports risk management efforts and ensures compliance with regulatory solvency requirements.

6. **Experience Rating and Claims Management:** Insurers use the formula to analyze historical claims data and update their claims distribution models based on actual experience. This process, known as experience rating, enables insurers to refine their risk assessments and improve their claims management practices, leading to more accurate and fair premium adjustments for policyholders.

The Panjer recursion formula is a versatile and efficient tool in actuarial practice, enabling insurers to accurately model the distribution of aggregate claims. Its applications in premium setting, reserving, reinsurance strategy, risk management, and solvency analysis are crucial for maintaining the financial health and competitiveness of insurance companies. By leveraging this formula, actuaries can ensure that their models reflect the underlying risks and support informed decision-making in managing insurance portfolios.

4.7 Summary

The collective risk model is a foundational framework in actuarial science used to quantify and manage the total risk associated with a portfolio of insurance policies over a fixed period, typically one year. This model treats the aggregate claims amount as a random variable, S , which is the sum of individual claims X_i, where N is the random number of claims. The model incorporates the variability in both the frequency (number of claims) and severity (amount of each claim), assuming that X_i are independent and identically distributed (i.i.d) random variables, and N is independent of X_i By modeling S, actuaries can derive the distribution function and moments of the aggregate claims, allowing for effective risk assessment, premium calculation, and reserve setting. The collective risk model is versatile and can be extended to account for reinsurance and other risk mitigation strategies, providing a robust tool for managing the financial stability of insurance operations.

4.8 References

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

4.9 Further Readings

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

UNIT: 5 SURVIVAL DISTRIBUTIONS AND LIFE TABLES

Structure

5.1 Introduction

A survival distribution, also known as a **survival function** or survival curve, is a fundamental concept in actuarial science and survival analysis. It represents the probability that a subject will survive beyond a given time t. Mathematically, it is defined as:

$$S(t) = P(T > t)$$

where T is a random variable representing the time until an event of interest, such as death or failure, occurs. The survival function decreases over time, reflecting the declining probability of survival as time progresses. It is closely related to other functions such as the probability density function (PDF) and the cumulative distribution function (CDF).

A life table, also known as a mortality table or actuarial table, is a statistical tool used to summarize the survival and mortality patterns of a population. It provides detailed information about the probability of surviving or dying at various ages. Life tables are essential for actuaries, demographers, and public health officials.

There are two main types of life tables:

1. **Cohort Life Table**: Tracks a specific group of individuals (cohort) from birth to death, providing data on survival and mortality rates over their lifetime.
2. **Period Life Table**: Provides a snapshot of the survival and mortality rates for a population during a specific time-period, based on age-specific death rates.

Key columns in a life table typically include:

x : Age

l_x: Number of individuals alive at age x

q_x: Probability of dying between ages x and x+1

p_x: Probability of surviving from age x to x+1

d_x: Number of deaths between ages x and x+1

e_x: Expected remaining years of life at age x (life expectancy)

Life tables are crucial for calculating premiums and reserves for life insurance, pensions, and annuities, as well as for understanding demographic trends and public health planning.

5.2 Objectives

By the end of this lesson, learners will be able to:

1. Understand key probability concepts related to the age at death, including survival functions and force of mortality.
2. Analyze life tables and their relationship to survival functions.
3. Interpret important life table characteristics such as complete expectation of life and median future lifetime.

4. Define and apply additional functions, including central death rate and temporary complete life expectancy.
5. Differentiate between select and ultimate life tables and their applications in actuarial science.

5.3 Probability for the Age at Death

The probability distribution of the age at death is characterized using survival functions, future lifetime distributions, and the force of mortality. These functions provide a mathematical framework for modeling mortality risk and estimating life contingencies.

5.3.1 The Survival Function

The survival function S(x) is defined as:

$$S(x) = P(T > x)$$

Where, T is a random variable representing the time until an event of interest occurs.

The survival function S(x) in terms of the cumulative distribution function (CDF) F(x) is defined as:

$$S(x) = 1 - F(x)$$

$$F_x(0) = 0$$

$$s(0) = 1$$

The function s(x) is called survival function.

For any positive x, s(x) is the probability that the new born will attain the age x. The distribution of X can be defined by specifying either the function $F_x(x)$ or s(x).

Using the laws of probability we can make probability statements about the age-at-death in terms of either the survival function or the distribution function.

For example, the probability that the new born dies between ages x and z, (x<z) is,

$$P_r(x < x \leq z) = F_Z(z) - F_X(x)$$

$$= s(x) - s(z)$$

5.3.2 Time until Death for a Person age X

The conditional probability that a new born will die between the ages x and z given survival to age x is:

$$Pr(x < X \leq z | X > x) = \frac{F_Z(z) - F_X(x)}{1 - F_X(x)}$$

$$= \frac{s(x) - s(z)}{s(x)}$$

The symbol (x) is used to denote a life-age x.

The future lifetime of (x), X-x is denoted by T(x). In actuarial science it is frequently necessary to make probability statements about T(x).

A single variate function that would be written as q(x) in probability notation is written as q_x in this system.

The probability statements about T(x),

$_tq_x$ – Probability that (x) will die within t years then $_tq_x$is the distribution of T(x)

$_tp_x$ – Probability that (x) will attain the age (x+t), i.e. the survival function for (x)

The special case of **life-age (0)** we have,

T(0) = X and

$_xp_0 = s(x); x \geq 0$

Now if t=1, convention permits us to omit the prefix and we have,

$_1q_x$ = Pr[(x) will die within one year]

$_1p_x$ = Pr[(x) will attain age (x+1)]

There is a special symbol for the more general event that (x) will survive t years and die within the following u years; i.e. (x) will die between ages (x+t) and (x+t+u)

$_{t|u}q_x = Pr[t < T(x) \leq t+u] = {}_{t+u}q_x - {}_tq_x = {}_tp_x - {}_{t+u}p_x$

(As before, in u=1 we have $_tq_x$)

At this point, it appears that there are two expressions for the probability that (x) will die between ages x and x+u.

At t=0, x →x+u

z= x+u

Which T equals to zero, it defines a probability that the life observed at age x will die between ages x and x+u. The observation on the life at age x might include information other than simply survival. Such information might be that the life has just passed a physical examination for insurance, or it might be that the life is commenced treatment for a serious illness. Life tables for situations where the observation of a life at x implies more than simply survival of a newborn to age x. Observation at age x will yield the same conditional distribution of survival as the hypothesis that a newborn has survived to age x, i.e.,

$$ {}_tp_x = {}_{x+t}p_0 / {}_xp_o = \frac{s(x+t)}{s(x)} $$

$$ {}_tq_x = 1 - \frac{s(x+t)}{s(x)} $$

Under this approach and its special cases can be expressed as follows:

$$ {}_{t|u}q_x = \frac{s(x+t)-s(x+t+u)}{s(x)} $$

$$ = \left[\frac{s(x+t)}{s(x)}\right]\left[\frac{s(x+t)-s(x+t+u)}{s(x)}\right] = {}_tp_x . {}_uq_{x+t} $$

5.3.3 Curtate Future Lifetime

A discrete random variable associated with the future lifetime is the number of future years completed by (x) prior to death. It is called carted future lifetime, denoted by K(x).

Because the K(x) is the greatest integer in T(x), its probability function is:

$$ Pr[K(x) = k] = Pr[k \le T(x) < k + 1] $$

$$ = Pr[k < T(x) \le k + 1] $$

$$ = {}_kp_x . q_{x+k} = {}_{k|}q_x \quad k = 0, 1, 2, \ldots $$

The switching of inequalities is possible since under our assumption that T(x) is a continuous type random variable,

$$ Pr[T(x) = k] = Pr[T(x) = k + 1] = 0 $$

The key observation is that if

$$ K(x)=k, \; k \le T(x) < k+1 $$

This leads to the following formula for probability function,

$$Pr[K(x) = k] = Pr[k \le T(x) < k + 1]$$

$$= {}_{k|1}q_x = \frac{dx+k}{lx}$$

Remark: K(x) = [t(x)] i.e. the integer part of t(x). Since it is a function of t(x), it is simple to calculate the probability function t(x) from what we know about k(x). The possible values of k(x) and numbers 0, 1, 2 and so on up to (w-x-1)

Example: If the life (70) eventually dies at age 85.8, then continuous lifetime is,

T(70) = 15.8 and curtate lifetime,

k(70)= [15.8] = 15

Example: A life table function lx = 100-x, 0 ≤ x ≤ 100, Compute k(75).

Solution: x =75

k[75] = Pr[k(75)=k]

$$= \frac{d_{75}+k}{l_{75}} = \frac{l_{75+k}-l_{75+k+1}}{l_{75}}$$

$$= \frac{[100-(75+k)]-[100-75-k-1]}{100-75} = 1/25$$

So, k(75) has 25 possible values 0, 1, 2,…, 24

Also, w – x – 1 = 24

w = 24+1+75 = 100

5.3.4 Force of Mortality

The force of mortality, often denoted by μ(x), is a fundamental concept in actuarial science and survival analysis. It represents the instantaneous rate of mortality at age x, essentially providing a measure of the risk of death at a very small age interval around x.

The **hazard function**, often used interchangeably with the force of mortality in certain contexts, is a broader concept applicable to various types of time-to-event data. It represents the instantaneous rate at which events occur, given that they have not yet occurred up to time t.

Mathematically, the force of mortality is defined as:

A function for instantaneous death can be obtained by using the density of probability of death at attained age x i.e. z = x + Δx,

$$Pr(x < X \leq \mathrm{x} + \Delta\mathrm{x} \mid \mathrm{X} > x) = \frac{F_X(\mathrm{x}+\Delta\mathrm{x}) - F_X(\mathrm{x})}{1-F_X(\mathrm{x})} = \frac{f_X(\mathrm{x})\Delta\mathrm{x}}{1-F_X(\mathrm{x})}$$

In this equation $F_X{}'(\mathrm{x}) = f_X(\mathrm{x})$ is the continuous age-at-death random variable. The function,

$$= \frac{f_X(\mathrm{x})}{1-F_X(\mathrm{x})}.$$

Has a conditional probability density interpretation. For each age, x, it gives the value of the conditional p.d.f of X at exact age x, given survival to that age and is denoted by μ(x). We have,

$$\mu(\mathrm{x}) = \frac{f_X(\mathrm{x})}{1-F_X(\mathrm{x})} = \frac{-s\prime(\mathrm{x})}{s(\mathrm{x})}$$

The properties of $f_X(x)$ and $1 - F_X(x)$ imply that:

$$\mu(x) \geq 0$$

Or we can say that,

The force of mortality $\mu(x)$ can be interpreted as a conditional density of failure at age X, given survival to age X while f(x) is the unconditional density of failure at age x.

Remark: In insurance sector, for determining the premium of an insured age x, the insurer is interested not only in the complete future lifetime but also in the individually curtate future lifetime.

The force of mortality can be used to specify the distributions of X. For the survival function,

$$\mu(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{-s'(\mathrm{x})}{s(\mathrm{x})}$$

Change x to y →

$$\mu(y) = \frac{-s'(\mathrm{y})}{s(\mathrm{y})}$$

$$\frac{s'(\mathrm{y})}{s(\mathrm{y})} = \frac{dlog(s(\mathrm{y}))}{d(\mathrm{y})}$$

because when we are differentiating the RHS of the above expression, we get

$$\mu(y) = -\frac{dlog(s(\mathrm{y}))}{d(\mathrm{y})}$$

$$-\mu(y)dy = dlog(s(y))$$

Integrating with respect to x and x+n,

$$=> -\int_x^{x+n} \mu(y)dy = \int_x^{x+n} dlog(s(\mathrm{y}))$$

Since this integration is under differentiation, RHS becomes,

$log\ [\frac{s(x+n)}{\Delta x}]$ and $\int_x^{x+n} dlog(s(y))$

$$=> log[s(y)]_x^{x+n}$$

$$=> log[s(x+n)] - log[s(x)]$$

$$=> log\ [\frac{s(x+n)}{s(x)}] = log\,{}_n p_x$$

$$=> \int_x^{x+n} \mu(y)dy = \log {}_n p_x$$

And on taking exponential,

$${}_n p_x = exp[\int_x^{x+n} \mu(y)dy\ ]$$

5.4 Life Tables

A life table, also known as a mortality table or actuarial table, is a statistical tool used to summarize the mortality and survival patterns of a population. It provides detailed information about the probability of survival and death at various ages, making it essential for actuaries, demographers, and public health officials.

A life table is typically constructed using census data, vital statistics, and mortality rates. Life tables are indispensable tools for analyzing survival and mortality patterns, providing crucial insights for various applications in actuarial science, demography, public health, and social planning.

5.4.1 Relation of Life Table Functions to Survival Function

The conditional probability that (x) will die in t years is that:

$${}_t q_x = 1 - \frac{s(\mathrm{x+t})}{s(\mathrm{x})}$$

and in particular we have,

$$q_x = 1 - \frac{s(\mathrm{x+1})}{s(\mathrm{x})}$$

Now, we consider a group of l_0 newborns, l_0 =100000 for instance. Each newborn's age at death has a distribution specified by survival function s(x). In addition, we let L(x) denote the group's number of survivors to age x.

We index these lines by j = 1,2,…,l_0 and observe that

$$L(x) = \sum_{j=1}^{l_0} I_j$$

And I_j is the indicator function of life j taking value 1 if life j survives to x and 0 otherwise

We denote E[L(x)] by l_x, representing the expected number of survivors to age x from start and we have,

$$l_x = l_0\, s(x)$$

Moreover, under the assumption that the indicators I_j are mutually independent and L(x) has a Binomial Distribution.

5.5 The Deterministic Survivorship Group

The deterministic survivorship group is a conceptual framework used in life tables and survival analysis to model the survival and mortality patterns of a population in a predictable and fixed manner. Here are the key characteristics of a deterministic survivorship group:

1. Fixed Initial Cohort:

- **Starting Population**: The model begins with a specific number of individuals, usually denoted as $\boldsymbol{l_0}$, representing the initial cohort at birth.
- **Example**: A common starting point might be 100,000 or 1,000,000 newborns.

2. Predefined Mortality Rates:

- **Age-Specific Mortality Rates**: Mortality rates $\boldsymbol{q_x}$ for each age are fixed and known in advance. These rates do not change over time or due to any external factors.
- **Consistency**: The mortality rates apply uniformly across the entire cohort without any variability.

3. Predictable Outcomes:

- **Deterministic Nature**: The outcomes, such as the number of survivors and deaths at each age, are precisely calculable without any randomness.

- **No Variability**: The survival and mortality patterns are completely predictable based on the initial assumptions.

4. **Simplified Calculations:**
   - **Mathematical Simplicity**: The deterministic approach allows for straightforward calculations of survival metrics, making it easier to construct and interpret life tables.
   - **Ease of Use**: The absence of stochastic elements simplifies the mathematical modelling process.

5. **Fixed Survival Function:**
   - **Survival Function**: The survival function S(x), representing the probability of surviving beyond age x, is deterministic and fixed.
   - **Predictable Decline**: The survival function declines in a predictable manner according to the predetermined mortality rates.

6. **Consistent Survivorship:**
   - **Survivorship Pattern**: The pattern of survivorship (number of individuals surviving to each age) is consistent and follows a predictable trajectory.
   - **No Random Fluctuations**: There are no random fluctuations in the number of survivors or deaths at any age.

7. **Calculation of Life Table Metrics:**
   - **Survivors l_x**: The number of individuals alive at each age x is calculated using the fixed mortality rates.
   - **Deaths d_x**: The number of deaths at each age x is also determined in a straightforward manner.
   - **Life Expectancy e_x**: Expected remaining years of life at each age x are computed based on the fixed survival function.

The deterministic survivorship group provides a clear and predictable model for understanding survival and mortality patterns within a population. Its fixed and non-variable nature makes it an excellent tool for straightforward analysis and calculation, though it may not capture the complexity and variability of real-world scenarios.

$$l_1 = l_0(1 - q_0) = l_0 - d_0$$

$l_2 = l_1(1-q_1) = l_1 - d_1 = l_0 - (d_0 + d_1)$

:

:

$l_x = l_{x-1}(1-q_{x-1}) = l_{x-1} - d_{x-1} = l_0 - \sum_{y=0}^{x-1} d_y$

$l_x = l_0(1- {}_xq_0)$

where l_x is the number of lives attaining age x in the survivorship group.

5.6 Other Life Table Characteristics

In this section we derive the expressions for commonly used characteristics of the distributions of T(x) and K(x) and introduce a general method for computing several of these characteristics.

5.6.1 Complete Expectation of Life

The expected value of T(x) denoted by e_x^0 is called complete expectation of life.

By definition and integrating by parts we have

$$e_x^0 = E[T(x)] = \int_0^\infty \mu(x+t)\,{}_tp_x dt$$

$$= \int_0^\infty t.dt\,(-\,{}_tp_x)$$

Thus, $e_x^0 = (\,{}_tp_x)\, dt$

The complete expectation of life at various stages is often used to compare levels of public health among different populations.

A similar integration of life at various stages is often used to compare levels of public health among different populations.

Now, $Var[T(x)] = 2\int_0^\infty t.\,{}_tp_x)\, dt - e_x^{0_2}$

5.6.2 Median Future Lifetime of (x)

The median future lifetime of (x) to be denoted by m(x) can be found by solving

$$Pr[\,T(x) > m(x)] = \frac{1}{2}$$

$$\text{or} \quad \frac{s[x+m(x)]}{s(x)} = \frac{1}{2}$$

For, m(x). In particular m(0) is given by solving s[m(0)] = ½

We can also find the mode of the distribution by locating the value of t that yields maximum value of ${}_tp_x . \mu(x+t)$

5.6.3 Curtate Expectation of Life

The expected value of K(x) is denoted by e_x and is called curtate expectation of life. By definition and use of summation by parts we have,

$$e_x = E[K(x)] = \sum_{k=0}^{\infty} k . q_{x+k} . {}_kp_x = \sum_{k=0}^{\infty} k . \Delta(-{}_kp_x)$$

Now, $E[K(x)^2] = \sum_{k=0}^{\infty}(2k+1) . {}_{k+1}p_x = \sum_{k=0}^{\infty}(2k-1)\, {}_kp_x$

Now, $\mathrm{Var}(K) = \sum_{k=0}^{\infty}(2k-1){}_kp_x - {e_x}^2$

5.7 Defining Additional Functions

To refine mortality analysis and life expectancy calculations, additional functions such as the central death rate and temporary complete life expectancy are introduced. These functions enhance the accuracy of life table estimates and provide deeper insights into population longevity. These functions help quantify mortality rates over specific time intervals and improve the modelling of future lifetimes. By incorporating these measures, actuaries can make more precise predictions for insurance and pension calculations.

5.7.1 Central Death Rate (L)

The symbol L_x denotes the total expected number of years lived between ages x and x+1 by survivors of the initial group of l_0 lives. We have

$$L_x = \int_0^1 t . \mu(x+t) l_{x+t} dt + l_{x+1}$$

Where the integral counts the years lived of those who die between ages x and x+1, and the term l_{x+1} counts the years lived between ages x and x+1. Integration by parts yields,

$$L_x = -\int_0^1 t\,.dl_{x+t} + l_{x+1}$$

$$= \int_0^1 l_{x+t}dt$$

The function L_x is also used in defining central death rate over the interval from x to x+1 denoted by m_x

$$m_x = \frac{l_x - l_{x+1}}{L_x}$$

5.7.2 n-temporary Complete Life Expectancy

The definition of m_x and L_x can be extended to age intervals of length other than one:

$${}_nL_x = \int_0^n t\,.\mu(x+t)l_{x+t}dt + nl_{x+n}$$

$$= \int_0^n l_{x+t}dt$$

Now, ${}_nm_x = \frac{l_x - l_{x+n}}{nLx}$

For the random survivorship group ${}_nL_x$ is the total expected number of years lived between ages x and x + n by the survivors of the initial group of L_0 lives, and ${}_nm_x$ is the average death rate experienced by the group over the interval (x, x + n)

The symbol T_X denotes the total number of years lived beyond age x by the survivorship group with L_0 initial members.

$$T_x = \int_0^\infty .l_{x+t}dt$$

The final expression can be interpreted as the integral of the total time lift between ages x + t and x + t + dt by l_{x+t} lives who survive to that age interval.

$$\frac{nLx}{l_x} = \int_0^n .{}_tP_xdt$$

This function is the **n-year temporary complete life expectancy** of x and id denoted by $e^0_{x:\overline{n|}}$

5.8 Select and Ultimate Tables

In actuarial science, select and ultimate tables are used to model the mortality rates of insured lives. These tables incorporate different mortality rates based on the duration since the policy was issued, recognizing that new policyholders (select lives) tend to have different mortality characteristics compared to those who have been insured for a longer period (ultimate lives). This distinction is crucial for accurately pricing insurance products and managing risk.

Select tables provide mortality rates for recently underwritten lives, reflecting the "selection effect," which is the observed phenomenon where newly insured individuals exhibit lower mortality rates due to the underwriting process. This process typically includes medical examinations and health assessments, leading to the selection of healthier individuals.

Ultimate tables provide mortality rates for lives that have been insured beyond the select period. These rates reflect the mortality experience of the broader insured population without the selection effect.

Select and ultimate tables are critical for various actuarial applications, including:

- **Pricing of Insurance Products**: Accurately assessing risk and setting premiums based on the insured's mortality risk profile.
- **Reserving**: Estimating the reserves needed to cover future claims, taking into account the different mortality rates during the select and ultimate periods.
- **Risk Management**: Understanding the mortality dynamics to manage the insurer's risk portfolio effectively.
- **Underwriting**: Informing the underwriting process to identify and select healthier individuals, thereby improving the insurer's risk pool.

Select and ultimate tables serve crucial purposes in the field of actuarial science, particularly in the accurate assessment and management of mortality risk for life insurance products. These tables help actuaries and insurers to better understand and predict the mortality rates of policyholders, which is essential for setting premiums, calculating reserves, and managing risk.

1. Reflecting Mortality Differentials

Purpose: To accurately reflect the differences in mortality rates between newly underwritten policyholders (select lives) and those who have been insured for a longer period (ultimate lives).

- **Select Mortality**: Newly underwritten individuals often undergo rigorous health screenings, resulting in lower initial mortality rates due to the selection effect.

- **Ultimate Mortality**: Over time, the mortality rates of these individuals tend to converge with the general insured population as the effects of initial selection wear off.

2. Accurate Pricing of Insurance Products

Purpose: To enable more precise pricing of life insurance policies by incorporating the varying mortality rates over different durations since policy issuance.

- **Premium Calculation**: By using select mortality rates for the initial period and ultimate rates thereafter, insurers can set premiums that are more aligned with the actual risk posed by the insured over the life of the policy.
- **Risk Assessment**: This distinction helps in assessing the long-term profitability and sustainability of life insurance products.

3. Reserving and Financial Planning

Purpose: To ensure adequate reserving for future claims and to support long-term financial planning and solvency of the insurance company.

- **Reserves Calculation**: Actuaries use selects and ultimate tables to calculate the reserves needed to cover future death benefits accurately.
- **Solvency Management**: These tables help insurers maintain sufficient capital to meet their policyholder obligations, thus ensuring financial stability.

4. Risk Management and Underwriting

Purpose: To improve the underwriting process and overall risk management strategies by distinguishing between different mortality experiences.

- **Underwriting Guidelines**: Insights from select and ultimate tables can refine underwriting criteria, leading to better risk selection and improved mortality experience.
- **Risk Segmentation**: By understanding how mortality rates change over time, insurers can better segment their risk portfolios and develop targeted risk management strategies.

5. Experience Analysis and Product Development

Purpose: To analyze mortality experience over time and support the development of new life insurance products.

- **Experience Studies**: Actuaries can conduct detailed experience studies to compare actual mortality against expected mortality derived from select and ultimate tables, identifying trends and areas for improvement.
- **Product Innovation**: These tables provide the data needed to design new insurance products that meet changing consumer needs and market conditions.

6. Regulatory Compliance and Reporting

Purpose: To comply with regulatory requirements and improve transparency in financial reporting.

- **Regulatory Standards**: Many regulatory frameworks require the use of select and ultimate mortality tables for statutory reporting and compliance.
- **Transparent Reporting**: Insurers can provide more accurate and transparent financial statements, which reflect their true risk exposure and financial health.

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Proportion Dying					Average Remaining Lifetime
Age Interval	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval	Of 100,000 Born Alive		Stationary Population*		Average Number of Years of Life Remaining at Beginning of Age Interval
Period of Life between Two Ages x to $x + t$	${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	$\mathring{e}_x$
Days						
0–1	0.00463	100 000	463	273	7 387 758	73.88
1–7	0.00246	99 537	245	1 635	7 387 485	74.22
7–28	0.00139	99 292	138	5 708	7 385 850	74.38
28–365	0.00418	99 154	414	91 357	7 380 142	74.43
Years						
0–1	0.01260	100 000	1 260	98 973	7 387 758	73.88
1–2	0.00093	98 740	92	98 694	7 288 785	73.82
2–3	0.00065	98 648	64	98 617	7 190 091	72.89
3–4	0.00050	98 584	49	98 560	7 091 474	71.93
4–5	0.00040	98 535	40	98 515	6 992 914	70.97
5–6	0.00037	98 495	36	98 477	6 894 399	70.00
6–7	0.00033	98 459	33	98 442	6 795 922	69.02
7–8	0.00030	98 426	30	98 412	6 697 480	68.05
8–9	0.00027	98 396	26	98 383	6 599 068	67.07
9–10	0.00023	98 370	23	98 358	6 500 685	66.08
10–11	0.00020	98 347	19	98 338	6 402 327	65.10
11–12	0.00019	98 328	19	98 319	6 303 989	64.11
12–13	0.00025	98 309	24	98 297	6 205 670	63.12
13–14	0.00037	98 285	37	98 266	6 107 373	62.14
14–15	0.00053	98 248	52	98 222	6 009 107	61.16
15–16	0.00069	98 196	67	98 163	5 910 885	60.19
16–17	0.00083	98 129	82	98 087	5 812 722	59.24
17–18	0.00095	98 047	94	98 000	5 714 635	58.28
18–19	0.00105	97 953	102	97 902	5 616 635	57.34
19–20	0.00112	97 851	110	97 796	5 518 733	56.40
20–21	0.00120	97 741	118	97 682	5 420 937	55.46
21–22	0.00127	97 623	124	97 561	5 323 255	54.53
22–23	0.00132	97 499	129	97 435	5 225 694	53.60
23–24	0.00134	97 370	130	97 306	5 128 259	52.67
24–25	0.00133	97 240	130	97 175	5 030 953	51.74
25–26	0.00132	97 110	128	97 046	4 933 778	50.81
26–27	0.00131	96 982	126	96 919	4 836 732	49.87
27–28	0.00130	96 856	126	96 793	4 739 813	48.94
28–29	0.00130	96 730	126	96 667	4 643 020	48.00
29–30	0.00131	96 604	127	96 541	4 546 353	47.06

Life Table for the Total Population: United States, 1979–81

(1) Age Interval: Period of Life between Two Ages x to $x+t$	(2) Proportion Dying: Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	(3) Of 100,000 Born Alive: Number Living at Beginning of Age Interval l_x	(4) Of 100,000 Born Alive: Number Dying during Age Interval ${}_td_x$	(5) Stationary Population*: Years Lived in the Age Interval ${}_tL_x$	(6) Stationary Population*: Years Lived in This and All Subsequent Age Intervals T_x	(7) Average Remaining Lifetime: Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
30–31	0.00133	96 477	127	96 414	4 449 812	46.12
31–32	0.00134	96 350	130	96 284	4 353 398	45.18
32–33	0.00137	96 220	132	96 155	4 257 114	44.24
33–34	0.00142	96 088	137	96 019	4 160 959	43.30
34–35	0.00150	95 951	143	95 880	4 064 940	42.36
35–36	0.00159	95 808	153	95 731	3 969 060	41.43
36–37	0.00170	95 655	163	95 574	3 873 329	40.49
37–38	0.00183	95 492	175	95 404	3 777 755	39.56
38–39	0.00197	95 317	188	95 224	3 682 351	38.63
39–40	0.00213	95 129	203	95 027	3 587 127	37.71
40–41	0.00232	94 926	220	94 817	3 492 100	36.79
41–42	0.00254	94 706	241	94 585	3 397 283	35.87
42–43	0.00279	94 465	264	94 334	3 302 698	34.96
43–44	0.00306	94 201	288	94 057	3 208 364	34.06
44–45	0.00335	93 913	314	93 756	3 114 307	33.16
45–46	0.00366	93 599	343	93 427	3 020 551	32.27
46–47	0.00401	93 256	374	93 069	2 927 124	31.39
47–48	0.00442	92 882	410	92 677	2 834 055	30.51
48–49	0.00488	92 472	451	92 246	2 741 378	29.65
49–50	0.00538	92 021	495	91 773	2 649 132	28.79
50–51	0.00589	91 526	540	91 256	2 557 359	27.94
51–52	0.00642	90 986	584	90 695	2 466 103	27.10
52–53	0.00699	90 402	631	90 086	2 375 408	26.28
53–54	0.00761	89 771	684	89 430	2 285 322	25.46
54–55	0.00830	89 087	739	88 717	2 195 892	24.65
55–56	0.00902	88 348	797	87 950	2 107 175	23.85
56–57	0.00978	87 551	856	87 122	2 019 225	23.06
57–58	0.01059	86 695	919	86 236	1 932 103	22.29
58–59	0.01151	85 776	987	85 283	1 845 867	21.52
59–60	0.01254	84 789	1 063	84 258	1 760 584	20.76
60–61	0.01368	83 726	1 145	83 153	1 676 326	20.02
61–62	0.01493	82 581	1 233	81 965	1 593 173	19.29
62–63	0.01628	81 348	1 324	80 686	1 511 208	18.58
63–64	0.01767	80 024	1 415	79 316	1 430 522	17.88
64–65	0.01911	78 609	1 502	77 859	1 351 206	17.19

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Age Interval	Proportion Dying	Of 100,000 Born Alive		Stationary Population*		Average Remaining Lifetime
Period of Life between Two Ages x to $x+t$	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
65–66	0.02059	77 107	1 587	76 314	1 273 347	16.51
66–67	0.02216	75 520	1 674	74 683	1 197 033	15.85
67–68	0.02389	73 846	1 764	72 964	1 122 350	15.20
68–69	0.02585	72 082	1 864	71 150	1 049 386	14.56
69–70	0.02806	70 218	1 970	69 233	978 236	13.93
70–71	0.03052	68 248	2 083	67 206	909 003	13.32
71–72	0.03315	66 165	2 193	65 069	841 797	12.72
72–73	0.03593	63 972	2 299	62 823	776 728	12.14
73–74	0.03882	61 673	2 394	60 476	713 905	11.58
74–75	0.04184	59 279	2 480	58 039	653 429	11.02
75–76	0.04507	56 799	2 560	55 520	595 390	10.48
76–77	0.04867	54 239	2 640	52 919	539 870	9.95
77–78	0.05274	51 599	2 721	50 238	486 951	9.44
78–79	0.05742	48 878	2 807	47 475	436 713	8.93
79–80	0.06277	46 071	2 891	44 626	389 238	8.45
80–81	0.06882	43 180	2 972	41 694	344 612	7.98
81–82	0.07552	40 208	3 036	38 689	302 918	7.53
82–83	0.08278	37 172	3 077	35 634	264 229	7.11
83–84	0.09041	34 095	3 083	32 553	228 595	6.70
84–85	0.09842	31 012	3 052	29 486	196 042	6.32
85–86	0.10725	27 960	2 999	26 461	166 556	5.96
86–87	0.11712	24 961	2 923	23 500	140 095	5.61
87–88	0.12717	22 038	2 803	20 636	116 595	5.29
88–89	0.13708	19 235	2 637	17 917	95 959	4.99
89–90	0.14728	16 598	2 444	15 376	78 042	4.70
90–91	0.15868	14 154	2 246	13 031	62 666	4.43
91–92	0.17169	11 908	2 045	10 886	49 635	4.17
92–93	0.18570	9 863	1 831	8 948	38 749	3.93
93–94	0.20023	8 032	1 608	7 228	29 801	3.71
94–95	0.21495	6 424	1 381	5 733	22 573	3.51
95–96	0.22976	5 043	1 159	4 463	16 840	3.34
96–97	0.24338	3 884	945	3 412	12 377	3.19
97–98	0.25637	2 939	754	2 562	8 965	3.05
98–99	0.26868	2 185	587	1 892	6 403	2.93
99–100	0.28030	1 598	448	1 374	4 511	2.82

Life Table for the Total Population: United States, 1979–81

(1)	(2)	(3)	(4)	(5)	(6)	(7)
	Proportion Dying					Average Remaining Lifetime
		Of 100,000 Born Alive		Stationary Population*		
Age Interval: Period of Life between Two Ages x to $x+t$	Proportion of Persons Alive at Beginning of Age Interval Dying during Interval ${}_tq_x$	Number Living at Beginning of Age Interval l_x	Number Dying during Age Interval ${}_td_x$	Years Lived in the Age Interval ${}_tL_x$	Years Lived in This and All Subsequent Age Intervals T_x	Average Number of Years of Life Remaining at Beginning of Age Interval $\mathring{e}_x$
Years						
100–101	0.29120	1 150	335	983	3 137	2.73
101–102	0.30139	815	245	692	2 154	2.64
102–103	0.31089	570	177	481	1 462	2.57
103–104	0.31970	393	126	330	981	2.50
104–105	0.32786	267	88	223	651	2.44
105–106	0.33539	179	60	150	428	2.38
106–107	0.34233	119	41	99	278	2.33
107–108	0.34870	78	27	64	179	2.29
108–109	0.35453	51	18	42	115	2.24
109–110	0.35988	33	12	27	73	2.20

(sourced from reference 3, section 5.9 Further Readings)

5.9 Summary

Survival distribution is a key concept in survival analysis, focusing on the time until an event of interest occurs, such as death or failure. The survival function, denoted by S(t), represents the probability that an individual survives beyond time t.

Life tables are a fundamental tool in actuarial science and demography, providing a summary of mortality patterns within a population. They are constructed based on survival distributions and offer insights into the survival rates, mortality rates, and life expectancy at various ages. The theories of survival distribution and life tables are essential for understanding and modelling mortality and survival patterns. They provide crucial tools for actuaries, demographers, and public health professionals in various applications.

By combining survival functions, hazard functions, and life table metrics, these theories offer a comprehensive framework for analysing and predicting the longevity and mortality characteristics of populations.

5.10 References

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.

- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

5.11 Further Readings

- Dickson, C.M.D. (2005): Insurance Risk and Ruin (International Series on Actuarial Science) Cambridge University Press.
- Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. and Nesbitt, C.J. (1997): Actuarial Mathematics, Society of Actuaries, Itasca, Illinois, U.S.A.
- Atkinson, M.E. and Dickson, D.C.M. (2011): An Introduction to Actuarial Studies, Elgar Publishing.

U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT – 404 (N)A/ MASTAT – 404(N)A

Actuarial Statistics

Block : 2 Reliability Estimations for Actuarial

Course Design Committee

Course Preparation Committee

Dr. Mahendra Saha **Writer**
Department of Statistics
Faculty of Mathematical Sciences
University of Delhi, New Delhi, India

Prof. Jitendra Kumar **Editor**
Department of Statistics
Central University of Rajasthan, Ajmer, Rajasthan

Prof. Shruti **Reviewer & Course Coordinator**
School of Sciences,
U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT–404 NA/MASTAT–404 NA **ACTUARIAL STATISTICS**

First Edition: ***July 2025.***
ISBN :

Printed By:

Block & Units Introduction

The ***Block - 2 –Reliability Estimations for Actuarial,*** is the second block of said SLM, which is divided into four units.

Unit – 6: Basic Concepts and Ageing: This unit lays the Reliability Theory with Applications in Insurance and Annuities, Failure and Failure Rates, Survival Function as a Reliability Measure in Insurance, Core Measures, Statistical Tools in Insurance and Annuity Modelling, Probability Distributions Used in Life Modelling, Force of Mortality Function, Importance in Insurance and Annuity Design. This unit also focus on the Ageing across the Policy Lifecycle: Stages and Claim Behaviour, Infant Mortality Period, Normal Life, Wear-Out-Phase, Bathtub Curve, Ageing in Actuarial and Insurance Context, Models and Strategies for Managing Ageing in Insurance.

Unit – 7: Reliability Estimation and Repairable Systems: This unit is dedicated to the Lifetime Data Analysis, Key distributions, Actuarial Applications of Lifetime Distributions, Censored Data Analysis, Types of Censoring, Kaplan-Meier Estimator, Reliability Block Diagrams, Common Configurations, Actuarial Applications of RBDs and Fault Tree Analysis. This unit also examines about the Maintenance Strategies, Preventive Maintenance, Corrective Maintenance, and Markov Models for Repairable Systems, Two-State Markov Model, Actuarial and Insurance Applications of Markov Models, Model Extensions, The Renewal Process, Availability Modeling, Instantaneous Availability, Steady State Availability, Steady State Unavailability, Mean Down Time, Impact of Repair Policies, and Maintenance Optimization.

Unit – 8: Growth Models, Accelerated Life Testing and Multiple Life Functions: This unit discuss about the, reliability growth models and accelerated life testing techniques are explored. The unit covers various reliability growth models, such as the Duane model and the AMSAA model. Accelerated life testing involves Arrhenius Model, Eyring Model, and Inverse Power Law Model. In this unit, we also discussed about the Introduction, Joint Distribution of Future life time, joint life and last survivor status, insurance and annuity benefits through multiple life functions evaluation for special mortality law.

Unit – 9: Decrement Theory and Application of Multiple Decrement Theory: In this unit, we discussed about the multiple decrement models, deterministic and random survivorship groups, associated single decrement tables, central rates of multiple decrements, net single premiums, and their numerical evaluations.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

UNIT: 6 BASIC CONCEPTS AND AGEING

Structure

6.1 Introduction

This unit focuses on the concept of reliability theory and its contribution towards the insurance sectors. Reliability theory is a mathematical and statistical framework that originated in the field of engineering. It was primarily developed to study the dependability of mechanical systems and components over time—specifically, to predict how likely a machine is to perform its intended function without failure for a specified duration. Over time, the versatility of reliability theory has allowed it to be adapted beyond engineering, particularly into fields where "failure" and "survival" are also core concerns. One of the most profound adaptations of reliability theory is found in insurance mathematics and actuarial science, especially in the design and evaluation of life insurance and annuity products.

In the context of insurance and annuities, the term "reliability" takes on a slightly different, though conceptually similar, meaning. Instead of focusing on the mechanical reliability of systems, we now focus on the biological or financial reliability—that is, the likelihood that a policyholder survives to a certain age or that a financial product delivers promised benefits over time. The concept of a system "failing" translates to a person's death (in life insurance) or the end of financial support (in annuities), and the functioning of a product is measured by its ability to make timely payouts, maintain value, and meet actuarial assumptions over the contract's duration.

The central role of reliability in insurance is perhaps most evident in life-contingent contracts. These are financial products whose payouts depend on the uncertain future lifetimes of individuals. Examples include:

- Term life insurance, where a benefit is paid if the insured dies within a certain period.
- Whole life insurance, where a benefit is paid upon the death of the insured, regardless of when it occurs.
- Deferred and immediate annuities, where payments continue if the annuitant is alive.

All these products require accurate modelling of the likelihood of survival over time, which directly aligns with the concept of reliability in a probabilistic sense. Just as engineers assess the probability that a system performs without failure, actuaries must estimate the survival function of individuals based on demographic data, health factors, and population studies.

Furthermore, we will define the foundational concepts of reliability theory and explain how they work in the field of life insurance and annuities with the mathematical and statistical tools used to analyse reliability of the data.

Ageing is a fundamental concept in both reliability theory and actuarial science, reflecting the dynamic nature of risk as individuals, policies, and systems evolve over time. In technical reliability, ageing typically refers to wear and deterioration in physical systems. In insurance, however, it manifests in the changing likelihood of insured events such as death, illness, or claims as time progresses. Understanding how risk profiles shift with age is crucial for developing sustainable insurance products and maintaining the financial soundness of insurers. From an actuarial perspective, ageing affects several key domains: the increasing probability of mortality in life insurance, the growing incidence of chronic diseases in health insurance, and the tendency for long-held policies to have different claim behaviours compared to newly issued ones. These changes are not random but follow identifiable patterns that can be modelled mathematically. By analyzing these trends, actuaries can design better pricing structures, determine adequate reserves, and predict future liabilities with greater confidence.

Moreover, the concept of reliability in insurance extends to the trust and dependability clients place in insurance products and services over long periods. Policies must remain viable and beneficial even as policyholders' age and risks increase. Therefore, ageing must be accounted for not only in the pricing and reserving processes but also in policy design and customer experience strategies.

This unit also explores the phenomenon of ageing within the context of insurance, drawing parallels to system reliability theory. It introduces the typical life cycle stages of policies and insured individuals, illustrates the impact of ageing on claim behaviour using the bathtub curve model, and discusses statistical tools used to quantify and manage age-related risks. The chapter also offers practical insights into how insurers respond to these challenges through dynamic underwriting, age-based premiums, and preventive programs.

6.2 Objectives

By the end of this chapter, learners will be able to:

1. Understand the terms of reliability and survival functions.
2. Uses of mathematical tools and statistical methods in insurance.
3. Apply reliability function with probability distributions and failure rate functions.
4. Examples are used to understand the practical knowledge of the reliability theory.
5. Understand how life cycle stages of policies and insured individuals impact ageing.
6. Different phenomena of ageing within the context of insurance.
7. Learn about impact of ageing on claim behavior using bathtub curve model.
8. Offers practical insight into how insurers respond to these challenges through dynamic underwriting, age-based premiums and preventive programs.

6.3 Reliability Theory with Applications in Insurance and Annuities

Reliability theory is a branch of statistics and engineering that deals with the study of how long systems or components perform their intended functions without failure. It is widely applied in fields like manufacturing, electronics, aerospace, and software engineering to design more dependable systems. In insurance, reliability refers to the predictability and dependability of risk assessment and claim occurrence over time. Although traditionally an engineering concept, reliability theory helps insurers evaluate the likelihood that insured entities (like people, properties, or machines) will function or survive without failure (e.g., accident, illness, or loss) over a period.

Key Concepts:

- **Reliability (R(t)):** Reliability models help insurers predict the probability of a policyholder filing a claim within a certain time frame. For example, life insurance companies may use survival analysis (related to reliability theory) to estimate life expectancy and set premium rates.

- **Failure Rate (λ):** The failure rate, denoted by λ(t), is a measure of how often a system or component fails per unit time, given that it has survived up to time t. It is a key concept in reliability engineering and also has analogues in insurance and survival analysis.

$$\lambda(t) = \frac{f(t)}{R(t)}$$

  Where, $f(t)$: Probability density function of failure time.

  $R(t)$: Reliability function = P(T>t), the probability the system survives beyond time t

- **Mean Time to Failure (MTTF):** Analogous to MTTF, it represents the expected time until an insurance claim is made, helping actuaries design more sustainable insurance plans

- **Mean Time Between Failures (MTBF):** Think of MTBF as representing the average time between claims (failures) for a policyholder or insured asset.

Example: Auto Insurance

MTBF = average time between car accidents (claims) for a specific driver or demographic group. If a driver files 4 claims over 8 years, then:

MTBF=$\frac{8\ years}{4} = 2\ years$, Meas on an average, a claim is made every 2 years.

- **Bathtub Curve**: The Bathtub Curve is a widely used model in reliability engineering that describes how the failure rate of a product or system changes over time. Its shape resembles a bathtub: high at the beginning, flat in the middle, and rising at the end. A common model showing failure rate over time in three phases—

- **Early Failures (Decreasing Rate):** It has high failure rate initially with mainly caused by design flaws, manufacturing defects or early- use stress.
- **Constant Failures (Random):** It has low and constant failure rate, failure occurs randomly, often due to unexpected external factors. This is the most stable and predictable period of operation.
- **Wear-Out Failures (Increasing Rate):** It increases failure rate over time. Caused by aging, fatigue, corrosion, or component degradation. It requires preventive maintenance or part replacement.

6.3.1 Failure and Failure Rates in Insurance

Failure refers to the event when a system or component no longer performs its intended function. Understanding failure behaviour is essential in predicting and mitigating system breakdowns.

The failure rate, denoted $\lambda(t)$, is defined as:

$$\lambda(t) = \frac{f(t)}{R(t)}$$

Where:

- $f(t)$: Probability density function of failure time
- $R(t)$: Reliability function

Failure rates can vary over time and are typically modelled using the bathtub curve, which describes three distinct periods:

1. Infant Mortality Phase: High but decreasing failure rate due to early defects.
2. Useful Life Phase: Low, relatively constant failure rate.
3. Wear-Out Phase: Increasing failure rate due to aging and wear.

6.3.2 Survival Function as a Reliability Measure in Insurance

The survival function, also known as the reliability function and denoted by $S(t)$, is a fundamental measure in reliability theory and actuarial science. It represents the probability that a system, individual, or insured risk continues to exist or operate without experiencing the defined event (e.g., failure, death, or claim) beyond a specific time t:

$$S(t) = P(T > t)$$

In the context of insurance:

- For life insurance, $S(t)$is the probability that an individual survives beyond age.
- In health insurance, it can represent the probability that no major health event occurs by time t.
- For property insurance, $S(t)$may represent the probability that no loss event (e.g., fire, theft) has occurred by time t.

This function is essential in actuarial modelling and premium calculations. It is closely related to the cumulative distribution function $F(t)$, which gives the probability that the event has occurred by time t:

$$F(t) = 1 - S(t)$$

The survival function is non-increasing, starting from 1 at time zero and decreasing as time increases. It is central to life tables, annuity pricing, and determining the expected present value of future claims in insurance and pension planning.

6.3.3 Core Measures

Maintainability and Availability Reliability engineering evaluates more than just the probability of failure; it includes measures that influence overall system performance and efficiency. Two such critical measures are maintainability and availability.

- Maintainability refers to the probability that a failed system or component can be repaired or restored to a functioning state within a specified time frame, using prescribed resources and procedures. High maintainability means that even if a failure occurs, the system can quickly return to service, minimizing downtime and cost. In operational terms, maintainability improves logistical support and reduces the need for extensive repair infrastructure. *Insurance Example*: In an insurance context, maintainability can refer to how efficiently an insurance provider resolves customer issues, such as processing claims or updating policy details. A maintainable insurance system enables quick claim settlement and customer service resolution. For example, if a policyholder submits a claim for medical expenses, a highly maintainable system would allow for minimal documentation and a fast turnaround, enhancing customer satisfaction.

- Availability integrates both reliability and maintainability and indicates the proportion of time a system is in a functioning condition. It reflects not only how often a system fails (reliability) but also how quickly it can be restored (maintainability). Availability is especially important for systems that must operate continuously, such as power plants, hospitals, or financial transaction networks. It is mathematically expressed as:

$$\text{Availability} = \frac{MTBF}{MTBF+MTTR}$$

Where:

- MTBF (Mean Time Between Failures) is the average time a system operates before experiencing a failure.
- MTTR (Mean Time To Repair) is the average time required to repair a system and return it to full functionality.

High availability is typically achieved by designing systems with redundancy, efficient maintenance protocols, and rapid fault detection and recovery mechanisms. In sectors such as aviation, defence, and telecommunications, availability is often contractually specified as a performance requirement.

Insurance Example: Availability in insurance can relate to the consistent accessibility of services and support. A highly available insurer provides 24/7 access to online policy management, responsive customer support, and real-time claims tracking. For instance, if a customer experiences a car accident at night, a high-availability insurance provider would offer mobile app access to file a claim, roadside assistance, and claim initiation without waiting for regular business hours.

In actuarial science and insurance operations, maintainability and availability have their parallels in policy servicing and claims management. For example, a highly "maintainable" insurance service would ensure quick policy updates and efficient handling of customer needs, while a highly "available" service would ensure that policyholders can access support and benefits when needed without interruption.

Example: Consider an insurance company that offers automobile coverage and uses an automated claims processing system. The company wants to ensure the system is reliable, maintainable, and highly available to provide continuous service to its customers.

- **Reliability:** The probability that the claim processing system will operate without failure for 12 months is 95%. This is based on historical data showing 5% of systems experience failure within a year.
- **Maintainability:** When failures do occur (e.g., server issues), the average time to restore the system is 2 hours. The company has streamlined support protocols and backup systems to minimize downtime.
- **Availability**: Given a Mean Time Between Failures (MTBF) of 6 months and a Mean Time to Repair (MTTR) of 2 hours, the system's availability is:

$$\text{Availability} = \frac{MTBF}{MTBF + MTTR} = \frac{4380}{4380 + 2} \approx 0.9995.$$

This high availability ensures customers can file claims anytime, improving satisfaction and operational efficiency.

6.4 Statistical Tools in Insurance and Annuity Modeling

In actuarial science, statistical tools are essential for modelling risk, estimating lifetimes, pricing insurance policies, and valuing annuities. Key tools and their meanings are as follows:

- ***Survival Function (S(t)):*** As explained earlier, it represents the probability that a life or insured event survives past time t. In insurance, it helps in computing premiums and reserves by indicating how likely a policyholder are to survive to a certain age.

- ***Hazard Rate or Force of Mortality ($\mu(t)$):*** This is the instantaneous rate of failure (or death) at age t. It is used in life tables to indicate the mortality pattern. It helps actuaries determine life insurance premiums and pension contributions.

- ***Probability Density Function (f(t)):*** Represents the likelihood of failure (or death) at a specific time t. It's useful in computing expected present values and in simulations.

- ***Cumulative Distribution Function (F(t)):*** Represents the probability that failure (or death) occurs by time t. It complements the survival function and is used in claim timing estimations.

- ***Expected Future Lifetime (e_x):*** The expected number of future years a person aged will live. It is crucial for calculating life annuities and pension liabilities.

- ***Life Tables:*** Tables that show the survival function, force of mortality, and other derived quantities for a cohort of individuals. Life tables are foundational tools in actuarial analysis for pricing life insurance and determining reserves.

- ***Actuarial Present Value (APV):*** The present value of expected future payments, discounted at a chosen interest rate. APV calculations form the basis of premium determination for life insurance and annuity contracts.

- ***Annuity Functions:*** These include values like , which represents the present value of an annuity payable annually for life beginning at age . Such functions are used to evaluate pension liabilities and design annuity products.

- ***Benefit Reserves:*** The amount set aside to meet future insurance liabilities. They are computed using expected values derived from the above functions, considering lapse, mortality, interest, and expenses.

These statistical tools allow actuaries to assess risk accurately, ensure solvency of insurance firms, and offer fair and competitive premiums. They form the quantitative backbone of insurance product development and financial reporting.

6.4.1 Probability Distributions Used in Life Modeling

Probability distributions play a central role in modelling the lifetime of individuals or systems. These distributions describe the statistical behaviour of time to failure or time until an event, such as death or claim occurrence. Commonly used distributions include:

- **Exponential Distribution**:
    - Assumes a constant failure (or hazard) rate.
    - Often used in contexts where the probability of an event is the same at any point in time.
    - Insurance Example: Used in modelling memory less processes such as time between claims when claim rate is constant.

- **Weibull Distribution**:
    - A flexible distribution that can model increasing, constant, or decreasing failure rates.
    - The shape parameter determines the behaviour of the hazard rate.

- Insurance Example: Used to model time until a policyholder files a claim, accounting for aging or other risk factors.

- **Gompertz Distribution**:
  - Frequently used in actuarial science to model human mortality.
  - Reflects an exponentially increasing force of mortality with age.

- **Log-Normal Distribution**:
  - Models time to failure when the logarithm of the variable follows a normal distribution.
  - Suitable when events are influenced by multiplicative factors.

- **Normal Distribution**:
  - Often used when lifetimes are symmetrically distributed around a mean.
  - Less common in reliability and mortality modelling but useful in aggregate loss modelling.

These distributions help actuaries and reliability engineers estimate survival probabilities, design products, calculate premiums, and manage risks effectively. The choice of distribution depends on empirical data and the nature of the risk being modelled.

6.4.2 Force of Mortality Function

The instantaneous rate of failure (or death) at age t. It is defined as force of mortality; it has one-to-one relation with the distribution function, survival function and the probability density function. The force of mortality at age t, denoted by μ_t, is defined as:

$$\mu_t = \lim_{\delta t \to 0} \left(\frac{1}{\delta t}\right) P[\, t < T < t + \delta t | T > t]$$

$$= \lim_{\delta t \to 0} \left(\frac{1}{\delta t}\right) \frac{S(t) - S(t + \delta t)}{S(t)}$$

$$= \left(-\frac{1}{S(t)}\right) \lim_{\delta t \to 0} \frac{S(t) - S(t + \delta t)}{\delta t}$$

$$= \left(-\frac{S'(t)}{S(t)}\right) = -\frac{d}{dt} \log \mathrm{S(t)}$$

$$= \left(\frac{f(t)}{1 - F(t)}\right)$$

From the formula of μ_t, it has a conditional probability density interpretation. For each age t, it gives the value of the conditional probability density function of T at exact age t, given survival to the age. It helps actuaries determine life insurance premiums, pension contributions, and mortality risk. A rising force of mortality with age is a key assumption in modelling human lifespan.

Example: Suppose X follows exponential distribution with parameter μ. Find the corresponding force of mortality.

Solution: For the exponential distribution with parameter μ, the probability density function is given by, $f(x) = \mu e^{-\mu x}$, for x > 0.

The survival function is given by $S(x) = e^{-\mu x}$, for x > 0.

Hence, the force of mortality is μ for all x > 0. It does not depend on x.

Example: For the uniform distribution over (0,100), find the force of mortality.

Solution: For uniform distribution probability density function is given by, $f(x) = \frac{1}{100}$, for x in (0,100) and the corresponding survival function is:

$$S(x) = \begin{cases} 1, & if \;\; x < 0 \\ 1 - x/100, & if \;\; 0 \le x < 100 \\ 0, & if \;\; x \ge 100 \end{cases}$$

Hence, the force of mortality is $\frac{1}{(100-x)}$.

6.5 Importance in Insurance and Annuity Design

Reliability concepts play a crucial role in system design and engineering. Designers use reliability data to:

- Optimize component selection and system architecture.
- Plan preventive maintenance schedules.
- Minimize downtime and repair costs.
- Ensure compliance with safety and performance standards.

By integrating reliability considerations from the early stages of design, systems can be made more robust, cost-effective, and dependable.

6.6 Ageing across the Policy Lifecycle: Stages and Claim Behavior

Insurance policies and the lives they cover typically follow a lifecycle with distinct phases and changing patterns of claims. This lifecycle can be generally divided into three main stages: the Early Stage (which includes Pre-Purchase and Underwriting), the Middle Stage (covering Active Policy Management and Claims), and the Late Stage (focused on Renewal and Termination). Gaining insight into these stages and their corresponding claim behaviors is essential for sound risk management and strategic policy development. Just like technical systems, insurance policies and insured lives follow a lifecycle that can be divided into three stages:

1. Infant Mortality Period
2. Normal Life (Stable Period)
3. Wear-Out-Phase

6.6.1 Infant Mortality Period

This stage marks the beginning of a policy's lifecycle immediately following its issuance. It is commonly associated with an elevated level of claim risk, often referred to as "early duration risk." This heightened risk may result from various factors such as incomplete or inaccurate disclosure of information by the policyholder, underwriting oversights, or misinterpretations of policy terms. For example, in health insurance, a policyholder might file a claim soon after purchasing the policy for a pre-existing condition that was either undisclosed or overlooked during underwriting. Similarly, in life insurance, an early claim may result from an unexpected death shortly after the policy is issued, which could raise concerns about possible fraud or misrepresentation.

Another example could involve critical illness insurance, where a policyholder is diagnosed with a serious illness like cancer within a few months of the policy's inception. If the condition existed but was not revealed during application, it challenges the insurer's underwriting process and highlights the importance of thorough risk assessment at the outset. This stage is thus critical for insurers, as it underscores the need for robust underwriting practices, clear communication with applicants, and vigilant claim review procedures.

6.6.2 Normal Life

Once a policy moves beyond the early phase, it enters a more stable and predictable stage of its lifecycle. During this period, the claim rate tends to level off, reflecting a lower overall risk profile among the remaining policyholders. This is partly because individuals who continue their coverage are typically those with fewer health issues and a greater commitment to maintaining the policy, which reduces the likelihood of adverse claims. This phase benefits both insurers and policyholders. For insurers, the predictability allows for more accurate pricing, better reserve planning, and reduced claim volatility. For policyholders, it means continued coverage with fewer complications and typically lower premium adjustments.

For example, life insurance policyholders in their 30s or 40s who have maintained their coverage are generally at a low and stable risk of mortality. In health insurance, this stage often reflects regular utilization patterns cantered around preventive care, such as annual checkups, vaccinations, and minor medical treatments, rather than high-cost or emergency interventions. Similarly, in disability insurance, claims during this period are more likely to relate to temporary conditions rather than severe or long-term disabilities. Overall, this middle stage represents a period of maturity and balance in the policy lifecycle.

6.6.3 Wear-Out-Phase

As the policy and the insured individual age, the policy enters its late stage—characterized by a significant increase in claim risk and frequency. This phase reflects the natural progression of aging, where insured individuals are more likely to experience serious health conditions and, in the case of life insurance, an increased likelihood of death. Consequently, the severity and cost of claims rise substantially, placing greater demand on insurers' resources and claim management systems.

For example, individuals over the age of 60 are at a higher risk of developing chronic illnesses such as diabetes, cardiovascular disease, arthritis, and cognitive impairments like dementia. These conditions often require long-term care and frequent medical interventions, resulting in higher health insurance claims. In life insurance, mortality rates increase steeply with age, leading to a greater number of death claims. As a result, insurers may need to adjust premiums, modify policy terms, or cap certain benefits to manage escalating costs. This late stage also demands robust financial planning on the part of insurers. They must maintain adequate reserves to cover the increasing liabilities and ensure the long-term sustainability of their products. Moreover, careful actuarial forecasting and policyholder communication become crucial to balancing risk and meeting obligations during this high-claims phase.

6.6.4 Bathtub Curve

The Bathtub Curve is a widely recognized conceptual model used to illustrate the lifecycle of insurance policies, especially in long-duration products. It highlights how claim rates evolve over time, aligning with the three major phases of a policy's life:

- ***Early Phase - Decreasing Claim Rate:*** Initially, claim rates may be elevated due to underwriting errors, non-disclosures, or misunderstandings. However, as these issues are identified and resolved, the rate typically declines. Insurers often manage this phase by introducing waiting periods, exclusions, or contestability clauses to reduce early, potentially fraudulent, or misinformed claims.
- ***Middle Phase - Stable Claim Rate:*** During this period, claim behavior becomes more predictable. Policyholders tend to be healthier and more engaged, leading to a relatively constant and manageable level of claims. This stability allows insurers to implement targeted renewal strategies, adjust benefits prudently, and conduct policy reviews to monitor emerging risk patterns.
- ***Last Phase - Increasing Claim Rate****:* As policyholder's age or the policy matures, claim frequency and severity increase, driven by biological aging, chronic conditions, or accumulated exposure. In this "wear-out" phase, insurers must rely on well-funded reserves and sophisticated actuarial projections to cover growing liabilities.

The Bathtub Curve is particularly useful in understanding and managing long-term insurance products such as whole life insurance, long-term care insurance, and annuities. For example, in annuities, payout obligations begin modestly, remain stable during the middle years, and rise sharply as annuitants live longer and continue drawing benefits.

Insurers apply insights from this lifecycle framework to enhance multiple aspects of policy management:

- ***Premium Structuring:*** Aligning premium schedules with the evolving risk profile across the policy's life.
- ***Underwriting and Product Design:*** Using early-stage insights to craft eligibility criteria, waiting periods, and exclusions.

- ***Risk Monitoring and Engagement:*** Leveraging the stable phase to build policyholder loyalty, update risk assessments, and ensure sustainable coverage.
- ***Financial Preparedness:*** Reserving and capital planning for the late stage, when claim burdens are highest.

Ultimately, understanding this ageing lifecycle is fundamental not just for accurate pricing and reserving, but also for effective marketing, underwriting, and policyholder communication. It reinforces the core principle that risk is dynamic, evolving alongside the policyholder's age, health, and life circumstances. Recognizing this evolution enables insurers to create more responsive, resilient, and customer-centric insurance solutions.

6.7 Ageing in Actuarial and Insurance Context

Aging significantly influences various dimensions of insurance, shaping how policies are priced, managed, and reserved. As individuals grow older, their risk profiles evolve, and insurers must adjust their strategies accordingly. Key areas affected by aging include:

- ***Mortality Rates:*** This rise progressively with age, directly affecting life insurance and annuity products. In life insurance, higher mortality risk leads to increased premiums for older applicants. Conversely, in annuities, where the insurer makes regular payouts until death, increased age results in shorter expected payout periods and thus lower annuity payments.
- ***Healthcare Utilization:*** Older policyholders typically consume more medical services due to age-related conditions such as chronic diseases, mobility issues, and cognitive decline. This results in higher claim frequencies and costs for health insurance and long-term care products.
- ***Persistency***: As individuals age, they are more likely to hold onto their insurance policies, especially when health conditions make new coverage difficult or expensive to obtain. This increased persistency impacts renewal strategies and requires insurers to maintain sufficient reserves to support the continued coverage and future claims of long-standing policyholders.

To effectively manage these age-related risks, actuaries employ a variety of analytical tools, including:

- ***Life Tables:*** These provide age-specific survival probabilities and form the foundation for life insurance and annuity pricing. They help estimate the likelihood that a policyholder will survive to a given age.
- ***Force of Mortality (μ(t)):*** This represents the instantaneous rate of death at a specific age t. It offers a continuous-time approach to mortality modelling, enabling more precise actuarial calculations for dynamic products and risk assessments.
- ***Hazard Functions***: These functions model the probability of an event (such as death, illness, or disability) occurring at a particular age, given survival to that age. They are especially valuable in products like critical illness or long-term care insurance, where the risk of claim is highly age-dependent.

Incorporating these actuarial tools allows insurers to better understand and anticipate the financial and operational implications of aging, leading to more accurate pricing, adequate reserving, and more robust product design. It also supports regulatory compliance and risk-based capital management, ensuring long-term sustainability in an aging population.

Example: Life Insurance For 50-Year-Old Male: You're an actuary at an insurance company. You're asked to price a term life insurance policy for a 50-year-old male that provides a death benefit of $100,000, payable if the policyholder dies within 10 years. The policyholder will pay an annual premium, and there is no cash value. You're provided with the following simplified life table data:

Age(x)	Probability of death within one year, q_x	Age(x)	Probability of death within one year, q_x
50	0.0045	55	0.0075
51	0.0050	56	0.0084
52	0.0055	57	0.0094
53	0.0060	58	0.0105
54	0.0067	59	0.0117

Assume: 1) Deaths occur at the end of each year. 2)The interest rate is 3% per year. 3)The policyholder pays premiums at the beginning of each year.

Solution: Present value of expected death benefit (PVDB): Use the probabilities provided to calculate the present value of the $100,000 death benefit over 10 years.

$$PVDB = \sum_{t=1}^{10}(100{,}000 \times q_{x+t-1} \times v^t \times \prod_{j=0}^{t-2}(1 - q_{x+j}))$$

Where $v = \frac{1}{1+i}$ and $i = 0.03$, p_x^{t-1}= Probability that a person aged x survives t-1 years.

Year(t)	Age	q_{x+t-1}	v^t	p_{50}^{t-1}	Term Value
1	50	0.0045	0.97087	1	0.004369
2	51	0.0050	0.94260	0.9955	0.004692
3	52	0.0055	0.91514	0.9905	0.004985
4	53	0.0060	0.88849	0.9850	0.005251
5	54	0.0067	0.86261	0.9791	0.005657
6	55	0.0075	0.83748	0.9725	0.006108
7	56	0.0084	0.81309	0.9652	0.006592
8	57	0.0094	0.78941	0.9571	0.007102
9	58	0.0105	0.76642	0.9481	0.007630
10	59	0.0117	0.74409	0.9381	0.008167

Now multiply each term value by $100,000 and sum up to get:

$$\text{PVDB} = 100{,}000 \times 0.060553 = 6055.30$$

Present value of premiums (PVP): We now calculate the present value of a level premium ***P*,** using: $PVP = P \times \sum_{t=1}^{10} v^{t-1} \times p_{50}^{t-1}$

Year(t)	v^{t-1}	p_{50}^{t-1}	Term Value
1	1	1	1
2	0.97087	0.9955	0.96650

3	0.94260	0.9905	0.93364
4	0.91514	0.9850	0.90141
5	0.88849	0.9791	0.86992
6	0.86261	0.9725	0.83889
7	0.83748	0.9652	0.80833
8	0.81309	0.9571	0.77821
9	0.78941	0.9481	0.74844
10	0.76642	0.9381	0.71898

If we set PVP = PVDB, then ***P***= $\frac{6055.3}{8.56432} = 707.038$

So, present value of expected death benefit = \$6055.3 and level annual premium = \$707.038.

6.8 Model and Strategies for Managing Ageing in Insurance

As policy holders age, their risk profile changes significantly, influencing both the frequency and severity of insurance claims. Actuaries use specific statistical models to accurately estimate these age-dependent risks, enabling insurers to design sustainable products and pricing strategies.

- ***Weibull Distribution:*** This distribution is widely used in health and life insurance when the likelihood of claims increases with age or policy duration. Its hazard function increases over time, making it ideal for modeling ageing-related events such as chronic illness or disability.
- ***Gompertz Law****:* Gompertz proposed that the human mortality rate increases exponentially with age. It is especially useful in life insurance to predict mortality rates among older individuals. The survival function for Gompertz's law is given by:

  $$S(x) = exp[-m(C^x - 1)], m = B/log_e C, \quad B > 0, \quad C > 1, \ x \geq 0.$$

  If C = 1, it reduces to the exponential distribution which is characterised by the fact that the force of mortality is constant.

- ***Makeham's Law***: A refinement of Gompertz, this model adds a constant hazard to account for risks that are not age-related, such as accidents or external shocks or any cause of death. It allows for a more comprehensive view of mortality by combining biological ageing with environmental risks. The force of mortality and survival function according to the law are:

$$\mu_x = A + BC^x \text{and } S(x) = exp[-Ax - m(C^x - 1)], B > 0, A \geq -B, C > 1, x \geq 0.$$

To maintain financial stability and ensure fair pricing, insurance companies apply various strategies:

- ***Age-Based Premiums:*** Premiums are adjusted upward as policyholders age to reflect their increasing risk. For example, a 65-year-old may pay double the premium of a 40-year-old for the same health plan.

- ***Policy Adjustments:*** Insurers may include age-based exclusions, offer optional riders, or impose benefit limits to control claim costs in older age brackets.

- ***Preventive Health and Wellness Programs:*** These initiatives encourage policyholders to adopt healthier lifestyles (e.g., free screenings, gym memberships), potentially delaying the onset of costly age-related conditions like diabetes or heart disease.

- ***Adequate Reserving:*** As claim severity increases with age, insurers allocate larger reserves to ensure they can meet future obligations. Actuarial models help forecast long-term liabilities with high accuracy.

These models and strategies are critical for calculating premiums, reserves, and benefits in age-sensitive insurance products, and for ensuring the solvency of insurance operations as their policyholder base matures.

Example: Suppose the life length random variable is modeled by: 1) Gompertz's force of mortality and 2) Makeham's force of mortality. In each case, find the probability density function $g(t)$ of $T(x)$. Check whether it is a density function.

Solution: The force of mortality and survival function for Gompertz's law is

$$\mu_x = BC^x \text{ and } S(x) = exp[-m(C^x - 1)], m = B/log_e C, \quad B > 0,$$
$$C > 1, \ x \geq 0.$$

The probability density function g(t) of T(x) is then given by,

$$g(t) = {}_tp_x\, \mu_{x+t} = exp[-mC^x(C^t - 1)]BC^{x+t}, for\ t > 0.$$

It is clearly a non-negative function. To verify that it integrates to 1 over (0,∞), we substitute, $mC^x(C^t - 1) = y$. Then $mC^xC^t \log C\, dt = dy$, that is , $BC^{x+t}\, dt = dy$. Hence,

$$\int_0^\infty g(t)\, dt = \int_0^\infty exp(-y)\, dy = 1.$$

The force of mortality and survival function for Makeham's law is

$$\mu_x = A + BC^x\ and\ S(x) = exp[-Ax - m(C^x - 1)],\ B > 0, A \geq -B, C > 1,\ x \geq 0.$$

The probability density function g(t) of T(x) for Makeham's Law are,

$$g(t) = {}_tp_x\mu_{x+t} = exp[-At - mC^x(C^t - 1)][A + BC^{x+t}],\ for\ t > 0$$

It is clearly a non-negative function. To verify that it integrates to 1 over (0,∞), we substitute, $At + mC^x(C^t - 1) = y$. Then $(A + mC^xC^t \log C)\, dt = dy$, that is , $(A + BC^{x+t})dt = dy$. Hence,

$$\int_0^\infty g(t)\, dt = \int_0^\infty exp(-y)\, dy = 1.$$

6.9 Summary

This unit introduced the essential principles of reliability theory and illustrated their applications in both engineering and actuarial science. We explored the definitions of reliability, failure rates, and the bathtub curve. Core measures such as maintainability and availability were defined with insurance-specific examples. Statistical tools like survival functions, hazard rates, and various lifetime distributions were examined in the context of insurance and annuity modelling. A practical insurance system example demonstrated how reliability concepts enhance service quality and operational efficiency. These foundational ideas support further exploration into system optimization, product design, and actuarial decision-making.

This unit examined the multifaceted impact of ageing on insurance systems, emphasizing how both insured lives and insurance policies evolve over time. The progression of risk across a policy's duration can be visualized through the bathtub curve, which highlights three distinct phases: a higher-risk early stage, a stable middle period, and a rising risk phase in later years due to ageing. Ageing significantly affects core insurance parameters such as mortality, morbidity, and persistency. As individuals grow older, their likelihood of experiencing health-related events or death increases, and they tend to retain their policies longer. These changes require careful and precise actuarial modelling to ensure sustainability and fairness in insurance operations. Actuaries employ statistical tools like the Gompertz law—which captures the exponential rise in mortality with age—and the Weibull distribution, suitable for modelling increasing hazard rates related to morbidity or long-term care needs. These models enable insurers to forecast future claims more accurately and set aside appropriate reserves.

To address the financial implications of an ageing insured population, insurers implement strategies such as:

- Age-based premium structures that align pricing with risk levels,
- Policy modifications like exclusions or riders,
- Preventive wellness programs aimed at improving health outcomes,
- And robust reserving practices to meet future claim obligations.

By integrating these actuarial methods and management strategies, insurers can not only maintain solvency but also offer products that adapt to the evolving risk profiles of their customers. Understanding and responding to the ageing lifecycle is, therefore, essential for effective policy design, underwriting, pricing, and long-term financial planning in the insurance industry.

6.10 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

6.11 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.

- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT: 7 RELIABILITY ESTIMATION AND REPAIRABLE SYSTEM

Structure

7.1 Introduction

Reliability refers to the probability that a system, component, or process will perform its intended function successfully, without failure, for a specific period under stated conditions. Originally rooted in engineering, reliability theory has grown into a vital interdisciplinary field that combines elements of mathematics, statistics, and applied science. In actuarial science, reliability theory plays a crucial role by bridging the gap between engineering principles and statistical modelling. Actuaries apply reliability concepts not only to mechanical or electronic systems but also to biological systems, such as human lifespan or health outcomes. This adaptation allows actuaries to evaluate the durability, longevity, and risk associated with a wide variety of insured entities—ranging from individual lives to manufactured goods. From an insurance perspective, understanding and modelling reliability is essential for estimating risk exposure, setting appropriate premiums, and designing policies that are both fair and financially sustainable. By analyzing the likelihood of failure or an adverse event occurring within a given timeframe, insurers can better manage liabilities and fulfil their financial commitments.

Practical Relevance in Insurance:

- ***Life Insurance:*** In this context, reliability models are used to estimate survival probabilities. The probability that an individual survives beyond a certain age is a direct analogue to system reliability, where the system is a human life.
- ***Product Warranty Insurance:*** Reliability estimation is crucial for predicting when products are likely to fail. This enables insurers to estimate the cost of warranty claims and set premiums that reflect the underlying risk.
- ***Health Insurance:*** Here, reliability models are used to estimate the time until the onset of a medical condition or hospitalization. This can aid in risk stratification, pricing of health plans, and projecting future healthcare needs.

By applying reliability concepts across these domains, actuaries can build robust models that support decision-making in underwriting, reserving, and long-term financial planning. This interdisciplinary approach not only improves the precision of risk assessments but also enhances the overall sustainability and fairness of insurance systems.

Unlike non-repairable systems which are characterized by a single failure event followed by replacement repairable systems are those that, upon failure, can be restored to a functional state through repair or maintenance. This distinction necessitates a shift in reliability modelling: rather than focusing solely on the prediction of time to first failure, the analytical emphasis moves toward understanding the frequency and distribution of recurrent failures, as well as evaluating the nature and effectiveness of the corresponding repair interventions.

In actuarial and insurance contexts, the modelling of repairable systems carries substantial significance:

- **Technical Insurance and Maintenance Contracts:** Insurers offering coverage for equipment breakdown, maintenance plans, or extended service warranties must rely on robust statistical models that capture the dynamics of system uptime, failure rates, and repair cycles. Accurate reliability estimates of repairable systems allow for the precise calculation of expected claim frequencies, pricing of premiums, and establishment of appropriate reserve levels. The analysis may involve techniques such as non-homogeneous Poisson processes (NHPP) or renewal processes, which are particularly suited for modelling recurrent event data.
- **Health Insurance and Chronic Disease Management:** A parallel framework can be drawn in the domain of health insurance, where the progression and treatment of chronic diseases can be conceptualized through a "failure-repair" paradigm. For instance, recurrent hospitalizations followed by treatment and recovery in patients with chronic illnesses (e.g., diabetes, heart disease) can be analyzed similarly to repairable systems. This perspective supports the development of predictive models for healthcare utilization, informs the design of disease management programs, and assists in estimating long-term healthcare costs and liabilities.

In both technical and health-related applications, actuarial models for repairable systems enhance the insurer’s ability to manage risk, price policies accurately, and ensure financial sustainability.

7.2 Objectives

By the end of this unit, students will be able to:

1. Understand the concept of reliability and its importance in actuarial science and insurance.
2. Apply statistical methods to analyse lifetime data and censored data.
3. Design and interpret reliability testing for systems and components.

4. Utilize tools such as Reliability Block Diagrams (RBDs) and Fault Tree Analysis (FTA) to model system reliability.
5. Make informed decisions about system design, maintenance, and risk management based on reliability estimates.
6. Understand the concept of repairable systems and their relevance in reliability modelling.
7. Differentiate between preventive and corrective maintenance strategies.
8. Apply the renewal process to model recurrent failures.
9. Use Markov models to describe the behaviour of repairable systems over time.
10. Compute and interpret availability as a measure of system performance.
11. Analyse the impact of different repair policies on reliability and system performance.
12. Optimize maintenance schedules to minimize downtime and operational cost.
13. Apply repairable system models in actuarial and insurance decision-making.

7.3 Lifetime Data Analysis

Lifetime data (also known as time-to-event data) refers to the observed time until the occurrence of a particular event of interest. In engineering, this often relates to the time until failure of a system or component. In the context of actuarial science and insurance, this typically refers to events such as:

- The time until death (in life insurance),
- The time until the onset of illness or hospitalization (in health insurance),
- The time until a product breaks down or fails (in warranty or product insurance).

Analysing lifetime data requires statistical models that can handle right-censored data (where the event has not occurred yet at the time of observation) and accommodate different types of hazard or failure rates.

7.3.1 Key Distributions

1. ***Exponential Distribution:*** This distribution assumes a constant failure rate over time, which is a useful simplification in many real-world scenarios, particularly for electronic components or systems with memory less properties.

- Probability density function: $f(t) = \lambda e^{-\lambda t},\ t \geq 0$
- Cumulative distribution function: $F(t) = 1 - e^{-\lambda t}$
- Reliability function: $R(t) = P(T > t) = e^{-\lambda t}$
- Hazard rate: $h(t) = \frac{f(t)}{R(t)} = \lambda$
- Mean (Expected lifetime): $E(T) = \frac{1}{\lambda}$

Example: A digital sensor in a monitoring device has a failure rate of 0.05 failures per hour. Using the exponential distribution, actuaries can calculate the probability that the sensor lasts beyond 20 hours:

$$R(20) = e^{-0.05\times 20} = e^{-1} \approx 0.3679$$

2. ***Weibull Distribution:*** The Weibull distribution is a versatile model that can represent increasing, decreasing, or constant failure rates depending on the shape parameter.

- **Probability Density Function:** $f(t) = \alpha\, \lambda^{\alpha} t^{\alpha-1} e^{-(\lambda t)^{\alpha}},\ t \geq 0$

  Where: $\alpha > 0$ is the shape parameter, and $\lambda > 0$ is the scale parameter.

- **Cumulative Distribution Function:** $F(t) = 1 - e^{-(\lambda t)^{\alpha}}$
- **Reliability Function:** $R(t) = P(T > t) = e^{-(\lambda t)^{\alpha}}$
- **Hazard Rate:** $h(t) = \frac{f(t)}{R(t)} = \alpha\, \lambda^{\alpha} t^{\alpha-1}$.

If $\alpha < 1$ it indicates about decreasing failure rate/ early-life failures; $\alpha = 1$ shows constant failure rate; $\alpha > 1$shows increasing failure rate.

Consider an example: Mechanical parts such as car engines tend to wear out with use. An actuary might model engine failure using a Weibull distribution with$\alpha = 2.5$ and $\lambda = 0.01$, indicating an increasing failure rate over time.

3. ***Log-normal Distribution:*** This model is appropriate when the logarithm of lifetime follows a normal distribution. It captures situations where failure arises from the product of many small, random effects.

- **Probability Density Function:** $f(t) = \frac{1}{t\sigma\sqrt{2\pi}} exp(-\frac{(log\ t - \mu)^2}{2\sigma^2}),\ t \geq 0.$
- **Reliability Function:** $R(t) = P(T > t) = 1 - \phi(\frac{log\ t - \mu}{\sigma})$

Where ϕ is the cumulative distribution function of standard normal distribution.

Example: In medical underwriting, the onset of certain chronic illnesses (like diabetes) may be modeled by a log-normal distribution, where environmental and genetic factors combine multiplicatively over time.

7.3.2 Actuarial Applications of Lifetime Distributions

- ***Life Insurance****:* Lifetime distributions help estimate the probability that an individual survives beyond a given age, essential for premium calculation, reserve setting, and product design.
- ***Health Insurance****:* Used to model the time until the onset of illness or hospital admission, which informs risk classification and policyholder segmentation.
- ***Product Warranty and General Insurance****:* Actuaries use these models to estimate expected claims due to product failures, determining optimal warranty periods and pricing strategies.
- ***Pensions and Annuities****:* Lifetime data models help in predicting the duration over which payments will be made, ensuring the sustainability of retirement products.

7.4 Censored Data Analysis

In real-world actuarial and reliability studies, it is often impractical to observe the complete lifetime of every subject or component. Individuals may leave a study early, or the study may end before some subjects experience the event of interest. In such cases, censored data becomes a critical component of survival analysis.

7.4.1 Types of Censoring

1. ***Right Censoring:*** The most common type in actuarial science. It occurs when the event of interest (e.g., death, failure) has not occurred by the end of the observation period. This is typical in life insurance, where a policyholder may still be alive at the end of a study. Example: A person is followed for 10 years, but they are still alive at the end. Their exact time of death is unknown — only that it exceeds 10 years.

2. ***Left Censoring:*** Occurs when the event has already happened before the individual or system enters the observation window, but the exact time of occurrence is unknown.

Example: A patient enters a medical study with a pre-existing condition, but the time of diagnosis is unknown.

3. ***Interval Censoring:*** The event of interest occurs between two known time points, but not exactly when. Example: A product is inspected every 6 months. If it fails between inspections, the exact time of failure is unknown, only that it happened in that interval.

   Understanding and accounting for censoring is essential for making accurate inferences about lifetime distributions and survival probabilities.

7.4.2 Kaplan-Meier Estimator

The Kaplan-Meier estimator, also known as the product-limit estimator, is a non-parametric method used to estimate the survival function from lifetime data, especially when the data include right-censored observations (subjects whose exact failure time is unknown). The Kaplan-Meier estimator is used to estimate the survival probability S(t), which is the probability that a subject survives beyond time t. It allows us to construct an empirical survival curve — a stepwise function that decreases only at times when actual events (e.g., deaths or failures) occur, making it ideal for handling censored data.

At each observed event time t_i, define: n_i: Number of individuals at risk just before time t_i and d_i: Number of events at time t_i, then the Kaplan-Meier estimator is defined as:

$$\hat{S}(t) = \prod_{t_i \leq t} (1 - \frac{d_i}{n_i})$$

This formula calculates the cumulative survival probability up to time t by multiplying the conditional survival probabilities at each time.

The graphs of the Kaplan-Meier curve is a stepwise survival function that drops at each event time. The height of each step reflects the proportion surviving beyond that time. Censored data is marked (usually with a tick or dot) but does not cause a drop; it only affects the number at risk.

Assumptions: To use the Kaplan-Meier estimator appropriately, the following assumptions are made: 1) Independent censoring: The censoring mechanism is unrelated to the likelihood of the event occurring. 2) Event times are precisely known, events occur at known, exact time points. 3) Subjects have similar survival prospects: The probability of survival does not differ across individuals with the same observed time.

Actuarial Applications:

- ***Life insurance**:* Estimating survival functions when not all policyholders have died by the end of the study.
- ***Medical underwriting**:* Estimating disease-free survival times based on partial data.
- ***Warranty insurance**:* Predicting time to product failure when not all products have failed yet.

7.5 Reliability Block Diagrams

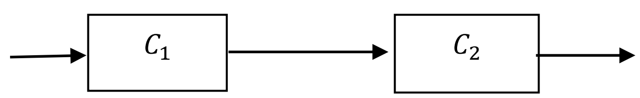
Reliability Block Diagrams (RBDs) are graphical models used to represent the logical interconnections and reliability dependencies of components within a system. They help quantify how the reliability of individual components affects the overall system performance. Each block in the diagram represents a component or subsystem, and the way these blocks are connected (in series, parallel, or mixed configurations) determines the system's ability to function without failure.

7.5.1 Common Configurations

1. ***Series System:*** In a series system, the entire system functions only if all components function. The system fails if any one component fails.

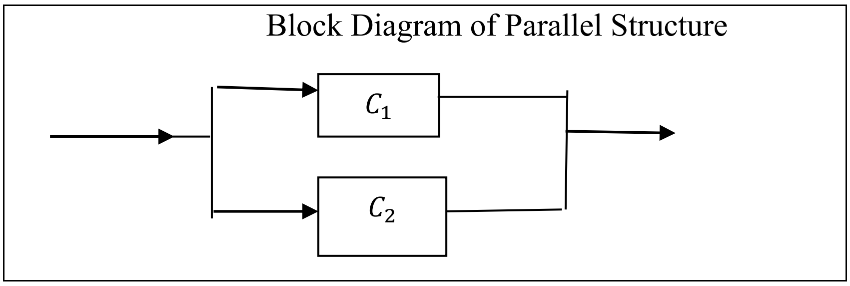
   System reliability formula: $R_{system} = R_1 \times R_2 \times \cdots R_n$. Where: R_i is the reliability of the i^{th} component, and n is the number of components in series. Reliability decreases as more components are added in series the system is only as reliable as its weakest link.

Block Diagram of a Series Structure

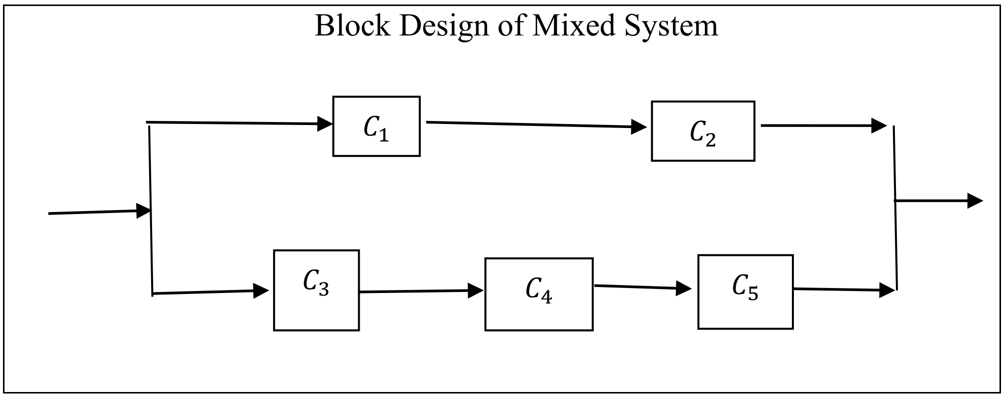


2. ***Parallel System:*** In a parallel system, the system continues to function as long as at least one component works. All components must fail for the system to fail. System

reliability formula: $R_{system} = 1 - \prod_{i=1}^{n}(1 - R_i)$. Where: $1 - R_i$ is the failure probability of the i^{th} component. Reliability increases with redundancy. Adding components in parallel improves the system's ability to withstand individual failures.



3. ***Mixed System:*** Mixed systems involve a combination of series and parallel configurations. These reflect more realistic system designs, where some components operate in parallel for redundancy, while others are in series due to functional dependency. These require step-by-step reduction: analyze smaller parallel or series groups, then combine. If a system consists of modules, we can compute first the reliability of each module, and then the reliability of the reliability of the system according to the structure function connecting the modules. The system reliability formula: $R_{system} = 1 - \prod_{i=1}^{n}(1 - \prod_{j=1}^{m} R_{ij})$. Where i^{th} component is in parallel and j^{th}component is in series.



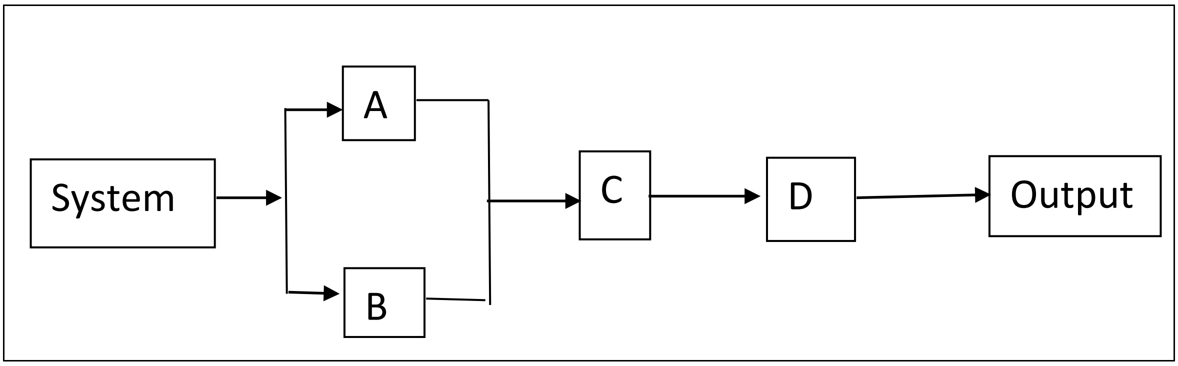
7.5.2 Actuarial Applications of RBDs

RBDs are particularly useful in insurance and risk modelling, where systems being insured (e.g., electronic devices, industrial machinery, smart infrastructure) consist of multiple interacting components.

Actuarial Uses Include:

- ***Modelling Insurance Portfolios**:* Represent multiple insured components or policies, where failure of any part could trigger a claim. For example, covering interconnected machinery in a factory.
- ***Assessing Risk Dependencies**:* Evaluate how the failure of one component (e.g., backup generator) increases the chance of system-wide failure, useful in modelling correlated risks.
- ***Product and Warranty Insurance***: Determine likelihood of system failure under warranty. RBDs help price policies that cover complex systems with both redundant and critical parts.
- ***Designing Resilience into Insurance Products**:* Suggest or evaluate designs with redundancy to minimize payout risk such as recommending backup systems for critical infrastructure.

Example: An insurance company covers a smart home system with backup power, sensors, and alarms. If any critical component fails, the house may be at risk. RBDs help assess the overall system reliability. A system consists of four components arranged as follows: Component A and component B are in parallel. Then their parallel combination is connected in series with Component C and Component D. Given the reliabilities as $R_a = 0.90, R_B = 0.85, R_c = 0.95, and\ R_D = 0.92$.



Solution: Calculate parallel reliability for A and B is:

$$R_{AB} = 1 - [(1 - R_A)(1 - R_B)] = 1 - [(1 - 0.90)(1 - 0.85)] = 1 - 0.015 = 0.985.$$

Reliability of series components C and D:

$$R_{CD} = R_C \times R_D = 0.95 \times 0.92 = 0.874.$$

Then the reliability of the system is: $0.987 \times 0.874 = 0.861$.

There is an 86.1% chance the system will function properly when needed and the insurer can quantify the risk of failure 13.9%, helping:1) Set appropriate premiums, 2) Evaluate cost-benefit of requiring backup systems, 3) Understand correlated component risks.

7.6 Fault Tree Analysis

Fault Tree Analysis (FTA) is a top-down, deductive failure analysis method used to determine the possible causes of a system-level failure (known as the Top Event). It is widely applied in reliability engineering and increasingly useful in actuarial science, especially for insuring complex systems.

- ***Top Event**:* The undesired event under investigation — for example, "device fails to deliver insulin."
- ***Basic Events**:* Root-level causes with known or estimated probabilities (e.g., "battery fails," "circuit breaks").
- ***Intermediate Events***: Events caused by combinations of other events. For example, a subsystem fails when both "sensor" and "controller" fail.
- ***Logic Gates**:*
  - ***AND Gate**:* All input events must occur for the output event to occur.
  - ***OR Gate**:* The output event occurs if any of the input events occur.

These gates structure the logic of the fault tree, helping actuaries and engineers understand which combinations of failures lead to system-level failure.

Quantifying Risk: FTA enables probabilistic modeling of the top event using the structure of basic and intermediate events.

For an AND gates: All input events must occur for the output to happen. The probability is:

$$P_{AND} = \prod_{i=1}^{n} P_i, \text{ where } P_i \text{ is the probability of the } i^{th} \text{ basic event.}$$

For OR gates: Only one event needs to occur. The formula becomes:

$$P_{OR} = 1 - \prod_{i=1}^{n}(1 - P_i).$$

This reflects the cumulative probability of failure due to any of the contributing factors.

Actuarial Use Case: In product liability insurance, especially for medical devices, smart home systems, or industrial machinery, FTA is essential to: 1) Identify critical failure pathways. 2) Quantify overall risk of product failure. 3) Determine how likely multiple failure are to occur together.

Example: An insulin pump may fail if 1) The battery dies (P = 0.02), 2) The software glitches (P = 0.01), or 3) The delivery motor jams (P =0.03).

Solution: If any one of these leads to failure (an OR gate), then:

$$P_{top} = 1 - (1 - 0.02)(1 - 0.01)(1 - 0.03) = 1 - 0.941 = 0.059$$

So, there's a 5.9% chance the pump will fail during a usage cycle.

Practical Applications in Actuarial Science and Insurance: Reliability methods like FTA and RBDs support core actuarial functions by modeling uncertainty, predicting loss occurrences, and optimizing decision-making.

1. **Insurance Product Design**

- ***Life Insurance****:* Use survival models (e.g., Kaplan-Meier, exponential) to estimate life expectancy and price policies accordingly.
- ***Warranty Insurance****:* Model failure time distributions (e.g., Weibull) to determine warranty coverage limits and terms.
- ***Health Insurance***: Predict time-to-illness using lifetime data, enabling optimal policy period design and reserve estimation.

2. **Premium Setting and Risk Classification**

- Reliable systems = lower premiums: If a system shows low failure probability, the insurer can offer better pricing.
- High-risk components = higher pricing or policy exclusions: Policies may exclude or rate-up specific components if their failure rates are high.

3. **Maintenance and Replacement Decisions:** Actuarial models help organizations and insurers make cost-effective decisions regarding upkeep.

   - ***Preventive Maintenance:*** Determine the optimal time to replace or service a component before it fails.

   - ***Cost-Benefit Analysis:*** Weigh the cost of proactive maintenance vs. the expected cost of failure and claim payouts.

4. **Portfolio Risk Assessment:** It is the process of analyzing and quantifying the collective risk associated with a group (or portfolio) of insured policies, systems, or components. Instead of focusing on a single item, actuaries examine how multiple risks interact, fail, or succeed together which is crucial for setting premiums, maintaining solvency, and managing reserves. In the real world, insurers do not cover just one policyholder or one device they insure thousands or millions of lives, machines, or health events. Assessing the risk in isolation can be misleading if it does not consider:
   - Dependency between risks (e.g., economic downturns affecting multiple insured businesses),
   - Simultaneous failures (e.g., multiple car parts failing due to poor manufacturing),
   - Catastrophic events (e.g., natural disasters impacting all homes in a region).

Survival models, Copulas, aggregate risk models and Monte Carlo Simulations are some of the tools used as actuarial tools to estimate the time to failure, to model dependency structure between risks, used to calculate the total expected claims and for simulating thousands of scenarios, especially when failure probabilities and dependencies are complex.

7.7 Maintenance Strategies

Maintenance strategies play a crucial role in ensuring the continued reliability and performance of systems over time. In reliability engineering, maintenance activities are broadly classified into two categories: Preventive Maintenance and Corrective Maintenance. From an actuarial and insurance standpoint, understanding these strategies is essential for accurate risk assessment, premium pricing, and claim forecasting.

7.7.1 Preventive Maintenance

Preventive maintenance refers to scheduled actions undertaken before a failure occurs, with the objective of reducing the likelihood of system breakdown. These actions may include routine servicing, timely replacement of components, regular inspections, and the application of condition-monitoring technologies.

- ***Proactive Nature*:** PM is initiated *before* a failure occurs. It relies on predefined schedules or condition-based triggers, making it inherently anticipatory rather than reactive.

- ***Predictive Potential*:** In advanced applications, preventive maintenance may include predictive elements through data analysis, sensor monitoring, and reliability-centred maintenance (RCM). This enables early detection of wear and degradation patterns.

- ***Goal-Oriented:*** The primary objectives of PM are to: Enhance system reliability and availability. It reduces unplanned downtime also lower long-term operational and repair costs and extend asset lifespan.

- ***Resource Planning:*** Preventive maintenance can be scheduled during non-peak hours or planned downtimes, making it more manageable in terms of labour, materials, and costs.

- ***Insurance and Actuarial Significance:*** Preventive maintenance is frequently incentivized within insurance frameworks due to its role in mitigating risk and reducing the frequency and severity of claims. For instance:

    1. ***Health Insurance:*** Policies may offer reduced premiums or additional benefits for policyholders who undergo regular preventive health screenings, vaccinations, or chronic disease monitoring.

    2. ***Automobile Insurance:*** Insurers may encourage routine vehicle inspections and servicing through discounts or reward programs, recognizing that well-maintained vehicles have a lower probability of accident or mechanical failure.

    3. ***Equipment Breakdown Insurance:*** Maintenance records are often a critical underwriting factor, with preventive maintenance practices reducing expected claim frequency.

From an actuarial modelling perspective, incorporating the effects of preventive maintenance into reliability and survival models can refine estimates of hazard functions and improve the accuracy of long-term cost projections.

7.7.2 Corrective Maintenance

Corrective maintenance involves actions taken after a failure has occurred, with the aim of restoring the system to a functional or operational state. This form of maintenance is typically unplanned and may be associated with emergency repairs or part replacements.

- **Reactive Nature:** CM is triggered only in response to actual failures. It does not involve planning or forecasting but rather immediate response to disruptions.
- **Potentially Costly:** Corrective actions may involve high repair or replacement costs, particularly if the failure is catastrophic or occurs at critical moments. The indirect costs (e.g., lost productivity, downtime) can also be significant.
- **Risk of Secondary Damage:** Delayed or insufficient maintenance may result in cascading failures, where a single malfunction leads to further component or system damage.
- **Unpredictability:** Unlike PM, corrective maintenance introduces variability and unpredictability in operations, making financial planning more challenging for both companies and insurers.

Insurance and Actuarial Significance: Corrective maintenance holds critical relevance in both insurance operations and actuarial modeling, as it is directly associated with the occurrence of claim events. A high frequency of corrective maintenance often correlates with elevated claim ratios, reflecting greater financial liability for insurers and increased operational costs for insured entities. This trend is particularly evident in sectors such as machinery breakdown insurance, health insurance—where emergency treatments or hospitalizations are treated as unplanned corrective events—and extended warranty programs, where each repair post-failure may generate a claim. From an actuarial perspective, modeling such events is essential for reliable loss forecasting, determining appropriate claim reserves, and conducting experience ratings. Accurate representation of corrective maintenance trends enables insurers to price products more effectively, manage claim volatility, and ensure the long-term sustainability of insurance offerings.

7.8 Markov Models for Repairable Systems

A Markov model is a type of stochastic process that describes a system which transitions between a set of discrete states over time, where the probability of moving to a future state depends solely on the current state, not on how the system arrived there. This defining feature is known as the Markov property or memory less property. Mathematically, for a process $\{X(t)\}_{t\geq 0}$, the Markov property implies:

$$P(X(t+\Delta t)\;=\;j|X(t)\;=\;i,\qquad past\;history)\;=\;P(X(t+\Delta t)\;=\;j|X(t)\;=\;i)$$

For all times t, states i, j and small-time intervals Δt.

Core components of a Markov Model:

1. **States:** The finite or countable set of conditions the system can be in (e.g., "Working", "Failed").
2. **Transition Rates or Probabilities:** In continuous-time Markov chains (CTMCs), which are often used in reliability and actuarial modelling, transitions occur according to exponential waiting times governed by transition rates (e.g., failure rate λ, repair rate μ).
3. **Transition Matrix:** A matrix that contains the probabilities (discrete-time) or rates (continuous-time) of moving from one state to another.
4. **Initial State Distribution:** The probability distribution describing where the system starts at time t=0.

Markov models provide a robust framework for analyzing the behavior of repairable systems, particularly when system transitions follow a memory less property where the future evolution of the system depends only on its present state and not on the sequence of events that preceded it. These models are highly valuable in reliability engineering and actuarial science, enabling the quantification of system performance, failure behavior, and repair dynamics over time.

7.8.1 Two- State Continuous-Time Markov Model

The two-state continuous-time Markov model is one of the simplest and most widely used stochastic models in reliability engineering and actuarial science. It is especially suitable for representing repairable systems, where a system alternates between working and failed states over time. The model assumes the system can exist in only two mutually exclusive states:

- **State 0 – Operational (Working):** The system is fully functional and performing its intended task.

- **State 1 – Failed (Non-operational):** The system has experienced a failure and is not functioning until it is repaired.

These states form a continuous-time Markov chain (CTMC) with exponential transition times, meaning the time spent in each state follows an exponential distribution, characterized by constant transition rates.

Let the transition rates between the two states be defined as:

- **λ Failure rate**: the rate at which the system transitions from working (State 0) to failed (State 1).
- **μ Repair rate**: the rate at which the system transitions from failed (State 1) back to working (State 0).

These rates from the generator matrix Q as follows:

$$\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

Each row sum to zero, as required for a continuous-time Markov chain. The steady-state (or long-run) probabilities represent the proportion of time the system spends in each state after a long period. Let:P_0 = Probability the system is in State 0 (Working), it represents system availability, and P_1 = Probability the system is in State 1 (Failed), it represents system unavailability. In steady state:

$$P_0 = \frac{\mu}{\lambda + \mu}, \qquad P_1 = \frac{\lambda}{\lambda + \mu}$$

These equations are derived by solving the balanced equations of the Markov chain.

Actuarial and Insurance Applications of Markov Models:

Markov models are widely employed in actuarial science and insurance for analysing stochastic processes related to:

1. ***Health Insurance:***
    - Modelling disease progression (e.g., healthy → ill → recovery → relapse).
    - Calculating expected number of hospitalizations and associated costs.
    - Projecting long-term care or chronic disease burdens using multi-state models.

2. ***Life Insurance and Survival Analysis:***
   - Representing transitions between states such as "alive", "disabled", and "dead".
   - Computing expected present values of future benefits and reserves.

3. ***Equipment and Maintenance Insurance:***
   - Estimating expected downtime and failure frequency for repairable systems.
   - Supporting pricing and reserving in service contracts, warranties, or machinery breakdown insurance.

4. ***Critical Illness Products and Long-Term Disability Insurance:***
   - Assessing transitions between healthy, ill, recovery, and death states with corresponding probabilities and financial outcomes.

The Markov model is a powerful mathematical tool for analysing systems that evolve unpredictably over time but do so with probabilistically defined patterns. Its memory less structure and analytical tractability make it especially suitable for modelling repairable systems, health transitions, and recurring events in actuarial and insurance contexts. By quantifying the likelihood and timing of state changes, actuaries can better forecast claims, price products, and manage risk.

Example: An insurance company offers coverage for a machine used in industrial production. The machine is modeled as a repairable system using a two-state continuous-time Markov model: State 0 (Operational): The machine is working, State 1 (Failed): The machine has failed and is under repair.The failure rate of the machine is $\boldsymbol{\lambda}$=0.02 failures per hour (i.e., the machine fails, on average, every 50 hours).The repair rate is μ=0.08repairs per hour (i.e., the average repair time is 12.5 hours). Calculate,

1. The steady-state probabilities that the machine is in:(a) Operational state, (b) Failed state
2. Calculate the expected availability and unavailability of the machine.
3. Interpret the results in an insurance context.

Solution:

Step 1: Steady-State Probabilities:

The steady-state probability that the system is operational (i.e., in state 0) is given by:

$$P_0 = \frac{\mu}{\lambda + \mu} = \frac{0.08}{0.02 + 0.08} = 0.80$$

The steady-state probability that the system is in the failed state (state 1) is:

$$P_1 = \frac{\lambda}{\lambda + \mu} = \frac{0.02}{0.02 + 0.08} = 0.20$$

Step 2: System Availability and Unavailability

These are just the steady-state probabilities:

- Availability (A) = Probability machine is operational $= A = P_0 = 0.80$
- Unavailability (U) = Probability machine is failed $= U = P_1 = 0.20$

Thus, over a long-time horizon, the machine is expected to be functioning 80% of the time and out of service 20% of the time.

Step 3: Insurance and Actuarial Interpretation

From the insurer's perspective:

- The 20% unavailability indicates that 1 out of every 5 hours, on average, the machine will be non-functional.
- This has direct implications for:

  - **Service Contract Pricing:** Downtime might be covered by the insurer under a loss-of-use clause.
  - **Claims Reserving:** If each downtime period leads to a claim, the expected number of claims over a policy period can be modelled using these probabilities.
  - **Premium Loading:** Higher failure rates lead to increased risk and should result in higher premiums or preventive maintenance incentives.

In a health insurance setting, a similar model could represent a chronic illness cycle (e.g., hospitalization and recovery), where the "operational" state is the person being healthy, and the “failed” state is the person being hospitalized. The probabilities help insurers model expected hospitalization days and determine reserves for such claims.

7.8.2 Model Extensions

While the two-state continuous-time Markov model provides a foundational framework for modelling systems that alternate between functioning and failed states, real-world systems often exhibit more complex behaviours that require richer modelling structures. To accommodate this, the basic model can be extended into multi-state Markov models, capturing degradation, delayed repairs, standby systems, and more. These model extensions are particularly valuable in actuarial science and insurance for accurately representing the dynamics of health conditions, technical failures, and service processes.

Beyond the standard two-state framework, Markov models can be extended to incorporate a broader range of real-world complexities that are essential for accurate actuarial and insurance analysis. One such extension is the multi-state operational model, which accounts for systems transitioning through multiple stages of performance degradation before eventual failure. This structure is especially relevant for modelling equipment covered under engineering insurance policies, where components may not fail instantaneously but instead deteriorate over time. In a similar vein, degradation-based Markov models provide a framework for representing irreversible transitions through worsening states, culminating in system failure. These models have critical applications in warranty analysis and chronic illness modelling, allowing actuaries to estimate the progression of risks over time and assess intervention strategies at various stages.

Incorporating repair delays introduces another level of realism by acknowledging that maintenance or recovery does not occur instantaneously. Markov models with intermediate states such as "awaiting repair" or "under repair"—are particularly valuable for representing systems subject to repair queues or logistical delays. Such models find applications in claim settlement modelling, hospital treatment delays, and equipment insurance, where delayed recovery impacts both service levels and financial outcomes. Additionally, redundant, and standby system models introduce parallel operational paths wherein backup units activate following the failure of primary systems. These extensions are vital in evaluating business continuity coverage, critical system reliability, and risk pricing in industries dependent on uninterrupted operations.

Moreover, semi-Markov models relax the memory less assumption of traditional Markov processes by allowing for non-exponential sojourn times. This flexibility enables a more accurate representation of time-dependent phenomena such as prolonged hospitalizations or extended equipment downtimes. For actuaries, semi-Markov models enhance the precision of loss modelling, experience rating, and duration-dependent reserving. Finally, the inclusion of absorbing states—such as death, permanent disability, or system retirement—is essential for modelling terminal outcomes. These states are instrumental in life insurance, disability coverage, and warranty product analysis, as they mark the cessation of liabilities or benefits. Altogether, these Markov model extensions provide a robust mathematical framework for capturing the dynamic behaviour of systems and individuals over time, significantly improving the accuracy and relevance of actuarial forecasts and insurance pricing models.

7.9 The Renewal Process

The renewal process is a fundamental stochastic model used to describe systems that, after each failure and subsequent repair, are restored to an "as-good-as-new" condition. Unlike non-repairable systems that fail only once, a renewal process captures the cyclical nature of failures and restorations, making it particularly applicable to repairable systems. The central quantity of interest is the renewal function, denoted by H(t), which represents the expected number of failures or renewals in a time interval [0, t]. Let F(t)be the cumulative distribution function of the time-to-failure of a component, and f(t)its corresponding probability density function. The renewal function satisfies the integral equation:

$$H(t) = F(t) + \int_0^t H(t - x) f(x)\, dx$$

This recursive relation reflects that the expected number of failures up to time t consists of an initial failure time governed by F(t), followed by subsequent failures whose timing is influenced by the convolution of f(x)with the renewal function itself.

In actuarial science, the renewal function has critical applications in modelling maintenance contracts and service insurance products. For instance, in equipment maintenance insurance, H(t)can be used to estimate the expected number of repairs (claims) during the insured period, which directly influences premium calculation and risk pricing. Similarly, in health insurance, particularly in modelling chronic or recurring illnesses, the renewal process framework can capture the repeated occurrence of hospital admissions or treatments over time. This enables actuaries to estimate not only expected claim counts, but also to structure reserving strategies and assess the financial sustainability of health plans. Therefore, the renewal process forms an essential tool for predicting future liabilities in various lines of insurance involving recurrent events.

Example: An insurance company provides maintenance coverage for a machine. The time to failure of the machine follows an exponential distribution with a constant failure rate $\lambda = 0.5$ failures per year (i.e., the average time between failures is 2 years). Each time the machine fails, it is repaired and returned to as-good-as-new condition, making it a classic renewal process. We want to compute the expected number of failures (repairs) over a 5-year policy period, i.e., calculate the renewal function H(t) for t=5.

Solution: For the exponential distribution with rate λ=0.5, we have:

- CDF:$F(t) = 1 - e^{-\lambda t} = 1 - e^{-0.5t}$
- PDF: $f(t) = \lambda e^{-\lambda t} = 0.5\, e^{-0.5t}$

For the exponential distribution, the renewal process becomes a Poisson process with intensity λ, and the renewal function simplifies to:

$$H(t) = \lambda t$$

If interracial (failure) times are exponentially distributed, the expected number of renewals (failures) in time t is simply proportional to time. Now calculate H(5)

$$H(5) \ = \ \lambda \times 5 \ = \ 2.5$$

So, the expected number of failures (and hence, expected number of insurance claims for repairs) over a 5-year period is 2.5. Over a 5-year period, the expected number of failures (repairs) is 2.5. In an actuarial context, this value can be used to:

- **Set premiums:** If the average cost per claim is, say, \$400, then the expected claim cost is 2.5×400=\$1000, which will inform the pure premium.
- **Reserve estimation:** If the policyholder renews their policy annually, the insurer may reserve funds assuming 0.5 expected claims per year.
- **Risk pricing:** Higher values of λ imply higher risk and justify higher premiums or preventive maintenance incentives.

7.10 Availability Modeling

Availability refers to the probability that a system is functioning (i.e., in an operational state) at a specific point in time. It is a key performance indicator in the analysis of repairable systems, particularly in engineering, reliability, and actuarial science. Unlike reliability, which typically measures the probability of survival over time without failure, availability explicitly accounts for both failures and repairs, making it highly relevant in scenarios where restoration is possible and frequent.

7.10.1 Instantaneous Availability

Instantaneous availability, denoted A(t), is defined as the probability that a system is operational (working) at a specific point in time *t.* It is a time-dependent function and is particularly important during the transient phase of system operation i.e., before the system reaches steady-state conditions. Let the system be modeled by a two-state process: State 0 means operational, and State 1 means Failed. The instantaneous availability at time t, $A(t)$, is given by:

$$A(t) = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} e^{-(\lambda+\mu)t}$$

Assuming, $\boldsymbol{\lambda}$ = Constant failure rate, μ = Constant repair rate. The system is initially operational at time t = 0.

- The first term, $\frac{\mu}{\lambda+\mu}$, is the steady-state availability. It represents the long-run probability that the system is operational.
- The second term, $\frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}$, is the transient term. It reflects how the system transitions over time from its initial condition toward the steady state.
- As $t \to \infty$, the exponential term decays to zero, and $A(t) \to \frac{\mu}{\lambda+\mu}$.

Instantaneous availability is particularly useful in:

- Short-term risk assessment: E.g., probability that a critical machine or system is operational at a specific point (like a deadline or peak season).
- Service-level contracts: Where guarantees are made about uptime at specific times.
- Health insurance: For modelling the probability a patient is in a healthy state at a specific time during treatment cycles.

- Reserving and pricing: Actuaries can estimate short-term liability exposure by understanding how availability evolves over time, especially for new or recently repaired systems.

Example: An insurance company provides coverage for a machine. The failure and repair rates are as follows: Failure rate λ=0.01 per hour, Repair rate μ=0.09 per hour and the machine starts in working condition at time t = 0. What is the instantaneous availability at time t=10 hours?

Solution: We have $\lambda = 0.01, \mu = 0.09 \; and \; t = 10$, need to calculate A(t)

$\mathrm{A}(10) = \frac{0.09}{0.01+0.09} + \frac{0.01}{0.01+0.09} \mathrm{e}^{-(0.01+0.09)10} = 0.9367$. So, the instantaneous availability at 10 hours is approximately 93.7%. This means at 10 hours into operation, the machine has a 93.7% probability of being operational, given the specified failure and repair rates. This estimate can help the insurer: Estimate expected claim exposure, Price risk more accurately, and design incentives for preventive maintenance (e.g., to improve availability).

7.10.2 Steady State Availability

Steady-state availability, denoted A_{ss}, represents the long-run proportion of time a system is expected to be operational. It is calculated when the system has reached equilibriummeaning that the rates of failure and repair are constant over time and the transient effects have stabilized. For a two-state continuous- time Markov model, where: **λ** = failure rate and μ = repair rate. The steady-state availability is given by: $A_{ss} = \frac{\mu}{\mu+\lambda}$, this indicates the fraction of time the system is up and running in the long term.

7.10.3 Steady State Unavailability

Steady-state unavailability is the long-run probability that a repairable system is not operational—i.e., the system is in a failed state. It complements steady-state availability and is critical when modelling the risk of downtime in reliability and insurance contexts. In steady state, the rates of failure and repair are constant, and the system cycles between working and failed conditions.

Steady-state unavailability = $U_{ss} = \frac{\lambda}{\lambda+\mu} = 1 - A_{ss}$.

Example: An insurer covers a power generator under an equipment breakdown policy. From historical data: The generator fails on average once every 50 hours with an average repair time is 2. What is the steady-state availability of the generator?

Solution: Failure rate of generator: $\lambda = \frac{1}{50} = 0.02$ failure/hour.

Repair rate $\mu = \frac{1}{2} = 0.5$ repairs/hour.

Steady-state availability = $A_{ss} = \frac{0.5}{0.02+0.5} = 0.9615$

Steady-state unavailability = $U_{ss} = 1 - 0.9615 = 0.0385.$

This means the generator is expected to be operational 96.15% of the time in the long run, and down only 3.85% of the time. From an actuarial or insurance perspective, this helps in: Estimating expected downtime-related claims, designing pricing models for service contracts and Incentivizing maintenance to improve availability (thus reducing risk and claims).

7.10.4 Mean Down Time

Mean Down Time (MDT) is the average amount of time a system remains in a non-operational (failed) state before it is repaired and restored to service. It's a measure of system downtime and is vital in reliability engineering, insurance pricing, and service-level guarantees. $MDT = \frac{1}{\mu}$. This assumes an exponential repair time distribution, which is common in continuous-time Markov models.

MDT is essential in contexts such as: 1) Downtime coverage insurance: Policies that pay when equ1ipment or systems are out of service. 2) Business interruption insurance: MDT helps model expected lost productivity. 3) Health insurance and service delivery: In modelling how long, a patient may stay untreated or how long a healthcare resource (e.g., MRI machine) remains unavailable. 4) Pricing warranties or service-level agreements (SLAs): A high MDT can imply higher expected claims or penalties.

7.11 Impact of Repair Policies

Repair policies define the strategies by which systems are restored after a failure and play a critical role in determining system longevity, reliability, and the economic burden associated with maintenance. Broadly, repair strategies are categorized into minimal repair, perfect repair, and imperfect repair. A minimal repair restores the system to the condition it was in just before failure essentially without improving its underlying reliability. In contrast, a perfect repair fully rejuvenates the system to an "as-good-as-new" state, resetting its failure distribution entirely. An imperfect repair lies between these two extremes, where the system is partially restored, and its reliability is improved but not fully reset.

From an actuarial and insurance standpoint, understanding the underlying repair policy is crucial for accurate risk modelling and pricing of maintenance or breakdown insurance products. For instance, under minimal repair assumptions, the failure intensity remains unchanged, leading to higher claim frequencies over the policy term. In contrast, perfect repair reduces future failure likelihood, resulting in lower expected claims. Imperfect repairs introduce intermediate complexities and often require stochastic modelling approaches to assess their long-term cost impact.

These distinctions directly influence expected cost per claim, reserving requirements, and even premium structuring. Furthermore, the perceived effectiveness of the repair policy affects customer satisfaction, which in turn impacts policyholder retention and long-term profitability. In summary, incorporating repair policy assumptions into actuarial models is essential for aligning insurance coverage with the actual performance dynamics of insured systems or services.

7.12 Maintenance Optimization

In reliability engineering and actuarial analysis, a key objective is to determine an optimal maintenance schedule that either minimizes the total expected cost over a planning horizon or maximizes system availability while managing risk exposure. Maintenance decisions involve balancing competing cost factors, including the cost of preventive maintenance(C_p), the cost of failure or corrective repair (C_f), and the penalty associated with system downtime (C_d). Actuarially, this leads to a cost minimization problem expressed as:

Optimal maintenance interval=$arg\ min_t(C_p + P_{fail}(t)\ \times C_f\ +\ D(t) \times\ C_d)$

Here, P(Failure)denotes the probability of system failure within a given time interval, which is influenced by the maintenance strategy chosen. This formula helps identify the interval at which preventive maintenance should be performed to strike a balance between frequent maintenance (which incurs higher direct costs) and infrequent maintenance (which increases the risk and cost of failure).

Various quantitative tools are employed in actuarial science to solve this optimization problem, including dynamic programming, stochastic simulation, and cost-benefit analysis. These methods allow actuaries to model complex systems and assess different maintenance strategies under uncertainty, especially when failure times follow non-exponential distributions or when external risk factors are present.

In practical terms, insurers offering fleet vehicle insurance or equipment breakdown coverage can apply these models to offer customized maintenance recommendations to policyholders. For example, a commercial transport company may receive a data-driven maintenance schedule that minimizes total operational risk, reduces claim frequency, and improves long-term profitability for both the insurer and the insured. Moreover, offering incentives for adherence to optimized maintenance schedules, such as premium discounts, can serve as an effective tool for both risk mitigation and customer retention.

7.13 Summary

Reliability estimation plays a vital role in modern actuarial science, serving as a powerful framework for evaluating the durability and risk of systems whether they are machines, human lives, or insured portfolios. It allows actuaries to move beyond deterministic assumptions and incorporate real-world uncertainty into their models, leading to more informed decisions. In this chapter, we've explored how lifetime data and censored observations are handled using statistical tools like the Kaplan-Meier estimator, and how important probability distributions (Exponential, Weibull, Log-normal) underpin reliability analysis. We have also examined the use of RBDs and FTA to model complex systems and identify critical points of failure.

These tools collectively allow actuaries to:

- Estimate survival and failure probabilities,
- Structure insurance products more effectively,
- Set premiums that reflect underlying risks,
- Assess entire portfolios with respect to correlated failures,
- Make informed maintenance and replacement decisions.

By integrating engineering concepts with statistical and financial modelling, reliability analysis enhances an actuary's ability to predict outcomes, price uncertainty, and maintain solvency especially in areas like life insurance, warranty and product liability insurance, health risk modelling, and enterprise risk management. Whether used to predict a product's failure or to model human longevity, reliability theory empowers actuaries to deliver more accurate and resilient solutions in a world where uncertainty is the only certainty.

Repairable system reliability models play a crucial role in accurately representing real-world systems that experience multiple failures and subsequent restorations over time. Unlike non-repairable models that focus solely on time to first failure, repairable system models incorporate the cyclical nature of failure and repair, making them indispensable in sectors such as manufacturing, healthcare, and transportation: -where systems are expected to recover and resume operation after breakdowns. For actuaries and insurers, a comprehensive understanding of both failure probabilities and the stochastic dynamics of repair and maintenance is essential. This knowledge enables more accurate estimation of risk exposure, claim frequency, and expected downtime, which directly influence product pricing and reserving strategies.

Analytical tools such as the renewal process, Markov chains, and availability models provide robust mathematical frameworks to capture these dynamics. The renewal process, for instance, models recurring failure-repair cycles, while Markov models offer state-based representations of system behaviour under uncertainty. Availability analysis quantifies the proportion of time a system is operational, which is vital for service-level agreements and uptime guarantees. By integrating these models, actuaries can design more efficient and equitable insurance products, optimize preventive maintenance schedules, and support data-driven decision-making for contract structures and premium incentives. Ultimately, this leads to enhanced sustainability in risk management, improved customer satisfaction, and more profitable insurance operations.

7.14 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

7.15 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT-8 GROWTH MODELS, ACCELERATED LIFE TESTING AND MULTIPLE LIFE FUNCTIONS

Structure

8.1 Introduction

In the domains of reliability engineering and actuarial science, a comprehensive understanding of system behaviour over time is essential for informed decision-making in insurance pricing, warranty design, and risk evaluation. This chapter focuses on three pivotal concepts that underpin such assessments: reliability growth models, accelerated life testing (ALT), and multiple life functions.

In actuarial science, the ability to assess and predict the reliability and longevity of systems or products is crucial in pricing insurance policies, designing warranty schemes, and estimating risk reserves. Two key tools that facilitate such evaluations are growth models and ALT. Growth models describe how a systems or product's reliability improves over time as a result of repeated testing, preventive maintenance, and corrective actions. This concept is particularly relevant in the actuarial domain for analysing long-term insurance products, such as extended warranties, health insurance for aging populations, and risk-based capital models. As products or systems are continually evaluated and repaired, their failure rates may decline over time, and this progressive improvement can be effectively modelled using statistical frameworks such as the Duane model and the AMSAA (Crow-AMSAA) model. These models are widely used to estimate the reliability trajectory of complex systems and to inform policy pricing and reserve strategies.

Accelerated Life Testing (ALT) serves as a complementary approach in which products or systems are subjected to elevated stress conditions such as increased temperature, voltage, or mechanical load—to induce failures more quickly than would occur under normal usage. This enables reliability analysts and actuaries to rapidly estimate failure probabilities and time-to-failure distributions, which are essential inputs for actuarial present value calculations, premium setting, and loss reserving. By extrapolating from ALT data, actuaries can predict how long a product or component is likely to last under typical operating environments, thereby supporting more accurate warranty pricing, insurance underwriting, and capital requirement assessments.

The third core topic of this unit, multiple life functions, pertains specifically to life-contingent actuarial models involving two or more individuals. These functions are foundational in valuing life insurance and annuity products such as joint life annuities, last survivor benefits, and multi-life endowment assurances. They capture the interdependence between lives and enable the computation of joint survival probabilities, last survivor probabilities, and corresponding actuarial present values. These models are essential in designing equitable products and assessing portfolio-level longevity risk.

Collectively, these methodologies growth modelling, accelerated life testing, and multiple life function analysis provide a rigorous and practical framework for evaluating the reliability and longevity of systems, whether mechanical or biological. Their integration allows for enhanced decision-making in insurance design, product development, financial planning, and risk management.

8.2 Objectives

By the end of this unit, students will be able to:

- Understand the concept and importance of reliability growth in systems.
- Describe and apply the Duane and AMSAA models for reliability growth analysis.

- Explain the need for and methodology behind accelerated life testing (ALT).
- Identify various types of stress used in ALT and their effects on system performance.
- Apply mathematical models (e.g., Arrhenius, Eyring, and Inverse Power Law) to ALT scenarios.
- Understand how Actuarial models involve two or more lives, vital in insurance product design such as joint annuities or survivor benefits.
- Analyse failure data from accelerated tests and extrapolate results to normal use conditions.
- Evaluate the advantages and limitations of growth models and ALT in practical engineering and actuarial contexts.

8.3 Reliability Growth Models

In many actuarial applications such as extended warranties, maintenance contracts, or insurance for technological products understanding how reliability evolves over time is critical for forecasting future claims and setting appropriate premiums. Reliability growth models are mathematical frameworks used to track and quantify improvements in a system's reliability during development or operation. These improvements typically result from failure identification, corrective actions, design enhancements, or better maintenance strategies. Actuaries apply these models to better estimate future liabilities and adjust risk exposures over time. Reliability growth affects key actuarial outcomes such as:

- Claim frequency over time (declining failure rate means fewer claims)
- Reserve estimates for future claims
- Premium calculations for long-term warranty and maintenance products
- Product lifetime distributions, which influence survival modelling and policy durations

Two widely used reliability growth models are the Duane Model and the AMSAA (Crow-AMSAA) Model.

8.3.1 Duane Model

The Duane model assumes that the Mean Time Between Failures (MTBF) improves following a power law as the system undergoes testing or field exposure. It's one of the simplest models but provides useful insights, especially during early phases of reliability improvement.

$$MTBF(t) = Kt^b$$

Or in failure rate terms: $\lambda(t) = \lambda_0 t^{-b}$.

Where, K: constant of proportionality, b: growth rate parameter, $\lambda(t)$: instantaneous failure rate. If 0<b<1, reliability is improving over time.

Example: An actuary is working with a manufacturer who offers a warranty on smart home devices. These devices have undergone field testing for reliability improvement. The manufacturer reports the cumulative Mean Time Between Failures (MTBF) at various cumulative test times. The actuary decides to use the Duane model:

$$MTBF(t) = Kt^b$$

Cumulative Time (t in hours)	Cumulative MTBF (hours)
100	50
1,000	100
10,000	200

Solution: We'll use logarithms to linearism the model for easier estimation of parameters. The Duane model becomes:

$$log\ (MTBF) = log\ (K) + b\ log\ (t)$$

Table formation according to transformation

t (hours)	MTBF(t)	$\log_{10}(t)$	$\log_{10}(MTBF)$
100	50	2.00	1.70
1,000	100	3.00	2.00
10,000	200	4.00	2.30

The estimated value of slope b is calculated with two points (1,000, 100) and (10,000, 200):

$$b = \frac{log(200) - log(100)}{log(10000) - log(1000)} = \frac{2.30 - 2.00}{4.00 - 3.00} = 0.30$$

Now solve for K using one data point, say at t = 1000:

$$MTBF = Kt^b \rightarrow 100 = K(1000)^{0.30}$$

$$K = \frac{100}{1000^{0.30}} = 6.31$$

Final model is: $MTBF = 6.31 \times t^{0.30}$. Estimate MTBF at t = 20,000 hours,

$$MTBF(20{,}000) = 6.31 \times (20{,}000)^{0.30} \approx 164\ hours$$

Interpretation:

1) **Improving Reliability:** Since b=0.30<1, reliability is improving over time. 2) Impact on Warranty Reserves: Fewer failures are expected in future periods. The actuary may reduce future claim reserves and adjust warranty pricing accordingly. 3) Premium Recalculation: Based on lower expected claim frequency, the net premium for the device warranty may be revised downward.

Example: An engineer is testing a new component and records the following cumulative data over time:

Time Interval (hrs)	Cumulative Failures	Cumulative MTBF = time/ failures
100	2	50
200	3	66.67
400	4	100
800	5	160

We'll use this data to check whether the system follows the Duane model, and estimate the learning curve parameter α.

Solution: The Duane model assumes that the cumulative MTBF follows a power-law relationship: $MTBF(t) = K \cdot t^{\alpha}$

Taking logarithms: $\log(MTBF) = \log(K) + \alpha \cdot \log(t)$, if we plot log(MTBF) against log(t), we should get a straight line. The slop gives the value of α.

Time (t)	Failures (F)	Cumulative MTBF	log(t)	log (MTBF)
100	2	50	2	1.699
200	3	66.67	2.301	1.824
400	4	100	2.602	2.000
800	5	160	2.903	2.204

Estimation of α using linear regression. Now, we use the least squares regression formula to estimate the slop α.

$$\alpha = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sum(x_i - \bar{x})^2}$$

Where: $x_i = \log(t)$ and $y_i = \log(MTBF)$

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	Product	$(x_i - \bar{x})^2$
2	1.699	-0.452	-0.233	0.105	0.204
2.301	1.824	-0.151	-0.108	0.016	0.023
2.602	2.000	0.150	0.068	0.010	0.023
2.903	2.204	0.451	0.272	0.123	0.203

Where, $\bar{x} = \frac{2.00+2.301+2.602+2.903}{4} = 2.452$ and $\bar{y} = \frac{1.699+1.824+2.000+2.204}{4} = 1.932$

$$\alpha = \frac{0.105 + 0.016 + 0.010 + 0.123}{0.204 + 0.023 + 0.023 + 0.203} = 0.561.$$

Interpretation: The learning curve parameter $\alpha = 0.56$, which lies between 0 and 1, including reliability improvement over time. The Duane model fits this data well, validating the assumption of improving reliability through learning and corrective actions.

8.3.2 AMSAA (Crow-AMSAA) Model

The AMSAA model extends the Duane approach using a Non-Homogeneous Poisson Process (NHPP) framework, allowing for statistical fitting of failure data. It is flexible and widely applied in defense, manufacturing, and increasingly in actuarial reliability modeling.

$$N(t) = \lambda t^{\beta}$$

N(t): cumulative number of failures by time t, λ: scale parameter, β: shape parameter

β>1: worsening reliability, β=1: constant failure rate, β<1: improving reliability

Example: An actuary is working with a company that sells five-year service contracts for high-end routers. During the first year, the company tracks cumulative failures over time and wants to know if product reliability is improving. The actuary decides to model the reliability using the AMSAA model (also called the Crow-AMSAA or NHPP model), which assumes:

$$\mathrm{N(t)} = \lambda t^{\beta}$$

Time (t in months)	Cumulative Failures N(t)	Time (t in months)	Cumulative Failures N(t)
1	5	4	14
2	9	5	15
3	12	6	16

Solution: We take logarithms of both t and N(t):

t (months)	N(t)	$log_{10}(t)$	$log_{10}(N(t))$
1	5	0.00	0.70
2	9	0.30	0.95
3	12	0.48	1.08
4	14	0.60	1.15
5	15	0.70	1.18
6	16	0.78	1.20

Now we fit a straight line to the log-log data using linear regression:

$$log\ (N(t)) = log\ (\lambda) + \beta log\ (t)$$

Using regression, we estimate 1) Slope β≈0.65 and 2) Intercept $log\ (\lambda) \approx 0.70 \rightarrow \lambda \approx 10^{0.70} \approx\ 5.01$. Final AMSAA model is as: $N(t)\ =\ 5.01 \times t^{0.65}$.

Forecast failure at t = 8 months: $N(8)\ =\ 5.01 \times 8^{0.65} \approx\ 22.05\ failures.$

Interpretation:

- Reliability is improving: β=0.65<1, which means the failure rate is decreasing.
- Future claims are expected to slow down, reducing risk exposure.
- The actuary uses this model to:

  1) Adjust reserve estimates for outstanding service contracts. 2) Recalculate expected claim frequency for pricing future contracts. 3) Update warranty premiums, considering the lower projected failure rate. 4) Improve IBNR (incurred but not reported) assumptions.

Example: An electronic component undergoes testing, and the number of cumulative failures is recorded at different time points:

Time(hrs)	Cumulative failures
100	2
200	3
400	4
800	5

We assume the data follows the AMSAA model:

$$N(t)\ =\ \lambda t^{\beta}$$

Taking logs: $\log\big(N(t)\big) = \log(\lambda) + \beta \log\ (t)$

Solution: We take log for both

Time t	Failures N(t)	$log_{10}(t)$	$log_{10}(N(t))$
100	2	2.000	0.301

200	3	2.301	0.477
400	4	2.602	0.602
800	5	2.903	0.699

Fit regression line: $Y = a + \beta X$

Where: $Y = \log\big(N(t)\big)$ and $X = \log{(t)}$

x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	Product	$(x_i - \bar{x})^2$
2	1.699	-0.452	-0.233	0.105	0.204
2.301	1.824	-0.151	-0.108	0.016	0.023
2.602	2.000	0.150	0.068	0.010	0.023
2.903	2.204	0.451	0.272	0.123	0.203

$\bar{x} = 2.452$ and $\bar{y} = 0.520$

$$\beta = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{(x_i - \bar{x})^2} = \frac{0.198}{0.453} = 0.44$$

$$\log(\lambda) = \bar{y} - \beta \cdot \bar{x} = 0.520 - (0.44 \times 2.452) = -0.561 \rightarrow \lambda = 0.275.$$

$$N(t) = 0.275 \cdot t^{0.44}$$

Interpretation: β = 0.44: Since β < 1, it suggests reliability degradation, i.e., the system is experiencing failures at a decreasing rate but not improving. λ = 0.275: Indicates the initial intensity of failure accumulation.

To predict the expected number of failures by 1000 hours:

$$N(1000) = 0.275 \cdot (1000)^{0.44} \approx 0.275 \cdot 52.48 \approx 14.43$$

Thus, we expect approximately 14 failures by 1000 hours.

8.4 Accelerated Life Testing (ALT)

Accelerated Life Testing (ALT) is a powerful technique used in reliability engineering and actuarial science to predict how long a product or system will last under normal operating conditions. The method involves testing the product under elevated stress conditions, such as increased temperature, voltage, mechanical vibration, or humidity, to induce failures in a shorter amount of time than would occur in regular use. The goal is to gather life data efficiently, enabling accurate life predictions and better-informed risk management decisions.

The primary objectives of Accelerated Life Testing include:

- ***Identifying Potential Failure Modes****:* ALT helps reveal weak points in the design or materials of a product by exposing it to intensified conditions. This allows engineers and actuaries to understand how and why a product may fail.
- ***Estimating Product Life under Normal Use Conditions****:* By collecting time-to-failure data under stress, ALT enables extrapolation to standard use conditions, helping to forecast reliability over the intended life cycle.
- ***Improving Product Design and Warranty Planning***: Insights gained from ALT can be used to enhance product durability, optimize component selection, and support actuarial modelling of warranty claims, loss reserves, and premium pricing strategies.

Different types of stress are applied depending on the product and its failure mechanisms:

- ***Thermal Stress***: Increasing the temperature to accelerate chemical or material degradation (e.g., batteries, electronics).
- ***Electrical Stress****:* Applying overvoltage or over current to simulate long-term electrical load (e.g., circuits, semiconductors).
- ***Mechanical Stress****:* Using repetitive mechanical motion or vibration to simulate years of usage in a short time (e.g., motors, fans).
- ***Environmental Stress****:* Raising humidity, corrosion, or exposure to UV to test environmental durability (e.g., outdoor sensors, enclosures).

These stress types are chosen based on the known or suspected failure mechanisms of the product. Several mathematical models are used to relate the accelerated failure data to normal use conditions. These models help quantify how stress affects the failure rate and provide acceleration factors for extrapolation.

8.4.1 Arrhenius Model

The Arrhenius model is widely used when temperature is the dominant stress factor, particularly for thermally activated failure mechanisms such as chemical degradation, oxidation, or diffusion.

$$AF = e^{\frac{E_a}{k}\left(\frac{1}{T_1}-\frac{1}{T_2}\right)}$$

Where:

- AF = Acceleration Factor: the ratio of time to failure at normal conditions to time at accelerated conditions
- E_a= Activation Energy (in eV)
- k = Boltzmann's Constant ≈ $8.617 \times 10^{-5} eV/K$
- T_1, T_2 = Temperatures in Kelvin (K = °C + 273.15)

Interpretation: This model calculates how much faster a product will fail at a higher temperature. It's used to extrapolate the Mean Time to Failure (MTTF) from high-temperature tests back to room temperature.

8.4.2 Eyring Model

The Eyring model expands on the Arrhenius equation by allowing for multiple stress factors (e.g., temperature and voltage). It is suitable for complex systems where more than one factor accelerates degradation.

$$\lambda(T) = A \cdot e^{\left(-\frac{E_a}{kT}\right)} \cdot T^n$$

Where:

- λ(T) = Failure rate at temperature T
- A = model constant
- E_a= activation energy
- k = Boltzmann's constant
- T = temperature in Kelvin

- n = empirical constant that adjusts for the additional stress factor (like humidity or voltage)

Interpretation: The Eyring model is more versatile than Arrhenius and is used in fields like semiconductor reliability, where both heat and electrical stress impact failure.

8.4.3 Inverse Power Law Model

This model is used for non-thermal stressors, such as voltage, pressure, or mechanical load. It assumes that the product's life is inversely proportional to the applied stress raised to a power.$L = k \cdot S^{-n}$. Where: 1) L = Life under stress, 2) S = Applied stress level, 3) k, n = model constants.

Interpretation: A small increase in stress results in a sharp decrease in life if the exponent nis large. This model is common in fatigue failure testing, such as materials subjected to repeated bending or loading.

The process of ALT involves both experimental testing and statistical modelling. The general steps are:

1. ***Conduct Tests at High Stress Levels:*** Expose test samples to elevated stress (e.g., high temperature, voltage). Record time-to-failure data for each unit.

2. ***Fit an Appropriate Statistical Life Distribution:*** Use Weibull, log-normal, or exponential distributions to model the failure data. Determine parameters like shape (β) and scale (η) for the distribution.

3. ***Apply Acceleration Models:*** Use Arrhenius, Eyring, or Inverse Power Law to calculate acceleration factors. Extrapolate the life data to normal use conditions.

4. ***Make Predictions:*** Estimate Mean Time To Failure (MTTF) or reliability over time under normal usage. Support actuarial functions like setting claim reserves, forecasting warranty payouts, or calculating premiums.

Applications in Insurance and Annuities

1. Integrating ALT into Insurance Modelling: Accelerated life testing principles enable insurers to predict claim timing and frequency. For pension plans, ALT aids in understanding the longevity risk inherent in deferred annuities.

2. ***ALT for Pricing Long-Term Care Policies:*** Suppose individuals are subjected to urban environmental stressors. ALT techniques help quantify deterioration in health and longevity, adjusting premiums accordingly to reflect the higher probability of early claims.

Example: An actuary is working with a manufacturer of medical devices (e.g., portable ECG monitors). These devices contain batteries that degrade over time. The company wants to offer a 3-year warranty, and the actuary must estimate the failure rate of the battery to determine expected warranty claims. Since real-time testing would take years, the manufacturer performs Accelerated Life Testing at higher temperatures. The manufacturer tests identical devices at three elevated temperatures:

Temperature (°C)	Temperature (K)	Average Time to Failure (hours)
60 °C	333.15K	2,000
80 °C	353.15K	1,000
100 °C	373.15K	500

Solution: Firstly, calculate ln (MTTF) and 1/T

T(K)	1/T (1/K)	MTTF (hours)	log (MTTF)
333.15	0.003001	2,000	7.601
353.15	0.002831	1,000	6.908
373.15	0.002679	500	6.215

We now fit the Arrhenius equation: $\log(MTTF) = A + \frac{B}{T}$. This is a linear regression of log (MTTF) vs. 1/T. Using linear regression, we find that: A = 19.10, and B = –5000 So, the equation becomes: $\log(MTTF) = 19.1 + \frac{5000}{T}$.

At T = 303.15 K (30 °C): $\log(MTTF) = 19.1 + \frac{5000}{303.15} = 2.61, s$

$$(MTTF) = e^{2.61} \approx 13.6\ hours.$$

If the device operates for 8 hours/day, then:

Expected life in days = $\frac{13.6}{8}$ ≈1.7 days.

Example: A manufacturer tests an electronic component at two elevated temperatures to estimate its expected life under normal operating conditions. The following test data is available:

- At 120°C (393 K), the component fails after 500 hours.
- At 80°C (353 K), it fails after 2000 hours.
- Estimate the expected life at 50°C (323 K) using the Arrhenius model, assuming the activation energy ($E_a = \ 0.6\ eV$).

Use Boltzmann's constant $k = 8.617 \times 10 - 5\ eV/K$.

Solution: Using the Arrhenius equation for two stress levels

$$ln\left(\frac{L_1}{L_2}\right) = \frac{E_a}{k}\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$$

Where:

- L_1: Life at low stress $(T_1 = 323K)$ = unknown
- L_2: Life at low stress $(T_2 = 393K)$ = 500 hours
- $E_a = 0.6eV, k = 8.617 \times 10^{-5}eV/K$

$$ln\left(\frac{L_1}{500}\right) = \frac{0.6}{8.617 \times 10^{-5}}\left(\frac{1}{323} - \frac{1}{393}\right)$$

$$ln\left(\frac{L_1}{500}\right) = \ 3.837$$

$$L_1 = \ 23{,}200\ hours$$

The expected life of the component at 50°C (323 K) is 23,200 hours.

8.5 Multiple Life Functions

Multiple life functions represent a fundamental class of actuarial tools designed to analyse the joint mortality and survival patterns of two or more individuals. These functions are particularly vital in the valuation and structuring of life insurance and annuity contracts that are contingent upon the lives of multiple people. Examples of such products include joint life annuities, last survivor annuities, and survivorship life insurance policies.

In a multiple life framework, each life denoted by ages x, y, and so forth it is associated with a corresponding future lifetime random variable, typically expressed as T_x, T_y, etc. The primary objective of multiple life analysis is to compute various actuarial quantities derived from these lifetime variables, which capture the joint survival behaviour of the individuals. These quantities include:

- The probability that all individuals survive beyond a given time period t,
- The probability that at least one individual survives beyond time t,
- The probability that deaths occur in a specific sequence, or that certain combinations of survival and death occur.

The theory of multiple life functions extends the classical single-life actuarial models by incorporating joint, marginal, and conditional survival and death probabilities. These models may be developed under the assumption of independent future lifetimes or may incorporate dependencies through advanced statistical structures, such as copula models or common shock models, which capture real-world correlations between lives. In actuarial mathematics, many insurance contracts depend on the lifetimes of two or more individuals. Multiple life models help evaluate such contracts through joint and last survivor distributions.

8.5.1 Joint Life Survival Probability

The Joint Life Survival Probability refers to the probability that both individuals in a multiple life model survive beyond a specified time t. It is used in insurance and annuity products where payments or coverage continue only until the first death among the lives involved. Mathematically, if T_x and T_y denote the future lifetime of two individuals aged x and y, then the joint life survival probability is defined as:

$${}_tp_{xy} = P(T_x > t, T_y > t)$$

If the lifetimes are assumed to be independent, this simplifies to:

$${}_tp_{xy} = P(T_x > t, T_y > t)$$

This function is commonly used in joint life annuities and first-to-die insurance products.

Example: Two individuals, aged 60 and 65 respectively, are covered under a joint-life insurance contract. The 5-year individual survival probabilities are given as follows:

${}_5p_{60}=0.85$ and ${}_5p_{65}=0.75$. Assuming that the future lifetimes of the two individuals are independent, calculate the probability that both individuals will survive the next 5 years.

Solution: We are asked to find the 5-year joint life survival probability, i.e.,

$$ {}_5p_{60,65}= P(T_{60} > 5,\ T_{65} > 5) $$

Under the assumption of independent lifetimes, we use:

$$ {}_5p_{60,65}={}_5p_{60}\cdot {}_5p_{65} $$

Substitute the given values:

$$ {}_5p_{60,65}= 0.85 \times 0.75 = 0.6375. $$

This means there is a 63.75% probability that both individuals will survive for the next 5 years.

8.5.2 Last Survivor Probability

The Last Survivor Probability represents the probability that at least one of the individuals survives beyond time t. It is relevant in contracts where payments continue until the second (last) death, such as last survivor annuities or second-to-die insurance.

It is defined as:

$$ {}_tp^{\overline{xy}} = P(T_x > t \text{ or } T_y > t) $$

Using the principle of inclusion-exclusion:

$$ {}_tp^{\overline{xy}}={}_tp_x+{}_tp_y-{}_tp_{xy} $$

Under the assumption of independence, this becomes:

$$ {}_tp^{\overline{xy}}=1-(1-{}_tp_x)(1-{}_tp_y) $$

This function ensures continued benefit payments as long as at least one individual remains alive.

Example: A man aged 60 and a woman aged 65 are covered under a last survivor annuity. The 5-year survival probabilities are:

${}_5p_{60}=0.85$ and ${}_5p_{65}=0.75$. Assume their lifetimes are independent. What is the probability that at least one of them survives the next 5 years?

Solution: We are asked to find:

$$ {}_5p_{\overline{60,65}} = P(T_{60} > 5 \; or \; T_{65} > 5) $$

Using the formula:

$$ {}_5p_{\overline{60,65}} = 1-(1-{}_5p_{60})(1-{}_5p_{65}) $$

Substitute the values we get:

$$ {}_5p_{\overline{60,65}} = 1-(1-0.85)(1-0.75) = 1-0.0375 = 0.9625 $$

This means there is a 96.25% probability that at least one of the two individuals will survive in the next 5 years.

Example: Given the following excerpt from a life table for males and females:

Age	$l_x(Males)$	$l_y(Females)$
60	85,000	88,000
61	84,000	87,200
62	82,800	86,300
63	81,500	85,300
64	80,000	84,000
65	78,000	82,500

Assume a radix (initial population) of $l_0 = 100{,}000$.

A male aged 60 and a female aged 62 are observed. Using the above life table:

1. Calculate the 2-year survival probability for:
   - The male aged 60.
   - The female aged 62.
2. Determine the joint life survival probability over 2 years for the pair.

3. Compute the last survivor probability over 2 years.
4. Find the probability that at least one dies within 2 years.
5. Calculate the probability that both die within 2 years.

Solution: Cohort size: $l_0 = 100{,}000$

Lives under observation: Male aged 60, Female aged 62 and time period t = 2 years

1). Individual 2-year survival probabilities

For male aged 60:

$$ {}_2p_{60}^{(M)} = \frac{l_{60+2}}{l_{60}} = \frac{82{,}800}{85{,}000} = 0.9741 $$

For female aged 62:

$$ {}_2p_{62}^{(F)} = \frac{l_{62+2}}{l_{62}} = \frac{84{,}000}{86{,}300} = 0.9734 $$

2). Joint Life Survival Probability (both survive 2 years)

$$ {}_2p_{60,62} = {}_2p_{60} \cdot {}_2p_{62} = 0.9741 \times 0.9734 = 0.9482 $$

There is a 94.82% chance both the male (60) and female (62) will be alive in 2 years.

3). Last Survivor Probability (At least one survives 2 years)

$$ {}_2p_{60,62}^{last} = 0.9741 + 0.9734 - 0.9482 = 0.9993 $$

There is a 94.82% chance both the male (60) and female (62) will be alive in 2 years.

3). Last Survivor Probability (At least one survives 2 years)

$$ {}_2p_{60,62}^{last} = 0.9741 + 0.9734 - 0.9482 = 0.9993 $$

There is a 99.93% chance that at least one of them will be alive in 2 years.

4). Probability that at least one dies (first death within 2 years)

$$ {}_2q_{60,62} = 1 - 0.9482 = 0.0518. $$

There is a 5.18% chance that at least one of them will die within 2 years.

5). Probability that both die (last survivor dies within 2 years)

$$ {}_{2}q_{60,62}^{last} = 1 - 0.9993 = 0.0007 $$

There is only a 0.07% chance that both the male and female die within 2 years.

8.8 Summary

Reliability growth models and accelerated life testing (ALT) are essential methodologies in the field of reliability engineering, offering valuable insights into product performance, durability, and long-term dependability. Reliability growth models, including the Duane and AMSAA (Crow) models, are particularly useful during the design and testing phases of a product lifecycle. These models track how reliability improves over time by analyzing observed failure data and quantifying the effectiveness of corrective actions. The Duane model assumes a power-law growth in reliability, while the AMSAA model uses a non-homogeneous Poisson process to evaluate how the rate of failures evolves. These models guide engineering teams in identifying trends, refining product designs, and predicting future failure rates—ultimately leading to more reliable systems and reduced operational risks.

On the other hand, Accelerated Life Testing (ALT) is a powerful experimental strategy used to estimate a product's lifetime under normal operating conditions by subjecting it to elevated stress levels such as higher temperatures, voltages, or mechanical loads. ALT drastically shortens the testing period, making it feasible to detect potential failure modes and validate design robustness in a fraction of the time. This methodology supports proactive product improvements, warranty analysis, and lifecycle cost estimation, making it indispensable for industries where reliability and safety are non-negotiable.

Complementing these engineering-focused models, Multiple Life Functions play a pivotal role in actuarial science, particularly in modelling the joint survival and mortality behaviours of two or more individuals. These functions, including joint life, last survivor, and first-to-die probabilities, are critical in the pricing and design of life insurance products, pensions, and annuities. By using life tables and survival functions, actuaries can accurately estimate the probability of different survival combinations, helping in the creation of financial instruments that align with longevity risk and demographic patterns.

Together, these concepts form a comprehensive toolkit for reliability analysts, engineers, and actuaries, enabling them to:

- Predict and improve system performance over time,
- Reduce testing costs and accelerate product validation,
- Design risk-resilient insurance and pension products,

- Make informed decisions in quality control, product development, and financial planning.

8.9 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

8.10 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT-9 DECREMENT THEORY AND APPLICATIONS OF MULTIPLE DECREMENTS THEORY

Structure

9.1 Introduction

In unit 9 we have studied how to find the premium for a variety of insurance products and some annuity products. In all these insurance and annuity products, the benefit is payable to a single individual. However, group insurance is a popular product wherein more than one life is involved in a policy contract. In this situation, it becomes necessary to decide when and how the benefit is to be paid. When a single life is involved in the insurance product, the benefit is payable at the moment of death or at the end of the year of death of the individual, that is when the status of the individual as 'being alive' terminates. In annuity products, annuity payments cease when the individual dies, that is when the status of the individual as 'being alive' terminates. While dealing with group insurance or annuity product, we have to define a so-called status of the group, termination of which determines the time of death benefit payment in insurance products and the time to stop survival benefits in annuity products. Thus, the first step while dealing with multiple lives is to define the status of a group and an extension of the definition of T(x) or K(x) to a group consisting of more than one individual. The following two are the commonly used approaches. Suppose $(x_1, x_2,...,x_m)$ denotes a group of m individuals, where x_i represents the age of i^{th} member at the time of policy issue.

Joint Life Status: The joint life status exists as long as all members of the group are alive and ends when the first death takes place. It is denoted by $(x_1, x_2, ..., x_m)$. The random variable $T = T(x_1, x_2, ..., x_m)$ denotes the time until failure of a status. It is defined as follows.

$$T = \min\{T(x_1), T(x_2), \ldots\ldots, T(x_m)\},$$

Where, T(xi) is the future life time of individual i.

Last Survivor Status: As the name indicates, this status exists as long as at least one member is alive and fails upon the last death. It is denoted by $(\bar{x}_1, \ldots, \bar{x}_3)$ $(\overline{x_1, \ldots, x_m})$. In the last survivor status, the random variable T(21···m) describing the time until failure of the status is defined as follows.

$$T(\overline{x_1, \ldots, x_m}) = \max\{T(x_1), T(x_2), \ldots\ldots, T(x_m)\}$$

Where, $T(x_i)$ is the time until death of individual i.

Many pension plans have joint and last survivor annuity options. In government pension plans, after the death of the husband, his dependent wife gets part of a pension.

After defining the status of the group, the next step is to obtain a distribution of time until failure of the status. Once it is defined, the entire theory can be easily generalized to define the actuarial present value of the benefit to be paid at the moment of termination of a status of the group. The main aim of this chapter is to express the distribution of time till termination of the status of the group, in terms of the distribution of time till failure of the status as 'being alive' for every one of the groups. With such a relation, actuarial present values of benefit in group insurance or group annuity can be obtained using the knowledge of distribution of T(x) corresponding to everyone in the group.

In Section 9.3, we derive the distribution of the future lifetime of the joint life status and in Section 9.4 that of the last survivor status. In both the cases, we express probability of survival or failure in terms of the probabilities for the individual lives in a group.

9.2 Objectives

After reading this unit, the learner should able to understand:

- Understand how different causes (e.g., disease, accident, resignation) affect the overall exit rate from a population.
- Provide a framework to calculate probabilities when multiple types of exits can happen, but only one occurs per individual.
- Break down the total exit rate into component probabilities for each decrement (e.g., probability of death due to cancer vs. heart disease).

- Predict future changes in population structure by accounting for various exit pathways.
- In life insurance or pensions, use the model to price products, set premiums, and calculate reserves based on cause-specific risk.

9.3 Joint Life Status

In joint life status

$$T(x_1, x_2,...,x_m) = \min\{T(x_i), i = 1,......,m\}$$

and for the last survivor status,

$$T(\overline{x_1, \ldots, x_m}) = \max\{ T(x_i), i = 1,......,m\}.$$

If Y_1,..., Yn are independent and identically distributed random variables, then the distribution function or survival function of

$$Y_{(1)} = \min\{Y_i, i = 1,\ldots, n\} \text{ and } Y_{(n)} = \max\{ Y_i, i = 1,......, n\}$$

can be obtained easily. However, in the group of individuals involved in the insurance contract, there is some association among the members of the group, for example, they may be members of a family, husband and wife or workers working in the same company in the same environment. As a result, $T(x_1)$, $T(x_2)$, ……, $T(x_m)$ cannot be considered as independent random variables. In practice, however, it is very difficult to model the dependence of these times until death random variables. Hence, in obtaining multiple life functions, an assumption is made that $T(x_1)$, $T(x_2)$, ……, $T(x_m)$ are independent random variables. Under this assumption, we express the distribution of $T(x_1 \cdots x_m)$ and $T(\overline{x_1, \ldots, x_m})$ in terms of the distribution of $T(x_1)$, ……, $T(x_m)$. To avoid the complexity of notation, we take m = 2. The extension to m > 2 is straightforward.

In a joint life status, the distribution function of time T(xy) to failure of a joint life status of a group of two individuals of ages x and y, in international actuarial notation is given by,

$${}_tq_{xy} = F_T(t) = P[\min\{T(x), T(y)\} \le t]$$

$$= 1 - P[\min\{T(x), T(y)\} \ge t]$$

$$= 1- P[T(x) > t]P[T(y) > t], \text{ (in view of independence assumption).}$$

$= 1 - {}_tp_x {}_tp_y.$

Thus, under the assumption of independence, the survival function of the time to failure of the joint life status, to be denoted by ${}_tP_{xy}$, is given by,

$${}_tP_{xy} = {}_tP_x {}_tp_y. \tag{9.3.1}$$

The probability density function of T(xy) is obtained from $F_T(t)$ by taking the derivative with respect to t. Thus,

$$f_T(t) = \frac{d}{dt}(1 - \text{tPxtPy}) = -{}_tP_x\ (-\,{}_tp_y\,\mu_{y+t}) - {}_tp_y(-{}_tp_x\mu_{x+t})$$

$$= {}_tP_x {}_tp_y\ (\mu_{x+t} + \mu_{y+t}). \tag{9.3.2}$$

From the expression of the survival function and the density function, it is easy to see that the force of failure of the joint life status at time t, denoted by μ_{xy} (t) is,

$$\mu_{xy}\ (t) = {f_{T(xy)}(t)}\Big/{(1 - F_{T(xy)}(t))} = \mu_{x+t} + \mu_{y+t} \tag{9.3.3}$$

Thus, the force of failure for the joint life status is the sum of forces of mortality for individuals if their future lifetimes are independent.

The curate future lifetime of the status, denoted by K(xy), is defined as follows. Note that K(xy) = k, if the joint life status terminates in year k + 1, that is if either x or y dies in year k + 1. In other words, K(xy) = k, if both lives survive k years, but both do not survive _k + 1_years. The latter definition is helpful to find the probability mass function of K(xy). It is obtained below.

$$P[K(xy) = k] = {}_kp_{xy} - {}_{(k+1)}P_{xy}$$

$$= {}_kP_{xy} - {}_kP_{xy}P_{x+k\ y+k}$$

$$= {}_kP_{xy}\ (1 - P_{x+k\ y+k})$$

$$= {}_kP_{xy}\ q_{x+k\ y+k}\ , \qquad k = 0,1,....$$

To express this probability in terms of single life function, we note that,

$$q_{x+k\ y+k} = (1 - P_{x+k\ y+k}) = 1 - P_{x+k}P_{y+k} = 1 - (1 - q_{x+k})(1 - q_{y+k}) = q_{x+k} + q_{y+k} - q_{x+k}\ q_{\ y+k}$$

Hence,

$$P[K(xy) = k] = {}_kP_{x\,k}P_y(q_{x+k} + q_{y+k} - q_{x+k}\,q_{y+k})\ , \qquad k = 0,1,.... \tag{9.3.4}$$

Once we have derived the formulae for the probability density function of T(xy) and probability mass function of K(xy), that is extension of the multiple lives, insurances and annuities for an individual life, can be defined easily for a group status. For example, with the definition of joint life status for multiple lives, an annuity is payable as long as the status survives, while an insurance is payable upon the failure of the status.

The actuarial present value of unit amounts payable immediately on the failure of the joint life status in whole life insurance, is denoted by $\bar{A}_{xy}$ and is given by,

$$\bar{A}_{xy} = E(v^{T(xy)}) = \int_0^\infty v^t \; {}_t p_{xy} \mu_{xy}(t)dt = \int_0^\infty v^t {}_t p_x {}_t p_y (\mu_{x+t} + \mu_{y+t})dt$$

With the knowledge of v , ${}_t p_x$, ${}_t p_y$ μ_{x+t} and μ_{y+t} ,$\bar{A}_{xy}$ can be evaluated. If the benefit of a unit amount is payable at the end of the year in which the joint life status fails, the actuarial present value of the benefit payment is given by,

$$A_{xy} = \sum_{k=0}^{\infty} v^{k+1} \{{}_k p_x \, {}_k p_y (q_{x+k} + q_{y+k} - q_{x+k} \, q_{y+k})\}.$$

It can be shown that under the uniformity assumption in unit age intervals for both ages,

$$\bar{A}_{xy} \cong \frac{i}{\delta} A_{xy}.$$

The actuarial present value of benefit in other insurance products is defined on similar lines. Thus, the actuarial present value of a unit amounts payable immediately on the failure of the joint life status in an n-year term insurance, is denoted by $\bar{A}^1_{xy:\bar{n}|}$ and is given by,

$$\bar{A}^1_{xy:\bar{n}|} = \int_0^n v^t \; {}_t p_{xy} \mu_{xy}(t)dt = \int_0^n v^t {}_t p_x {}_t p_y (\mu_{x+t} + \mu_{y+t})dt.$$

If the benefit of a unit amount is payable at the end of the year in which the joint life status fails in an n-year term insurance, then the actuarial present value of the benefit payment is given by,

$$A^1_{xy:\bar{n}|} = \sum_{k=0}^{n-1} v^{k+1} \{{}_k p_x \, {}_k p_y (q_{x+k} + q_{y+k} - q_{x+k} \, q_{y+k})\}.$$

For an n-year endowment insurance, the corresponding formulae are as follows.

$$\bar{A}^1_{xy:\bar{n}|} = \bar{A}^1_{xy:\bar{n}|} + v^n {}_n p_{xy} \text{ and } A^1_{xy:\bar{n}|} = A^1_{xy:\bar{n}|} + v^n {}_n p_{xy} .$$

Various types of annuities and their actuarial present values are easily extendable to the group contracts. The notations are very similar to single life cases; we just need to replace x by xy. The relation between Z and Y as given by,

$$Y = \frac{1-Z}{d} \qquad \text{or} \qquad Y = \frac{1-Z}{\delta}$$

in earlier, is also applicable in this setup. This relation is also useful to find the variance of present value random variable of annuity. Following are the formulae for actuarial present value of some annuities. $\bar{a}_{xy}$ denotes the actuarial present value of an annuity of amount 1 unit per annum, payable continuously so long as both (x) and (y) are alive. It is given by

$$\bar{a}_{xy} = E(\bar{a}_{T(xy)}) = \int_0^\infty \bar{a}_{\overline{t}|}\ t\,p_{xy}\mu_{x+t}\,\mu_{y+t}\,dt = \int_0^\infty v^t\ t\,p_{xy}dt = \frac{1-\overline{A_{xy}}}{\delta}$$

The actuarial present value of an annuity of amount 1 unit, payable at the beginning of each year so long as both (x) and (y) are alive, is denoted by $\ddot{a}_{xy}$ and is given by,

$$\ddot{a}_{xy} = \sum_{t=0}^{\infty} v^t\ {}_tp_{xy} = \frac{1-A_{xy}}{d}.$$

The actuarial present value of an annuity of amount 1 unit payable at the end of each year so long as both (x) and (y) are alive is denoted by a_{xy} and is given by,

$$a_{xy} = \sum_{t=0}^{\infty} v^t\ {}_tp_{xy} = \ddot{a}_{xy} - 1$$

The actuarial present value of an annuity of amount 1 in an n-year temporary annuity due, in a joint life status, denoted by, $\ddot{a}_{xy:\overline{n}|}$ is defined as

$$\ddot{a}_{xy:\overline{n}|} = \ddot{a}_{xy} - {}_np_{xy}\, v^n\ \ddot{a}_{x+n\ y+n} = \sum_{t=0}^{n-1} v^t\ {}_tp_{xy} = \frac{1-A_{xy:\overline{n}|}}{d}$$

Other annuity functions are defined analogously.

With the extension of two components of premium calculation to multiple lives, premiums for multiple life contracts can be defined on similar lines as those for single lives studied in previous units. As an illustration, suppose a whole life insurance of unit benefit, is purchased by (xy) with annual premiums payable as whole life annuity due. Thus, benefit is payable at the moment of first death and premium payments will cease when the first death occurs. Using equivalence principle, the annual premium P is given by,

$$P = \frac{\bar{A}_{xy}}{\ddot{a}_{xy}}$$

If the premiums are payable as an n-year temporary annuity due, the premium P will be given by,

$$P = \frac{\bar{A}_{xy}}{\ddot{a}_{xy:\overline{n}|}}$$

Premiums for other insurance and annuity products are computable on similar lines.

Example 9.3.1: Prove that ${}_np_y\bar{A}^{1}_{x:\bar{n}|} + {}_np_x\bar{A}^{1}_{y:\bar{n}|} < \bar{A}^{1}_{xy:\bar{n}|} < \bar{A}^{1}_{x:\bar{n}|} + \bar{A}^{1}_{y:\bar{n}|}$ and $\bar{A}_{xy} < \bar{A}_x + \bar{A}_y$

Solution: From equation (9.3.2) the probability density function of *T(xy)* is given

by, $f_T(t) = = {}_tP_{x\,t}p_y(\mu_{x+t} + \mu_{y+t})$

Thus, using the formula of $\bar{A}^{1}_{xy:\bar{n}|}$ we get,

$$\bar{A}^{1}_{xy:\bar{n}|} = \int_0^n v^t\,{}_tp_{x\,t}p_y\,(\mu_{x+t}+\mu_{y+t})\,dt$$

$$\leq \int_0^n v^t\,{}_tp_x\,\mu_{x+t}\,dt + \int_0^n v^t\,{}_tp_y\,\mu_{y+t}\,dt$$

$$= \bar{A}^{1}_{x:\bar{n}|} + \bar{A}^{1}_{y:\bar{n}|}$$

The second step follows from the fact that being a probability function, both ${}_tP_x$ and ${}_tp_y \leq 1$. If both ${}_tP_x$ and ${}_tp_y$ are 1, then in this case, both μ_{x+t} and μ_{y+t} are 0. As a consequence the probability density function of *T(xy),T(x)* and *T(y)*at t will be 0.Ifit is true for all $t \leq n$, then $\bar{A}^{1}_{xy:\bar{n}|}$, $\bar{A}^{1}_{x:\bar{n}|}$ and $\bar{A}^{1}_{y:\bar{n}|}$are all 0.To provethe first part of the inequality, we use the result that being a survival function, ${}_tP_x$ is a decreasing function of *t.* Thus,

$$t<n \Rightarrow {}_tP_x > {}_nP_x \text{ and } {}_tP_y > {}_nP_y \,.$$

Thus, we get

$$\bar{A}^{1}_{xy:\bar{n}|} = \int_0^n v^t\,{}_tp_{x\,t}p_y\,(\mu_{x+t}+\mu_{y+t})dt$$

$$= \int_0^n v^t\,{}_tp_x\mu_{x+t}\,{}_tp_y\,dt + \int_0^n v^t\,{}_tp_y\,\mu_{y+t\,t}p_x\,dt$$

$$> {}_nP_y \int_0^n v^t\,{}_tp_x\mu_{x+t}\,dt + {}_nP_x \int_0^n v^t\,{}_tp_y\,\mu_{y+t}\,dt$$

$$= {}_nP_y\,\bar{A}^{1}_{x:\bar{n}|} + {}_nP_x\,\bar{A}^{1}_{y:\bar{n}|}.$$

Inequality for a whole life insurance follows on similar lines as in the proof of $\bar{A}^{1}_{xy:\bar{n}|} < \bar{A}^{1}_{x:\bar{n}|} + \bar{A}^{1}_{y:\bar{n}|}$. It can also be obtained by allowing n in the inequality

$$\bar{A}^{1}_{xy:\bar{n}|} < \bar{A}^{1}_{x:\bar{n}|} + \bar{A}^{1}_{y:\bar{n}|}$$

In Example 9.3.5 we verify these results when the underlying mortality pattern governed by Gompertz's law.

In the next section we will derive the results for the last survivor status on similar lines as those proved for joint life status and also establish link among the random variables *T(xy)*, $T(\overline{xy})$, *T(x)* and *T(y)*. This link is the avail used to derive formulae for the actuarial present values in insurance and annuity products in the setup of last survivor status.

9.4 Last Survivor Status

In the last survivor status, the distribution function of $T(\overline{xy})$, the time of failure of the last survivor status of a group consisting of two lives, is given by,

$$ {}_tq_{\overline{xy}} = P[T(\overline{xy}) \le t] = P[\max\{T(x), T(y)\} \le t] \qquad (9.4.1)$$

$$= P[T(x) \le t]\, P[T(y) \le t] = {}_tq_y \,.\, {}_tq_x$$

Hence,

$$ {}_tp_{\overline{xy}} = 1-(1-{}_tp_x)(1-{}_tp_y) = {}_tp_x + {}_tp_y - {}_tp_x \,.\, {}_tp_y \qquad (9.4.2)$$

Further, the probability density function is given by,

$$f_{T(\overline{xy})}(t) = \frac{d}{dt}(1-{}_tp_x)(1-{}_tp_y)$$

$$= (1 - {}_tp_y)\,(\mu_{y+t} \,.\, {}_tp_y) + (1-{}_tp_y)\,({}_tp_x \,.\, \mu_{x+t})$$

$$= {}_tp_x\, \mu_{x+t} + {}_tp_y\, \mu_{y+t} - {}_tp_x\, {}_tp_y\, (\mu_{x+t} + \mu_{y+t}) \qquad (9.4.3)$$

$$= f_{T(x)}(t) + f_{T(y)}(t) - f_{T(xy)}(t) \qquad (9.4.4)$$

The probability density function of $T(\overline{xy})$ is expressed in terms of single life functions in (9.4.3), while (9.4.4) establishes a relation among the probability density functions of T(x), T(y) and T(xy). The force of mortality corresponding to this status is obtained as,

$$\mu_{\overline{xy}}(t) = \frac{f_{T(\overline{xy})}(t)}{1- f_{T(\overline{xy})}(t)} = \{{}_tp_x\, \mu_{x+t} + {}_tp_y\, \mu_{y+t} - {}_tp_{xy}\, \mu_{xy}(t)\} / \{{}_tp_x + {}_tp_y - {}_tp_{xy}\}.$$

Example 9.4.1: It is given that

(i) ${}_5p_{50} = 0.9$,

(ii) ${}_5p_{60} = 0.8$,

(iii) $q_{55} = 0.03$,

(iv) $q_{65} = 0.05$ and

(v) T(50) and T(60) are independent. Calculate $_{5|}q_{\overline{50:60}}$.

Solution: Under the assumption of independence, $_tq_{\overline{xy}} = {}_tq_x \; {}_tq_y$ for the last survivor status. Hence,

$$_{5|}q_{\overline{50:60}} = {}_6q_{\overline{50:60}} - {}_5q_{\overline{50:60}} = {}_6q_{50} \; {}_6q_{60} - {}_5q_{50} \; {}_5q_{60} .$$

From the given information, $_6p_{50} = {}_5p_{50} \, p_{55} = (0.9)\,(0.97) = 0.873$

and $_6p_{60} = (0.8)\,(0.95) = 0.76$, so that

$$_{5|}q_{\overline{50:60}} = (0.127)(0.24) - (0.1)(0.2) = 0.01048.$$

The probability mass function of $K(\overline{xy})$ is as follows. For a non-negative integer k, applying basic probability laws and the assumption of independence of K(x) and K(y) we have,

$$P[K(\overline{xy}) = k] = P[\,K(x) = k, K(y) < k] + P[\,K(y) = k, K(x) < k] + P[\,K(x) = k, K(y) = k]$$

$$= (1 - {}_kp_y)\, {}_kp_x q_{x+k} + (1 - {}_kp_x)\, {}_kp_y q_{y+k} + {}_kp_x {}_kp_y \, q_{x+k} \, q_{y+k}$$

$$= {}_kp_x q_{x+k} + {}_kp_y \, q_{y+k} - + {}_kp_x {}_kp_y (q_{x+k} + q_{y+k} - q_{x+k} \, q_{y+k}).$$

Equation (9.4.4) exhibits a relationship among the distribution of T(xy), $T(\overline{xy})$, T(x) and T(y) under the assumption of independence of T(x) and T(y). In the following we show that such a relationship exists among T(xy), $T(\overline{xy})$, T(x) and T(y) irrespective of the assumption of independence. It is to be noted that,

$$T(xy) = \begin{cases} T(x) \; if \; (x) dies \; before \; (y) \\ T(y) \; if \; (y) dies \; before \; (x) \end{cases}$$

and

$$T(\overline{xy}) = \begin{cases} T(y) \; if \; (x) dies \; before \; (y) \\ T(x) \; if \; (y) dies \; before \; (x) \end{cases}$$

So, even if T(x) and T(y) are not independent,

$$T(xy) + T(\overline{xy}) = T(x) + T(y). \qquad (9.4.5)$$

Equation (9.4.5) essentially follows from the following mathematical identity: For any two numbers a and b, min{a, b} + max{a, b} = a + b. Further,

$$F_{T(xy)}(t) + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t) \text{ if (x) dies before (y),}$$

$$= F_{T(y)}(t) + F_{T(x)}(t) \quad \text{if (y) dies before (x).}$$

Either (x) dies before (y) or (y) dies before (x), hence,

$$F_{T(xy)}(t) + F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t). \tag{9.4.6}$$

Taking derivatives with respect to t, we get,

$$f_{T(xy)}(t) + f_{T(\overline{xy})}(t) = f_{T(x)}(t) + f_{T(y)}(t), \tag{9.4.7}$$

This equation has been derived in (9.4.4) under the assumption of independence of T(x) and T(y). From (9.4.6),

$$F_{T(\overline{xy})}(t) = F_{T(x)}(t) + F_{T(y)}(t) - F_{T(xy)}(t)$$

$$= 1 - {}_tp_x + 1 - {}_tp_y - 1 + {}_tp_{xy}.$$

In international actuarial notation,

$${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}.$$

From (9.4.7),

$$f_{T(\overline{xy})}(t) = f_{T(x)}(t) + f_{T(y)}(t) - f_{T(xy)}(t)$$

$$= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x\, {}_tp_y(\mu_{x+t} + \mu_{y+t})$$

The last equality follows under the assumption of independence of T(x) and T(y). A relationship similar to that among T(xy), $T(\overline{xy})$, T(x) and T(y) exists among K(xy), $K(\overline{xy})$, K(x) and K(y). Thus, using the arguments similar to that for T(xy), $T(\overline{xy})$

we get,

$$K(xy) + K(\overline{xy}) = K(x) + K(y). \tag{9.4.8}$$

Relations developed in (9.4.7) and in (9.4.8) can be exploited to obtain expectations, variances and covariances of the joint and last survivor future lifetimes.

Suppose expectations of T(xy) and $T(\overline{xy})$ are denoted by e_{xy}^0 and $e_{\overline{xy}}^0$ respectively.

Then, $e_{xy}^0 = \int_0^\infty {}_tp_{xy}\, dt$, $\quad e_{\overline{xy}}^0 = \int_0^\infty {}_tp_{\overline{xy}}\, dt$.

Since ${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_{xy}$, we get a relation

$$e^0_{\overline{xy}} = e^0_x + e^0_y - e^0_{xy}.$$

It also follows from (9.4.5). Similarly, for curate future life time, the expectation of K(xy) is given by, $e_{xy} = \sum$ ${}_kp_{xy}$ and that of K($\overline{xy}$) is given by, $e_{\overline{xy}} = \sum \ {}_kp_{\overline{xy}}$.

Further, we get a relation,

$$e_{\overline{xy}} = e_x + e_y - e_{xy}$$

Variances of T(xy) and T($\overline{xy}$) , again it is noted that,

$$V(T(xy)) = 2\int_0^\infty t\,{}_tp_{xy}\,dt - (e^0_{xy})^2,\ V(T(\overline{xy})) = 2\int_0^\infty t\,{}_tp_{\overline{xy}}\,dt - (e^0_{\overline{xy}})^2.$$

To find Cov (T(xy), T($\overline{xy}$) , again it is noted that,

$$T(xy)\,T(\overline{xy}) = T(x)\,T(y),$$

Hence, under the assumption of independence of T(x) and T (y)

$$\begin{aligned} Cov\,(T(xy), T(\overline{xy})) &= E(T(xy)\,T(\overline{xy})) - e^0_{xy}e^0_{\overline{xy}} = E(T(x)\,T(y)) - e^0_{xy}e^0_{\overline{xy}} \\ &= E(T(x))\,E(T(y)) - e^0_{xy}e^0_{\overline{xy}} \\ &= e^0_x e^0_y - e^0_{xy}e^0_{\overline{xy}} \\ &= e^0_x e^0_y - e^0_{xy}\,(e^0_x + e^0_y - e^0_{xy}) \\ &= e^0_x e^0_y - e^0_{xy}e^0_x + e^0_{xy}e^0_y - e^0_{xy}e^0_{xy}) \\ &= e^0_x(e^0_y - e^0_{xy}) - e^0_{xy}\,(e^0_y - e^0_{xy}) \\ &= (e^0_x - e^0_{xy})\,(e^0_y - e^0_{xy}). \end{aligned}$$

Note that $e^0_x \geq e^0_{xy}$ and $e^0_y \geq e^0_{xy}$, consequently, Cov (T(xy), T($\overline{xy}$)) ≥ 0.

Thus, T(xy) and T($\overline{xy}$) are positively correlated except in trivial cases where $e^0_x or\ e^0_y$ equals e^0_{xy}.

Example 9.4.2: It is given that

(i) T(x) and T(y) are independent,

(ii) E(T(x)) = E(T(y)) = 4.0,

(iii) Coυ (T(xy), T($\overline{xy}$)) = 0.09. Calculate E(T(xy)).

Solution: If T(x) and T(y) are independent,

$$\text{Cov}\,(T(xy),\, T(\overline{xy})\,) = (e_x^0 - e_{xy}^0)\,(e_y^0 - e_{xy}^0)$$

Therefore,

$$0.09 = (4 - e_{xy}^0)\,(4 - e_{xy}^0) = (4 - e_{xy}^0)^2 \Rightarrow 4 - e_{xy}^0 = \pm\, 0.3 \Rightarrow e_{xy}^0 = 3.7 \text{ or } 4.3.$$

Since T(xy) = min{T(x), T(y)}, it follows that $e_{xy}^0 \le e_x^0$ and therefore, we choose the smaller root, $e_{xy}^0 = 3.7$ (note that $e_{\overline{xy}}^0 = 4.3$).

Example 9.4.3: Show that the probability of two lives (x) and (y) dying in the same year can be expressed as

$$1 + e_{xy} - p_x(1 + e_{x+1\,y}) - p_y(1 + e_{x\,y+1}) + p_{xy}(1 + e_{x+1\,y+1}).$$

Assume that T(x) and T(y) are independent and identically distributed.

Solution: Under the assumption of independence of T(x) and T(y), and using the formula $e_{xy} = \sum {}_kp_{xy}$, the probability that both die in the same year can be written as follows and summation goes from t=1 to ∞.

$$\sum P[\,t-1 < T(x) < t,\ t-1 < T(y) < t]$$

$$= \sum ({}_{t-1}p_x - {}_tp_x) - ({}_{t-1}p_y - {}_tp_y)$$

$$= \sum \{{}_{t-1}p_{xy} - p_y\,{}_{t-1}p_{x\,y+1} - {}_{t-1}p_y - p_x\,{}_{t-1}p_{x+1\,y} + {}_tp_{xy}\}$$

$$= \sum \{{}_{t-1}p_{xy} - p_y\,{}_{t-1}p_{x\,y+1} - p_x\,{}_{t-1}p_{x+1\,y} + p_xp_y\,{}_{t-1}p_{x+1\,y+1}\}$$

$$= 1 + e_{xy} - p_x(1 + e_{x+1\,y}) - p_y(1 + e_{x\,y+1}) + p_{xy}(1 + e_{x+1\,y+1}).$$

Example 9.4.4: Show that the probability of two lives (30) and (40) dying at the same age from their last birthday can be expressed as

$${}_{10}p_{30}\,(1 + e_{40:40}) - 2\ {}_{11}p_{30}\,(1 + e_{40:41}) + p_{40}\,{}_{11}p_{30}\,(1 + e_{41:41}) .$$

Assume that T (30) and T(40) are independent and identically distributed.

Solution: Under the assumption of independence of T(30) and T(40), and using the formula $e_{xy} = \sum {}_kp_{xy}$, the probability that both die in the same year can be written as follows and summation goes from t=1 to ∞.

$\sum P[\, 10+t< T(30) <11+t,\ t< T(y) < t+1]$

$$= \sum ({}_{10+t}p_{30} - {}_{11+t}p_{30}) - ({}_{t}p_{40} - {}_{t+1}p_{40})$$

$$= \sum {}_{10}p_{30}\,{}_{t}p_{40:40} - {}_{10}p_{30}\, p_{40}\,{}_{t}p_{40:41} - {}_{11}p_{30}\,{}_{t}p_{41:40} + {}_{11}p_{30}\, p_{40}\,{}_{t}p_{41:41}\}$$

$$= {}_{10}p_{30}(1+ e_{40:40}) - 2\ {}_{11}p_{30}(1+ e_{40:41}) + p_{40}\,{}_{11}p_{30}(1+ e_{41:41}) \ .$$

Actuarial present values of benefit in insurance products and of annuity payments in annuity products for the last survivor status are defined exactly on similar lines as those for joint life status. However, instead of computing these separately, the general relationship among T(xy), T($\overline{xy}$), T(x) and T(y) and also among K(xy), K($\overline{xy}$), K(x) and K(y), is useful to establish relations among the actuarial present values of benefit in insurance products and of annuity payments in annuity products. Thus, from the relations (9.3.5) and (9.3.8), we get relationships among the actuarial present values. Some of these are listed below.

$$\bar{A}_{\overline{xy}} + \bar{A}_{xy} = \bar{A}_{x} + \bar{A}_{y}$$

$$A_{\overline{xy}} + A_{xy} = A_{x} + A_{y}$$

$$\bar{A}^{1}_{\overline{xy}:\bar{n}|} + \bar{A}^{1}_{xy:\bar{n}|} = \bar{A}^{1}_{x:\bar{n}|} + \bar{A}^{1}_{y:\bar{n}|}$$

$$A^{1}_{\overline{xy}:\bar{n}|} + A^{1}_{xy:\overline{n}|} = A^{1}_{x:\overline{n}|} + A^{1}_{y:\overline{n}|}$$

$$\bar{a}_{\overline{xy}} + \bar{a}_{xy} = \bar{a}_{x} + \bar{a}_{y}$$

$$a_{\overline{xy}} + a_{xy} = a_{x} + a_{y}$$

$$\bar{a}^{1}_{\overline{xy}:\bar{n}|} + \bar{a}^{1}_{xy:\bar{n}|} = a^{1}_{x:\bar{n}|} + \bar{a}^{1}_{y:\bar{n}|}$$

$$a^{1}_{\overline{xy}:\bar{n}|} + a^{1}_{xy:\overline{n}|} = a^{1}_{x:\overline{n}|} + a^{1}_{y:\overline{n}|} \ .$$

These formulae allow us to express the actuarial present values of last survivor annuities and insurances in terms of those for the individual lives and the joint life status. Computation of premiums is similar to that in joint life status. Following examples illustrate the computations.

Example 9.4.5: Suppose the future lifetimes T(x) and T(y) are independent and each has the distribution defined by Gompertz's law with the force of mortality, $\mu_x = BC^x$ and $\mu_y = BC^y$ with B = 0.0001151 and C = 1.096. It is given that $\delta = 0.05$, x = 25 and y = 30.

i. Determine the survival function, the force of mortality for the joint life status T(xy) and the probability density function of T(xy).

ii. Calculate $A^{1}_{\overline{xy}:\overline{n}|}$ and $A^{1}_{xy:\overline{n}|}$ for n =1 to 10. Verify the result proved in example 9.3.1.

iii. Calculate $\bar{A}_{\overline{xy}}$ and $\bar{A}_{xy}$.

iv. Calculate the annual premium, payable as a whole life continuous annuity, for the benefit of1000, payable at the moment of first death in a whole life insurance issued to (xy).

iv. Calculate the annual premium, payable as a whole life continuous annuity for the benefit of 1000, payable at the moment of last death in a whole life insurance issued to $\overline{xy}$.

vi. Calculate the annual premium, payable as an n-year temporary continuous life annuity, for the benefit of 1000, payable at the moment of first death in an n-year term insurance, for n = 1 to 10.

vii. Calculate the annual premium, payable as an n-year temporary continuous life annuity, for the benefit of 1000, payable at the moment of last death in an n-year term insurance, for n = 1 to 10.

Solution: We know that, for Gompertz's law for t > 0, the survival function of T(x) is given by, ${}_{t}p_{x} = \exp[-\, mC^{x}(C^{t} - 1)]$. Hence, in joint life status, the survival function of T(xy) is given by,

$$ {}_{t}p_{xy} = {}_{t}p_{x}\, {}_{t}p_{y} = \exp[-\, m(C^{x} + C^{Y})(C^{t} - 1)]. $$

By the definition of force of mortality,

$$ \mu_{x+t} = BC^{x+t} $$

thus, from equation (9.3.3),

$$ \mu_{xy}(t) = \mu_{x+t} + \mu_{y+t} = BC^{t}\,(C^{x} + C^{y})\ . $$

Hence the probability density function of T(xy) is given by ,

$$ g_{T(xy)}\,(t) = {}_{t}p_{xy}\mu_{xy}(t) = e^{-m(\,C^{t}\,-1\,)(C^{x} + C^{y}\,)}\, BC^{t}\,(C^{x} + C^{y}\,) $$

Suppose $m(C^x + C^y)$ is denoted by α_{xy} . Then g(t) is expressible as ,

$$g_{T(xy)}(t) = \alpha_{xy} \log C\, e^{\alpha_{xy}} e^{-\alpha_{xy} C^t}\, C^t \text{ , for } t > 0.$$

The probability density function of T(x) is given by,

$$g_{T(x)}(t) = \alpha_x \log C\, e^{\alpha_{xy}} e^{-\alpha_{xy} C^t}\, C^t \text{ , for } t > 0 \text{ , with } \alpha_x = m\, C^x .$$

The two densities are exactly of the same form with α_{xy} replaced by α_x .

This observation is useful to find using the same approach of computing probabilities of certain intervals in gamma distribution with appropriate parameters.

(ii) Thus, we have,

$$\overline{A}\,^{1}_{xy:\overline{n}|} = \alpha_{xy}\, e^{\alpha_{xy}} \Gamma(\lambda)\, \alpha_{xy}^{-\lambda}\, P[1 \le W \le C^n],$$

where W follows gamma distribution with shape parameter A and scale parameter $1/\alpha_{xy}$. We find the values of 1000 $\bar{A}^1_{xy:\bar{n}|}$ for x =25 and y=30 using the formulae derived above and R software. These are presented in the fifth column of Table 9.3.1. Using the relation $\bar{A}^1_{\overline{xy}:\bar{n}|} + \bar{A}^1_{xy:\bar{n}|} = \bar{A}^1_{x:\bar{n}|} + \bar{A}^1_{y:\bar{n}|}$ we find the values of 1000 $\bar{A}^1_{\overline{xy}:\bar{n}|}$. These are reported in the last column of Table 9.3.1. Fourth and sixth column report the upper limit $1000\bar{A}^1_{\bar{x}:\bar{n}|}$ + 1000 $\bar{A}^1_{\bar{y}:\bar{n}|}$ and the lower limit ${}_np_y\bar{A}^1_{x:\bar{n}|}$ + ${}_np_x\bar{A}^1_{y:\bar{n}|}$ denoted by L, respectively, as derived in Example 9.3.1. Thus, the result proved in Example 9.3.1 is verified.

Table 9.4.1 *Gompertz's law:* $1000\bar{A}^1_{xy:\bar{n}|}$ and 1000 $\bar{A}^1_{\overline{xy}:\bar{n}|}$

n	$1000\bar{A}^1_{x:\bar{n}|}$	$1000\bar{A}^1_{y:\bar{n}|}$	$1000\bar{A}^1_{\bar{x}:\bar{n}|}$ + 1000 $\bar{A}^1_{\bar{y}:\bar{n}|}$	$1000\bar{A}^1_{xy:\bar{n}|}$	L	1000 $\bar{A}^1_{\overline{xy}:\bar{n}|}$
1	1.16	1.84	3.00	3.00	3.00	0.00
2	2.37	3.75	6.12	6.11	6.10	0.01
3	3.63	5.74	9.37	9.35	9.32	0.02
4	4.94	7.80	12.74	12.71	12.65	0.03
5	6.31	9.95	16.26	16.19	16.12	0.07
6	7.73	12.19	19.92	19.81	19.70	0.11
7	9.21	14.51	23.72	23.57	23.40	0.15
8	10.74	16.93	27.67	27.46	27.22	0.21

9	12.34	19.43	31.77	31.49	31.16	0.28
10	14.00	22.04	36.04	35.67	35.23	0.37

From the values reported in the table, we note that for the underlying mortality pattern, values of 1000 $\bar{A}^{1}_{\overline{xy}:\bar{n}|}$are very low for all n, in view of the fact that the chance of death of both (25) and (30) within the tenure of 10 years is very less. But it is to be noted that

$$\bar{A}^{1}_{\overline{xy}:\bar{n}|} > \bar{A}^{1}_{\bar{x}:\bar{n}|} \text{ and } \bar{A}^{1}_{\overline{xy}:\bar{n}|} > \bar{A}^{1}_{\bar{y}:\bar{n}|}$$

It is logically appealing as the chance of death of one out of two in a group is higher than that of a single individual.

To find $\bar{A}_{xy}$, we again use the same approach, and hence, $\bar{A}_{xy}$ is given by,

$$\bar{A}_{xy} = \alpha_{xy}\ e^{\alpha_{xy}}\Gamma(\lambda)\ \alpha_{xy}^{-\lambda}\ P[W \geq 1],$$

Where, W follows gamma distribution with shape parameter λ and scale parameter l/ α_{xy} .

For the given mortality and interest pattern we have calculated $\bar{A}_x$ for x = 25. Hence, we get the value of $\bar{A}_{xy}$. These are as given below.

$$\bar{A}_{xy} = 235.98, \bar{A}_x = 152.54 \quad \bar{A}_y = 189.12 \quad \bar{A}_{\overline{xy}} = 105.68 .$$

It is to be noted that,

$$\bar{A}_{\overline{xy}} < \bar{A}_x < \bar{A}_y < \bar{A}_{xy} \quad , \text{for } x < y$$

Annual premium P(joint) for whole life insurance in a joint life status and P(Last) for a whole life insurance in last survivor status are given by,

$$\text{P(joint)} = \frac{\delta \bar{A}_{xy}}{1- \bar{A}_{xy}} = 15.44 \qquad \text{and} \qquad \text{P(Last)} = \frac{\delta \bar{A}_{\overline{xy}}}{1- \bar{A}_{\overline{xy}}} = 5.90.$$

It is to be noted that the premium in the joint life status is higher than that in the last survivor status. In term insurance, annual premium, payable as an n-year temporary annuity, which ceases at the first death in the joint life status, if it occurs before the end of the term, are obtained using the formula $1000 \bar{A}^{1}_{xy:\bar{n}|} / \bar{a}^{1}_{xy:\bar{n}|}$. These are reported in the second column of Table 9.4.2. For the setup of last survivor status, annual premium payable as an n-year temporary annuity ceases at the last death. It is obtained using the formula $1000^{\bar{A}^{1}_{\overline{xy}:\overline{n}|}} \Big/ \bar{a}_{\overline{xy}:\overline{n}|}$.

these are reported in the second column of table 9.4.2.

Table 9.4.2: *Gompertz's Law: Premium for Term Insurance*

n	Joint Life Status	Last Survivor Status
1	3.08	0.00
2	3.22	0.00
3	3.37	0.01
4	3.53	0.01
5	3.69	0.01
6	3.86	0.02
7	4.04	0.02
8	4.22	0.03
9	4.42	0.04
10	4.62	0.04

From the values reported in the table, we note that for the underlying mortality pattern, values of the premium for the last survivor status are very low for all n, for the same reason as stated above.

Example 9.4.6: Suppose the curate future lifetimes K(x) and K(y) are independent and each has the distribution defined by Gompertz's law with the force of mortality, $\mu_x = BC^x$ and $\mu_y = BC^y$ with B = 0.0001151 and C = 1.096. It is given that $\delta = 0.05$, x = 25 and y = 30.

i. Determine the probability mass function of K(xy). Calculate e_{xy}.

ii. Calculate $1000A^{1}_{xy:\overline{n}|}$, $1000A_{xy:\overset{1}{\overline{n}|}}$ and $1000\ A_{xy:\overline{n}|}$ for n = 1 to 10, for joint life status and last survivor status.

iii. Calculate premiums for insurance products specified in (ii).

iv. Calculate $1000A_{xy}$ and $1000\ A_{\overline{xy}}$

v. Calculate the annual premium, payable as a whole life annuity due, payable till the first death, for the benefit of 1000, payable at the end of the year of first death in whole life insurance issued to (xy).

vi. Calculate the annual premium, payable as a whole life annuity due, payable till the last death, for the benefit of 1000, payable at the end of the year of last death in a whole life insurance issued to $\overline{xy}$.

Solution: To find the required actuaria1 present values we need to find the probability mass function of K. (25:30) corresponding to Gompertz's law. The probability mass function of K(xy) 1s given by,

$$P[K(xy)=k] = {}_kp_{x\,k}p_y(q_{x+k} + q_{y+k} - q_{x+k}\, q_{y+k}).$$

For Gompertz's law, we can obtain,

$${}_kp_{xy} = {}_kp_{x\,k}p_y = \exp[-m(C^x + C^y)(C^k - 1)] = e^{-\alpha_{xy}(C^k - 1)},$$

Where, α_{xy} is as defined above. Further $q_{x+k} = 1 - e^{-mC^x(C-1)C^k}$. Substituting these expressions in the expression of the probability mass function of K(xy) and after some algebra we get,

$$P[K(xy)=k] = e^{\alpha_{xy}}(e^{-\alpha_{xy}C^k} - e^{-\alpha_{xy}C^{k+1}}).$$

We know the probability mass function of K(x) and it is given by,

$$P[K(x)=k] = e^{\alpha_x}(e^{-\alpha_x C^k} - e^{-\alpha_x C^{k+1}}).$$

Where, $\alpha_x = mC^x$. It is to be noted that the form of the probability mass function of K(xy) and of K(x) is a exactly the same with only a change of α_{xy} and α_x. Table 9.4.3, displays the probabilities.

Table 9.4.3: *Gompertz's law: Probability Distribution of K(25:30)*

k	Probability	*k*	Probability	k	Probability
0	0.00307	26	0.02394	50	0.01166
1	0.00336	27	0.02533	51	0.00.932
2	0.00367	28	0.02672	52	0.00724
3	0.00400	29	0.02808	53	0.00544
4	0.00437	30	0.02939	54	0.00394
5	0.00477	31	0.03063	55	0.00274

6	0.00520	32	0.03177	56	0.00183
7	0.00566	33	0.03277	57	0.00116
8	0.00617	34	0.03361	58	0.00070
9	0.00672	35	0.03425	59	0.00040
10	0.00731	36	0.03466	60	0.00021
11	0.00795	37	0.03481	61	0.00011
12	0.00863	38	0.03467	62	0.00005
13	0.00937	39	0.03421	63	0.00002
14	0.01016	40	0.03342	64	0.00001
15	0.01101	41	0.03229	65	0.00000
16	0.01191	42	0.03082	66	0.00000
17	0.01287	43	0.02903	67	0.00000
18	0.01389	44	0.02696	68	0.00000
19	0.01497	45	0.02463	69	0.00000
20	0.01611	46	0.02212	70	0.00000
21	0.01731	47	0.01949		
22	0.01855	48	0.01681		
23	0.01985	49	0.01418		
24	0.02118	39	0.03421		
25	0.02255	40	0.03342		

$P[K(25:30) = 0] = 0.00307$ implies that the chance of death of one of (25) and (30) in the first year after signing the contract is 0.00307. For $k \geq 65$, these are almost 0. Here, $P[K(30) = 0] = 0.00188$. In fact, $P[K(x) = k] < P[K(xy) = k]$, for all k. Thus, in a group if two lives are involved then the chance of death of one of the two in any year is higher than the chance of death of a single individual. From the distribution of K(25:30) we get, $e_{25:30}$ = E(K(25 : 30)) = 32.11. From the probability distribution of K(25:30), we note that the mode of the distribution is at 37.

For an n- year term insurance 1000 $A^{1}_{xy:\overline{n|}}$ for n = 1 to 10 are obtained using the following formula and the probability mass function of K(xy) derived below.

The values of $A^{1}_{xy:\overline{n|}}$ and those of other two products are reported in Table 9.4.4.

Table 9.4.4: Actuarial present values of benefit of 1000 in joint life status

n	Term	Pure Endowment	Endowment
1	2.92	948.31	951.23
2	5.96	899.02	904.98
3	9.12	852.02	861.13
4	12.39	807.19	819.58
5	15.80	764.42	780.21
6	19.33	723.61	742.93
7	22.99	684.65	707.64
8	26.79	647.46	674.25·
9	30.72	611.95	642.67
10	34.08	578.03	612.83

When these values are compared to those of $1000\bar{A}^{1}_{xy:\overline{n|}}$ reported in Table 9.3.1, we note that, I000$A^{1}_{xy:\overline{n|}}$< 1000 $\bar{A}^{1}_{xy:\overline{n|}}$ as expected.

Using the actuarial present values displayed in Table 9.3.4, we find the premiums for all these products in a joint life status. These are reported in Table 9.3.5. The general formula for finding the premium is,

$$\text{Premium} = \frac{1000\ d\ Actuarial\ present\ value\ of\ benefit}{1-Actuarial\ present\ value\ of\ benefit\ in\ endowment\ insurance}.$$

Table 9.4.5 *Premiums for benefit of 1000 in joint fife status*

n	Term	Pure Endowment	Endowment
1	2.92	948.31	951.23
2	3.06	461.44	464.50
3	3.20	299.23	302.44
4	3.35	218.20	221.55
5	3.51	169.62	173.13
6	3.67	137.28	140.95
7	3.84	114.21	118.05
8	4.01	96.94	100.95
9	4.19	83.52	87.72
10	4.38	72.81	77.20

The actuarial present values of benefit of 1000 in an n-year term insurance, n- year pure endowment and n-year endowment for last survivor status are obtained using the relations stated in equation (9.4.9). These are reported in Table 9.4.6.

Table 9.4.6 *Actuarial present value so f benefit of 1000 in last survivor* status

n	Term	Pure Endowment	Endowment
1	0.00	951.23	951.23
2	0.01	904.83	.904.84
3	0.02	860.69	860.71
4	0.04	818.69	818.73
5	0.07	778.74	778.80
6	0.10	740.72	740.82
7	'0.15	704.55	704.70
8	0.21	670.13	670.34
9	0.28	637.38	637.66
10	0.36	606.21	606.57

Using the actuarial present values, we find the premiums for these entire products in the last survivor status. These are reported in Table 9.4.7. The general formula for finding the premium is as stated above.

Table 9.4.7 *Premiums for benefit of 1000 In last survivor status*

n	Term	Pure Endowment	Endowment
1	8.61	951.23	951.23
2	8.95	463.72	463.73
3	9.30	301.36	301.36
4	9.67	220.27	220.28
5	10.05	171.70	171.72
6	10.44	139.39	139.41
7	10.85	116.36	116.39
8	11.28	99.14	99.17
9	11.72	85.79	85.83
10	12.17	75.15	75.19

The actuarial present values of benefit in a whole life insurance under both the setup are obtained using the appropriate formulae and the probability mass function of K(xy). These are as given below.

$$A_{xy} = 230.17, \quad A_x = 148.78, \quad A_y = 184.47, \quad A_{\overline{xy}} = 103.08.$$

Here also $A_{\overline{xy}} < A_x < A_y < A_{xy}$. Annual premium P(joint) for a whole life insurance in joint life status and P(Last) for a whole life insurance in last survivor status are given by,

$$P(\text{Joint}) = \frac{d\,A_{xy}}{1 - A_{xy}} = 14.58 \qquad \text{and} \qquad P\,(\text{Last}) = \frac{dA_{\overline{xy}}}{1 - A_{\overline{xy}}} = 5.61.$$

Again, it is to be noted that the premium in the joint life status is higher than that in the last survivor status.

Example 9.4.7: Z is a present-value random variable for a discrete whole life insurance of 1 issued to (x) and (y) which pays 1 at the first death and 1 at the second death. You are given (i) $a_x = 9$, (ii) $a_y = 13$, (iii) $i = 0.04$. Calculate E(Z).

Solution: From the given information we get, $E(Z) = A_{xy} + A_{\overline{xy}} = A_x + A_y$

$$= 1 - d\ddot{a}_x + 1 - d\,\ddot{a}_y = 1 - d(a_x + 1) + 1 - d(a_y + 1) = 2 - \frac{.04}{1.04}[9 + 1 + 13 + 1] = 1.08.$$

9.5 Summary

The theory is formulated with the solitary eventuality of death in mind. Many times, a single life or a group of lives—that is, several lives—are vulnerable to various circumstances.

The industrial field likewise experiences similar circumstances. For instance, there are several ways that a metal strip can fail under test conditions, including shearing, bucking, and cracking. Time to failure is therefore one of the variables of importance. Mortality and survival function are the terms we employ here. The removal from a particular status is referred to as a decrement in actuarial science. In the insurance industry, multiple decrement models are those that consider different reasons of decrement. A general theory of multiple decrement models that considers the impact of numerous causes of decrement on a group of people can be created from the theory that has been developed thus far. Many decrement models take into account a large number of lives that are impacted by multiple decrement causes.

9.6 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An Introduction using R. Universities Press (India). Hyderabad.

9.7 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT – 404 (N)A/ MASTAT – 404(N)A

Actuarial Statistics

Block : 3 Insurance and Annuities

Course Design Committee

Course Preparation Committee

Dr. Mahendra Saha **Writer**
Department of Statistics
Faculty of Mathematical Sciences, University of Delhi, New Delhi

Prof. Jitendra Kumar **Editor**
Department of Statistics
Central University of Rajasthan, Ajmer, Rajasthan

Prof. Shruti **Reviewer & Course Coordinator**
School of Sciences,
U. P. Rajarshi Tandon Open University, Prayagraj

MScSTAT– 404 NA/MASTAT– 404 NA **ACTUARIAL STATISTICS**

First Edition: ***July 2025***
ISBN :

Printed By:

Block & Units Introduction

The ***Block - 3 –Insurance and Annuities,*** is the third block of said SLM, which is divided into four units.

Unit – 10: Fundamentals of Computation of Interest Rate: In this unit, we discussed about the principles of compound interest. Nominal and effective rates of interest and discount, force of interest and discount, compound interest, accumulation factor, continuous compounding.

Unit – 11: Life Insurance & Life Annuities: In this unit, we discussed about the Insurance payable at the moment of death and at the end of the year of death-level benefit insurance, endowment insurance, deferred insurance and varying benefit insurance, recursions, commutation functions. Single payment, continuous life annuities, discrete life annuities, life annuities with monthly payments, commutation functions, varying annuities, recursions, complete annuities-immediate and apportion able annuities-due.

Unit – 12: Net Premiums & Net Premium Reserves: In this unit, we discussed about the ccontinuous and discrete premiums, true monthly payment premiums, apportion able premiums, commutation functions, and accumulation type benefits. Payment premiums, apportion able premiums, commutation functions, accumulation type benefits. Continuous and discrete net premium reserve, reserves on a semi continuous basis, reserves based on true monthly premiums, reserves on an apportion able or discounted continuous basis, reserves at fractional durations, allocations of loss to policy years, recursive formulas and differential equations for reserves, commutation functions.

Unit – 13: Some Practical Considerations: In this unit, we discussed about the practical considerations of premiums that include expenses-general expenses types of expenses, per policy expenses. Claim amount distributions, approximating the individual model, stop-loss insurance.

At the end of every unit, the summary, self-assessment questions, and further readings are given.

UNIT:10 FUNDAMENTALS OF COMPLUTATIONS OF INTEREST RATES

Structure

10.1 Introduction

Here, we use the assumption that the interest rate stays constant and that the sole source of investment earnings uncertainty is the random investment duration, which is determined by the insured's future life expectancy. Consequently, a benefit function and a discount function define each life insurance model. The only random variable in these models is the discount function, which is the interest discount factor from the time of payment to the time of policy issue, with t being the duration between issue and death. We shall examine the concept of time value of money in order to comprehend the interest discount component. It is a basic idea in the business and financial world and is frequently used in the actuarial profession to determine the present value of future premiums and benefit payments.

Common insurance products include life, term, and endowment insurance; many of these have just recently entered the market, but they are only minor variations or combinations of these; in all of these, benefit payments are contingent on the policyholder's death or survival; in all of these insurance products, the number of premiums due is also frequently determined by the insured's death or survival; in summary, the time period for which the policy is in effect is the time between the policy's issuance and the payment of claims; in all insurance products, this varies randomly.

10.2 Objectives

- Define and apply concepts like present value (PV), future value (FV), and accumulated value.
- Distinguish between effective, nominal, real, discount, and force of interest.
- Convert between different interest rate forms (e.g., nominal to effective).
- Calculate present and accumulated values for various annuity types (e.g., immediate, due, continuous).
- Work with survival function, cumulative distribution function and probability density function and Use life table functions

10.3 Compound Interest and Discount Factor

There are many situations in everyday life when we come across the concept of interest. For example, a deposit in a bank account earns interest. When one borrows money from a bank, money has to return after some period with interest. Thus, the payment of interest as a reward for the use of capital is an established part of economic life in any country. Interest rates reflect economic conditions of a country. The most common types of interest are (i) simple and (ii) compound interest. Simple interest is paid on the principal while compound interest is com-putted on both the principal and the accumulated interest. In the following, we will see why compound interest is preferred to simple interest. Suppose a(n) denotes the accumulated value of 1 rupee at the end of n years. Then the effective rate in of interest in the nth year is defined as the actual rate of increase per unit invested during that time. It is given by,

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{I(n)}{a(n-1)}$$

where I(n) denote the interest in the nth year. With simple interest, a(n) = 1 + i_n, where i denotes the rate of interest per rupee per annum. Thus, with simple interest, the effective rate in of interest in the nth year is given by,

$$i_n = \frac{1 + in - 1 - i(n-1)}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}$$

It is clear that, i_n is a decreasing function of n, that is, as the number of years of investment increases, the effective rate of interest decreases. This is certainly not desirable.

Consequently, simple interest is rarely used in practice. The formula for compound interest rate is,

$$A = P(1 + i)^n,$$

where A is the accumulated value, known as the amount, of principal P after n years. With compound interest

$$a(n) = (1 + i)^n.$$

Hence, the effective rate in of interest in the nth year is given by,

$$i_n = \frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}} = \frac{(1+i)^{n-1}(1+i-1)}{(1+i)^{n-1}} = i,\ \forall n \geq 1.$$

Thus, with compound interest, the effective rate of interest remains the same every year. This is definitely a noteworthy feature of compound interest and is why the compound interest system is always preferred to the simple interest system. With compound interest, the sequence $\{a(n) = (1 + i)^n, n \geq 1\}$ is in geometric progression with the common ratio $(1 + i)$. Thus, the principal at the end of the j^{th} year is $(1 + i)^j$. When it is multiplied by $(1 + i)$, we get the principal at the end of the $(j + 1)^{th}$ year. In this sense, $(1 + i)$ is known as an accumulation factor.

The system of compound interest can be viewed from the angle of discount and is discussed below.

To get the amount $1 + i$ at the end of one year, one has to invest 1 rupee at the beginning of the year. Thus, the discount on $1 + i$ is i or the discounted value of $1 + i$ is 1. So the discounted value of 1 is $(1 + i)^{-1}$. This is known as the present value of 1, due at the end of the year and is denoted by v. Thus $v = (1 + i)^{-1}$ and is known as the discount factor. Sometimes it is convenient to work with a rate of discount. Analogous to the effective rate of interest in the nth year, one can define the effective rate of discount in the nth year. To get amount a(n) at the end of the nth year, one has to invest $a(n - 1)$ at the beginning of the nth year. Thus, the discount is $a(n) - a(n - 1)$ on the amount a(n). So effective rate d_n of discount in in the nth year is defined as follows.

$$d_n = \frac{a(n) - a(n-1)}{a(n)}$$

$$= \frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}} \frac{1}{(1+i)^n}(1+i)^{n-1}(1+i-1)$$

$$= i(1+i)^{-1}$$

Thus $d_n = i(1 + i)^{-1}$, for all n > 1. As it is free from n, we denote it by d. Thus d = i/(1 + i). Since interest and discount are merely different ways of looking at the same problem, it follows that for every given rate of interest there is a corresponding rate of discount and vice versa. Following is a list of relations among v, d and i.

i. $v = (1 + i)^{-1}$,

ii. $d = i/(1 + i) = iv$

iii. $1 - d = v = (1 + i)^{-1}$

iv. $i = d/(1 - d)$

v. $i = (1 - v)/v$.

From (ii) we note that d < v and d < i. For i from 0 to 0.6, $i + i^2 < 1$, which implies that i < v, for these values of i. In practice, i ranges from 0.03 to 0.15, so practically i < v always. Many a times, interest is paid several times a year. Monthly interest schemes, quarterly or six-monthly interest schemes are common schemes of interest payment. In such cases, it is necessary to find the rate of interest for the fraction of the year. The common financial practice is to express rate of interest as rate per annum, even though the interest is paid more frequently than once per year. Such a rate is referred to as nominal rate of interest. There is a relation between the nominal rate of interest i(m), when interest is paid m times per annum, and i, the effective rate of interest in a year. Suppose the interest is paid at the end of each m^{th} part of the year. Then i(m)/m denotes the rate of interest paid in the m^{th} part. It is to be noted that i(m) /m cannot be i/m as the interest in the previous m^{th} part is added in the principal for the next m^{th} part. Thus, i(m) is slightly less than i. To find its relation with i, we proceed as follows. With the interest rate $i^{(m)}/m$, the accumulated value of 1 at the end of m parts, that is one year, is $(1 + \frac{i^{(m)}}{m})^m$. With the annual rate of interest i, the accumulated value of 1 at the end of one year is 1 + i. To find the relation between i and i(m), we set

$$(1 + \frac{i^{(m)}}{m})^m = 1 + i.$$

Hence, $i^{(m)} = m[(1 + i)^{1/m} - 1] = \frac{(1+i)^{1/m} - 1}{1/m}$

Table 10.3.1 specifies $i^{(12)} < i^{(4)} < i^{(2)}$ for various values of i. It is to be noted that $i^{(12)} < i^{(4)} < i^{(2)} < i$, as expected.

Table 10.3.1 *Effective Rate of Interest and Nominal Rate of Interest*

i	$i^{(2)}$	$i^{(4)}$	$i^{(12)}$
0.05	0.0494	0.0491	0.0489
0.06	0.0591	0.0587	0.0584
0.07	0.0688	0.0682	0.0678
0.08	0.0785	0.0777	0.0772
0.09	0.0881	0.0871	0.0865
0.10	0.0976	0.0965	0.0957

Extending such a division of a year into m parts, for the payment of interest, in the limit, the nominal rate becomes the annual rate of continuous growth and is known as the force of interest or instantaneous rate of interest. We proceed to define the concept of instantaneous rate of interest at time t. In some sense it is parallel to the concept of force of mortality. It is denoted by δ_t and is defined as,

$$\delta_t = \lim_{h \downarrow 0} \frac{a(t+h) - a(t)}{h\, a(t)}$$

$$= \frac{1}{a(t)} \frac{d}{dt} a(t)$$

$$= \frac{1}{a(t)} \frac{d}{dt} (1+i)^t$$

$$= \frac{1}{a(t)} (1+i)^t \log(1+i)$$

$$= \log(1+i), \quad \forall\, t.$$

Thus, δ_t is also free from t and we denote it by δ. Thus $\delta_t = \delta = \log(1 + i)$ and $e^{\delta} = (1 + i)$ or $e^{-\delta} = (1 + i)^{-1} = v$, that is, the discount factor and the instantaneous rate of interest are related by the equation $v = e^{-\delta}$. In the next section, we use this relation heavily while calculating the moments of the distribution of the present value random variable. Note that

$$\lim_{m\to\infty} i(m) = \lim_{m\to\infty} \frac{(1+i)^{1/m} - 1}{1/m} = \log(1+i) = \delta,$$

as is expected. Table 10.3.2 specifies $i^{(50)}$, $i^{(100)}$, $i^{(200)}$ and $i^{(400)}$ for values of i used in Table 10.3.1. The last column of the table displays values of δ obtained using the formula $\delta = \log(1 + i)$.

Table 10.3.2 *Force of Interest*

i	$i^{(50)}$	$i^{(100)}$	$i^{(200)}$	$i^{(400)}$
0.05	0.0488	0.0488	0.0488	0.0488
0.06	0.0583	0.0583	0.0583	0.0583
0.07	0.0677	0.0677	0.0677	0.0677
0.08	0.0770	0.0770	0.0770	0.0770
0.09	0.0863	0.0862	0.0862	0.0862
0.10	0.0954	0.0954.	0.0953	0.0953

It is to be noted that $i^{(50)} = i^{(100)} = i^{(200)} = i^{(400)}$, up to three decimals. Thus, rounding values of $i^{(400)}$ to 4 decimals, we get $\lim_{m\to\infty} i(m)$. These values coincide with the values of δ displayed in the last column, obtained using the formula $\delta = \log_e(1 + i)$. Further note that $i > \delta$ for various values of i. This inequality can be proved mathematically as follows. From the formula $\delta = \log(1 + i)$, we get using the Taylor series expansion of e^x,

$$i = e^{\delta} - 1 = 1 + \delta + \delta^2/2 + ... - 1 = \delta + \delta^2/2 + \cdots > \delta.$$

To sum up, in actuarial language, the continuous rate at which a sum of money grows under the operation of interest is the force of interest and the increase per unit due to the effect of the operation of this force of interest in any given period is the effective rate of interest.

Example 10.3.1: Find the amount to which Rs.10,000/- will accumulate after 10 years if the rate of interest is

a. 5% as the force of interest,

b. 5% as the effective rate of interest and

c. 5% per annum payable quarterly.

Solution:

a. $\delta = .05 \Rightarrow i = e^{.05} - 1 = 0.051271$ (slightly higher than δ). Hence the amount of Rs. 10,000/- after 10 years will be

$$10,000(1 + i)^{10} = 10,000(e^{.05})^{10} = 10,000\ e^{0.5} = 16,487/-.$$

b. It is given that i = 0.05, hence the amount is

$$10^4(1 + 0.05)^{10} = 16,289/-.$$

c. We are given that i(4) = 0.05. So the amount at the end of 10 years will be

$$10^4 \times (1 + 0.05/4)^{40} = 16,436/-.$$

Note that accumulation of 10,000 after 10 years is highest with 5% as the force of interest and least with 5% as the effective rate of interest.

Example 10.3.2. Find the amount to which Rs. 1,000/- will accumulate at

a. 4% per annum payable quarterly for 10 years.

b. 6% per annum payable half yearly for 5 years.

c. the rate of interest corresponding to an effective rate of discount at 3% per annum for 8 years.

d. 5% effective for 10 years, 4% effective for 5 years and 2.5% effective for 3 years.

Solution:

a. The effective rate of interest is 0.01 per quarter and the money is invested for 40 quarters, hence the answer is $1000(1.01)^{40} = 1488.90$.

b. As in (a) above, the answer is, $1000(1.03)^{10} = 1343.90$.

c. We are given that d = 0.03. Hence the effective rate of interest i is given by,

$$1/(1 + i) = 1 - 0.03 \text{ (since } v = 1 - d\text{), that is, } (1 + i) = 1/0.97$$

The corresponding amount is therefore given by,

$$1000\ (1/0.97)^8 = 1275.90.$$

After 10 years, the amount is $1000(1.05)^{10}$. This accumulates for 5 years at 4% and therefore becomes $1000(1.05)^{10}(1.04)^5$. This further accumulates for 3 years at 2.5% and therefore amounts to, $1000\ (1.05)^{10}(1.04)^5(1.025)^3 = 2134.20$.

Example 10.3.3: In how many years will a sum of money double itself at compound interest with effective rate $i = 0.05$?

Solution: If the effective rate is $i = 0.05$, the value of the required number n of years, is given by the equation

$$(1 + i)^n = 2 \text{ and hence } n = \log 2/\log (1.05) = 14.21 \text{ years.}$$

With compound interest we have seen that 1 rupee accumulates to $1 + i$ if the effective rate of interest is i per rupee per annum. Thus, the present value of $1 + i$ to be received after 1 year is 1. Further, 1 rupee accumulates to $(1 + i)^2$ after two years. Thus, the present value of $(1 + i)^2$ to be received after 2 years is 1. Continuing in this manner the present value of $(1 + i)^n$ to be received after n years is 1. Using simple rule of threes, if the present value of $(1 + i)^n$ to be received after n years is 1, the present value of 1 to be received after n years is $(1 + i)^{-n} = v^n$. In other words, if v^n is deposited today at the effective rate of interest i, then after n years it will accumulate to 1. Thus v^n is the discounted value of 1 to be received after n years. v^n is known as the present value of 1 to be received after n years and v, as stated earlier, is known as the discount factor or discount function. In general, present value can be obtained from the formula for compound interest rate. The accumulated value A of principal P after n years is given by $A = P(1 + i)^n$, i being the annual interest rate. From this

$$P = A(1 + i)^{-n} = Av^n \qquad (10.3.1)$$

is the present value of amount A to be received after n years with effective rate of interest i. We heavily use the concept. Present value of 1 unit to be received after n years = v^n in the entire further development. Table 10.3.3 displays the present value of 1000 for values of n from 1 to 10 and values of $i = 0.05$ to 0.10, with the increment of 0.01. These are obtained using the formula $P = A(1+i)^{-n}$, with $A = 1000$.

Table 10.3.3 *Present Value of 1000*

n	0.05	0.06	0.07	0.08	0.09	0.10
1	952.38	943.40	934.58	925.93	917.43	909.09
2	907.03	890.00	873.44	857.34	841.68	826.45
3	863.84	839.62	816.30	793.83	772.18	751.31
4	822.70	792.09	762.90	735.03	708.43	683.01

5	783.53	747.26	712.99	680.58	649.93	620.92
6	746.22	704.96	666.34	630.17	596.27	564.47
7	710.68	665.06	622.75	583.49	547.03	513.16
8	676.84	627.41	582.01	540.27	501.87	466.51
9	644.61	591.90	543.93	500.25	460.43	424.10
10	613.91	558.39	508.35	463.19	422.41	385.54

From the table we note that the present value of 1000 to be received after 1 year is 952.38 if the rate of interest is 5% and 909.09 if the rate of interest is 10%. It is clear that as the rate of interest increases, present value decreases. It also decreases when n increases. If 385.54 is deposited today and if the rate of interest is 10%, then in 10 years, it will accumulate to 1000. From the table we note that if the rate of interest is 8%, then in 9 years, the amount deposited will double itself.

Example 10.3.4: How much money should be deposited today so that after 5 years, the investor will get 10,000 rupees? Suppose the effective rate of interest is 0.06.

Solution: From the formula $P = A(1 + i)^{-n} = Av^n$ we get $P = 10000(1 + 0.06)^{-5} = 7472.58$. Thus, 7472.58 invested today will accumulate to 10000 after 5 years.

Example 10.3.5: Suppose a dealer wants to start a scheme of selling bikes costing 60,000 in 10 equal annual installments. The schedule of installments is such that the first installment will be paid at the time of purchase, second will be paid after 1 year and the subsequent 8 installments will be paid similarly. What will be the installment if the current rate of interest is 0.06.

Solution: It is obvious that the installment cannot be 6000, as it is not at all afford-able to the dealer. Installment 1 has to be such that the total present value of ten installments should equal 60000. The present value of the first installment to be paid at the time of delivery of the bike is I itself. The present value of the second installment to be paid after 1 year is Iv, present value of the third installment to be paid after 2 years is Iv2. Continuing in this manner, total present value of ten installments is $I + Iv + Iv^2 + \cdots + Iv^9$. Equating it to 60000, we get $I = 7690.22$. The actual installment will be 7690.22 plus some amount to account for administrative expenses.

Remark: Procedure similar to that outlined in Example 10.3.5 can be used to find EMI (equal monthly installments) for a housing loan.

Example 10.3.6: Find the present value on April 1, 2002 of the following cash flow: 5000 received on April 1, 2003, 7000 paid on April 1, 2004, 4000 received on April 1, 2005, 3000 paid on April 1, 2006 and 6000 received on April 1, 2007, when the effective rate of interest is 0.04.

Solution: The present value on April 1, 2002 is

$$5000/(1.04) - 7000/(1.04)^2 + 4000/(1.04)^3 - 3000/(1.04)^4 + 6000/(1.04)^5 = 4258.93.$$

In financial mathematics, one studies cash flows of payments whose timings and size are fixed in advance, as in examples discussed above. The methods discussed above are not directly applicable if payments depend on death or survival of the concerned individual. Such cases arise in many situations in practice. For example, a retired employee receives regular pension payment until death and the employer has to decide quite in advance how much to pay into the pension fund in order to provide the employee with life pension. In such situations, the exact number of payments cannot be foreseen. To get such a benefit he has to make a one time or periodical payments to the insurance company and the company has to make a decision about the premium rates at the time of issue of the policy. Determining the premiums optimally is an important task of any insurance company. It depends mainly on the insurance product which involves three aspects the amount of benefit, the time at which the benefit is to be paid and the mode of premium payments. The most important issue is that the company has to decide the premium amount at the time of issue of the policy and for that it has to take into account the time value of the money involved in the premium input and the benefit output.

More precisely, suppose a life insurance contract with a benefit value of 1 lakh is signed by an insured with a single premium paid at the time of issue of the policy. The prominent question is what should be the amount of single premium. The insurer will invest the premium amount to earn interest on it and will pay the benefit from the accumulated value of the premium amount over the period for which the policy is in force. It is a straightforward approach, wherein expenses incurred and the profit margins of the company are ignored. With this view, the premium amount should be such that it will accumulate to 1 lakh, with the specific rate of interest, over a specific period, which is of course random in this setup.

In all insurance products, the period for which the policy is in force, or the period involved in the calculation of present values is a random variable. Hence, the distribution theory from statistics comes in the picture. In the next two sections, we will study a present value random variable corresponding to the benefit payable at the moment of death or at the end of year of death for a variety of insurance products. The next chapter discusses the present value function of various types of modes of premium payments. These two chapters together help to decide the premium to be charged for a variety of insurance products as the main guiding principle in the Expected present value of inflow = Expected present value of outflow.

Inflow to the company is via premiums and interest and outflow of the company is via payment of claims and administrative expenses.

In any insurance product, the most important factor is the amount of benefit to be paid whenever a claim is made. Periodical premiums highly depend on the benefit amount, along with the interest rate. The main consideration in fixing the premium installments is to decide how much amount has to be collected from the insured which will grow to a sufficient fund to pay the benefit promised in the contract. Further these premium installments are to be fixed at the time of issue of the policy, that is, at the time of signing the contract. As a consequence, premiums are mainly based on the present value of the benefit. The important factor involved in deciding the present value is the rate of interest. However, complexity increases when the period for interest calculations is not fixed but is a random variable governed by the future life time of the insured. Hence, it is absolutely essential to take into account the random mortality pattern along with the interest rate.

Suppose an insurance product is such that the benefit is paid to the beneficiary after the death of the insured. The benefit is paid either at the moment of death or at the end of the policy year. Although it is said that the benefit is paid at the moment of death, it is not to be taken in the strict sense of the word. It may take a couple of days to report a death to the insurance company and some more time to settle the policy, but such delay is usually small compared to the time for which the policy is in force and does not have a significant effect on the finance of the insurance companies. So, it can be ignored and, in all derivations, one can treat the death benefit as being paid at the moment of death. Such treatment simplifies the mathematics to a great extent without losing much accuracy. With this assumption, the time period for which the policy is in force is the future life time, time from signing the contract to death of the insured. In this set up the continuous random variable T studied in Section 4.2, comes handy.

In some products benefit is not paid as soon as the claim is made but at the end of the year of death. More precisely, suppose a policy is written on November 2, 1950. The policy year is taken as November 2, 1950 to November 1, 1951. Suppose the insured died on February 7, 2007. The policy year ends at November 1, 2007. So, the benefit is paid on November 1, 2007, that is at the end of the policy year. Thus, the time period for which the policy is in force is November 2, 1950 to November 1, 2007, that is 57 years. This period is nothing but curtate, future life time of insured +1 year.

10.4 Future Life Time Random Variable

Death is a certain event. However, the age at which an individual will die cannot be predicted with certainty. Hence, the life time of an individual is modelled by a random variable. Suppose a random variable X denotes the new born's age at death, that is, the life length of an individual. Then X is a continuous random variable. Being a life length, X is a non-negative valued random variable. If F denotes its distribution function, then it is 0 for $x < 0$. Suppose $S(x) = 1 - F(x)$ denotes the survival function. S(x) specifies the probability that the new born will attain age x. Further, the probability that the life length is between ages x and z is given by,

$$P[x < X < z] = F[z] - F[x] = S(x) - S(z) = \int_x^z f(u)du$$

f being the probability density function of X. Any function F satisfying the following four properties is a distribution function of non-negative random variable. (i) $0 \leq F \leq 1$, (ii) F is non-decreasing, (iii) F is right continuous and (iv) $F(0) = 1$, $F(\infty) = 1$. Similarly, any function S satisfying the following four properties is a survival function of a non-negative random variable. (i) $0<S<1$, (i) S is

Non-decreasing, (iii) S is right continuous and (iv) $S(0) = 1$, $S(\infty) = 0$.

Example 10.4.1: Which of the following can serve as survival functions for $x \geq 0$?

(a) $S(x) = \exp(x - 0.7(2^x - 1))$

(b) $S(x) = 1/(1+x)^2$

(c) $S(x) = \exp(-x^2)$.

Solution:

(a) $S(1)= \exp(1 - 0.7(2^1 - 1)) = \exp(.3) > 1$, hence it cannot serve as a survival function.

(b) $S(0) = 1$, $S(\infty)=0$, S is decreasing and continuous. Hence S can serve as a survival function.

(c) In this case, all four requirements for a function to be a survival function are satisfied.

It is known that the distribution of a continuous random variable can be specified either in terms of its distribution function, the survival function or the probability density function. There is one more function, often used in actuarial science and demography, which has a one-to-one relation with all the above three functions. It is called the force of mortality. In the following sections, we will discuss this concept and derive its relation with other functions. the force of mortality, at age x, denoted by μ_x, is defined as follows

$$\mu_x = \lim_{\delta x \to 0} \frac{1}{\delta x} P\{x < X < x + \delta x, \qquad X > x]$$

$$= \lim_{\delta x \to 0} \frac{1}{\delta x} \frac{S(x) - S(x + \delta x)}{S(x)}$$

$$= -\frac{1}{S(x)} \lim_{\delta x \to 0} \frac{S(x + \delta x) - S(x)}{\delta x}$$

$$= \frac{f(x)}{1 - F(x)}$$

From the formula of μ_x, it is clear that it has a conditional probability density interpretation. For each age x, T_t gives the value of the conditional probability density function of X at exact age x, given survival to that age. In actuarial science and demography, μ_x is called the force of mortality. In reliability theory, μ_x is known as the failure rate or hazard rate or hazard rate function. μ_x, essentially specifies the rate at which there is a failure or death at age x given that the individual has survived up to age x. The force of mortality is sometimes referred to as the mortality function or mortality law.

In the following, we derive the relations among the distribution function, survival function, probability density function and the force of mortality. It is known that,

$$F'(x) = f(x) = -S'(x).$$

As defined above,

$$\int_x^z \mu_t\, dt = \frac{f(x)}{1-F(x)} = -S'(x)/S(x) = -\, d/dx \log S(x).$$

Hence,

$$\int_x^z \mu_t\, dt = -\log S(x), \text{and } S(x) = \exp(-\int \mu_t\, dt) \tag{10.4.1}$$

Further, we have

$$f(x) = -S'(x) = \left(\exp \int_x^z \mu_t\, dt\right)\mu_x = \mu_x\, S(x)$$

To summarise, we have,

$$\mu_x = \frac{f(x)}{S(x)} = -\, d/dx \log S(x),$$

$$S(x) = \left(\exp\left(-\int_x^z \mu_t\, dt\right)\right) \text{ and } f(x) = \mu_x\, S(x)$$

We know that a function has to satisfy certain conditions for it to qualify as a distribution function, a survival function or a probability density function. Analogously, we have some conditions on μ_x for it to be a proper force of mortality. We derive these conditions in the following.

We have $(x) = \left(\exp\left(-\int_x^z \mu_s\, ds\right)\right)$and S(x) is a decreasing function, that is $exp\left(-\int_x^z \mu_s\, ds\right)$ is a decreasing function, so $\mu_s \geq 0$ $\forall$s.

Further $\lim_{x\to\infty} S(x) = 0.$

Hence, we get,

$$\lim_{x\to\infty} \exp\left(-\int_x^z \mu_s\, ds\right) = 0 \Rightarrow \int_0^\infty \mu_s\, ds) = \infty.$$

Thus, (i) $\mu_s \geq 0$ for all s and (ii) $\int \mu_s\, ds = \infty$ are the two conditions on μ_s for it to be a proper force of mortality.

Example 10.4.2: It is given that $\mu_x = A + e^x$ and $S(0.50) = 0.50$, calculate A.

Solution:

$$S(0.50) = 0.50$$

implies,

$$0.5 = \exp\left(-\int_0^{0.5} \mu_x \, dx\right) = \exp\left(-\int_0^{0.5} (A + e^x) dx\right) = \exp(-0.5A) - e^{(0.5)} + 1 .$$

Hence, $\log(0.5) = -0.5A - \exp(0.5) + 1,$

which gives A = 0.089.

Example 10.4.3: Suppose X follows exponential distribution with parameter μ. Find the corresponding force of mortality.

Solution: For the exponential distribution with parameter μ, the probability density function is given by,

$$f(x) = \mu e^{-\mu x}, \text{ for } x > 0 \text{ and } 0 \text{ otherwise.}$$

The survival function is given by

$$S(x) = e^{-\mu x}, \text{ for } x > 0.$$

Hence, the force of mortality is μ for all x > 0. It does not depend on x.

Remark: In the class of all continuous distributions, exponential distribution is the only distribution having a constant force of mortality. Thus, the force of mortality at age 10 is the same as at age 80, which is unrealistic in the case of the human life length random variable. As a consequence, exponential distribution is not a good model for human life length.

Example 10.4.4: For the uniform distribution over (0,100), find the force of mortality.

Solution: For uniform distribution over (0,100), probability density function is given by, f(x) = 1/100, for x in (0,100) and the survival function is

$$S(x) = \begin{cases} 1, & if\ x < 0 \\ 1 - \dfrac{x}{100}, & if\ 0 \le x < 100 \\ 0, & if\ x \ge 100 \end{cases}$$

Hence, the force of mortality is 1/(100 −x).

Remark: For uniform distribution over (0,100), P[10 ≤ X ≤ 20] = P[80 ≤ X ≤ 90] = 1/10. Thus, if the human life length is modeled by uniform distribution on (0, 100), then the chance of death in age group (10, 20) is the same as the chance of death in the age group (80, 90), which is again unrealistic. Hence, a uniform distribution is also not a good model for human life length.

The most commonly encountered probability laws which model human life length are (i) De Moivre' law (1729), (ii) Gompertz's law (1825), (iii) Makeham's law (1860) and Weibull law (1939).

In one of the first attempts to algebraically describe the mortality experience of human lives, Abraham De Moivre proposed, early in the eighteenth century, that the human life length be modeled by the uniform distribution over the interval (0, w) with force of mortality $\mu_x = (w - x)^{-1}$ and survival function $S(x) = 1 - x/w$, w being the limiting age. With this form of survival function,

$$P[x \leq X \leq x + 1] = S(x) - S(x + 1) = 1/w.$$

Thus, the chance of death in a unit interval (x, x+1) does not depend on x, as noted in the above remark. Hence it is not a realistic model and is rarely used.

Benjamin Gompertz, actuary of the Alliance of London, reasoning on physio-logical grounds, proposed in 1825, the most discerning mathematical formulation of changes in mortality with age.

He postulated that μx satisfies the differential equation $d\mu_x/dx = k\mu_x$, $x > 0$. Its solution is given by the force of mortality

$$\mu_x = BC^x,\ B > 0,\ C > 1,\ x \geq 0.$$

This formula implies that the force of mortality increases with age. It thus represents the deterioration in the ability of human beings to withstand death. The survival function for Gompertz's law is given by

$$S(x) = \exp[-m(C^x - 1)],\ m = B/\log_e C,\ B > 0,\ C > 1,\ x \geq 0.$$

Gompertz's law is often found to be quite accurate, at least as a first approximation, for ages over 25 or 30, the value of C usually being between 1.07 and 1.12. With C = 1, Gompertz's law reduces to the exponential distribution which is characterized by the fact that the force of mortality is constant a major factor for not considering it a good model for human life length.

Some 35 years later, William Makeham, proposed to add to the geometric component of the force of mortality postulated by Gompertz, a constant component intended to reflect the causes of death due to chance irrespective of age. Thus the forces of mortality and survival function according to Makeham's law are

$$\mu_x = A + BC^x \text{ and } S(x) = \exp[-Ax - m(C^x - 1)],\ B>0,\ A\geq -B,\ C > 1,\ x \geq 0.$$

The constant has been interpreted as the term which captures the accident hazard. Both Gompertz's and Makeham's formulae produce excessively high mortality rates at ages over 90, probably because the population surviving to these ages is intrinsically heterogeneous.

Weibull's law of mortality is given by

$$\mu_x = kx^n \text{ and } S(x) = \exp(-ux^n+1),\ u = k/(n + 1),\ k > 0,\ n > 0,\ x \geq 0.$$

It is usually less successful than those of Gompertz and Makeham to represent human mortality as its support for any combination of parameters does not cover the human life span.

In the insurance business, the random variable of interest is the time until death random variable for a person aged x. Symbol (x) is used to denote a life aged x. Suppose T(x) denotes the time until death of (x). It is also known as a future life time of (x) or residual life of (x) or remaining life of (x). In particular, T(0) = X. It is to be noted that the distribution of T(x) is the same as the conditional distribution of X - x given that X > x. If Gx(t) denotes the distribution function of T(x), then for $t \geq 0$,

$$\begin{aligned} G_x(t) &= P[T(x) \leq t] \\ &= P[X - x \leq t \mid X > x] \\ &= P[x < X \leq x + t] / P[X > x] \\ &= S(x) - S(x + t) / S(x). \qquad (10.4.2) \end{aligned}$$

In actuarial science, it is frequently necessary to make probability statements about T(x). Hence, the International Actuarial Congress in 1898 adopted the international actuarial notation to describe the distribution function and the survival function of T(x). These notations are revised or extended as necessary by the International Actuarial Association's permanent committee on notation. The distribution function and the survival function of T(x) are respectively denoted by ${}_tq_x$ and ${}_tp_x$ and are given by,

$${}_tq_x = P[T(x) \leq t],\ t > 0 \text{ and } {}_tp_x = 1 - {}_tq_x = P[T(x) > t],\ t\geq 0.$$

${}_tq_x$ is the probability that a person aged (x) dies in (x, x+t) while ${}_tp_x$is the probability that (x) will survive up to age x + t. In particular, if x = 0, then T(0) = X, ${}_tq_0$ = F(t), t > 0 and ${}_tp_0$ = S(t), t > 0. If t = 1, as per the international actuarial notation, t is omitted. So that, ${}_1q_x$ = q_x which denotes the probability that (x) will die within 1 year and ${}_1p_x = p_x$ is the probability that (x) will attain age x + 1. ${}_tq_x$ and ${}_tp_x$ are essentially functions of the survival function of X. In the following, we express ${}_tq_x$ and ${}_tp_x$ in terms of S(x).

$${}_tq_x = P[T(x) < t] = (S(x) - S(x + t))/S(x) = 1-S(x+t)/S(x)$$

and $${}_tp_x = 1 - {}_tq_x = S(x + t)/S(x).$$

${}_tp_x$ can be expressed as ${}_tp_x = (S(x + t)/S(0))(S(0)/S(x)) = {}_{x+t}p_0/\ {}_xp_0$. It then follows that ${}_{x+t}p_0 = {}_xp_0\ {}_tp_x$. The left-hand side of this equation is the chance that a new born will survive for x + t years, while the right-hand side expresses the same probability as a product of two probabilities, the first is the probability of survival to age x. The attained age of the individual is then x. The second factor is the probability that the individual of age x survives for t more years. In general, for 0 < s <t,

$${}_tp_x = S(x+t)/S(x) = S(x + s)/S(x) * S(x + t)/S(x+s) = {}_sp_x\ {}_{t-s}p_{x+s}.$$

Using the formula of S(x) in terms of μx, as given in (10.4.1), tpx can be expressed as

$${}_tp_x = S(x + t)/S(x) = \exp(- \int_0^{x+t} \mu_s\, ds) / \exp(-\int_0^x \mu_s ds) = \exp(- \int_x^{x+t} \mu_s\, ds) = \exp(-\int_0^t \mu_{x+s} ds).$$

From this survival function of T(x), we obtain the probability density function of T(x). The probability density function g(t) of T(x) is the derivative of its survival function with a minus sign. Thus,

$$g(t) = -d/dt\ {}_tp_x = \exp(- \int_x^{x+t} \mu_s ds)\ \mu_{x+t} = {}_tp_x\ \mu_{x+t}\ , t\geq 0. \qquad (10.4.3)$$

The formula $g(t) = {}_tp_x\ \mu_{x+t}$ for probability density function of T(x) can be interpreted as follows. It is a product of two factors. The first factor is the probability that the individual (x) survives for t units of time. The attained age then will be x + t. The second factor is the force of mortality at age (x + t), which is the value of the conditional probability density function of X at the exact age x + t, given survival to that age. We heavily use this expression in the following chapters. By the definition of force of mortality, the force of mortality of T(x) at t is given by $g(t)/{}_tp_x = \mu_{x+t}$. Thus, if μ_t is the force of mortality of life length random variable

X at t, then μ_{x+t} is the force of mortality of T(x) at t. In international actuarial notation E(T(x)) is denoted by e_x^0.

Example 10.4.5: It is given that F(x) = 0 for x < 0 and F(x) = 1 - 1/(x+1) for x ≥ 0. Which of the following are true?

(i) ${}_xp_0 = 1/(x+1)$, (ii) $\mu_{49} = 0.02$, (iii) ${}_{10}p_{39} = 0.8$.

Solution:

i. ${}_xp_0 = S(x)/S(0) = 1/(x+1)$.

ii. By definition $\mu_x = f(x)/(1 - F(x)) = (x+1)/(x + 1)^2 = 1/(x+1)$. Hence, $\mu 49 = 1/50 = 0.02$.

iii. ${}_{10}p_{39} = S(49)/S(39) = 1/50/1/40 = 4/5 = 0.8$.

Thus (i), (ii) and (iii) are true.

Example 10.4.7: Suppose the life length random variable X is modelled by a uniform distribution over (0,100). Find the probability density function g(t) of T(25).

Solution: To find the probability density function g(t) of T(x), we first find ${}_tp_x$ and μ_{x+t}. When X has a uniform distribution over (0,100), then the survival function is given by

$$S(x) = \begin{cases} 1, \; if \; x < 0 \\ 1 - \dfrac{x}{100}, \; if \; 0 \leq x < 100 \\ 0, \; if \; x \geq 100 \end{cases}$$

Hence,

$${}_tP_x = S(x+t)/S(x) = (100\text{-}x\text{-}t)/\,(100\text{-}x)$$

and

$$\mu_x = f(x)/S(x) = (1/100)/((100 - x)/100) = 1/(100\text{-}x).$$

The probability density function g(t) of T(x) is given by,

$$g(t) = {}_tP_x \; \mu_{x+t} = 1/(100 - x).$$

It is the probability density function of a random variable with uniform distribution over (0, 100 - x). Thus, distribution of T(25) is uniform over (0,75).

Example 10.4.9: Suppose the life length random variable is modeled by a distribution with force of mortality as specified below.

$$\mu_s = \begin{cases} 0.01, & if 0 < s < 15 \\ 0.02, & if\ 15 \le s < 25 \\ 0.03, & if\ s \ge 25 \end{cases}$$

Find corresponding S(x), f(x) and probability density function g(t) of T(20). Check whether f(x) and g(t) are density functions.

Solution: To find the survival function from the given force of mortality we use the formula,

$S(x) = \exp -\int_0^x \mu_s\, ds$. For $0 < x < 15$, we get $S(x) = \exp- \int_0^x 0.01\ ds = \exp(-0.01x)$.

For $15 \le x < 25$, we get

$$S(x) = \exp- \int_0^x \mu_s\, ds = \exp-\int_0^{15} 0.01\ ds - \int_{15}^x 0.02\ ds = \exp(-0.02x + 0.15)$$

For $x \ge 25$,

$$S(x) = \exp- \int_0^x \mu_s\, ds = \exp- \int_0^{15} 0.01\ ds - \int_{15}^{25} 0.02\ ds - \int_{25}^x 0.03\ ds = \exp(-0.03x + 0.4).$$

Thus, the survival function is given by,

$$S(x) = \begin{cases} e^{-0.01x}, & if\ \ 0 \le x < 15 \\ e^{-0.02x+0.15}, & if\ 15 \le x < 25 \\ e^{-0.03x+0.4}, & if\ x \ge 25 \end{cases}$$

To find f(x), we use the formula, $f(x) = -S'(x)$ and we have,

$$f(x) = \begin{cases} 0.01e^{-0.01x}, & if\ \ 0 \le x < 15 \\ 0.02e^{-0.02x+0.15}, & if\ 15 \le x < 25 \\ 0.03e^{-0.03x+0.4}, & if\ x \ge 25 \end{cases}$$

It is easy to check that f(x) is always non-negative and $\int f(x)\, dx = 1$. To find probability density function g(t) of T(20), we use the formula,

$$g(t) = \mu_{20+t}\ {}_tp_{20} = \mu_{20+t} S(20+t) / S(20)$$

The form of $S(20 + t)$ changes for $0 < t < 5$ as $(20 + t)$ varies between 20 to 25 and for $t \ge 5$, $(20+t) > 25$. Thus, with appropriate choice of $S(20 + t)$, we get

$$g(t) = \begin{cases} 0.02e^{-0.02t}, if\ 0 < t < 5 \\ 0.03e^{-0.03t+0.05}, if\ t \geq 5. \end{cases}$$

It is easy to check that g(t) is always non-negative and $\int_0^\infty g(t)$ dt = 1.

Remark: For the life length random variable modeled by the force of mortality, it is found that P[T(20) ≤ 100] = 0.9477. It means that the future life time of (20) can be larger than 100. Thus, this distribution may not be a good model for human life length. The following example is similar to Example 10.4.9, except for the fact that P[T(30) ≤ 70] is almost 1. So it is a feasible model for human life length.

Example 10.4.10: Suppose the life length random variable is modeled by

i. Gompertz's force of mortality and

ii. Makeham's force of mortality. In each case, find the probability density function g(t) of T(x). Check whether it is a density function.

Solution: The forces of mortality and survival function for Gompertz's law are

$$\mu_x = BC^x \text{ and } S(x) = \exp[-m(C^x - 1)], B > 0, C > 1, x > 0, m = B/\log_e C.$$

The probability density function g(t) of T(x) is then given by,

$$g(t) = \mu_{x+t}\,{}_tp_x = \exp[-mC^x(C^t - 1)]\, BC^{x+t}, \text{ for } t > 0.$$

It is clearly a non-negative function. To verify that it integrates to 1 over (0,∞), we substitute,

$$mC^x(C^t - 1) = y.$$

Then $mC^xC^t \log C dt = dy$, that is, $BC^{x+t}dt = dy$. Hence,

$$\int_0^\infty g(t) = \int_0^\infty \exp(-y)\, dy = 1.$$

The forces of mortality and survival function for Makeham's law are

$$\mu_x = A + BC^x \text{ and } S(x) = \exp[-Ax - m(C^x - 1)], B > 0, A \geq -B, C > 1, x \geq 0.$$

The probability density function g(t) of T(x) for Makeham's law is given by,

$$g(t) = \mu_{x+t}\,{}_tp_x = \exp[-At - mC^x(C^t - 1)]\, [A+BC^{x+t}], \text{ for } t > 0.$$

It is clearly a non-negative function. To verify that it integrates to 1 over (0,∞), we substitute,

$At+mC^x (C^t-1) = y$. Then $(A+mC^xC^t \log C)dt = dy$, that is, $(A+BC^{x+t})dt = dy$.

Hence, $\int_0^\infty g(t) = \int_0^\infty \exp(-y)\, dy = 1$.

We will use these probability density functions of T(x) corresponding to Gompertz's law and Makeham's law in further chapters to evaluate premiums if the underlying mortality pattern is one of these two.

Many insurance and annuity products involve a deferment of benefit. For example, in a 5-year deferred whole life insurance, benefit is not paid if claim is made in the first five years. In such cases, probabilities of the more general event that (x) will survive t years and die within the following u years are involved. The corresponding probabilities are known as deferred probabilities. The symbol t indicates deferment for t years. Suppose ${}_{t|u}q_x$ denotes the probability that (x) will survive t years and die within the following u years. Then it is given by,

${}_{t|u}q_x$ = probability that (x) will die between ages x + t and x + t + u

$= P[t < T(x) \leq t + u] = P[t < X - x < t + u \mid X > x]$

$= P[x + t < X < x + t + u \mid X > x] = P[x + t < X < x + t + u, X > x]/P[X > x]$

$= P[x + t < X < x + t + u]/P[X > x] = (S(x + t) - S(x + t + u))/S(x)$

$= (S(x + t) - S(x + t + u))/S(x + t)\ S(x + t)/S(x) = uqx{+}t\ tPx.$

Alternatively,

${}_{t|u}q_x = (S(x + t) - S(x + t + u))/S(x) = S(x + t)/S(x) - S(x + t + u)/S(x)$

$= {}_tp_x - {}_{t+u}p_x = 1 - {}_tq_x - 1 + {}_{t+u}q_x = {}_{t+u}q_x - {}_tq_x.$

As before, if u = 1, the prefix is deleted in ${}_{t|u}q_x$ and is denoted by ${}_{t|}q_x$. The relation ${}_{t|u}q_x = {}_tp_{x\,u}q_{x+t}$ has a nice interpretation. The left-hand side of the equation is the probability that (x) will survive for the next t units of time and death will occur in the next u units of time. The right-hand side of the equation is the product of two probabilities. The first factor is the probability that (x) will survive for the next t units of time. Attained age then will be x + t. The second factor is the probability that (x + t) will die in the next u units of time.

Example 10.4.11: The life of a television set is specified by the following survival function.

$$S(x) = \begin{cases} 1, & if\ x < 0 \\ 1 - \dfrac{x}{w}, & if\ 0 \leq x < w \\ 0, & if\ x \geq w \end{cases}$$

and $e_0^0 = 10$. For a proposed new type of television, with the same w, the new survival function is,

$$S^*(x) = \begin{cases} 1, \ if\ 0 \le x \le 4 \\ \dfrac{w-x}{w-4}, \ if\ 4 < x <\le \end{cases}$$

Calculate the increase in life expectancy at time 0.

Solution: For the old type of television set,

$$e_0^0 = 10 = \int_0^w S(x)dx = \int_0^w \left(1 - \frac{x}{w}\right) dx = w/2$$

Hence w = 20. For the proposed new type, the life expectancy is

$$\tilde{e}_0^0 = \int_0^4 dx = \int_4^{20} \left(\frac{20-x}{16}\right) dx = 4 + 8 = 12$$

Thus, increase in life expectancy of the proposed television is 12 − 10 = 2 years.

In the above derivation, expectation is obtained using the survival function. Alternatively, from S(x), one can find the probability density function f(x) as as f(x) = − d/dx[S(x)]. Thus, for the old type of television, the probability density function of life time is constant over the range (0,20). Hence its expectation is 10. For the proposed type, the probability density function of life time is

$$f(x) = 1/16 \text{ for } 4 < x < 20.$$

Hence its mean is (20 + 4)/2 = 12.

In many problems of life insurance, the exact age at death is not important. It is enough to know the number of complete one year long periods an individual survives after signing the contract. For example, according to a certain annuity plan if the insured gets Rs. 10,000/- on each birthday between the ages 60 and 70, it does not make a difference whether death occurs at the exact age 65.2 or 65.7. The only thing that counts is that the death occurs between 65 and 66. In some insurance products, the benefit payments are made at the end of the policy year. For example, if the policy is signed on November 2, 1980 and death occurs in February 2007, then benefit payment is made on November 1, 2007, that is at the end of a policy year. Hence, it is worth studying the number of complete one year long periods an individual aged x survives after signing the contract. Such considerations lead to the concept of curtate future life time random variable. It is a discredited version of a random variable T(x) and is discussed in the next section.

10.5 Curtate Future Life Time

A discrete random variable K(x) associated with future life time T(x) is defined as the largest integer strictly smaller than T(x). For example, if T(x) = 7.5, then K(x) = 7; if T(x) = 8, then K(x) = 7; if T(x) = 0.5 then K(x) = 0 and if T(x) = 1.3 then K(x) = 1. Thus, K(x) is an integer valued random variable with possible values 0,1,2,..., and it specifies the number of complete future years of (x). It is known as the curtate future life time at age x. The probability mass function of K(x) is obtained from the distribution of T(x). Thus,

$$P[K(x) = k] = P[k < T(x) \leq k + 1]$$

$$= {}_{k+1}q_x - {}_kq_x$$

$$= {}_kp_x - {}_{k+1}p_x$$

$$= {}_kp_x - {}_kP_x {}_1p_{x+k}$$

$$= {}_kp_x \, q_{x+k}$$

$$= {}_{k|}q_x, \; k = 0, 1, 2, \ldots,$$

Where, k|qx is the deferred probability defined earlier. E[K(x)] is the expected number of complete future years of life of (x) and is denoted by ex. It is given by,

$$e_x = E(K(x))$$

$$= \sum_1^\infty k \, {}_kp_x \, q_{x+k}$$

$$= \sum_1^\infty k^{p_x}.$$

It is referred to as a curtate expectation while e_x^0 is referred to as a complete expectation. e_x is often used as a measure of the standard of living and health care in a country.

Example 10.5.1: Using S(x) = 1 − x²/100 for 0 ≤ x ≤ 10, find the distribution of K(4). Also obtain its expectation e_4.

Solution: $P[K(4) = k] = k|q_4$, k = 0, 1, ... Hence,

$$0|q_4 = q_4 = 1 - S(5)/S(4) = 9/84,$$

$$1|q_4 = {}_1P_4 \, q_5 = S(5)/S(4) \, (1 - S(6)/S(5)) = S(5) - S(6)/S(4) = 11/84,$$

$$2|q_4 = {}_2P_4\, q6 = S(6)/S(4)\,(1 - S(7)/S(6)) = S(6) - S(7)/S(4) = 13/84.$$

Similarly, $3|q_4 = 15/84$, $4|q_4 = 17/84$, $5|q_4 = 19/84$ and $k|q_4 = 0$ for $k = 6,7, \ldots$ From this distribution, we get $e_4 = 2.916$.

Example 10.5.2: Suppose $e_{30} = 58.72$ and $\mu_{30+t} = 0.004$ for $0 \le t \le 2$. Find e_{32}.

Solution: Using the definition of expectation in terms of survival function, we get,

$$e_x = \sum_1^\infty k^{p_x} = P_x + P_x\, P_{x+1} + Px\; {}_2P_{x+1} + Px\; {}_3P_{x+1} + \ldots$$

$$= P_x\,(1 + P_{x+1} + {}_2P_{x+1} + {}_3P_{x+1} + \ldots) = Px\,(1 + \sum_1^\infty k^{p_{x+1}} = p_x\,(1 + e_{x+!}).$$

This is a recurrence relation between e_x and e_{x+1}. We use it repeatedly to find e_{31} from $e_{30}$and e_{32} from e_{31}. For this we first find p_{30} and p_{31} using the formula ${}_tP_x = \exp\,(-.\int_0^t \mu_{x+s} ds)$ Thus, $p_{30} = p_{31} = \exp(-0.004) = 0.996008$. Using the recurrence relation, we get $e_{31} = 57.95$ and $e_{32} = 57.19$.

10.6 Life Tables

A life table is useful in many actuarial calculations. The importance of life tables in actuarial science has been recognized by marking 1693 as the year that actuarial science began. In that year, Edmund Halley published a paper entitled 'An estimate of the degrees of the mortality of mankind, drawn from various tables of births and funerals at the city of Breslau'. The life table contained in Halley's paper is known as the Breslau table. Surprisingly, some quite modern notation and ideas are found in the Breslau table.

A life table describes the mortality experience of a certain group of individuals. A published life table usually contains tabulations, by individual ages, of the basic functions p_x, q_x, l_x, d_x and some additional derived functions, which are discussed in this section. Suppose we have a group of l_0 new borns. l_0 is known as the radix and is usually taken as 1,00,000. It is further assumed that each new born's age at death random variable X has the same distribution and is specified by the survival function S(x). Suppose (x) denotes the cohort's number of survivors to age x, the values of which are 0, 1, 2, ... Suppose l_x lives are labeled as j = 1, 2, ..., l_0. The indicator random variable is defined as, $I_j(x) = 1$ if the life j survives to age x and 0 otherwise. Thus, $I_j(x)$ is an indicator of the survival to age x of life j and we have $\sum I_j(x)$. Further $E(I_j(x)) = 1 \times S(x)$. Therefore, $E(l_x) = l_0S(x) = l_x$, say. Under the assumption that indicators $I_j(x)$ are mutually independent, lx has binomial $B(l_0, S(x))$ distribution for each $x = 0, 1, 2, \ldots$. From it also we have $l_x) = l_0S(x)$. Thus, lx represents the expected number of survivors to age x from the l0 new borns. From the relation $l_x = l_0S(x)$, it is clear that if the values of l_x are specified for all $x \geq 0$, then the distribution of X gets completely specified. In many practical situations, the survival function is specified using l_x instead of S(x).

Example 10.6.1: With $S(x) = 1 - x^2/100$, for $0 \leq x \leq 10$, and $l_0 = 1{,}00{,}000$, find l_1, l_4 and l6.5.

$$l_1 = l_0S(l) = l_0S(1) = 99{,}000,$$

$$l_4 = l_0S(4) = 84{,}000$$

$$\text{and } l_{6.5} = l_0S(6.5) = 57,\ 750.$$

In analogy with l_x, suppose ${}_nD_x$ denotes the number of deaths between ages x and x + n from among the initial l0 lives. Since a new born has probability $S(x) - S(x + n)$ of death between ages x and x + n from among the initial l_0 lives, , the expected value of ${}_nD_x$ is given by

$${}_nd_x = E({}_nD_x) = l_0(S(x) - S(x + n)) = l_x - l_{x+n}$$

When n = 1, ${}_nD_x$ and ${}_nd_x$ are written as D_x and d_x respectively.
$l_0\ \mu_x$ is interpreted as the expected density of deaths in the age-interval (x, x + dx).

For human lives, there have been few observations of age at death beyond 110. Death being a certain event, it is reasonable to assume that there is an upper limit to age. Age w such that $S(x) > 0$ for $x < w$ and $S(x) = 0$ for $x \geq w$ is called the limiting age. The group of l_0 new borns, each with survival function S(x), is usually referred to as a random survivorship group.

Thus, number of deaths in a unit interval (x, x + 1) does not depend on x, that is, equal numbers would die each year. It has of course been found unrealistic and hence this model is rarely used.

There are interesting inter-relations between μx and lx. We have $l_x = l_0S(x)$. Therefore, $\log l_x = \log l0 + \log S(x)$. S(x) is a differentiable function of x and hence l_x is also a differentiable function of x. Taking derivative with respect to x, we get $-d/dx \log l_x = -d/dx \log S(x)$, that is, $(-1/l_x)(dl_x/dx) = \mu_x$, which can be expressed as $-dl_x = \mu_x\ l_x d_x$. Thus, decrement in l_x is proportional to the force of mortality and lx. If force of mortality is high (low), then decrease in lx is high (low). Similarly, if l_x is high (low), then decrease in l_x is high (low).

Example 10.6.2: Suppose a survival model is defined by the following values of p_x.

x :	0	1	2	3	4
p_x :	0.9	0.8	0.6	0.3	0

i. What are the corresponding values of S(x) for x = 0, 1, 2, 3, 4 and 5?

ii. Using a radix l0 = 10,000, find values of l_x and d_x for x = 0, 1,·2, 3, 4.

iii. What is the value of w in this case?

iv. Verify that I: $\sum_{x=0}^{w-1} d_x = l_0$.

v. Find ${}_2d_0$, ${}_2q_1$, ${}_2p_1$ and ${}_3q_2$.

Solution:

i. We know that px = S(x+1)/S(x) and S(0) = 1. Using the recursive relation S(x + 1) = p_xS(x) with S(0) = 1, we get S(1) = 0.9, S(2) = 0.72, S(3) = 0.432, and so on.

ii. l_x is found using the relation $l_x = l_0S(x)$ and d_x is found using, $d_x = l_x - l_{x+1}$. The results are presented in the following table.

X	*Px*	*S(x)*	*lx*	*dx*
0-	0.9	1	10,000	1000
1	0.8	0.9	9,000	1800
2	0.6	0.72	7,200	2880
3	0.3	0.432	4,320	3024
4	0	0.1296	1296	1296
5	-	0	0	0

iii. S(x) = 0 is first reached at x = 5. Hence w = 5.

iv. Note that addition of the last column is 10,000.

v. ${}_3d_0 = l_0 - l_3 = 5680$, ${}_2q_1 = (l_1-l_3)/l_1 = 0.52$, ${}_3p_1 = l_4/l_1 = 0.144$, ${}_3q_2 = (l_2-l_5)/l_2 =$ 1.

In the following, we will discuss in detail the construction of a life table. In a life table, the first column is of age x, which are usually taken as integers 0, 1, 2, ... The second is of q_x, third gives values of l_x and the fourth column gives values of dx. When the survival function S(x) is known, which is assumed to be the same for all individuals, one can compute the four columns of the life table using the formulae, developed above.

The approach of constructing a life table using the specific survival function is known as a random survivorship approach. The support for such analytic functions and such an approach has declined in recent years. The other approach to interpret life table is known as the deterministic survivorship approach. This is rooted mathematically in the concept of decrement rates. An important point to be noted is that the life tables are not constructed by observing l0 new borns until the last survivor died. In the deterministic approach, the functions in the life table are based on the estimates of q_x, which are obtained from age specific death rates m_x. Age specific death rates are obtained using vital statistics records. Under the assumption of uniformity of deaths in a unit interval, a concept to be discussed in the next section, q_x and m_x are related by the equation $q_x = m_x/(1 + 1/2m_x)$.

Further, it is assumed that all the members of the group, at each age of their lives, are subject to effective annual rates of mortality specified by the values of qx in the life table. With these quantities qx, the progress of the group will be determined as follows:

$$l_1 = l_0(1 - q_0) = l_0p_0 = l_0 - d_0 = l_0p_0.$$

$$l_2 = l_1(1 - q_1) = l_1p_1 = l_1 - d_1 = l_0 - (d_0 + d_1) = l_0p_1p_0.$$

In general,

$$l_x = l_x - 1(1 - q_x - 1) = l_x - 1\ p_x - 1 = l_x - 1 - d_x - 1 = l_0 - \sum_{y=0}^{x-1} d_y$$

$$= l_0\ (1 - {}_xd_0/\ l_0) = l_0(1 - {}_xq_0) = l_0{}_xp_0.$$

Thus, l_x is the number of lives attaining age x in the survivorship group. Values of x running from 0 to the limiting age w, corresponding values of q_x, l_x and d_x, constitute the first four columns of a life table. There are three more functions derived from lx, which we will discuss in the following paragraphs.

The fifth column in the life table is that of L_x. L_x denotes the total expected number of years lived between ages x and x + 1 by survivors of the initial group of l0 lives. L_x is computed as the addition of two terms, one corresponding to the number of years lived by l_x+1 survivors in x to x + 1 and the second corresponding to the number of years lived by those who die in (x, x + 1). Thus, L_x is given by,

$$L_x = 1 * l_x + 1 + \int_0^1 1 *_t \mu_{x+t}\, l_{x+t}\, dt.$$

The first term, gives the number of years lived by l_{x+1} survivors in x to x + 1 and the integral in the second term counts the years lived by those who die between x and x + 1. Here μ_{x+t} is the instantaneous death rate and l_{x+t} is the number of survivors at x + t who live fraction t of years. Thus, number of years of these survivors is t l_{x+t} μ_{x+t} dt, where 0 < t < 1. Integrating t between 0 and 1 gives the number of years lived in (x, x + 1) by those who die in (x, x + 1). The word 'expected' is used in view of the fact that l_x are expected values. Now, we have a result that $\mu_x\, l_x = -\, dl_x$.

Therefore,

$$\frac{d\, l_{x+t}}{dt} = -\, l_{x+t}\mu_{x+t}$$

Hence, using integration by parts we get,

$$L_x = l_{x+1} + + \int_0^1 t\mu_{x+t}\, l_{x+t}\, dt = l_{x+1} - \int_0^1 t\ \frac{d\, l_{x+t}}{dt}$$

$$= l_{x+1} - [tlx + t]_0^1 + 1 * l_{x+t}\, dt = l_{x+1} - l_{x+1} + \int_0^1 1 * l_{x+t}\, dt$$

Thus, with the usual interpretation of definite integrals it is to be noted that L_x represents the area below the curve of l_x bounded by the ordinates at x and x + 1.

Under the assumption of uniformity of deaths in a unit interval (a concept discussed in the next section), l_x is a straight line in the interval (x, x + 1). Hence the region below the curve of l_x bounded by the ordinates at x and x + 1 becomes a trapezium. Using the formula for the area of a trapezium it follows that

$$L_x = (l_x + l_{x+1})/2$$

L_x is usually interpreted as the expected number of individuals in the age group (x, x + 1).

The function L_x is also used to define the age specific death rate or the central death rate at age x, denoted by m_x. It is defined as

$$m_x = \left(\int_0^1 1 * \mu_{x+t}\, l_{x+t}\, dt\right) / \int_0^1 1 * l_{x+t}\, dt.$$

It can also be represented in terms of d_x and L_x as follows.

$$\int_0^1 1 * \mu_{x+t}\, l_{x+t}\, dt = \int_0^1 t\ \frac{d\, l_{x+t}}{dt}\, dt$$

$$= [-lx + t]_0^1 + \int_0^1 l_{x+t} \frac{d1}{dt} dt$$

$$= l_x - l_{x+1}.$$

Hence,

$$m_x = (l_x - l_{x+1})/L_x = d_x/L_x.$$

Note that, m_x is the weighted average of μ_{x+t}, weights being l_{x+t}.

The next column is of T_x, which denotes the expected number of years lived beyond age x, by the survivorship group with l_0 initial members. It is given by,

$$T_x = \int_0^\infty t\, \mu_{x+t} l_{x+t} dt = -\int_0^\infty t\, \frac{d\, l_{x+t}}{dt} dt$$

$$= \int_0^\infty l_{x+t} dt = \int_0^\infty l_t\, dt$$

$$= L_x + L_{x+1} + L_{x+2} + ...$$

$$T_x = L_x + T_{x+1}.$$

It gives the backward recurrence relation between T_x and T_{x+1}, which is useful in the calculation of T_x values for all x. We define $T_w = L_w$. From the formula of T_x

To indicates the expected total population. Such an interpretation follows from the fact that $T_0 = L_0 + L_1 + ...$ ……. the interpretation of L_x as the expected population in the age group (x, x + 1).

The last column of the life table is of e_x, It denotes the average number of years of future life time of an individual of the l_x survivors of the group at age x. It given by,

$$e_x = T_x/l_x = l_x^{-1} \int_0^\infty l_{x+t} dt$$

$$= l_x^{-1} \int_0^\infty l_x\, {}_tp_x\, dt$$

$$= \int_0^\infty 1 * {}_tp_x\, dt = \sum_{k=1}^\infty 1 * {}_kp_x,$$

Where, k runs from 1 to ∞.

The expectation e_0^1 analogous to e_x and defined in Section 10.3 is given by,

$$E(T(x)) = \int_0^\infty t\, \mu_{x+t}\, {}_tp_x\, dt$$

Thus, a life table consists of eight columns specifying x, q_x, p_x, l_x, d_x, L_x, T_x and e_x where x = 0, 1, 2, ... We summaries below the formulae used in the construction of a life table.

$$p_x = 1 - q_x$$

$$l_{x+1} = l_x p_x$$

$$d_x = l_x - l_{x+1}$$

$$L_x = (l_x + l_{x+1})/2$$

$$T_x = L_x + T_{x+1}$$

$$e_x = T_x/l\_$$

While using the formula for L_x as $L_x = (l_x + l_{x+1})/2$, we make the assumption that deaths are uniform over a unit interval. The calculation of T_x values is easy if we proceed in the reverse direction, i.e., from age w to 0, using the backward recurrence relation derived above. It is to be noted that $T_w = L_w$. Then

$$T_{w-1} = L_{w-1} + T_w,\ T_{w-2} = L_{w-2} + T_{w-1}.$$

Continuing in this direction we find the T_x column.

10.7 Summary

To understand the notion of interest discount factor, we have studied the notion of time value of money. It is a fundamental concept throughout the business and financial world and is widely used in the actuarial profession to obtain the value of the benefit payment and premiums to be received in future. In summary, the time period for which the policy is in force, is the period of the policy issue to the payment of claims. The random variable specifying time until death after a certain age is a basic building block. This chapter demonstrated how this distribution can be summarized in a life table Life tables also help to solve certain problems in hospital, transport and hotel management. In the insurance business, life tables are useful to calculate premiums and reserves.

10.8 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.

- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

10.9 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial Statistics: An Introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT:11 LIFE INSURANCE AND LIFE ANNUITIES

Structure

11.1 Introduction

The insurance system as a mechanism meant to reduce the adverse financial impact of some types of random events, such as untimely death, accident, fire, theft, or natural calamities. Depending on the events they cover, insurance products are classified as life insurance products and non-life or general insurance products. Life insurance products include whole life insurance, term insurance, endowment insurance and their variants. Non-life or general insurance products include vehicle insurance, property insurance and medical insurance. Here, we will study some standard models for life insurance designed to reduce the financial impact of the random event of untimely death. In the life insurance products discussed in this unit, the benefit amount is fixed and the time of payment depends on the time of death of the insured. Hence, the products are developed as functions of T, the insured's future life time random variable, more specifically in terms of time from the issue of policy to the time of payment.

As stated, insured's pay certain premium periodically to get the benefit. Insurance companies invest the money collected via premiums in interest-earning enterprises. In view of the long-term nature of these insurances, the amount of investment earnings, up to the time of payment, is also uncertain. Such uncertainty is due to two reasons:

i. The random period for which the policy is in force and

ii. The variation in the rate of interest depending on the economic and financial status of the country.

Here, we assume that the rate of interest remains the same and uncertainty in investment earnings is only due to the random period of investment, which is dictated by the future life time of the insured. As a consequence, all life insurance models are specified by a benefit function b_t and a discount function v_t. In such models, v_t is the interest discount factor from the time of payment back to the time of policy issue, t is the length of the interval from issue to death and that is the only random quantity.

To understand the notion of interest discount factor, we will study the notion of time value of money. It is a fundamental concept throughout the business and financial world and is widely used in the actuarial profession to obtain the present value of the benefit payment and premiums to be received in future. Here the problem of interest is to find what sum of money must be invested to accumulate a given sum after a specified time at a specified rate of interest. The process of solving this problem is known as finding the present value of a given sum. In the calculation of present values, money flows are discounted, that is, valued in the current time frame by taking into account the time value of money. Actuaries use the concept of time value of money together with the concept of random variability in the calculation of actuarial present values. The theory of compound interest is highly used to find the time value of money. We briefly discuss the theory of compound interest and a concept called force of interest, parallel to the force of mortality.

11.2 Objectives

1. Calculate the expected present value (EPV) of future death benefits.
2. Use mortality tables and interest rates to set fair premiums.
3. Use survival models to compute the expected present value of future annuity payments.
4. Account for life expectancy, interest rates, and payment frequency.
5. Price annuities fairly and set reserves to ensure long-term payment ability.

11.3 Benefit Payable at the Moment of Death

Suppose bt is the benefit to be paid after t units of time. To begin with we assume that t is fixed. From (5.2.1), its present value is given by baut. Suppose Z_t denotes the present value, at policy issue, of the benefit payment to be made after t units of time. We define the present value function Z_t, by

$$Z_t = b_t v^t, \text{ P} \qquad (11.3.1).$$

b_t is the benefit function and v^t is the discount function. Equation (11.3.1) is the basic equation in the calculation of expected present values of benefit for a variety of insurance products. In any insurance product, the time for which the policy is in force is a random variable. If the benefit is to be paid at the moment of death, then the time from policy issue to the death of the insured is the insured's future life time random variable, T, defined in Section 4.2. We use the function Z_T to define the present value random variable Z_T. Thus, the present value, at policy issue, of the benefit payment is the random variable Z_T defined as

$$Z_T = b_T v^T = b_T e^{-\delta T} = b_T\,(1+i)^{-T}.$$

Whenever clear from the context, the suffix T in Z_T is omitted. From the definition of Z_T, it is clear that to study the random variable Z_T, we need information on three aspects:

i. Benefit function, depending on the insurance product,
ii. Interest rate reflecting the financial status of the country and
iii. Distribution of the future life time random variable reflecting the mortality pattern of the population under study.

In practice, distribution of Z_T is always summarized by two main characteristics—mean and variance of the distribution. The expectation $E(Z_T)$ of the present value random variable Zr is called the actuarial present value. It is also termed as the net single premium, as it is expected that this amount will accumulate over the period T to the required benefit amount at the specific rate of interest. It is net because it does not take into account administrative expenses and profit. It is a single premium in contrast to annual, semi-annual, quarterly, monthly or other types of premiums that are acceptable in life insurance practice. With this view, the purchase price of the insurance is at least the net single premium. The actuarial present value $E(Z_T)$ of the benefit can be interpreted as the average present value of the cash outflow of the insurance company corresponding to number of policies issued to many individuals of the same age. For any portfolio it is essential to know how much the present value deviate from the expected present value. Var(Z_T) is used to gain knowledge about such a spread. Thus, for each insurance product we find E(Z_T) and Var(Z_T). Notation for the net single premium differs from product to product. We use international actuarial notation to denote net single premiums.

All life insurance policies pay a benefit if death of the insured occurs while the policy is in force. However, the features of life insurance policies vary, depending on the type of policy. In this chapter, we will discuss the following three major types of life insurance policies.

i. Term life insurance
ii. Endowment insurance and
iii. Whole life insurance.

We will also discuss the variations of these models when there is a deferment period and also when the benefit amount is the same for the entire period of policy (known as level benefit in insurance literature) or when it varies with time. The first step is to define the benefit function bt. It is free from t if it is a level benefit. In the entire book it is assumed that the present value is based on compound interest at some constant annual effective rate of interest, say i, so that the discount function is taken as $v^t = (1+i)^{-t} = e^{-\delta t}$.

Term Life Insurance: The term life insurance provides coverage for a specified period of time, called the policy term. The policy benefit is payable only if the insured makes claim, that is, dies during the specified term. The length of the term varies considerably from policy to policy. The term may be as short as the time required to complete an airplane trip or as long as 40 years or more. The most common plan of term insurance is the level term life insurance, which provides a death benefit that remains the same over the term of the policy. For example, under a five year level term policy that provides Rs.1,00,000/- coverage, the insurer agrees to pay Rs.1,00,000/- if the insured makes the claim at any time during the five year period. If claim is not made in 5 years, the insurer has no liability to pay the benefit. The amount of each renewal premium payable for a level term life insurance policy usually remains the same throughout the stated term of coverage. Term life insurance is also known as temporary insurance.

Term life insurance costs less than other types of life insurance for the same amount of coverage. For this reason, many insurance experts recommend term insurance for people on a limited budget or who require coverage for only a short time. Many businesses provide life insurance to their employees through a group life insurance plan. Most of these plans offer term insurance. In the following, we will define the present value random variable corresponding to level benefit, n year term life insurance.

n Year Term Life Insurance: An n year term life insurance provides benefit only if the insured dies within the n year term of an insurance commencing at issue. If a unit is payable at the moment of death of (x), then the benefit function is given by,

$$b_t = \begin{cases} 1, & if\ t \leq n \\ 0, & if\ t > n \end{cases}$$

The present value random variable is then defined as,

$$Z = \begin{cases} v^T, & if\ T \leq n \\ 0, & if\ T > n \end{cases}$$

b_t = 1 means 1 unit, which may be 10,000 or 1 lakh or its multiple. In the theoretical derivations, we take b_t = 1 and adopt appropriate modifications. In the description of n year term life insurance, it is assumed that a claim is made if the insured dies in the specified term. With this assumption we can work with the random variable T. In practice, death is not the only reason to make the claim. An occurrence of a random event leading to financial loss may be taken as sufficient reason to make the claim. However, such events need to be clearly defined and stated in the policy contract. Random variable T in such cases will be the time from the issue of the policy to occurrence of such events. It is necessary to model its distribution. As an example, in a foreign tour, loss of passport or baggage, accidents leading to serious injury, illness requiring hospitalization etc. are some random, unpredictable events resulting in financial loss.

The net single premium for the n year term insurance with a unit benefit payable at the moment of death of (x) is E(Z). In international actuarial notation it is denoted by $\bar{A}^1_{x:\,\bar{n}|}$ and is given by,

$$\bar{A}^1_{x:\,\bar{n}|} = E(Z) = E(Z_T) = \int_0^\infty z_t g(t)dt = \int_0^\infty v^t\, {}_tp_x \mu_{x+t}\, dt. \qquad (11.3.2)$$

Here the letter A stands for 'assurance' and the bar indicate that the amount is payable immediately on death of (x). We use the probability density function g(t) of T. If the benefit amount is b, then the net single premium will be b$*\,\bar{A}^1_{x:\,\bar{n}|}$. The second moment of the distribution of Z is given by,

$$E(Z^2) = \int_0^n (v^t)^2 \, {}_tp_x\mu_{x+t} \, dt = \int_0^\infty e^{-2\delta t} {}_tp_x\mu_{x+t} \, dt$$

The second integral shows that the second moment of Z is equal to the net single premium for an n year term insurance for a unit benefit payable at the moment of death of (x), calculated at a force of interest equal to 2 times the given force of interest, that is, 2δ. Thus, the variance of Z_T is given by

$$Var(Z) = {}^2\bar{A}^1_{x:\,\bar{n}|} - (\bar{A}^1_{x:\,\bar{n}|})^2, \text{ 1 x: η} \qquad (11.3.3)$$

Where ${}^2\bar{A}^1_{x:\,\bar{n}|}$ is the net single premium for an n year term insurance for a unit amount, calculated at the force of interest 2δ. In the following example, we find $\bar{A}^1_{x:\,\bar{n}|}$when the mortality pattern is specified.

Example 11.3.1: Suppose the life length random variable is modeled by a distribution with force of mortality as specified below.

$$\mu_s = \begin{cases} 0.04, & if 0 \le s < 15 \\ 0.08, & if\ 15 \le s < 25 \\ 0.12, & if\ 25 \le s > 35 \\ 0.18 & if\ \ s \ge 35\ . \end{cases}$$

i. Find $\bar{A}^1_{x:\,\bar{n}|}$ take δ= 0.06.

ii. Find the net single premium for an n year term insurance for a benefit of 1000 to be paid at the moment of death, for n = 1, 2,..., 10 and for δ = 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12. Comment on the changes in values as n changes and as δ changes.

Solution: The probability density function g(t) of T(30) is given by,

$$g(t) = \begin{cases} e^{-0.12t}, & if\ \ 0 \le t < 5 \\ e^{-0.18t+0.3}, & if\ \ t \ge 15 \end{cases}$$

i. By definition of $\bar{A}^1_{x:\,\bar{n}|}$,

$$\bar{A}^1_{x:\,\bar{n}|} = \int_0^5 v^t g(t)dt = 0.12 \int_0^5 e^{-0.06t-0.12t}\, dt\ = \frac{0.12(1-e^{-0.9})}{0.18} = 0.3956.$$

Thus, the actuarial present value of 1 unit benefit is 0.3956 in a 5-year term insurance when the force of interest is 0.06. It is interpreted as, if (30) signs a 45-year term insurance contract for a benefit of 1 unit and if he is willing to pay a onetime premium at the time of policy issue then the net single premium is 0.3956.

ii. To find the net single premium in an n year term insurance for a benefit of 1000, for various values of n and δ, we first find the formula for A as a function of n and δ. Note that the form of g(t) changes depending on whether t≤ 5 or t > 5. So, the formula for $\bar{A}^{1}_{30:\,\bar{n}|}$ also changes accordingly.

For n ≤ 5, $\bar{A}^{1}_{30:\,\bar{n}|} = 0.12 \int_0^n e^{-\delta t - 0.12t}\, dt$

$$= \frac{0.12(1 - e^{-(\delta+0.12)n})}{(\delta + 0.12)}$$

For n > 5, $\bar{A}^{1}_{30:\,\bar{n}|} = 0.12 \int_0^n e^{-\delta t - 0.12t}\, dt + 0.18 \int_0^n e^{-\delta t - 0.3 - 0.18t}\, dt$

Calculating the integrals, we get for n > 5,

$$\bar{A}^{1}_{30:\,\bar{n}|} = \frac{0.12(1-e^{-5\delta-0.6})}{(\delta+0.12)} + \frac{0.18(e^{-5\delta-0.6} - e^{-n\delta-0.18n-0.3})}{(\delta+0.18)}$$

Values of 1000 $\bar{A}^{1}_{30:\,\bar{n}|}$ for various values of n and d are reported in Table 11.3.1

Table 11.3.1 1000 $\bar{A}^{1}_{30:\,\bar{n}|}$

n	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
1	110.35	109.82	109.29	108.76	108.24	107.72	107.20	106.69
2	203.46	201.55	199.67	197.81	195.97	194.16	192.37	190.61
3	282.00	278.17	274.40	270.71	267.09	263.54	260.05	256.62
4	348.27	342.17	336.21	330.40	324.74	319.21	313.82	308.55
5	404.18	395.62	387.32	379.27	371.46	363.89	356.54	349.40
6	472.91	460.68	448.92	437.58	426.67	416.15	406.01	396.24
7	527.51	511.86	496.88	482.54	468.81	455.64	443.03	430.94
8	570.90	552.12	534.24	517.21	500.97	485.50	470.73	456.65
9	605.37	583.79	563.34	543.94	525.53	508.06	491.46	475.69
10	632.76	608.71	586.00	564.55	544.28	525.11	506.97	489.80

With δ = 0.05 for a 1-year term insurance, the net single premium or the purchase price is 110.35 for a benefit of 1000 rupees. It is to be noted that as the force of interest increases, the value of the net single premium decreases, as expected. For example, if the force of interest is doubled, the values of the net single premium are not halved. Further as the term of insurance contract increases, the net single premium increases. It is in view of the fact that as term increases, the chance of claiming the benefit increases and to compensate it the premium amount increases.

We will now study an n year pure endowment insurance product which is complementary to the n year term insurance.

n Year Pure Endowment Insurance: In an n year term insurance, no benefit is paid after n years. The n year pure endowment insurance is complementary to the n year term insurance in the sense that no benefit is paid before the specified term of n years is over. More specifically, an n year pure endowment provides a benefit at the end of the n years if and only if the insured survives at least n years from the time of policy issue. If the amount payable is 1 unit, then the benefit function and the present value random variable are given by,

$$b_t = \begin{cases} 1, & if\ t \le n \\ 0, & if\ t > n \end{cases}$$

$$Z_T = \begin{cases} v^T, & if\ T \le n \\ 0, & if\ T > n \end{cases}$$

The net single premium is denoted by $A_{x:\frac{1}{n}|}$ or by ${}_nE_x$ and is given by,

$$A_{x:\frac{1}{n}|} = {}_nE_x = E(Z) = v^n\, {}_np_x \qquad (11.3.4)$$

Variance of Z_T is given by,

$$Var(Z_T) = v^{2n}\, {}_np_x - (v^n\, {}_np_x)^2 = v^n\, {}_np_{x\,n}q_x = {}^2A_{x:\frac{1}{n}|} - (A_{x:\frac{1}{n}|})^2 \qquad (11.3.5)$$

Pure endowment insurance is not a popular product. We will use it in theoretical developments. We will now proceed to discuss one more model for life insurance the endowment insurance.

Endowment Insurance: It is a combination of an n year pure endowment insurance and an n year term insurance. It provides a specified benefit amount whether the insured lives to the end of the term or dies during that term. Each endowment policy specifies a maturity date, which is the date on which the insurer will pay the policy's face amount to the policy owner if the insured is still living or to the beneficiary at the moment of death, if the insured dies before that date. Beneficiary is the one to whom the benefit amount is payable. It is essential to enter the name of the beneficiary in the contract. Thus, n year endowment insurance provides a specified benefit to the insured at the end of the n year term if the insured is alive or the benefit is paid to the beneficiary if death occurs before the term ends.

In endowment insurance, premiums are usually level throughout the term of the policy. A policy holder can purchase an endowment policy with a single premium or with a series of premiums over a limited period of time. Endowment insurance is mainly a means of saving money. Policyholders often use endowment policies to finance the education of their children.

If the insurance is for a unit amount and the death benefit is payable at the moment of death, then the present value random variable is given by,

$$Z_T = \begin{cases} v^T, & if\ T \leq n \\ 0, & if\ T > n\,. \end{cases}$$

It can be rewritten as,

$$Z_T = \begin{cases} v^T + 0\,, & if\ T \leq n \\ 0 + v^n, & if\ T > n. \end{cases}$$

Note that the distribution of Z_T is a mixture of continuous and discrete distributions as with probability ${}_np_x$, it takes value v^n and with probability 1 - ${}_np_x$. It is distributed like v^T. We obtain its expected value as follows. Recalling the definitions of a present value random variable for an n year term insurance and n year pure endowment insurance, it is easy to see that,

Z_T (endowment insurance) = Z_T (term insurance) + Z(pure endowment insurance). (11.3.6)

We use this relation to find the formula for the net single premium for n year endowment insurance. It is denoted by and is given by the formula

$$\bar{A}_{x:\overline{n|}} = \bar{A}^{1}_{x:\,\overline{n|}} + \bar{A}_{x:\overset{1}{\overline{n|}}}$$

We studied an n year pure endowment essentially to find this formula. Var (Z_T) is also obtained from this relation, which simplifies to,

Var (Z_T) = 2 $\bar{A}_{x:\overline{n|}}$ - ($\bar{A}_{x:\overline{n|}}$)^2 (11.3.8)

Whole Life Insurance: In this product, the benefit is paid to the beneficiary at the moment of death of the insured. Whole life insurance products offer lifetime coverage, that is, provides payment following the death of the insured at any time in the future. This is in contrast with the term life insurance which provides protection for a certain period of time and provides no benefits after that period ends.

Whole life policies can be classified on the basis of the length of the policy's premium payment period such as (i) continuous premium policies or (ii) limited payment policies. Under a continuous premium whole life policy, premiums are payable until the death of the insured and as such the amount of each premium payment is lower than the premium amount required under any other premium payment schedule. A limited payment whole life policy is a policy for which premiums are payable only until some stated period or until the insured's death, whichever occurs first. If the insured dies before the end of the specified premium payment period, the insurer will pay the death benefit to the named beneficiary and no further premiums are payable. Limited payment policies are designed to meet a policy owner's need for whole life insurance protection that is funded over a limited time period. The policy owner, for example, may expect that his income will drop considerably when he retires and he will still need life insurance coverage after retirement.

The present value random variable for the whole life insurance with benefit of 1 unit payable at the moment of death of (x), is given by

$$Z = v^T, \quad \text{for } T \geq 0.$$

The net single premium is denoted by $\bar{A}_x$ and is given by,

$$\bar{A}_x = E(Z) = E(v^T) = E(e^{-\delta T}) = \int_0^{\infty} v^T \, {}_tp_x \mu_{x+t} \, dt \qquad (11.3.9)$$

Whole life insurance is the limiting case of an n year term insurance, as n → ∞. It is to be noted that $= \bar{A}^1_{x:\,\bar{n}|} \rightarrow \bar{A}_x$ as $n \rightarrow \infty$. Further, $\bar{A}_{x:\overline{n|}} = \bar{A}^1_{x:\,\bar{n}|} + v^n {}_np_x \rightarrow \bar{A}_x$

as $n \rightarrow \infty$ as v^n converges to 0. From the expression of $\bar{A}_x$, it is to be noted that $\bar{A}$ is the moment generating function of T at (-δ).

Example 11.3.4: Suppose the life length random variable X is modeled by a uniform distribution over (0, 100). Find 50000 $\bar{A}_x$ for ages x = 25, 30, 35, 40, 45, 50 and δ = 0.05.

Solution: As the life length random variable X has a uniform distribution over (0, 100), then T(x) also has uniform distribution over (0, 100 - x). For uniform distribution over (0, a), the moment generating function Mx(t) is given by,

$$M_x(t) = E(e^{tx}) = \frac{1}{a}\int_0^{\infty} e^{tx}\, dx = (e^{at} - 1) / at.$$

Hence,

$$\bar{A}_x = M_{T(x)}(-\delta) = \left(1 - e^{-\delta(100-x)}\right) \Big/ \delta(100 - x).$$

Table 11.3.5 displays the values of $50000\bar{A}_x$ for various values of x. As age increases, the net single premium for whole life increases, as expected.

Table 11.3.5: 50000 $\bar{A}_x$ *when T(x) has U(0,*100*x)*

Age*(x)*	$50000\bar{A}_x$
25	13019:76
30	13854.32
35	14788.09
40	15836.88
45	*l7019A9*
50	18358.30

Example 11.3.5: Suppose an individual of age 30 signs a whole life insurance contract. He is insured for a death benefit of 1 lakh, payable at the moment of death. Assuming that he is subject to a constant force of mortality μ = 0.04, find the net single premium. Also find the variance of corresponding present value random variable. The benefit payments are to be withdrawn from an investment fund earning interest at a rate δ = 0.06.

Solution: From the given information we have,

$$Z_T = 10^5 v^T, \text{ for } T \geq 0.$$

For constant force of interest d and mortality μ, the net single premium for the unit benefit whole life insurance is

$$\bar{A}_x = \int_0^{\infty} e^{-\delta t} e^{-\mu t} \mu dt = \frac{\mu}{\mu+\delta} \text{ and } {}^2\bar{A}_x = \frac{\mu}{\mu+2\delta}$$

Thus, for this example,

$$E(Z_T) = {}^{10}\bar{A}_x = 10^5 \frac{0.04}{0.1} = 40000 = 4 * 10^4$$

$$E(Z_T^2) = 10^{10}\,{}^2\bar{A}_x = 10^{10}\frac{0.04}{0.04+2(0.06)} = 25 \times 10^8 \text{ and}$$

$$Var(Z_T) = 9 \times 10^8.$$

Thus, the individual has to pay Rs. 40000/- as a net single premium. Note that the variability of the present value random variable is quite high.

Remark: From Example 11.3.5, we note that when the mortality is modeled by the constant force of mortality, the net single premium $\bar{A}_x$ is $\mu / (\mu+\delta)$. It is free from x. Thus, premium for an individual of age 30 is the same as that of age 80, which is not reasonable. It again shows that the constant force of mortality is not a good model for modeling human life length in the insurance field.

11.4 Benefit Payable at the End of the Year of Death

In the previous section, we discussed how to find the actuarial present value of the benefit in variety of models for life insurance when the death benefits are payable at the moment of death. In all these models the basic random quantity is T, the future life time random variable of the insured. In practice, in some insurance products, benefits are payable at the end of the year of death, that is, at the end of the policy year after death. The first policy year is taken as August 20, 1970 to August 19, 1971. Suppose the insured died on May 15, 2007. The policy year ends at August 19, 2007, So the benefit is paid on August 19, 2007, that is at the end of the policy year. Thus, the time period for which the policy is in force is August 20, 1970 to August 19, 2007, that is 37 years. This period is nothing but the curtate future life time, K(30), of the insured +1 year. Thus, in insurance products where benefit is payable at the end of the year of death, the underlying mathematics is based on curtate random variable K.

In this section, we will discuss all the insurance products with the appropriate modification to account for the change in time of payment of the benefit. The main modification is that the size and time of payment of the death benefits depend only on the number of complete years lived by the insured from policy issue to the time of death. These insurances are referred to as payable at the end of the year of death. The insurance model in such a setup is in terms of functions of the curtate future life time of the insured. Suppose b_{k+1} is the benefit function representing the benefit amount payable at K+ 1. We again assume that the theory of compound interest is used to find the present value of the money. The basic equation defining the present value random variable Z_{K+1}, at policy issue, of the benefit payment, is given by,

$$Z_{K+1} = b_{k+1}\, v^{K+1} = b_{k+1}e^{-\delta(K+1)}. \quad (5.4.1)$$

At the time of policy issue, the insurance year of death is 1 plus the curtate future lifetime random variable K. Note that this equation is similar with the only change that T is replaced by K + 1. As in the previous section, we obtain the actuarial present value of the benefit corresponding to the n year term insurance, n year endowment insurance, whole life insurance, with level benefit or varying benefit, with or without deferment. The terms and conditions of all these insurance products remain the same except that the benefit is paid at the end of the year of death. For example, for an n year term insurance with benefit of a unit amounts payable at the end of the year of death, we have

$$b_{k+1} = \begin{cases} 1, \text{if k} \; = \; 0,1,\ldots,\text{n} - 1 \\ \quad 0, elsewhere \end{cases}$$

$$Z_{K+1} = \begin{cases} v^K, \text{if K} \; = \; 0,1,\ldots,\text{n} - 1 \\ \quad 0, elsewhere \end{cases}$$

To find the expectation and variance of this present value random variable we have to use the distribution of the curtate future life time random variable. It is a discrete random variable with the probability mass function given by,

$$P[K = k] = {}_kP_x\, q_{x+k}, \text{ for } k = 0,1,\ldots.$$

The net single premium for this insurance is denoted by, $A^1_{x:\,\bar{n}|}$ and is given by the following formula.

$$A^1_{x:\,\bar{n}|} = E(Z_{K+1}) = \sum_{k=0}^{n-1} v^{k+1} P[K = k] = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x\, q_{x+k}. \quad (11.4.2)$$

Further, $Var(Z) = {}^2A^1_{x:\,\bar{n}|} - (A^1_{x:\,\bar{n}|})^2$, where,

$${}^2A^1_{x:\,\bar{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x\, q_{x+k}$$

In the formula for actuarial present values and variance of present value random variables, the integrals are replaced by summation. Further note that there is no bar above A, to distinguish the net single premium when benefit is payable at the end of the year of death. The formulae for net single premiums for other insurance products discussed in Section 11.3 can be derived on similar lines.

From the formula for $A_{x:\,\bar{n}|}$, it is to be noted that the net single premium for an n year endowment insurance is again the addition of that for an n year term insurance and an n year pure endowment insurance. Thus,

$$A_{x:\,\bar{n}|} = A^{1}_{x:\,\bar{n}|} + A_{x:\,\frac{1}{n}|}^{1}$$

From the formula for net single premium for a whole life insurance issued to (x), it is clear that it may be obtained by letting n → ∞ in the formula for an n year term insurance model. From the formula for A_x we get a recursion relation for A_x as follows.

$$A_x = \sum_{k=0}^{\infty} v^{k+1}{}_{k}p_{a}\, q_{x+k} = vq_x \sum_{k=1}^{\infty} v^{k+1}{}_{k}p_{x}\, q_{x+k}$$

$$= vq_x + vp_x \sum_{k=1}^{\infty} v^{k}{}_{k-1}p_{x+1}\, q_{x+k}$$

$$= vq_x + vp_x \sum_{j=0}^{\infty} v^{j+1}{}_{j}p_{x+1}\, q_{x+1+j}$$

$$= vq_x + vp_x A_{x+1} \qquad (11.4.3)$$

Using the backward recurrence relation $A_x = vq_x + vp_xA_{x+1}$ and with $A_w = 1$, we can find A_x values for all x.

Example 11.4.1: Prove that

i. ${}_{m|}A^{1}_{x:\,\bar{n}|} = {}_{m}E_x A^{1}_{x+m:\,\bar{n}|}$

ii. ${}_{m|}A_{x:\,\bar{n}|} = {}_{m}E_x A_{x+m:\,\bar{n}|}$

iii. ${}_{m|}A_x = {}_{m}E_x\, A_{x+m}$.

Solution:

i. ${}_{m|}A^{1}_{x:\,\bar{n}|} = \sum_{k=m}^{m+n-1} v^{k+1}{}_{k}p_x\, q_{x+k}$

$= \sum_{r=0}^{n-1} v^{m+r+1}{}_{m+r}p_x\, q_{x+m+r}$

$= {}_{m}p_x\, v^{m} \sum_{r=0}^{n-1} v^{m+r+1}{}_{r}p_{x+m}\, q_{x+m+r}$

$= {}_{m}E_x A^{1}_{x+m:\,\bar{n}|}$

ii. $_{m|}A_{x:\,\overline{n}|} = {}_{m|}A^{1}_{x:\,\overline{n}|} + v^{m+n}{}_{m+n}p_x$

$= {}_mE_{x\,m|}A^{1}_{x+m:\,\overline{n}|} + v^m{}_mp_xv^n{}_np_{x+m}$

$= {}_mE_{x\,m|}A^{1}_{x+m:\,\overline{n}|} + {}_mE_{x\,m}E_{x+m}$

$= {}_mE_x\,({}_{m|}A^{1}_{x+m:\,\overline{n}|} + {}_mE_{x+m})$

$= {}_{m|}A_{x:\,\overline{n}|} = {}_mE_xA_{x+m:\,\overline{n}|}$

iii. can be obtained from (i) by allowing n to tend to ∞ or by adopting similar steps as in (i) with the change in the upper limit to ∞.

Example 11.4.3: It is given that $q_{28} = 0.135$, $q_{29} = 0.146$, $q_{30} = 0.159$, $q_{31} = 0.173$, $q_{32} = 0.188$ and $i = 0.05$.

i. Find the actuarial present value of benefit of 1000 in a 5-year term insurance, 5-year endowment insurance, issued to (28).

ii. Find the actuarial present value of benefit of 1000 in a 2-year deferred 2-year term insurance, 2-year deferred 2-year endowment insurance, issued to (28).

Solution:

i. By definition

$A^{1}_{28:\,\overline{5}|}\ / = \sum_{k=0}^{4} v^{k+1}{}_kp_x\ q_{x+k}$

$= v^1q_{28}+v^2q_{28}q_{29}+v^3q_{28}q_{29}q_{30}+v^4q_{28}q_{29}q_{30}q_{31}+v^5q_{28}q_{29}q_{30}q_{31}q_{32}$

$=0.5086848$

For a 5-year endowment insurance,

$A_{28:\,\overline{5}|}=A^{1}_{28:\,\overline{5}|} + v^5{}_5p_{28}$

$= A^{1}_{28:\,\overline{5}|} + v^5p_{28}p_{29}p_{30}p_{31}p_{32}$

$= 0.8355623$

Hence, $1000\ A^{1}_{28:\,\overline{5}|}= 508.68$ and $1000\ A_{28:\,\overline{5}|}= 835.56$.

ii. We use the results to find the actuarial present value of benefit in a deferred insurance. Thus,

$$ {}_{2|}A^{1}_{28:\overline{2|}} = {}_{2}E_{28}A^{1}_{30:\overline{2|}} \text{ and } {}_{2|}A^{1}_{28:\overline{2|}} = {}_{2}E_{28}A_{30:\overline{2|}} $$

Now ${}_{2}E_{28} = v^2{}_{2}p_{28} = v^2\, p_{28}p_{29} = 0.6700317$.

Further $A^{1}_{30:\overline{2|}} = vq_{30} + v^2p_{30}q_{31} = 0.283395$

and $A_{30:\overline{2|}} = A^{1}_{30:\overline{2|}} + v^2{}_{2}p_{30} = 0.9142404$.

Hence, we get,

$$ 1000\ {}_{2|}A^{1}_{28:\overline{2|}} = 189.88 \text{ and } 1000\ {}_{2|}A_{28:\overline{2|}} = 612.57. $$

11.5 Relation between A and $\bar{A}$

In all the examples in the previous section, it is to be noted that to calculate the actuarial present value of the benefit payable at the end of the year of death, it is sufficient to have the information on p_x or q_x for integer values of x. These values are usually estimated from age specific death rates which are decided on the basis of vital statistics registry of a state or the region under study. As a consequence, data to model the random mortality of the individuals on p_x or q_x for integer values of a is easily available and can be revised periodically. Thus, calculation of the actuarial present value of the benefit payable at the end of the year of death is possible for a variety of insurance products. On the other hand, calculation of actuarial present value of the benefit payable at the moment of death requires the knowledge of the probability density function of T(x) and in turn a suitable model for life length X. In practice, given the data on life lengths of various individuals from a specific region, one can estimate S(x) using either the empirical survival function, the Kaplan Meier estimator, the Anderson Nelson estimator or by estimating parameters in some suitable parametric model. Such an estimator then can be used to model the mortality pattern of the individuals in that region and in the calculation of actuarial present values. However, estimating S(x) is comparatively more difficult than estimating p_x or q_x. So attempts are made to find a relationship between net single premiums corresponding to the insurance payable at the moment of death and at the end of the year of death. Of course, such a relation is possible under a certain assumption—the assumption of pattern of deaths at fractional ages. We begin with the assumption of a uniform distribution of deaths over a unit interval. Under the assumption of uniformity, for integer value of (x + k) we have the following result, given as

$$ {}_{s}p_{x+k}\ \mu_{x+k+s} = q_{x+k},\ 0 < s < 1. \qquad (11.5.1) $$

We use it to derive the relation between net single premiums corresponding to the insurance payable at the moment of death and at the end of the year of death for the insurance products. We begin with the n year term insurance. The actuarial present value of a unit benefit payable at the moment of death is given by,

$$\bar{A}^{1}_{x:\,\bar{n}|} = \int_0^n v^t\, {}_tp_x \mu_{x+t} dt$$

$$= \sum_{k=0}^{n-1} \int_k^{k+1} v^t\, {}_tp_x \mu_{x+t} dt$$

$$= \sum_{k=0}^{n-1} \int_0^1 v^{k+s}\, {}_{k+s}p_x \mu_{x++k+s} ds$$

$$= \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x \int_0^1 v^{k+s}\, {}_sp_{x+k} \mu_{x+k+s} ds$$

$$= \sum_{k=0}^{n-1} v^{k+1}\, {}_kp_x q_{x+k} \int_0^1 (1+i)^{1-s}\, ds \text{ ds}$$

The last line follows by (11.5.1). But,

$$\int_0^1 (1+i)^{1-s}\, ds = \frac{-[(1+i)^{1-s}]_0^1}{\log(1+i)} = i/\delta.$$

Thus, $\bar{A}^{1}_{x:\,\bar{n}|} = (i/\delta) A^{1}_{x:\,\bar{n}|}$ (5.5.2)

Allowing n to tend to ∞ we get a similar relation for whole life insurance.

Thus, $\bar{A}_x = (i/\delta)\, A_x.$ (11.5.3)

Example 11.5.1: Suppose the force of mortality follows Gompertz's law given by $\mu_x = BC^x$. Find the values of the net single premium for a benefit of 1000 for a whole life insurance for ages 25, 30, 35, 40 and for an n year term insurance for n = 1,2,..., 10 and for age 25 when the benefit is paid at the moment of death and when the benefit is paid at the end of the year of death. Compare 1000 $(i/\delta)A_x$ and $1000\bar{A}_x$. Also compare 1000 $A^{1}_{x:\,\bar{n}|}$ / and $1000\bar{A}^{1}_{x:\,\bar{n}|}$.Comment on your findings. For both insurance products, verify the relations between two net single premiums under the assumption of uniformity of deaths in unit age interval.

Take $\delta = 0.05$, B = 0.0001151 and C = 1.096.

Solution: Here, we find $1000A_x$, $1000\ (i/\delta)A_x$ and $1000\ \bar{A}_x$. The values are reported in Table 11.5.1.

Table 11.5.1 Gompertz's law: Relation between $1000\ \mathbf{A_x}$ and $1000\bar{A}_x$.

Age	**.-1000A_x**	**1000$(i/\delta)A_x$**	**1000$\bar{A}_x$.**
25	148.78	152.56	152.54
30	184.47	189.15	189.12
35	226.97	232.74	232.70
40	276.73	283.76	283.72

From the table we note that the values in the third and fourth column are almost the same. Thus, the relation $\bar{A}_x = (i/\delta)\ A_x$ is satisfied when the mortality pattern is governed by Gompertz's law with the specified values of the parameters. It is easy to verify that ${}_tq_x \neq t\ q_x$ under Gompertz's law, but we still have, $\bar{A}_x = (i/\delta)\ A_x$. Table 11.5.2 reports the values for term insurance. Here also we note that the relation $\bar{A}^1_{x:\,\overline{n}|} = (i/\delta)A^1_{x:\,\overline{n}|}$ holds good.

Table 11.5.2 Gompertz's law: Relation between 1000 A

n	$1000A^1_{x:\,\overline{n}\vert}$	$1000\ (i/\delta)A^1_{x:\,\overline{n}\vert}$	$1000\bar{A}^1_{x:\,\overline{n}\vert}$
1	1.13	1.16	1.16
2	2.31	2.37	2.37
3	3.54	3.63	3.63
4	4.82	4.94	4.94
5	6.15	6.31	6.31
6	7.54	7.73	7.73
7	8.98	9.21	9.21
8	10.48	10.75·	10.74
9·	·12.04·	12.35 '	12.34
10	13.66	14.01	14.00

For an n year endowment insurance we have a relation,

$$\bar{A}_{x:\bar{n}|}=\bar{A}^{1}_{x:\,\bar{n}|}+A^{1}_{x:\,\frac{1}{n}|}\ =(\mathrm{i}/\delta)\ A^{1}_{x:\,\bar{n}|}+A_{x:\,\frac{1}{n}|}$$

We now find the actuarial present value of the annually increasing n year term insurance payable at the moment of death defined in Section 5.3. For this insurance, the present-value random variable is

$$Z=\begin{cases}[T+1]v^{T}, \text{if T} < n\\ 0, \qquad if\ T\geq n\end{cases}$$

Note that $T+1]=K+1$. Further, we use the relation T=K+S to rewrite Z as

$$Z=\begin{cases}[K+1]v^{K+1}v^{S-1}, \text{for K} = 0,1,\dots,\text{n}-1\\ \quad 0, \qquad \text{for K} = \text{n},\text{n}+1,\dots,\end{cases}$$

Suppose W denotes the present value random variable for the annually increasing n year term insurance payable at the end of the year of death, then

$$W=\begin{cases}[K+1]v^{K+1}, \text{for K} = 0,1,\dots,\text{n}-1\\ \quad 0, \qquad \text{for K} = \text{n},\text{n}+1,\dots,\end{cases}$$

Thus, Z = W$(1+i)^{1-S}$ and E(Z) = E[W$(1+i)^{1-S}$]. Since W is a function of K +1 alone and K +1 and 1-S are independent,

$$E(Z) = E(W)E(1+i)^{1-S} = i/\delta\ (IA)\ x{:}n.$$

Thus, we have,

$$(IA)\ 1/x{:}n = i/\delta\ (IA)^{1}_{x:\,\bar{n}|}.$$

Careful study of the general model provides the basis of the similarities. Suppose $b_T = b^*_{K+1}$, we can write the present value random variable Z for these insurances as

$$Z = b_T v^T = b^*_{K+1} v^{K+1}\ (1+i)^{1-S}$$

and

$$E(Z) = E[b^*_{K+1} v^{K+1}\ (1+i)^{1-S}] \qquad (11.5.4)$$

Under the assumption of a uniform distribution of deaths over each year of age, we can use the independence of K and S and the fact that 1-S also has a uniform distribution. Then we can write (11.5.4) as

$$E(Z) = E[b^*_{K+1} v^{K+1}]\ E[(1+i)^{1-S}] = E[b^*_{K+1} v^{K+1}]\ i/\delta.$$

Example 11.5.3: Calculate for an individual of age 50, the actuarial present value for an annually decreasing 5-year term insurance, paying 5,000 at the moment of death, in the first year, 4,000 in the second year, and so on. It is given that $q_{50} = 0.0123$, $q_{51} = 0.0138$, $q_{52} = 0.0151$, $q_{53} = 0.0165$, $q_{54} = 0.018$. Use the assumption of uniform distribution of deaths over each year of age and i = 0.06.

Solution: For an annually decreasing 5-year term insurance, we have in the units of 1000,

$$b_t = \begin{cases} 5 - |t|, \text{if T} \leq 5 \\ 0, \quad\quad if\ T \geq 5 \end{cases}$$

It is a function of only k, the integral part of t, and hence we may write it as

$$b_k = \begin{cases} 5 - k, \text{if k} = 0,1,2,3,4,5 \\ \quad 0, \quad\quad if\ k = 6, \dots \end{cases}$$

The discount function is v^t, so we have

$$(D\overline{A})^1_{50:\,\overline{5}|} = i/\delta\ (DA)_{50:\,\overline{5}|} = (1.0297087) \sum_{k=0}^{4} (5-k) v^{k+1} {}_kp_{50}q_{50+k} = 0.1867452.$$

Then, $1{,}000(D\overline{A})^1_{50:\,\overline{5}|} = 186.75$.

11.6 Annuities Certain

Annuity: A series of periodic payments or a special type of cash flow where payments occur at regular intervals is known as an annuity.

Annuities are common in our day-to-day transactions. Most of us make and receive such periodic payments; for example, monthly rent, monthly telephone bills, salaries paid on a regular basis, monthly scholarships, equal monthly installments for repayment of housing loan or car loan etc.

In the financial services industry, the term 'annuity' has a technical meaning. In this setup, annuity means saving plans sold by insurance companies to provide regular income to the insured. Thus, in financial services industry annuity is defined as follows:

Annuity: A contract, in which one party, the insurer, promises to make a series of periodic payments in exchange for a premium or a series of premiums, to the insured, is known as an annuity.

Annuity contract is often described as the flip side of life insurance because annuities provide benefit as long as the individual survives, whereas life insurance provides benefit after death. In this chapter, our focus shifts from life insurance policies to annuities. With the interpretation of annuity, as an annuity contract, it is again the outflow of the insurance company, but a periodic outflow. The techniques discussed in this chapter enable us to find its actuarial present value. An insurer uses a combination of factors to calculate the amount of the periodic annuity benefit payments that it will be liable to pay under an annuity policy. Every annuity calculation, however, is based on the following basic mathematical concept.

A certain amount of money, known as the principal, which is invested for a certain period of time at a certain rate of interest, can be paid out in a series of periodic payments over a stated period of time.

There are various forms of annuities. Depending on the nature of the period of annuity, it is dichotomized as annuity certain and life annuity.

Annuity Certain: If the period of annuity payments is fixed, then such an annuity is known as annuity certain. For example, monthly scholarship for 24 months of post graduate study forms annuity certain. In some insurance products, the numbers of premiums are fixed.

Life Annuity: An annuity in which the number of payments depends on the survival of the individual is known as a life annuity. Whole life insurance is usually purchased by paying the premiums till the insured's death. Premium payments then form a life annuity. Monthly pension that a government employee receives after retirement till he is alive forms a life annuity. Thus, in life annuity, the period of annuity payments is a random variable while in annuity certain it is deterministic. Each of these two types of annuities is further classified as continuous or discrete annuities depending on the mode of payments.

Continuous Annuity: An annuity in which payments are made continuously at a rate of 1 unit per annum is known as continuous annuity.

Discrete Annuity: An annuity in which payments are made at discrete time points at a rate of 1 unit per annum is known as discrete annuity.

The discrete time points may be the beginning of a year or the end of a year. Such a distinction gives rise to following two types of discrete annuities.

Annuity Due: An annuity due, also known as annuity payable annually in advance, is an annuity in which payments are made at the beginning of the year.

Annuity Immediate: An annuity immediate, also known as annuity payable annually in arrear, is an annuity in which payments are made at the end of the year.

The terms, annuity due and annuity immediate do not seem to be logical but these are conventionally used in the insurance field as defined above. The first payment of an annuity immediate is not made immediately at the beginning of the first payment period, rather it is due at the end of it. The other corresponding terms, annuity payable annually in advance and annuity payable annually in arrear, for annuity due and immediate respectively are more logical.

An annuity period may be a month or a quarter or six months instead of a year. Such division of a year leads to the following type of annuity.

m-thly (payable) Annuity: An annuity which is payable in m parts in a year with a rate of 1 unit per year, that is 1/m in each m^{th} part, is known as mthly annuity. When m = 2, it is six monthly, when m = 4, it is quarterly and when m = 12, it is a monthly annuity. If the payments are made at the beginning of the m^{th} part, it is mthly annuity due while if payments are made at the end of the m^{th} part, it is mthly annuity immediate. Equal monthly installments in the repayment of a housing loan form a monthly annuity due. Monthly salary of a working person forms monthly annuity immediate. Quarterly interest on the savings in a bank or post office forms a quarterly annuity immediate. It may be a certain or a life annuity.

Deferred Annuity: An annuity in which periodic payments begin after a certain period is known as deferred annuity. People often purchase deferred annuities during their working years in anticipation of the need for income after retirement.

Level Annuity: An annuity in which periodic payments are of the same amount is known as level annuity. In level annuity it is assumed that the payment is of 1 unit per annum. Any other level annuity can be obtained from this by a simple multiplication. Annuities which are not level, are known as varying annuities.

Life annuities are further classified as n year temporary life annuity, whole life annuity and n year certain and whole life annuity depending on the terms and conditions of payments. We study these in detail in different Sections.

Periodical premiums in life insurance form an annuity where payments are from insured to insurer. As stated above in financial services industry, annuity means a contract in which the insurer promises to make a series of periodic payments to the insured. Annuities may be purchased as single premium annuities by paying a single lump-sum premium or by paying periodic premiums over a period of years. Periodic premiums can be paid on either a level premium basis or a flexible premium basis. Under a periodic level premium annuity, the contract holder pays equal premium for the annuity at regularly scheduled intervals. Under a flexible premium annuity, the amount of each premium payment can vary between a fixed minimum amount and a fixed maximum amount. In such a contract, the periodical benefit payments from the insurer to the insured form an annuity and periodical premium payments from the insured to the insurer also form an annuity. In any setup to decide the purchase value of the annuity contract, it is absolutely essential for the insurer to find out the expected present value of the annuity. With the main aim of calculating premiums, which are to be fixed at the issue of the policy, our entire discussion in this chapter is focused on finding the present value of the various types of annuities in annuity certain and expected present value of the various types of annuities in life annuity.

In the next section, we first discuss level annuity certain and then in the following sections proceed to level life annuities. The theory of varying annuities is analogous and is not discussed here. Life annuity theory is analogous to the theory of annuity certain but brings in survival as a condition for payment.

Annuities Certain: In annuity certain, an annuity is payable for a fixed period of time. The fixed period of time is known as the period certain. At the end of the period certain, annuity payments cease. The theory of compound interest is used to find the present value of the annuity. In the following, we find formulae for the present value of discrete annuity immediate and due, with annual payments and mthly payments and then for continuous annuity.

Annuity Certain Immediate: Suppose the period certain is n years and the payment of 1 unit is to be made at the end of each year for n years. Present value of these payments is denoted by a symbol $a_{\overline{n|}}$, the general symbol for the present value of any type of annuity being a. We now derive the formula for $a_{\overline{n|}}$, on the assumption of a constant effective annual rate of interest i or equivalently the constant force of interest d. Present value of a unit paid at the end of the first year is v, that of paid at the end of the second year is v^2 and so on. Thus,

$$a_{\overline{n|}} = v + v^2 + v^3 + \cdots + v^n$$

$$= v(1 - v^n)/(1 - v) = (1 - v^2)/i, \qquad (11.6.1)$$

is a present value of the n level payments, each of one unit for n years paid at the end of the year, that is, $a_{\overline{n|}}$ is the present value of annuity certain immediate. If annuity certain for n years is purchased by a single lump-sum premium, the insurer has to charge at least $a_{\overline{n|}}$ amount of money to the annuitant. $a_{\overline{n|}}$, thus, represents the purchase price of an n-year annuity immediate with 1 unit payment per annum. This amount, along with the interest on this, with effective annual rate of interest i, will accumulate to a fund from which the insurer is able to pay one unit at the end of each year for n years. Table 11.6.1 shows the value of 1000 $a_{\overline{n|}}$, the present value of annuity certain immediate with payment of 1000 per annum, for values of n from 1 to 10 and i = 0.05, 0.06, ……, 0.10.

Table 11.6.1: Present value of annuity certain immediate of 1000 per annum

n	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**	**0.10**
1	952.38	943.40	934.58	925.93	917.43	909.09
2	1859.41	1833.39	1808.02	1783.26	1759.11	1735.54
3	2723.25	2673.01	2624.32	2577.10	2531.29	2486.85
4	3545.95	3465.11	3387.21	3312.13	3239.72	3169.87
5	4329.48	4212.36	4100.20	3992.71	3889.65	. 3790.79
6	5075.69	4917.32	4766,54	4622.88	4485.92	4355.26
7	5786.37	5582.38	5389.29	5206.37	5032.95	4868.42
8	6463.21	6209.79	5971.30	5746.64	5534.82	5334.93
9	7107.82	6801.69	6515.23	6246.89	5995.25	5759.02
10	7721.73	7360.09	7023.58	6710.08	6417.66	6144.57

From the table we note that if the rate of interest is 5%, then the present value of a year annuity immediate of 1000 is 952.38, while for 10 years the value is 7721.73.Thus the amount 952.38 will accumulate to 1000 in 1 year while amount 7721.73will accumulate in 10 years to a fund from which it is possible to pay Rs 1000 at the end of every year for 10 years. Thus, the purchase price of a 10-year annuity immediate of 1000 per year is 7721.73. Note that as the rate of interest increases, the present value of annuity decreases. With a high-interest rate fewer amounts will accumulate to a fund that will be sufficient to pay periodical payments. Further as increases, the present value also increases as the period of payments increases. The present value of a series of payments in a certain time period looks at the value of cash flow in that period at the beginning of the period. Sometimes, we need to find out the value of the cash flow at the end of the period, that is, we want to find the amount to which all the payments would accumulate at a given rate of interest, at the end of the period. The general symbol for accumulated amount is S. There are two, approaches to find the accumulated amount. We first find the present value of the cash flow in the given period and its accumulation over the specific period will give the accumulated value. a is the present value of the n level payments, each of one unit for n years paid at the end of the year. The accumulated value of $a_{\bar{n}}$ ata rate of interest i, at the end of n years is $(1 + i)^n a_{\bar{n}}$. It is denoted by $S_{\bar{n}|}$. Thus

$$S_{\overline{n|}} = (1 + i)^n a_{\bar{n}} .$$

This expression can be derived alternatively as follows. The accumulated value of1 unit paid at the end of 1 year is $(1 + i)^{n-1}$, that of paid at the end of second year is $(1+i)^{n-2}$ and so on, the accumulated value of 1 unit paid at the end of nth year is 1. Thus,

$$S_{\overline{n|}}= (1 + i)^{n-1}+(1 + i)^{n-2} \cdots + (1 + i) + 1$$

$$= ((1+i)^n - 1)/I$$

$$= (1 + i)^n (1 - (1 + i)^{-n})/i$$

$$= (1 + i)^n (1 - v^n)/i = (1 + i)^n a_{\bar{n}}.$$

$S_{\bar{n}}$ thus denotes the value of the series of 1-unit payments at the time of last payment. It is clear that $S_{\bar{1}} = 1$.

It is to be noted that a represents the value of the cash flow in the period of n years at the beginning of the period while $S_{\bar{n}}$ represents the value of the cash flow in the period of n years at the end of the period.

Example 11.6.1: Find the present value and the accumulated value of a 10-year annuity immediate of Rs. 1000/- per annum if the effective rate of interest is 5%.

Solution: With usual notation, i = 0.05, the present value 1000 $a_{\bar{n}|}$ is given by,

$$10000a_{\bar{n}|} = 1000(1 - v^{10})/i = 1000(1 - (1+0.05)^{-10})/0.05 = 7721.73.$$

The accumulated value is 1000 $S_{\overline{10|}}$ and is given by,

$$1000S_{\overline{10|}} = 1000 \left(\frac{(1+i)^{10}-1}{i}\right) = \frac{1000}{0.05}\{(1 + 0.05)^{10} - 1\} = 12577.89.$$

Note that, $1000S_{\overline{10|}}$= 12577.89 = $(1 + i)^{10}$ 7721.73 = $(1 + i)^{10}$ $1000a_{\bar{n}|}$.

Thus, the value of the cash flow in the period of 10 years, at the beginning of the period is 7721.73 and at the end of the period are 12577.89.

Example 11.6.3: An annuity of Rs.12000 per year is payable for 15 years. What is the price of this annuity purchased on January 1, 2000 and if the first payment is made on December 31, 2000? Suppose the effective annual rate of interest is 5%.

Solution: The purchase price is nothing but the present value of the annuity. Since the first payment is made at the end of the year, it is annuity immediate for 15 years. Hence,

$$\text{Purchase price} = 12000\ a_{\overline{15|}} = 12000\frac{1-(1+0.05)^{-15}}{0.05} = 124555.90.$$

Annuity Certain Due: For an annuity certain due, the payments are made at the beginning of each year for n years. The present value of this annuity is denoted by $\ddot{a}_{\overline{n|}}$. To find its expression, note that the present value of the first payment of 1 unit at the beginning of the year is 1 itself that of the second payment is v and so on. The present value of the last payment of 1 unit at the beginning of the nth year is v^n-1. Hence $\ddot{a}_{\overline{1|}}$= is given by,

$$\ddot{a}_{\overline{n|}} = 1+v+v^2+\ldots+v^{n-1}$$

$$= (1 - v^n)/(1 - v) = (1 - v^n)/d. \qquad (11.6.3)$$

it can be easily verified that,

$\ddot{a}_{\overline{1|}}=1$

$a_{\bar{n}|}= v\ddot{a}_{\overline{n|}}$,

$\ddot{a}_{\overline{n|}}= 1+\ a_{\overline{n-1|}}$ with $a_{\bar{0}|} = 0$.

From the second relation it is clear that

$$a_{\bar{n}|}= v\ddot{a}_{\overline{n|}}<\ddot{a}_{\overline{n|}}\ ,\ \text{for all n.} \qquad (11.6.4)$$

The inequality in (11.6.4) appeals logically also, as in annuity immediate payments are made at the end of the year while in annuity due these are made at the beginning of the year.

We know that $1000\ddot{a}_{\overline{1|}}$ = 1000 for all the values of i, as it is annuity due and the first payment is made at the beginning of the one-year interval. The interpretation of all the values is similar to that for annuity immediate. Thus, if the rate of interest is 5%, then the present value of a 10-year annuity due of 1000is 8107.82. This amount will accumulate in 10 years to a fund from which it is possible to pay Rs 1000 at the beginning of every year for 10 years. It is to be noted that the present value of an n-year annuity due of 1000 is larger than that of ann. year annuity immediate of 1000 for all n, verifying the relation in (11.6.4). Note that here also as rate of interest increases, the present value of annuity decreases, as expected.

The equation $\ddot{a}_{\overline{n|}} = (1 - v^n)/d$ can be rewritten as $1 = d\ddot{a}_{\overline{n|}} + v^n$. This identity has the following nice interpretation. Suppose unit 1 is invested at the beginning of the first year. The annual interest of d units is paid at the beginning of each year. Thus, interest payments form an annuity due and its present value is $d\ddot{a}_{\overline{n|}}$. At the end of year n, the outstanding capital is still unit 1 and its present value is V^n. Thus, originally invested unit 1 is equivalent to the addition of present values of annuity payment of interest and the capital of 1 unit. The following example illustrates the practical application of the present value of annuity due to decide on the equal installments to repay a loan.

Example 11.6.4: A loan of Rs.50,000/- is taken on January 1, 2000. It has to be repaid in 5 equal installments payable yearly at the beginning of the year. Based on a 6 % annual rate of interest, determine the amount of the installment.

Solution: Suppose the annual installment is I. It should be such that the total present value of all the 5 installments should be 50000. Hence, we get the equation as,

$$50{,}000 = I\ddot{a}_{\overline{5|}} == (1 - (1.06)^{-5})\ I/d = 4.465106\ I.$$

Solving the equation, we get I = 11197.94.

The accumulated value corresponding to the annuity due is denoted by $\ddot{S}_{\overline{n|}}$ and is given by,

$$\ddot{S}_{\overline{n|}} = (1+i)^n + (1 + i)^{n-1} + ... + (1+i)$$

$$= (1+i)\{(1 + i)^n - 1\}/i = ((1+i)^n - 1)/d$$

$$= (1 + i)^n \{1 - (1 + i)^{-n}\}/d$$

$$= (1+i)^n\ddot{a}_{\overline{n|}}. \qquad (11.6.5)$$

$\ddot{S}_{\overline{n|}}$ thus denotes the value of the series of 1-unit payments, one unit time after the last payment. From the formulae of $S_{\overline{n|}}$ and $\ddot{S}_{\overline{n|}}$, it follows that,

$\ddot{S}_{\overline{n|}}$= $(1 + i)^n + (1 + i)^{n-1} + ... + (1 + i) = (1 + i)^n + (1+i)^{n-1} + ... + (1 + i) + 1 – 1 = S_{\overline{n+1|}} - 1$.

m-thly (payable) Annuity Certain: The complexity of the present value expression of annuities increases when an annuity payment is made m times a year, with a rate of 1 unit per annum, for n years. If the amount of annuity is 1 unit, then $\frac{1}{m}$ unit amount is paid in every m[th] part. In n years there are m_n, m parts. The present value of mthly annuity immediate is denoted by $a_{\overline{n|}}^{(m)}$ and is given by,

$$a_{\overline{n|}}^{(m)} = \frac{1}{m}\sum_{k=1}^{mn}(v^{1/m})^k$$

$$= \frac{1}{m}\left[v^{1/m} + v^{2/m} + \cdots + (v^{1/m})^{mn}\right]$$

$$= \frac{1}{m}v^{1/m}\left\{1 + v^{1/m} + \cdots + (v^{1/m})^{mn-1}\right\}$$

$$= \frac{1}{m}v^{1/m}\left\{1 - (v^{1/m})^{mn}\right\}\Big/(1 - v^{1/m})$$

$$= \{1 - (v)^n\}\Big/m\{(1+i)^{1/m} - 1\} = \{1 - (v)^n\}\Big/(i)^m$$

$$= \frac{i}{(i)^m}a_{\overline{n|}},$$

Where, $i^{(m)} = m\{(1 + i)^{1/m} – 1\}$, the nominal rate of interest when interest is paid mthly. It is to be noted that a(m) denotes the present value of periodical payments where one unit is paid per annum in m parts at the end of the m[th] part and for n years.

The present value of an annuity due, where $\frac{1}{m}$ unit amount is paid at the beginning of every mth part for n years, is denoted by $\ddot{a}_{\overline{n|}}^{(m)}$ and is given by,

$$\ddot{a}_{\overline{n|}}^{(m)} = \frac{1}{m}\sum_{k=1}^{mn-1}(v^{1/m})^k$$

$$= \frac{1}{m}\left[v^{1/m} + v^{2/m} + \cdots + (v^{1/m})^{mn-1}\right]$$

$$= \frac{1}{m}\left\{1 - (v^{1/m})^{mn}\right\}\Big/(1 - v^{1/m})$$

$$=\{1-(v)^n\}\Big/ m(1-v^{1/m})$$

$$=\{1-(v)^n\}\Big/ (d)^m$$

$$= (1+i)^{1/m} a_{\overline{n|}}^{(m)}$$

$$=\frac{i}{(d)^m} a_{\bar{n}|}$$

$$=\frac{d}{(d)^m} \ddot{a}_{\overline{n|}}$$

Here $d^{(m)} = m(1 - v^{1/m})$ is the nominal rate of discount paid m times a year. In this setup too, the amount paid per year is 1 unit. It is paid in m parts at the beginning of the m^{th} part. It is to be noted that, as $m \to \infty$, $d^{(m)}$ converges to $-\log v = \delta$ and

$$\ddot{a}_{\overline{n|}}^{(m)} = \frac{d}{(d)^m} \text{ and } \ddot{a}_{\overline{\infty|}}^{(m)} = \frac{1}{(d)^m}$$

Example 11.6.7: Prove that $a_{\bar{n}|} < a_{\overline{n|}}^{(m)} < \ddot{a}_{\overline{n|}}^{(m)} < \ddot{a}_{\overline{n|}}$ and hence, $i > i^{(m)} > d^{(m)} > d$.

Solution: We have

$$\ddot{a}_{\overline{n|}}^{(m)} = (1+i)^{1/m} a_{\overline{n|}}^{(m)} > a_{\overline{n|}}^{(m)}, \qquad (11.6.8)$$

Now to prove that $a_{\bar{n}|} < a_{\overline{n|}}^{(m)}$, suppose $u_m = \sum_{k=1}^{mn} (v^{k/m})$.

Note that $0 < v < 1$ and $0 < k/m < 1$ for k = 1, 2, ….., m. Hence, $v^{k/m} > v$ for all k = 1, 2, ..., m. As a consequence, $u_m > mv$. Grouping the terms for each of n years, the expression for $a_{\overline{n|}}^{(m)}$ given by $(\sum_{k=1}^{mn} (v^{k/m}))/m$ can be rewritten as

$$a_{\overline{n|}}^{(m)} = \frac{1}{m} \sum_{k=1}^{n-1} (v^k)\, u_m$$

$$> \frac{1}{m} \sum_{k=1}^{n-1} (v^k)\, u\, m = \sum_{k=1}^{n} (v^k)$$

$$= a_{\bar{n}|}, \qquad (11.6.9)$$

To prove that $\ddot{a}_{\overline{n}|}^{(m)} < \ddot{a}_{\overline{n}|}$ suppose that $w_m = \sum_{k=1}^{m-1}(v^{k/m})$. Note that $0 < v < 1$. Hence, each term in w_m, except the first, is < 1. As a consequence, $w_m < m$. Grouping the terms for each of n years, the expression for $\ddot{a}_{\overline{n}|}^{(m)}$ given by $\frac{1}{m}\sum_{k=1}^{mn-1}(v^{k/m})$ can be rewritten as

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}\sum_{k=1}^{n-1}(v^k)\, w_m$$

$$< \frac{1}{m}\sum_{k=1}^{n-1}(v^k)\, m = \frac{1}{m}\sum_{k=1}^{n-1}(v^k)$$

$$= \ddot{a}_{\overline{n}|} \qquad (11.6.10)$$

Combining the above results, we get the following inequality among the present values of annuities discussed so far.

$$a_{\overline{n}|} < a_{\overline{n}|}^{(m)} < \ddot{a}_{\overline{n}|}^{(m)} < \ddot{a}_{\overline{n}|} \qquad (11.6.11)$$

From the formulae of these present values, we note that all of them have the same numerator $(1 - v^n)$. The denominator is i, $i^{(m)}$, $d^{(m)}$ and d respectively according to the order in (11.6.11). Hence, we have

$$i > i^{(m)} > d^{(m)} > d \qquad (11.6.12)$$

Example 11.6.9: A loan of Rs. 10 lakhs are to be repaid over 10 years by a level annuity payable monthly in arrears. The nominal rate of interest (12) is 0.06. Find the monthly installment.

Solution: The nominal rate of interest i(12) is 0.06, hence interest rate per month is $i^{(12)}/12 = 0.005$. The total number of monthly installments payable in arrears in 10 years is $10 \times 12 = 120$. Hence, we get the following equation to find the monthly installment I.

$$10{,}00000 = \mathrm{I}a_{\overline{120}|} = \mathrm{I}\ {\{1-(v)^{120}\}}\big/{((i)^{12}/12)}$$

$$= \mathrm{I}\ {\{1-(1.005)^{-120}\}}\big/{0.005} == \mathrm{I}(90.07345).$$

Solving we have, I = 11102.05 is the monthly installment.

Remark: Here, the nominal rate of interest is given. Hence, we have calculated the rate of interest per month and the total number of monthly payments. Thus, unit of time is taken as 1 month. Discount factor v in the above formula is taken as v = $(1 + .005)^{-1}$. Note that the installments are close, the marginal difference is attributable to change in the nature of interest rate and nature of annuity due and immediate.

Following are the formulae for the accumulated values corresponding to mthly annuities.

$$S_{\overline{n|}}^{(m)} = (1+i)^n a_{\overline{n|}}^{(m)} = \frac{i}{(i)^m} S_{\bar{n}|}$$

$$\ddot{S}_{\overline{n|}}^{(m)} = (1+i)^n \ddot{a}_{\overline{n|}}^{(m)} = \frac{d}{(d)^m} \ddot{S}_{\overline{n|}} = (1+i)^{(1/m)} S_{\overline{n|}}^{(m)} \qquad (11.6.12)$$

From (11.6.11), multiplying each term in the inequality by (1+i)", we get a following relation among the accumulated values, similar to that in (11.6.11).

$$S_{\bar{n}|} < S_{\overline{n|}}^{(m)} < \ddot{S}_{\overline{n|}}^{(m)} < \ddot{S}_{\overline{n|}} \qquad (11.6.13)$$

Continuous Annuity Certain Suppose the partitioning of a period of 1 year in m parts is done finer and finer and payments are done in each m^{th} part with a rate of 1 unit per annum. With the finer partition, the beginning and end of the m^{th} part does not make any difference. Hence to find the present value of continuous annuity certain, we allow m to tend to ∞, in the expression for $a_{\overline{n|}}^{(m)}$ or $\ddot{a}_{\overline{n|}}^{(m)}$. The present value of annuity payment made continuously at the rate of 1 unit per year for n years is denoted by $\bar{a}_{\overline{n|}}$and is given by,

$$\bar{a}_{\overline{n|}} = \int_0^n v^t = [v^t / \log v]_0^n = \frac{1-v^n}{-\log v} = \frac{1-v^n}{\delta} = \frac{i}{\delta} = a_{\bar{n}|} = \frac{d}{\delta} = \ddot{a}_{\bar{n}|}$$

It is to be noted that both $(1 - v^n)/i^{(m)}$ and $(1 - v^n)/d^{(m)}$ tend to $(1 - v^n)/\delta$ as $m \to \infty$, in view of the fact that both $i^{(m)}$ and $d^{(m)} \to \delta$ as $m \to \infty$.

In annuities payable continuously, the distinction between annuity due and annuity immediate has no meaning. The concept of annuity payable continuously, at the rate of one per year, is of course an abstraction but makes use of familiar mathematical tools and as a practical matter, closely approximates annuities payable on a monthly basis. We need to go for such an abstraction because in life annuity, the future life time random variable is a continuous random variable.

To find a relation between $\bar{a}_{\overline{n}|}$ and $a_{\bar{n}|}$, we note that

$$\frac{i}{\delta} = (e^{\delta} - 1)/\delta = (1 + (\delta/2) + (\delta^2/2) + \ldots -1)/\delta$$

$$= 1 + (\delta/2) + (\delta^2/6) + \ldots > 1$$

Hence, $\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\bar{n}|} > a_{\bar{n}|}$ (11.6.15)

The inequality is reasonable as $\bar{a}_{\overline{n}|}$ is the present value of annuity payment madecontinuously while a| is the present value of the annuity payment made at the end of the year. To find a relation between $\bar{a}_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$, we have to check whether d/ δ > 1or < 1. To decide it, suppose a function f is defined as $f(\delta) = d - \delta = 1 - e^{-\delta} - \delta$.

It follows that f(0) = 0 and the first derivative of f is negative for any value of δ. Hence the function f is a decreasing function. Consequently δ > 0 implies that f(δ) = d - δ < f(0) = 0, that is d < δ or d/ δ < 1.

Hence, $\bar{a}_{\overline{n}|} = \frac{d}{\delta} \ddot{a}_{\overline{n}|} < \ddot{a}_{\overline{n}|}$ (11.6.16)

This inequality is reasonable as ā| is the present value of annuity payment made continuously while ä| is the present value of annuity payment made at the beginning of a year. Following is the formula for the accumulated value corresponding to annuities payable continuously.

$$\bar{S}_{\overline{n}|} = \int_0^n (1 + i)^t dt = \frac{(1+i)^n - 1}{\log(1+i)} = \frac{(1+i)^n - 1}{\delta}$$

$$= (1+i)^n \bar{a}_{\overline{n}|} = \frac{i}{\delta} S_{\overline{n}|} = \frac{d}{\delta} \ddot{S}_{\overline{n}|}. \quad (11.6.17)$$

Hence, we have

$$a_{\bar{n}|} < \bar{a}_{\overline{n}|} < \ddot{a}_{\overline{n}|} \text{ and } S_{\bar{n}|} < \bar{S}_{\overline{n}|} < \ddot{S}_{\overline{n}|} \quad (11.6.18)$$

Table 11.6.3 shows the values of 1000 $a_{\bar{n}|}$, 1000 $a_{\overline{n}|}^{(m)}$, 1000 $\bar{a}_{\bar{n}|}$, 1000 $\ddot{a}_{\overline{n}|}^{(m)}$ *and* 1000 $\ddot{a}_{\overline{n}|}$ for m= 12, for various values of n and i = 0.05.

Table 11.6.3: Comparison of Present Values of Annuities

n	**$10^3 a_{\bar{n}|}$**	**$10^3 a_{\overline{n}|}^{(12)}$**	**$10^3 \bar{a}_{\overline{n}|}$**	**$10^3 \ddot{a}_{\overline{n}|}^{(12)}$**	**$10^3 \ddot{a}_{\overline{n}|}$**

1	952.38	974.01	976.00	977.98	1000.00
2	1859.41	1901.65	1905.52	1909.39	1952.38
3	2723.25	2785.11	2790.78	2796.45	2859.41
4	3545.95	3626.50	.3633.88	3641.27	3723.25
5	4329.48	4427.82	4436.83	4445.86	4545.95
6	5075.69	5190.99	5201.55	5212.13	5329.48
7	5786.37	5917.81	5929.86	5941.92	6075.69
8	6463.21	6610.02	6623.48	6636.95	6786.37
9	7107.82	7269.27	7284.07	7298.89	7463.21
10	7721.73	7897.13	7913.21	7929.31	8107.82

From the table it is to be noted that 1000 $a_{\bar{n}|}$< 1000 $a_{\overline{n}|}^{(12)}$, as proved above, in view ofthe nature of periodical payments. In annuity immediate, payment is made once atthe end of the interval while in mthly annuity immediate, of payments are made at the end the *m*th part of the interval. On the other hand, $10^3\ddot{a}_{\overline{n}|}$>$10^3\ddot{a}_{\overline{n}|}^{(12)}$, because in annuity due, payment is made once at the beginning of the interval while in m-thly payable annuity due, payments are made at the beginning of the m-th part of the interval.

Thus, the m payments are spread over the interval and its present value is less than that of payment made once at the beginning of the interval. Column 3 and column 5 denote the present value of a periodical payment of 1000, when 1000 rupees are paid in 12 installments. Thus, equal monthly installments will be 1000/12. Further note that $1000\ddot{a}_{\overline{n}|}^{(12)}$and $1000\bar{a}_{\overline{n}|}$ are close to each other for all n. This observation isconsistent with the remark made above those annuities payable continuously, at the rate of one per year closely approximates annuities payable on a monthly basis. From the table, we note that $a_{\overline{n}|}^{(m)}$<$\bar{a}_{\overline{n}|}$<$\ddot{a}_{\overline{n}|}^{(m)}$.

Example 11.6.12: A loan of Rs.10 lakhs is to be repaid over 10 years by a continuous annuity. The force of interest is 0.06. Find the annual installment.

Solution: We are given δ = 0.06. Hence the corresponding value of v is v = exp (-δ) =0.9417645. Suppose I is the annual installment to be paid continuously at a rate of I per annum. Then we have the equation

$$10^6 = I\, \bar{a}_{\overline{10|}} = I\,(1 - v^{10}) / \delta = I\,(7.519806).$$

Solving it we get, I = 132982.20.

In the following, we find the formulae for the present value for all the annuities studied so far when payments are deferred for a certain period.

Deferred Annuity: Present values for deferred annuities are obtained on similar lines. Suppose payment of 1 unit is made for n years at the end of the interval, after a deferment period of m years. The present value of such deferred annuity immediate is denoted by $m|a_{\overline{n|}}$ and given by,

$$m|a_{\overline{n|}} = v^{m+1} + \ldots + v^{m+n} = v^{m+1}\left(\frac{1-v^n}{1-v}\right) = v^m\, a_{\overline{n|}} = a_{\overline{m+n|}} - a_{\overline{n|}}\ .$$

The present value of an m year deferred annuity due, denoted by, $m|\ddot{a}_{\overline{n|}}$, is defined on similar lines. Thus,

$$m|\ddot{a}_{\overline{n|}} = v^m + \ldots + v^{m+n-1} = v^m\left(\frac{1-v^n}{1-v}\right) = v^m\, \ddot{a}_{\overline{n|}} = \ddot{a}_{\overline{m+n|}} - \ddot{a}_{\overline{n|}}\ .$$

In an m year deferred continuous annuity, certain annuity payments are deferred form years and then made continuously at the rate of 1 unit per year for n years. Its present value is denoted by $m|\bar{a}_{\overline{n|}}$ and given by,

$$m|\bar{a}_{\overline{n|}} = \int_m^{m+n} v^t\, dt = \frac{v^m - v^{m+n}}{\delta} = v^m \bar{a}_{\overline{n|}} = \frac{i}{\delta} m|a_{\overline{n|}} = \frac{d}{\delta} m|\ddot{a}_{\overline{n|}}\ .$$

From the above formulae it is clear that,

$$m|a_{\overline{n|}} = v^m\, a_{\overline{n|}} < a_{\overline{n|}}\ ,\quad m|\ddot{a}_{\overline{n|}} = v^m\, \ddot{a}_{\overline{n|}} < \ddot{a}_{\overline{n|}} \text{ and } m|\bar{a}_{\overline{n|}} = v^m \bar{a}_{\overline{n|}} < \bar{a}_{\overline{n|}}.$$

Such a relation seems reasonable as in an m year deferred annuity; the insurer has no liability of payment in the first m years. Hence,

$$m|a_{\overline{n|}} < m|\bar{a}_{\overline{n|}} < m|\ddot{a}_{\overline{n|}}\ .$$

Accumulated values corresponding to the above three deferred annuities are obtained by multiplying the present values by $(1 + i)^{m+n}$. These are thus given by,

$$m|S_{\overline{n|}} = S_{\overline{n|}}\ ,\quad m|\bar{S}_{\overline{n|}} = \bar{S}_{\overline{n|}}\ ,\quad m|\ddot{S}_{\overline{n|}} = \ddot{S}_{\overline{n|}}\ .$$

There is no effect of deferment on the accumulated values as there are no payments in the first m years. The present value for the deferred annuities with m-thly payable payments can be obtained on similar lines.

So far, we have discussed calculation of present values of a variety of level annuities. The same arguments are useful to find present values of varying annuities. The following example illustrates the procedure.

In the next section, we will discuss the theory of life annuities, in which the phenomenon 'period certain' is replaced by a random quantity, as the period of annuity payments depends on the survival of the individual. Life annuities are also classified into two categories—continuous life annuities and discrete life annuities, depending on the mode of periodical payments. If life annuities are paid continuously at a rate of 1 unit per annum, these are termed as continuous life annuities. In this case, the period of payments is governed by the future life time random variable T(x). A life annuity where the payments are made either at the beginning of the payment intervals or at the end of the payment intervals until the individual's death, are known as discrete life annuity. For continuous annuities, there is no distinction between payments at the beginning of payment intervals or at the end of the intervals, that is, between annuity due and annuity immediate. For discrete annuities, the distinction is meaningful. In discrete life annuities, the period of payments is governed by the curtate future life time random variable K(x). Discrete life annuity due has a more prominent role in actuarial applications. For example, in most individual life insurances, periodic premiums form an annuity due.

11.7 Continuous Life Annuities

Life annuities play a major role in life insurance operations as life insurances are usually purchased by a life annuity of premiums rather than by a single premium. The amount payable at the time of claim may be converted through a settle mentation into some form of life annuity for the beneficiary. For example, there may be a monthly income payable to a surviving spouse. A continuous life annuity can be any one of the following three types.

i. n-year temporary life annuity
ii. whole life annuity
iii. n-year certain and life annuity.

Each of these is further classified depending upon the deferment or otherwise of payments. We will discuss each in detail in the following sections.

n-Year Temporary Life Annuity: In this type of annuity, payment of one unit per annum is payable continuously for at most n years. Thus, the annuity payments cease when an individual dies during the term of n years or at the end of n years. If the individual survives the n year term. Such a situation arises if an n year term insurance or an n year endowment insurance, with benefit payable at the moment of death, is purchased by paying periodical premiums continuously at a certain rate per annum. Payments of premiums will stop if the death occurs before the term of n years is over or definitely at the end of n years if the individual survives the term. To decide the amount of premium to be fixed at the issue of policy, we have to find the expected present value of premium payments which form an n year temporary life annuity and balance it with the actuarial present value of the benefit.

In life annuity, the present value of the annuity becomes a random variable. The present value random variable Y for an n year temporary life annuity of 1 per year, payable continuously while (x) survives during the next n years, is defined by noting the random period of annuity payments. It is T(x) if death occurs within a span of n years after signing the contract at age x and n years if the individual survives n years. Thus, we get,

$$Y = \begin{cases} \bar{a}_{\overline{T|}} & if\ 0 \leq T < n \\ \bar{a}_{\overline{n|}} & if\ T \geq n \end{cases} \tag{11.7.1}$$

The distribution of Y in this case is a mixed distribution as in the case of a present value random variable of benefit in an n year endowment insurance. With probability ${}_nq_x$, it is distributed as $\bar{a}_{\overline{T|}}$ and with remaining probability it takes value $\bar{a}_{\overline{n|}}$.The maximum value of Y is $\bar{a}_{\overline{n|}}$. Note that the random variable Y can be written as,

$$Y = \begin{cases} \bar{a}_{\overline{T|}} = \dfrac{1 - v^T}{\delta} = \dfrac{1 - Z}{\delta} & if\ 0 \leq T < n \\ \bar{a}_{\overline{n|}} = \dfrac{1 - v^n}{\delta} = \dfrac{1 - Z}{\delta} & if\ T \geq n \end{cases}$$

Where,

$$Z = \begin{cases} v^T & if\ 0 \leq T < n \\ v^n & if\ T \geq n \end{cases} \tag{11.7.2}$$

In (11.7.2), Z is the present value random variable of the unit benefit, to be paid at the moment of death, for an n year endowment insurance. Hence,

$$E(Y) = E\left[\frac{1-Z}{\delta}\right] = \frac{1-\bar{A}_{x:\bar{n}|}}{\delta} = \bar{A}^2_{x:\bar{n}|}$$

and

$$\mathrm{Var}(Y) = \frac{\mathrm{Var}(Z)\,/\,\delta^2}{\delta^2} = ({}^2\bar{A}^2_{x:\bar{n}|} - \bar{A}^2_{x:\bar{n}|})/\delta^2 \ - \qquad (11.7.3)$$

In international actuarial notation, the actuarial present value of an n year temporary life annuity is denoted by $\bar{a}_{x:\bar{n}|}$. Thus,

$$\bar{a}_{x:\bar{n}|} = \frac{1-\bar{A}_{x:\bar{n}|}}{\delta}$$

We know that

$$\bar{A}_{x:\bar{n}|} = \int_0^n v^t g(t)\,dt + v^n {}_np_x$$

Using the relation $g(t) = -(d/dt)_tp_x$, we solve the integral $\int_0^n v^t g(t)\,dt$ by parts and get $\bar{A}_{x:\bar{n}|} = 1 - \int_0^n v^t {}_tp_x\,dt$. Substituting in (11.3.4), we get

$$\bar{a}_{x:\bar{n}|} = \int_0^n v^t {}_tp_x dt \qquad (11.7.5)$$

This is known as the current payment integral for the actuarial present value for the n year temporary annuity. This formula is similar to that for $\bar{a}_{\bar{n}|}$, except the factor ${}_tp_x$. This factor reflects the fact that payment is made at time t if (x) survives for t units, $t \leq n$.

From (11.7.4), we get $1 = \delta\ \bar{a}_{x:\bar{n}|} + \bar{A}_{x:\bar{n}|}$. Hence, in terms of annuity values, Var(Y)as given in (11.7.3) can be rewritten as,

$$\mathrm{Var}(Y) = 1 - 2\delta\ {}^2\bar{a}^2_{x:\bar{n}|} - (1-\delta\ \bar{a}_{x:\bar{n}|})^2 / \delta^2 = (2/\delta)*(\bar{a}_{x:\bar{n}|} - {}^2\bar{a}^2_{x:\bar{n}|}) - (\bar{a}_{x:\bar{n}|})^2$$

In this expression, ${}^2\bar{a}^2_{x:\bar{n}|}$ denote the present value of an n-year temporary life annuity at the rate of 2δ.

As we have defined the accumulated value $\bar{S}_{\bar{n}|}$ of an n-year continuous annuitycertain, we define the accumulated value of an n-year temporary annuity as follows.

It is denoted by $\bar{S}_{x:\bar{n}|}$ and is defined as $\bar{S}_{x:\bar{n}|} = \bar{a}_{x:\bar{n}|}\ /\ {}_nE_x$. It denotes the accumulated value of 1 unit benefit per annum, paid continuously for at most n years. Observe that

$$\bar{S}_{\bar{n}|} = (1+i)^n \bar{a}_{\bar{n}|} \text{ and } \bar{S}_{x:\bar{n}|} = \bar{a}_{x:\bar{n}|}\ /\ {}_nE_x = (\ (1+i)^n \bar{a}_{x:\bar{n}|}) / {}_np_x$$

Thus, $\bar{S}_{x:\bar{n}|}$ has a similar expression to that of $\bar{S}_{\bar{n}|}$ with one additional factor ${}_np_x$, as the annuity payments are conditional on survival of (x).

Example 11.7.1: It is given that the force of interest is δ = 6 % and the actuarial present value at age 27 of a unit benefit to be paid at the moment of death in a5-year endowment insurance with force of interest δ is 0.7395 and with force of interest 2δ is 0.6006. Find the actuarial present value at age 27 of a continuous 5-year temporary life annuity payable at the rate of 10000 per annum. Also find the standard deviation of the annuity.

Solution: We are given that $\bar{A}_{27:\overline{5}|} = 0.7395$ and ${}^2\bar{A}_{27:\overline{5}|} = 0.6006$. Then,

$$\bar{a}_{27:\overline{5}|} = (1 - \bar{A}_{27:\overline{5}|}) / \delta = 4.341667 \text{ and } 10000\bar{a}_{27:\overline{5}|} = 43416.67.$$

Thus, 43416.67 is the actuarial present value of the annuity payable continuously, at the rate of 10000 per year, for at most 5 years. Further variance of this annuity is given by

$$\frac{{}^2\bar{A}_{27:\overline{5}|} - (\bar{A}_{27:\overline{5}|})^2}{\delta^2} = 14.92771.$$

Hence the standard deviation of the annuity of 10000 is 10000 × 3.863639 = 38636.39.

Note that the standard deviation is very high, indicating a large spread in the present value random variable of the annuity. Thus, if we have a portfolio of 5-year temporary life annuity contracts, all issued to individuals of age 27, then there is large variation in the mortality behavior of these individuals and hence in annuity payments.

Whole Life Annuity: This annuity, as the name implies, provides payments through- out the future life time of individual, that is, until death of the individual. Whole life insurance is usually purchased by whole life annuity of premiums, that is, premiums are paid continuously at a certain rate per annum till the individual is alive.

The present value random variable corresponding to payments made continuously at the rate of 1 per year for future lifetime T of (x) is defined as

$$Y = \bar{a}_{\overline{T}|} = \frac{1 - v^T}{\delta}$$

The support of this random variable is (0,1/δ). The actuarial present value for a continuous whole life annuity is denoted by ax. We know that the probability density function of T is ${}_tp_x\ \mu_{x+t}$, so the actuarial present value can be calculated by,

$$\bar{a}_x = E[Y] = \int_0^\infty \bar{a}_{\overline{t|}}\, {}_tp_x\, \mu_{x+t}\, dt \qquad (11.7.6)$$

We have noted that, $\frac{d}{dt}\, {}_tp_x = -\, {}_tp_x\, \mu_{x+t}$. Hence using integration by parts, we obtain

$$\bar{a}_x = \int_0^\infty v^t\, {}_tp_x\, dt. \qquad (11.7.7)$$

This integral can be interpreted as involving a momentary payment of 1 dt made at time t, discounted at interest back to time zero by multiplying with v^t and further multiplied by ${}_tP_x$ to reflect the probability that (x) has survived up to x + t and hence payment is made at time t. this is the current payment form of the actuarial present value for the whole life annuity. Formula for $\bar{a}_x$ as given in (11.7.7) can also be obtained by allowing n to tend to ∞ in the formula for $\bar{a}_{x:\overline{n|}}$ given in 11.7.5.

The present value random variable corresponding to whole life annuity can be expressed as

$$Y = \bar{a}_{\overline{T|}} = \frac{1-v^T}{\delta} = \frac{1-Z}{\delta},$$

Where $Z = v^t$ is a present value random variable corresponding to a unit benefit payable at the moment of death in whole life insurance. Taking expectations on both sides, we get a relation between the actuarial present values of benefit function and the annuity as given below.

$$\bar{a}_x = \frac{1-\bar{A}_x}{\delta} \quad . \qquad (11.7.8)$$

From 11.7.8 , we obtain

$$1 = \delta\, \bar{a}_x + \bar{A}_x. \qquad (11.7.9)$$

This also has the same kind of interpretation as that for annuity certain due.

To measure the mortality risk in a continuous life annuity, we are interested in $\mathrm{Var}(\bar{a}_{\overline{T|}})$. We get ,

$$\mathrm{Var}(\bar{a}_{\overline{T|}}) = \mathrm{Var}\left(\frac{1-v^T}{\delta}\right) = \frac{Var(v^T)}{\delta^2} = \frac{{}^2\bar{A}_x-(\bar{A}_x)^2}{\delta^2}.$$

Further, we observe that since $1 = \delta\, \bar{a}_{\overline{T|}} + v^T$, $\mathrm{Var}(\delta\, \bar{a}_{\overline{T|}} + v^T) = 0$. Thus, there is no mortality risk for the combination of a continuous life annuity of δ per year and a life insurance of 1 payable at the moment of death.

11.8 Discrete Life Annuities

The theory of discrete life annuity, also known as annuities payable annually, is analogous to the theory of continuous life annuities. In continuous life annu1hes, present value random variables are defined in terms of future life time random variable T(x), while in discrete life annuities, present value random variables are defined in terms of curtate future life time random variable K(x). Thus, integrals are replaced by sums and integrands by summands. For continuous annuities, there was no distinction between payments at the beginning of payment intervals or at the end of the intervals, that is, between annuity due and annuity immediate. For discrete annuities, the distinction is meaningful. Discrete life annuity due is often used in applications. For example, most individual life insurances are purchased by an annuity due of periodic premiums. As in continuous life annuities, discrete life annuities can be any one of the following three types

i. n-year temporary life annuity
ii. whole life annuity
iii. n-year certain and life annuity.

Each of these is further classified depending upon the deferment or otherwise of payments and as due and immediate. We will discuss each in detail in the following.

n-Year Temporary Life Annuity Due: In this annuity, payment of one unit per annum is payable at the beginning of the year for at most n years. Thus, the annuity payments cease when an individual dies during the term of n years or at the end of n years if the individual survives the n-year term. Such a situation arises if an n-year term insurance or an n-year endowment insurance, with benefit payable at the end of the year of death is purchased by paying periodical premiums at the anniversary of the policy year. Payments of premiums will stop if the death occurs before the term of the n-years is over or definitely at the end of n years if the individual survives the term. Thus, the total number of payments will be 1 + K(x), if death occurs before the end of an n-year term and n if the (x) survives the n-year term. Hence, the present value random variable of this annuity is defined as follows.

$$Y = \begin{cases} \ddot{a}_{\overline{K+1}|}, & \text{if } \mathrm{K} = 0,1,\dots,\mathrm{n}-1 \\ \ddot{a}_{\overline{n}|}, & \text{if } \mathrm{K} = \mathrm{n},\mathrm{n}+1,\dots, \end{cases}$$

It actuarial its actuarial present value is denoted by $\ddot{a}_{x:\overline{n}|}$ and is given by,

$$\ddot{a}_{x:\bar{n}|} = E(Y) = \sum_{k=0}^{n-1} \ddot{a}_{\overline{K+1}}\,{}_kp_xq_{x+k} + \ddot{a}_{\bar{n}|}\,{}_np_x.$$

Using the relation $\ddot{a}_{\overline{K+1}} - \ddot{a}_{\bar{n}|} = v^k$ and expressing

$$P[K(x) = k] = {}_kp_xq_{x+k} = {}_kp_x(1-p_{x+k}) = {}_kp_x{}_{k+1}P_x$$

the above sum can be simplified as follows.

$$\ddot{a}_{x:\bar{n}|} = E(Y) = \sum_{k=0}^{n-1} \ddot{a}_{\overline{K+1}}\,{}_kp_xq_{x+k} + \ddot{a}_{\bar{n}|}\,{}_np_x.$$

$$= \sum_{k=0}^{n-1} \ddot{a}_{\overline{K+1}}\,{}_kp_x\,(1-{}_{k+1}P_x) + \ddot{a}_{\bar{n}|}\,{}_np_x.$$

$$= 1 + \sum_{k=1}^{n-1} \ddot{a}_{\overline{K+1}}\,{}_kp_x - \sum_{k=0}^{n-2} \ddot{a}_{\overline{K+1}}\,{}_kp_x{}_{k+1}P_x) - \ddot{a}_{\bar{n}|}\,{}_np_x + \ddot{a}_{\bar{n}|}\,{}_np_x.$$

$$= 1 + \sum_{k=1}^{n-1} \ddot{a}_{\overline{K+1}}\,{}_kp_x - \sum_{k=1}^{n-1} \ddot{a}_{\bar{n}|}\,{}_kp_x$$

$$= 1 + \sum_{k=1}^{n-1}(\ddot{a}_{\overline{K+1}} - \ddot{a}_{\bar{n}|})\,{}_kp_x$$

$$= 1 + \sum_{k=0}^{n-1} v^k\,{}_kp_x \qquad (11.7.1)$$

It is the actuarial present value of Y in the current payment form. Expressing Y in a different way, we get the relation of $\ddot{a}_{x:\bar{n}|}$ with the actuarial present value of the benefit function studied earlier. Note that Y can be written as Y = (1 – Z)/d, where,

$$Y = \begin{cases} v^{K+1}, & \text{if } K = 0,1,\dots,n-1 \\ v^n, & \text{if } K = n, n+1,\dots, \end{cases}$$

Taking expectations on both sides, we have,

$$\ddot{a}_{x:\bar{n}|} = (1-E[Z])/d = (1 - A_{x:\bar{n}|})/d \qquad (11.7.2)$$

Thus, the actuarial present value of an n-year temporary discrete life annuity due, with unit payment and the actuarial present value of one unit benefit to be paid at the end of the year of death in an n-year endowment insurance are related to each other. This relation is useful to find values of one from the other and in the calculation of premiums can be rearranged as

$$1 = \ddot{a}_{x:\bar{n}|} + A_{x:\bar{n}|} \qquad (11.7.2)$$

Its interpretation is similar to that of annuity certain due in Section 6.2. We use the relation between Y and Z to calculate the variance of Y. Thus,

$$\mathrm{Var(Y)} = \mathrm{Var(Z)}/\, d^2 = ({}^2A_{x:\bar{n}|} - (A_{x:\bar{n}|})^{2)}/d^2$$

The actuarial accumulated value at the end of the term of an n-year temporary life annuity due of 1 unit per year, payable till (2) survives, or up to n, whichever is earlier, is denoted by$\ddot{S}_{x:\bar{n}|}$ and is given by

$$\ddot{S}_{x:\bar{n}|} = \ddot{a}_{x:\bar{n}|}/{}_nE_x$$

This formula is analogous to the formula

$$\ddot{S}_{\bar{n}|} = \ddot{a}_{\bar{n}|}/v^n$$

for accumulated values in annuities certain. The additional factor ${}_np_x$ comes in the denominator as we are concerned with life annuities and payments are made if (x) survives for the next n years.

n-Year Temporary Life Annuity Immediate: In this annuity, payment of one unit per annum is payable at the end of the payment intervals for at most n years. If death occurs in the first year after signing the contract, then there is no payment. 1 unit will be paid at the end of the first policy year, if (x) survives the first year. If death occurs in the second year after signing the contract, then there is only one payment of 1 unit at the end of the first year. Thus, the total number of payments will be K(x), if death occurs during an n-year term and n if the (x) survives the n-year term. The present value random variable of an n-year temporary life annuity immediate is defined as follows.

$$Y = \begin{cases} a_{\bar{K}}, & \text{if } \mathrm{K} = 0,1,\dots,\mathrm{n}-1 \\ a_{\bar{n}|}, & \text{if } \mathrm{K} = \mathrm{n},\mathrm{n}+1,\dots, \end{cases}$$

Here we define $a_{\bar{0}|} = 0$. The actuarial present value of Y is denoted by $a_{x:\bar{n}|}$ and is given by,

$$a_{x:\bar{n}|} = \mathrm{E(Y)} = \sum_{k=0}^{n-1} a_{k|}\, {}_kp_x q_{x+k} + a_{\bar{n}|}\, {}_np_x.$$

Proceeding on similar and using the fact that $a_{\bar{0}|} = 0$, the above sum simplifies as follows.

$$a_{x:\bar{n}|} = \mathrm{E(Y)} = \sum_{k=0}^{n-1} a_{\bar{k}|}\, {}_kp_x q_{x+k} + a_{\bar{n}|}\, {}_np_x.$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|}({}_{k}p_x - {}_{k+1}p_x) + a_{\overline{n}|}\,{}_{n}p_x.$$

$$= \sum_{k=0}^{n-1} a_{\overline{k}|}\,{}_{k}p_x - \sum_{k=0}^{n-1} a_{\overline{k}|}\,{}_{k+1}p_x + a_{\overline{n}|}\,{}_{n}p_x.$$

$$= v\, p_x + \sum_{k=2}^{n-1} a_{\overline{k}|}\,{}_{k}p_x - \sum_{k=2}^{n-1} a_{\overline{k-1}|}\,{}_{k+1}p_x + a_{\overline{n}|}\,{}_{n}p_x.$$

$$= v\, p_x + \sum_{k=2}^{n-1} a_{\overline{k}|}\,{}_{k}p_x - \sum_{k=2}^{n-1} a_{\overline{k-1}|}\,{}_{k}p_x$$

$$= v\, p_x + \sum_{k=2}^{n-1} (a_{\overline{k}|} - a_{\overline{k-1}|})\,{}_{k}p_x$$

$$= v\, p_x + \sum_{k=2}^{n-1} v^k\,{}_{k}p_x \qquad (11.7.4)$$

It is the actuarial present value in the form of current payment integral. The relation of $a_{x:\overline{n}|}$ with the actuarial present value of the benefit function is not obvious in this type of annuity. We write Y as $Y = (1 - Z_1 - Z_2)/i$, where,

$$Z_1 = \begin{cases} (1+i)v^{K+1}, & \text{if } K = 0,1,\dots,n-1 \\ 0, & \text{if } K = n, n+1, \dots, \end{cases}$$

and

$$Z_2 = \begin{cases} 0, & \text{if } K = 0,1,\dots,n-1 \\ n, & \text{if } K = n, n+1, \dots, \end{cases}$$

$Z_1/(i+1)$ is the present value random variable for a unit benefit in an n-year term insurance, payable at the end of the year of death while Z_2 is the present value random variable for a unit benefit, n-year pure endowment insurance. Taking expectations on both sides and after some algebra, we have,

$$E(Y) = a_{x:\overline{n}|} = \frac{(1+i)A^1_{x:\overline{n}|} - A_{x:\overset{1}{\overline{n}|}}}{i} = v\, \ddot{a}_{x:\overline{n}|} - A^1_{x:\overline{n}|} \qquad (11.7.5)$$

Here, we can derive one more relation between $a_{x:\overline{n}|}$ and $\ddot{a}_{x:\overline{n}|}$ as follows.

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k\,{}_{k}p_x = 1 + \sum_{k=1}^{n} v^k - {}_{k}p_x - v^n\,{}_{n}p_x = v\, a_{x:\overline{n}|} - {}_{n}E_x \qquad (11.7.5)$$

Example 11.7.1: Prove that $a_{x:\overline{n}|} < \ddot{a}_{x:\overline{n}|}$

Solution: It follows immediately from (11.7.5) We have,

$$a_{x:\overline{n}|} = v\, \ddot{a}_{x:\overline{n}|} - A^1_{x:\overline{n}|} < v\, \ddot{a}_{x:\overline{n}|} < \ddot{a}_{x:\overline{n}|}$$

as $A^{1}_{x:\overline{n}|}$ is positive and v is a positive fraction. Thus, the expected present value of an n-year temporary life annuity immediate is less than that for due. It is logical in view of the nature of time points of the payment.

Example 11.7.5 Suppose the force of mortality follows Gompertz law given by $\mu_x = BC^x$. Find the values of (i) n- year temporary life annuity due and (ii) n-year temporary life annuity immediate for payment of 100 rupees per annum, form = 1, 2, ..., 10 and for ages 25, 30, 35, 40. Take $\delta = 0.05$, B = 0.0001151 and C = 1.096.

Solution:

i. From (11.7.2) we have

$$\ddot{a}_{x:\bar{n}|} = (1 - A_{x:\bar{n}|})/d$$

We have obtained in Example 11.7.2 the values of $A_{x:\bar{n}|}$ for the specified values of z and n for Gompertz law with the given values of parameters. Using those values and equation (11.7.2) we find the values of 100 $\ddot{a}_{x:\bar{n}|}$. These are reported in Table 11.4.1 and can be obtained directly from the formula in (6.4.1) and the values of ${}_kp_x$ for Gompertz law.

Table 11.7.1: Gompertz's law: 100 $\ddot{a}_{x:\bar{n}|}$.

n	**25**	**30**	**35**	**40**
1 2	100.00 195.01	100.00 194.94	100.00 194.84	100.00 -194.68
3	285.27	285.07	284.76	284.27
4	371.00	370.61	369.99	"369.01
5	452.42	451.77	450.74	449.12
6	529.74	528.77	527.23	524.81
7	603.15	601.79	599.64	596.27
8	672.84	671.02	-668.17	663.69
9	738.98	736.65	732.99	727.26
10	801.73	798.82	794.26	-787.13

The first row of this table is 100 for all ages as by the formula for $\ddot{a}_{x:\bar{n}|}\ddot{a}_{x:\bar{1}|}= 1$. Annuity due is paid at the beginning of the interval. Hence for a one yearterm its present value is the same as the amount paid at the beginning of theinterval. The entry $\ddot{a}_{25:\overline{10}|}$ = 801.73 is interpreted as follows: Present value of the payments made at the beginning of the year at the rate of 100 per annum for at most 10 years is 801.73. In other words, if mode of premium payments is such that premium is paid at the beginning of the year at the rate of 100per annum for at most 10 years, then its present value, that is expected cash inflow to the insurance company corresponding to this policy, is 801.73. From the table we note that values of 100 $\ddot{a}_{x:\overline{n}|}$ decrease marginally as age increases for the given mortality law with given parameters, however they increase as the term n increases because the period of annuity payments increases. The values in this table are marginally higher than those reported in Table 11.7.1 specifying the values of 100 $\bar{a}_{x:\overline{n}|}$ for the same mortality law. Here the annuity payments are at the beginning of the interval while in the setup of Table 11.7.1, annuity payments are continuous at the same rate and hence such a marginal difference.

ii. From (11.7.5) we have

$$a_{x:\overline{n}|} = \frac{1-(1+i)A^{1}_{x:\overline{n}|}-A_{x:\overline{n}|}^{\;\;\,1}}{i}$$

We have obtained in Example 10.7.2 the values of $A^{1}_{x:\overline{n}|}$and $A_{x:\overline{n}|}^{\;\;\,1}$for the specified values of x and n for Gompertz law with the given values of parameters. Using those values and equation (11.4.5) we find the values of 100 $a_{x:\overline{n}|}$. These are reported in Table 10.7.2.

Table 11.7.2: Gompertz's law: 100 $a_{x:\overline{n}|}$.

n	**25**	**30**	**35**	**40**
1	95.01	94.94	94.84	94.68
2	185.27	185.07	184.76	184.27
3	271.00	270.61	269.99	269.01
4	352.42	351.77	350.74	349.12
5	429.74	428.77	427.23	424.81
6	503.15	501.79	499.64	496.27
7	572.84	571.02	568.17	563.69
8	638.98	636.65	632.99	627.26
9	701.73	698.82	694.26	687.13
10,	761.27	757.72	752.15	743.47

For a comparative study of the actuarial present values of annuity payments which are either continuous or at the beginning of the payment interval or at the end of the interval, we combine the results of Example 11.7.2 and 11.7.5 for age 25 and display them in the Table 11.7.3.

Table 11.7.3: Gompertz law: n-year temporary life annuity

n	$\mathbf{100\ddot{a}_{25:\overline{n}|}}$	$\mathbf{100\bar{a}_{25:\overline{n}|}}$	$\mathbf{100a_{25:\overline{n}|}}$
1	100.00	97.48	95.01
2	195.01	190.10	185.27
3	285.27	278.08	271.00
4	371.00	361.64	352.42
5	452.42	440.99	429.74
6	529.74	516.34	503.15
7	603.15	587.87	572.84
8	672.84	655.77	638.98
9	738.98	720.20	701.73
10,	801.73	781.34	761.27

From the table it is clear that

$$100\ddot{a}_{25:\overline{n}|} > 100\bar{a}_{25:\overline{n}|} > 100a_{25:\overline{n}|}$$

for all n. Such inequality follows from the nature of the time points of payments in these three cases. In $100\ddot{a}_{25:\overline{n}|}$, itis at the beginning of the interval, in $100\bar{a}_{25:\overline{n}|}$, it is paid continuously while in $100a_{25:\overline{n}|}$, it is paid at the end of the interval.

11.9 Summary

In actuarial statistics, life insurance refers to financial products providing a benefit upon death. The benefit payable at the moment of death is modeled using continuous-time functions, reflecting payment immediately upon death, while the benefit payable at the end of the year of death assumes a discrete structure, with payment delayed until the end of the year. The relation between these two valuation approaches is often expressed using a conversion factor involving the force of interest. Life annuities, on the other hand, provide periodic payments while the annuitant is alive. Annuities certain guarantee payments for a fixed period regardless of survival. Continuous life annuities assume constant payment flows throughout life, while discrete life annuities make regular, usually annual, payments until death. These concepts are fundamental in evaluating and pricing life-contingent financial products.

11.10 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

11.11 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT:12 NET PREMIUMS AND NET PREMIUMS RESERVES

Structure:

12.1 Introduction

In all the previous units of insurance and annuities the discussion of various topics was mainly motivated by the calculation of premiums for a variety of insurance products. Here, we will proceed with this main aim of determining premiums corresponding to various modes of premium payments and various types of insurance products. The premiums charged by an insurance company for an insurance policy serve two purposes. Firstly, these generate funds needed to meet the benefit paying liabilities and the expenses of running of business. Secondly, premium calculations also have to take into account the profit margin to the company, as after all it is business. An important issue in the determination of the premiums is that these are to be fixed at the timing of signing the contract. As a consequence, premium calculations depend heavily on the actuarial present values of the benefit to be paid, whenever claim arises and the actuarial present values of the payments for various modes of payment of the premiums. There are a number of approaches to determine the premium rates, which may result in different numerical values for the premium. However, all the principles are based on the impact of the insurance on the wealth of the insurance company. One approach requires that the chance of loss to the insurance company should be as small as possible while the second approach requires that the expected loss to the company should be 0. Regardless of the approach, we need to define a random variable that gives the present value at issue of the insurer's loss, when the insurance contract specifies the benefit function, the time of its payment, premium level and its mode of payment. Hence we begin the next section by defining a loss at issue random variable.

Insurance policies are mainly classified as fully continuous or fully discrete policies, depending on the time of payment of benefit and the mode of premium payment. If the death benefit is payable at the moment of death and the premiums are payable as a continuous annuity, the policy is referred to as a fully continuous policy. If death benefit is payable at the end of the year of death and premiums are payable as a discrete life annuity due, the policy is referred to as a fully discrete policy. In fully discrete policy the premium payments may be made annually, semi-annually, quarterly or monthly. If the death benefit is payable at the moment of death and the premiums are payable as mthly annuity due, m = 1, 2, 4, 12, then the policy is referred to as a semi-continuous policy. This chapter also discusses the determination of premiums in a fully continuous policy and determination of premiums in a fully discrete policy and a semi-continuous policy. The modifications needed in these two types when the premium payments form a monthly annuity.

The premiums are determined only by the pattern of benefits and the mode of premium payments and do not consider expenses, profit or contingency margins. The last section illustrates how expenses are incorporated into the model for premiums. Premiums calculated without an allowance for expenses are called net premiums, while premiums which include expenses and profit and which are actually charged are called office or gross premiums or expense loaded premiums or contract premiums. It is important to note that the net premiums are uniquely determined by the mortality pattern of the group under study and the interest patterns dictated by the financial market conditions. Both the factors are beyond the control of the insurance company.

12.2 Objectives

After reading this unit the learner should able to understand:

1. To ensure that sufficient funds are collected to pay for the expected future claims or benefits promised under the policy.
2. To ensure that policyholders contribute fairly according to the risk they bring to the insurance pool (e.g., age, gender, health status).

3. To set a premium level that ensures the long-term financial sustainability of the insurer without excessive profit or loss.
4. To demonstrate that the insurer can meet its future obligations, thus maintaining financial solvency.
5. To avoid sudden spikes in required funding by building up reserves gradually over time.

12.3 Loss at Issue Random Variable

In this unit, we begin with the definition of a present value of loss to the insurance company related to a specific portfolio of policies, as loss to the company is the main governing principle in the determination of premiums. An insurance policy usually involves a contract in which a life annuity of premiums will be paid to avail the policy's benefits. If death occurs t units after the policy issue, then the loss-at-issue function l(t) is defined as,

$$l(t) = \text{present value of benefits paid} - \text{present value of premiums received}$$

where the present values are calculated as per the rate at the policy issue date. For all insurance policies, l(t) is a decreasing function of t, since the later the death occurs, the smaller will be the present value of the death benefit and the more premiums will have been received. In practice, the time from signing the contract to death is a future life time random variable T. Hence, in a fully continuous policy, the loss at issue random variable L = L(T) for a unit benefit is defined as

$$L(T) = ZT - PY,$$

where the policy has level premiums at annual rate P. Zr as defined in before, represents the present value random variable for one unit death benefit to be paid at the moment of death and Y represents the present value random variable of the continuous annuity of premiums. In the case of a fully discrete policy, wherein death benefit is payable at the end of the year of death and premiums are payable as a discrete life annuity due, the loss at issue random variable for a unit benefit is of the form

$$L(K) = ZK - PY,$$

where K is a curtate future life time random variable. The policy has level premiums at annual rate P, Z as defined in Chapter 5, represents the present value random variable for one unit death benefit to be paid at the end of year of death and Y represents the present value random variable of the discrete annuity of premiums. For a policy with discrete premiums, unless specified otherwise, it is assumed that the premiums are payable as annuity due. In a semi-continuous policy loss at issue random variable is defined similarly, where Z is a function of T while Y is a function of K, which in turn is a function of T. The following two examples illustrate the concept of a loss function.

Example 12.3.1: On May 6, 1996, (65) bought a Rs. 1,00,000/- whole life insurance policy with death benefit payable at the end of the year of death. The policy is purchased by means of annual premiums, payable at the start of each year the policy remains in force. The policy holder died on August 6, 2003 and the loss to the insurer was Rs. 30,000/-. If i = 0.06, what was the annual premium paid?

Solution: The benefit is payable on May 6, 2004, which is the end of the policy year in which death occurs. This is 8 years after the policy is issued. Suppose the annual premium is P, which is paid for 8 years at the beginning of the year, thus premium payments form an 8-year annuity certain due. Hence the loss to the insurer corresponding to benefit of 1 lakh is given by

$$l(7) = 10^5 v^8 - P\, \ddot{a}_{\overline{8}|} = 62741\text{-}6.58P$$

Equating it to 30,000, we have P = (62741 - 30,000)/ 6.58 = 5000.

Example 12.3.2: A fully continuous 10-year term insurance of face amount Rs.10,000/- has an annual premium rate of Rs. 100/- and the force of interest is 0.05. Find the value of the loss at issue random variable

i. if death occurs exactly 5 years after issue and
ii. if death occurs exactly 15 years after issue.

Solution: (i) If an annual premium of 100 corresponds to a benefit of 10000, for 1 unit benefit it will be 0.01. For a one-unit benefit, the value of loss at issue random variable is given by,

$$l(5) = e^{-5(0.05)} - (0.01)\bar{a}_{\overline{5}|} = 0.735,$$

hence for a 10000 benefit it is 7350. In the second case, the policy ends after 10 years and death occurs after 15 years, hence there is no death benefit or the corresponding Z has value 0. Hence for one unit benefit, value of loss at issue random variable is given by,

$$l(15) = 0 - (0.01)\bar{a}_{\overline{10}|} = 0.079.$$

For a 10000 benefit, the value is -790. The negative loss is of course the gain to the insurer.

The following three principles are generally used for premium calculation.

i. **Percentile Premium:** In this approach, the value of the annual premium rate P is found so that probability statement P[L > 0] = c is satisfied for some specified c. For instance, if c is .05, then P is found so that there is only 5% probability that the loss is positive. The premium calculated with this approach is referred to as 100 c-th percentile premiums.

ii. **Exponential Premium:** Using an exponential utility function, P is found so that the expected utility of wealth with insurance is equal to the utility of wealth without insurance, that is, by solving the equation U(w) = E(U(w – L)), where U(w) is the exponential utility function.

iii. **Equivalence Principal Premium:** In equivalence principal premium, P is found so that the expected loss E(L) to the company is 0, that is by solving the equation E(Z) = P E(Y). It leads to the following principle to determine premium.

Actuarial present value of benefit = Actuarial present value of premiums.

This result also follows from the equation U(w) = E(U(w – L)), where U(w) = w. Insurance companies usually have large number of contracts. Hence, by the law of large numbers, the expected loss describes the average actual loss quite accurately. In this approach, it is not necessary to derive the distribution of L. It is enough to have knowledge of E(Z) and E(Y).

Premiums found on the basis of the third principle are called benefit premiums or net annual premiums. As a special case, if the policy is purchased by a single premium on the issue date, the annuity random variable is the constant 1, since only one premium is to be paid. From the equivalence principle we get,

$$E(Z) = PE(Y) = P.$$

For this reason, E(Z) is called as the net single premium, defined earlier. It is also known as the single benefit premium. Benefit premiums are thus obtained by solving an expression of the form A = Pä (in the fully discrete case), or $\bar{A}$ = Pā (in the fully continuous case), or $\bar{A}$ = Pä (in the semi-continuous case), where post- or pre-subscripts to A and a are as per the policy under consideration. Premiums are usually calculated using this approach. The following two examples illustrate the application of the first two approaches. At last, we solve the problems of deciding premiums using the equivalence principle.

Example 12.3.3: For a fully continuous whole life insurance of 1000 issued to (30), it is given that

i. $\delta = 0.05$ and

ii. mortality follows Gompertz's law with B = 0.001151 and C = 1.096. Calculate P, the 5th percentile premium for this insurance, that is, the least annual premium such that the insurer's probability of a positive financial loss is at most 0.05.

Solution: Here, we wish to find the smallest premium such that $P[L(T) > 0] \leq 0.05$, where $L(T) = e^{-\delta T} - P\bar{a}_{\overline{T}|}$. Thus, we find the premium for which $P[L(T) > 0] = 0.05$. So we must have a knowledge of the distribution of L. In general, it is very difficult to find the distribution of L. We can avoid this derivation by noting the fact that L(T) is a decreasing function of T. Hence, $P[L(T) > 0] = P[T < t_0]$ for some t_0. We first find the point t_0 such that ${}_{t0}q_{30}$ $P[L(30) \leq t_0] = 0.05$. With Gompertz Law, ${}_{t0}q_{30} = 1 - {}_{t0}q_{30} = 1 - \exp{-mC^{30}(C^{t0}-1)} = 0.05$

where $m = B/\log_e C$. with given value of B and C, we get $t_{0=}14.0084$. Since the loss at issue decreases as time of death increases, we choose P so that L-0 if l(15) = 0. Thus,

$$l(15) = 1000e^{-05(15)} - P\bar{a}_{\overline{15}|} = 0 \rightarrow P = (1000e^{-0.05(15)})/\ \bar{a}_{\overline{45}|} = 44.76.$$

It is the 5th percentile premium paid continuously at a rate of 44.76 per annum till the individual dies.

12.4 Fully Continuous Premiums

As outlined in the previous section, fully continuous premiums using equivalence principle are obtained by solving the equation,

$$E(Z_T) = PE(Y).$$

Expressions for $E(Z_T)$ and E(Y) will change as insurance product changes and mode of premium payments changes, respectively. We begin with the first insurance product discussed in previous Section and it is the n-year term insurance with level, one unit benefit payable immediately on the death of (x). Suppose the premiums are paid continuously till the claim is made in n years after the issue of the policy or for n years, if claim is not made in n years. Thus, the premium payments form an n-year temporary continuous life annuity. The corresponding loss at issue random variable is defined as,

$$L(T) = \upsilon^{T} - P\,\bar{a}_{\overline{T|}} \quad \text{if } T \le n \qquad \text{and} \qquad 0 \;- P\,\bar{a}_{\overline{n|}}\ , \text{ if } T > n.$$

Recall that in term insurance, if claim is not made in the term of n years, insurer has no liability to pay the benefit. But insured has paid the premiums as a n-year temporary continuous life annuity. Thus, when T > n the loss random variable takes value – $P\bar{a}_{\overline{n|}}$, which is negative, resulting in gain for the insurer. In this set up $E(Z_T) = \bar{A}^{\,1}_{x:\overline{n|}}$ and $E(Y) = \bar{a}_{x:\overline{n|}}$. Hence, E(L(T)) = 0 implies $P = \frac{\bar{A}^{\,1}_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}}$. In international actuarial notation benefit premium, for 1 unit benefit, in this set up is denoted by $\bar{P}(\bar{A}^{\,1}_{x:\overline{n|}})$. Thus,

$$\bar{P}(\bar{A}^{\,1}_{x:\overline{n|}}) = \frac{\bar{A}^{\,1}_{x:\overline{n|}}}{\bar{a}_{x:\overline{n|}}} \qquad (12.4.1)$$

For whole life insurance with unit benefit, the loss random variable is given by

$$L(T) = v^{T} - P\,\bar{a}_{\overline{T|}}\ .$$

The benefit premium is denoted by $\bar{P}(\bar{A}_x)$. With E(L) = 0, we get,

$$\bar{P}(\bar{A}_x) = \frac{\bar{A}_x}{\bar{a}_x}\ . \qquad (12.4.2)$$

The symbols $\bar{P}(\bar{A}^{\,1}_{x:\overline{n|}})$ and $\bar{P}(\bar{A}_x)$ refer to the premium corresponding to unit benefit. It is to be noted that $\bar{P}(\bar{A}_x)$ is the limit of $\bar{P}(\bar{A}^{\,1}_{x:\overline{n|}})$ as n tends to ∞.

L is a random variable due to the random nature of time-until-death. To measure the variability of losses on an individual whole life insurance, variance of L is used as a measure of the variability. When, E(L) = 0, Var(L) = E(L^2). However, instead of obtaining the second moment of L, it is simple to find the variance using the following approach.

$$\mathrm{Var}(\upsilon^{T} - \bar{P}\bar{a}_{\overline{T|}}) = \mathrm{Var}\left[v^{T} - \frac{\bar{P}(1-v^{T})}{\delta}\right] = \mathrm{Var}\left[v^{T}\left(1+\frac{\bar{P}}{\delta}\right) - \frac{\bar{P}}{\delta}\right]$$

$$= \mathrm{Var}\left[v^{T}\left(1+\frac{\bar{P}}{\delta}\right)\right] = \mathrm{Var}(v^{T})\left(1+\frac{\bar{P}}{\delta}\right)^{2}$$

$$= [{}^2\bar{A}_x - (\bar{A}_x)^2]\left(1+\frac{\bar{P}}{\delta}\right)^2. \qquad (12.4.3)$$

For the premium determined by the equivalence principle, we can use the identity, $\delta\bar{a}_x + \bar{A}_x = 1$, to rewrite $1+\frac{\bar{P}}{\delta}$ as

$$1+\frac{\bar{A}_x}{\bar{a}_x\,\delta} = \frac{\bar{a}_x\,\delta + \bar{A}_x}{\bar{a}_x\,\delta} = \frac{1}{\bar{a}_x\,\delta}\ .$$

Thus (12.4.3) reduces to

$$\mathrm{Var}(L) = {}^2\bar{A}_x - (\bar{A}_x)^2 / (\delta\bar{a}_x)^2\ . \qquad (12.4.4)$$

From (12.4.3), we note that

$$\mathrm{Var}(\upsilon^T - \bar{P}\bar{a}_{\overline{T}|}) > {}^2\bar{A}_x - (\bar{A}_x)^2$$

Thus, variance of the loss associated with a single premium whole life insurance is less than the variance of the loss associated with an annual premium whole life insurance. The result is reasonable as in single premium whole life insurance; the entire premium is already paid at the issue of the policy.

Example 12.4.3: The actuarial present value at age 27 of a unit benefit to be paid at the moment of death in a 5-year endowment insurance with force of interest & is 0.7395 and with force of interest 25 is 0.6066. The actuarial present value at age 27 of a unit benefit to be paid at the moment of death in a whole life insurance with force of interest d is 0.5321.

i. Write the expression for the loss at issue random variable to find the premium per annum, payable as a continuous 5-year temporary life annuity, when the benefit amount is Rs.1000, payable at the moment of death, in a 5-year endowment insurance. Find the variance of the loss at issue random variable.

ii. Find the premium per annum, payable as a continuous life annuity when the benefit amount is Rs.1000, payable at the moment of death in a whole life insurance issued to (27).

iii. Find the annual premium, payable continuously for at most 5 years, for a 5-year deferred continuous whole life annuity payable to (27) at the rate of 7000 per annum.

iv. Find the premium per annum, payable as a 5-year continuous annuity certain, for a 5-year certain and continuous whole life annuity payable at the rate of

7000 per annum, issued to (27). Compare the premiums in (iii) and (iv). It is given that the force of interest is δ = 6 %

Solution:

(i) The expression for loss at issue random variable when the premiums are payable as a continuous 5-year temporary life annuity and when the benefit amount is 1 unit, payable at the moment of death, in a 5-year endowment insurance is as follows.

$$L(T) =. \upsilon^{T} - P\,\bar{a}_{\overline{T}|} \quad \text{if } T \leq n \qquad \text{and} \quad \upsilon^{5} - P\,\bar{a}_{\overline{5}|} \quad \text{if } T > 5\ .$$

From Table 12.4.1, for unit benefit, the premium is given by

$$\bar{P}(\bar{A}_{27:\overline{5}|}) = \frac{\bar{A}_{27:\overline{5}|}}{\bar{a}_{27:\overline{5}|}} = \frac{\delta \bar{A}_{27:\overline{5}|}}{1 - \bar{A}_{27:\overline{5}|}} = 0.141938.$$

Hence for a benefit of 1000, the annual premium is 141.94. It will be paid continuously for at most 5 years at the rate of 141.94 per annum. Further, variance of loss at issue random variable can be derived as follows. Suppose the premium $\bar{P}(\bar{A}_{27:\overline{5}|})$ is denoted by $\bar{P}$.

$$\mathrm{Var}(L(T)) = \mathrm{Var}\left[Z - \frac{\bar{P}(1-Z)}{\delta}\right] = \mathrm{Var}\left[Z\,(1 + \frac{\bar{P}}{\delta}) - \frac{\bar{P}}{\delta}\right]$$

$$= \mathrm{Var}\left[Z\,(1 + \frac{\bar{P}}{\delta})\right] = \mathrm{Var}(Z)\,(1 + \frac{\bar{P}}{\delta})^2$$

$$= [{}^2\bar{A}_x - (\bar{A}_x)^2\,]\,(1 + \frac{\bar{P}}{\delta})^2 \quad .$$

Here Z denotes the present value random variable for a unit benefit to be paid at the moment of death in a 5-year endowment insurance issued to (27). From the given values we find the variance. It is 0.6767013 for a 1-unit benefit and 10^6 x 0.6767013 = 676701.3 for a 1000-unit benefit.

ii. For unit benefit, the premium in whole life insurance is given by

$$\bar{P}(\bar{A}_{27}) = \frac{\bar{A}_{27}}{\bar{a}_{27}} = \frac{\delta \bar{A}_{27}}{1 - \bar{A}_{27}} = 0.05686.$$

Hence for a benefit of 1000, the annual premium is 56.86. It will be paid continuously at the rate of 56.86 per annum till (27) dies. Note that the premium for whole life insurance is less than that for 5-year endowment insurance, as in the latter the period for accumulation is comparatively less.

iii. For the given information, in Example 12.4.3, we have calculated the actuarial present value at age 27 of a 5-year deferred continuous life annuity. It is 3.4567 for a 1-unit benefit. Hence, for the payment of 7000 per annum, it is 24196.90. This is the expected present value of the outflow of the company. Now if the premium P per annum is paid as a 5-year temporary continuous life annuity, then the present value of the inflow to the company is $P\bar{a}_{27:\overline{5}|} = P(1-\bar{A}_{27:\overline{5}|})/\delta = 4.341667$ P. Equating the expected present value of the outflow to the expected present value of the inflow, we get $7000\ {}_{5|}\bar{a}_{27} = P\bar{a}_{27:\overline{5}|}$
=> P = 573.18.

Thus, P = 5573.18 per annum is paid continuously for at most 5 years and in return (27) is paid Rs 7000 per annum continuously, after 5 years, till (27)'s death.

iv. For the given information, in Example 12.4.3 we have calculated the actuarial present value at age 27 of a 5-year certain and continuous whole life annuity payable at the rate of 7000 per annum. It is 54434.77. This is the expected present value of the outflow of the company and can be interpreted as the net single premium for the said annuity. Now if the premium P per annum is paid as a 5-year continuous annuity certain, then the present value of the inflow to the company is $P\bar{a}_{\overline{5}|} = 4.319696$ P. Equating the expected present value of the outflow to the expected present value of the inflow, we get $7000\ \bar{a}_{\overline{27:\overline{5}|}} = P\bar{a}_{\overline{5}|}$ => P = 12601.53.

Thus, premium P = 12601.53, per annum, will be paid continuously for 5 years and in return (27) will receive from the insurer, a guaranteed payment of 7000 per annum for the first five years and then a payment of 7000 per annum till (27) is dead.

The premium for a 5-year certain and continuous whole life annuity is more than double of that for a 5-year deferred whole life annuity. It seems reasonable as in a 5-year certain and continuous whole life annuity, payment of 7000 per annum is guaranteed while in a 5-year deferred whole life annuity there is no payment for the first 5 years after signing the contract.

12.5 Fully Discrete Premiums

In this section, we discuss how to derive the annual benefit premium in fully discrete policies, that is when the sum insured is payable at the end of the policy year in which death occurs and the first premium is payable when the insurance is issued. Subsequent premiums are payable on the anniversaries of the policy issue date while the insured survives during the contractual premium payment period. The payment of annual premiums thus forms a life annuity due. The procedure to find fully discrete premiums is similar to that adopted for fully continuous policies, in the previous section. The loss at issue random variable L is now a function of K, the curtate future life time random variable. As an illustration we consider whole life insurance when the unit benefit is payable at the end of year of death and premiums form whole life discrete annuity due. In this set up, according to international actuarial notation, the level annual premium for a unit benefit is denoted by p_x, where the absence of $(\bar{A}_x)$ means that the insurance is payable at the end of the policy year of death. The loss random variable for this insurance is $L = v^{K+1} - P_x\ddot{a}_{\overline{K+1}}$, where the curtate random variable K takes values 0, 1, 2, ………. The equivalence principle requires that E(L) = 0, that is, $E(v^{K+1}) - P_x\, E(\ddot{a}_{\overline{K+1}}) = 0$, which gives, $P_x = A_x/\ddot{a}_x$.

This is the discrete analogue of (12.4.2). Using steps parallel to those taken in the derivation of (12.4.4), we obtain,

$$Var(L) = {}^2A_x - (A_x)^2/(d\ddot{a}_x)^2 \qquad (12.5.2)$$

Table 12.5.1 exhibits the values of $\ddot{a}_x$, A_x and P_x for integer ages x, running from 0 to 100. The underlying mortality law is taken to be the Gompertz's law with force of mortality

$$\mu_x = BC^x, > 0 \text{ and } S(x) = \exp- m(c^x - 1).$$

We have taken B = 0.0001151 and C = 1.096, so that m = B/log C = 0.00125. Further $\delta = 0.05$, hence v = 0.952380952 and d = 0.04761905. The table is constructed using Excel worksheet. The column of är values is constructed using a backward recurrence relation given by,

$$\ddot{a}_x = 1 + vp_x\ddot{a}_{x+1}), \text{ with } \ddot{a}_w) = 0.$$

Further, $A_x = 1 - d\ddot{a}_x$ and $P_x = A_x / \ddot{a}_x$

Thus, with the knowledge of distribution of K(x) one can find P_x values. Under the assumption of uniform distribution of deaths between integral ages, we know that

$$\bar{P}_x = (i/\delta)\ A_x.$$

Hence, we can find A_x. From it $\bar{a}_x$ can be obtained as $\bar{a}_x = (1 - A_x)/\delta$, so that

$$P_x = A_x / \bar{a}_x$$

can be calculated. Such a procedure emphasizes the significance of the assumptions for fractional ages.

Table 12.5.1: Premium for Unit Benefit Whole Life Insurance: Gompertz's law

X	*S(x)*	p_x	$\ddot{a}_x$	A_x	P_x
0	1	0.99987941	19.93444582	0.050741624	0.002545424
1	0.99987941	0.999867776	19.88356489	0.053164524	0.002673792
2	0.999747201	0.99985502	19.83036419	0.055697888	0.002808717
3	0.999602258	0.999841034	19.77474835	0.058346258	0.002950544
4	0.999443355	0.999825698	19.71661906	0.061114317	0.003099635
5	0.999269151	0.999808884	19.65587508	0.064006885	0.003256374
6	0.999078174	0.999790447	19.59241228	0.067028919	0.003421167
7	0.998868814	0.999770232	19.52612368	0.070185517	0.003594442
8	0.998639306	0.999748067	19.45689946	0.073481905	0.00377665
9	0.998387717	0.999723765	19.38462708	0.076923443	0.00396827
10	0.998111927	0.999697118	19.30919135	0.080515617	0.004169808
11	0.997809617	0.999667901	19.23047452	0.084264034	0.004381797
12	0.997478245	0.999635867	19.14835644	0.088174415	0.004604803
13	0.997115031	0;999600743	19.06271467	0.09225259	0.004839426
14	0.996716925	0.999562232	·18.97342473.	0.096504488	0.005086298

15	0.996280595	0.999520007	18.88036024	0.100936126	0.005346091
16	0.995802387	.0.99947371•	18.78339321	0.105553599	0.005619517
17	0.995278306	0.999422949	18.68239429	0.110363066	0.00590733
18	0.99470398	0.999367294	18.57723308	0.115370738	0.006210329
19	0.994074625	0.999306273	18.46777848	0.120582857	0.006529364
20	0.993385009	0.999239369	18.35389908	0.12600568	0.006865336
21	0.99262941	0.999166015	18.23546358	0.13164546	0.0072192
22	0.991801572	0.999085591	18.11234126	0.137508421	0.007591974
23	0.99089466	0.998997415	17.98440253	0.143600736	0.007984738
24	0.989901203	0.99890074	17.85151944	0.149928496	0.008398641
25	0.988813045	0.998794749	17.71356634	0.156497685	0.008834905
26	0.987621277	0.998678546	17.57042054	0.163314144	0.009294834
27	0.986316181	0.998551147	17.42196302	0.170383543	0.009779813
28	0.984887154	0.998411475	17.26807922	0.177711336	0.01029132
29	0.983322636	0.998258351	17.10865986	0.185302726	0.010830932
30	0.981610033	0.99809048	16.94360182	0.193162625	0.011400328
31	0.979735629	0.997906446	16.77280909	0.201295604	0.012001305
32	0.977684499	0.997704696	16.59619374	0.20970585	0.01263578
33	0.975440416	0.997483528	16.41367699	0.218397115	0.013305801
34	0.972985747	0.997241078	16.22519028	0.227372664	0.014013559

Table 12.5.1 Continued

X	$S(x)$	p_x	$\ddot{a}_x$	A_x	P_x
35	0.970301355	0.996975305	16.03067638	0.236635222•	0.0147614
36	0.96736649	0.996683972	15.8300906	0.246186916	0.015551832
37	0.964158676	0,99636463•	15.62340198	0.256029221	0.016387546
38	0.960653603	0.996014596	15:41059452	0.2661629	0.017271423

39	0.95682501	0.995630933	15.19166842	0.276587941	0.018206555
40	0.952644577	0.995210424	14.96664136	0.287303505	0.019196258
41	0.948081814	0.99474955	14.73554973	0.298307857	0.020244094
42	0.943103957	0.994244456	14.49844989	0.309598315	0.021353891
43	0.937675881	0.993690927	14.25541937	0.321171185	0.022529761
44	0.931760015	0.993084348	14.00655804	0.333021713	0.023776128
45	0.925316287	0.992419673	13.75198923	0.345144025	0.025097753
46	0.918302087	0.991691383	13.49186073	0.357531084	0.026499761
47	0.910672267	0.990893444	13.22634575	0.370174642	0.027987673
48	0.902379179	0.990019258	12.95564369	0.383065203	0.029567439
49	0.893372765	0.989061619	12.67998087	0.396191991	0.031245472
50	0.883600713	0.988012653	12.39961097	0.409542925	0.033028691
51	0.873008685	0.986863764	12'.11481542	0.423104605	0.034924561
52	0.861540637	0.985605567	11.82590347	0.436862303	0.036941136
53	0.849139248	0.984227825	11.53321212.	0.450799972	0.039087114
54	0.835746475	0.982719375	11.23710577	0.46490026	0.041371886
55	0.821304254	0.981068049.	10.93797558	0.479144541	0.043805596
56	0.805755362	0.979260596	10.63623858	0.49351°2955	0.046399199
57	0.789044476	0.97728259	10.3323365	0.507984468	0.04916453
58	0.771119429	0.975118341	10.02673428	0.522536941	0.05211437
59	0.751932698	0.972750797	9.719918208	0.537147215	0.055262524
60	0.731443131	0.970161444	9.412393888	0.551791215	0.058623898
61	0.709617925	0.967330196	9.104683755	0.566444064	0.062214579
62	0.686434847	0.964235291	8.797324377	0.58108021	0.066051925
63	0.661884704	0.960853174	8.490863451	0.595673573•	0.070154652
64	0.635974018	0.95715839	8.185856534	0.610197698	0.074542926
65	0.608727867	0.953123467	7.882863552	0.624625921	0.079238454

66	0.580192815	0.948718811	7.582445103	0.638931547	0.084264579
67	0.550439838	0.943912592	7.285158612	0.653088032	0.089646371

Table 12.5.1 Continued

X	$S(x)$	p_x	$\ddot{a}_x$	A_x	P_x
68	0.519567094	0.93867066	6.991554372	0.667069172	0.095410711
69	0.487702387	0.932956448	6.702171534	0.680849294	0.101586372
70	0.455005087	0.926730913	6.417534093	0.694403444	0.108204091
71	0.421667279	0.919952487	6.138146937	0.707707581	0.115296618
72	0.387913862	0.912577062	5.864492017	0.720738755	0.122898753
73	0.354001293	0.904558015	5.597024703	0.733475281	0.131047355
74	0.320214707	0.895846273	5.336170389	0.745896902	0.139781313
75	0.286863152	0.886390443	5.082321395	0.757984937	0.149141481
76	0.254272756	0.876137008	4.835834237	0.769722409	0.159170553
77	0.222777772	0.865030615	4.597027302	0.781094157	0.169912882
78	0.192709593	0.853014451	4.36617897	0.792086924	0.181414213
79	0.164384068	0.840030753	4.143526217	0.802689425	0.193721334
80	0.138087672	0.826021443	3.929263722	0.812892391	0.206881606
81	0.114063378	0.81092893	3.723543474	0.822688583	0.220942387
82	0.092497293	0.794697082	3.526474885	0.832072792	0.235950296
83	0.073507329	0.777272407	3.338125379	0.841041808	0.251950335
84	0.057135219	0.758605444	3.158521393	0.84959437	0.268984839
85	0.043343088	0.738652396	2.98764973	0.857731108	0.287092258
86	0.032015476	0.717377002	2.825459071	0.865454464	0.306305787
87	0.022967166	0.694752669	2.67186141	0.872768632	0.32665191
88	0.0159565	0.67076484	2.52673285	0.879679508	0.348148997
89	0.010703059	0.645413587	2.389912705	0.886194747	0.370806325

90	0.0069079	).618716399	2.261198549	0.892323986	0.39462434
91	0.004274031	0.590711078	2.140331843	0.898079538	0.41959827
92	0.002524717	0.561458678	2.026961098	0.903478139	0.445730379
93	0.001417524	0.531046355	1.920549349	0.908545361	0.473065355
94	0.000752771	0.499589964	1.820136339	0.913326928	0.501790392
95	0.000376077	0.467236217	1.723699783	0.91791914	0.53252843
96	0.000175717	0.434164167	1.626339539	0.922555337	0.567258752
97	7.62899E-05	0.400585741	1.514764537	0.927868428	0.612549611
98	3.05607E-05	0.366745052	1.349281019	0.935748587	0.693516453
99	1.1208E-05	0.332916188	1	0.952381	0.952381
100	3.73131E-06	0	0	1	-

Formulae for the annual benefit premium in other fully discrete policies and semi-continuous policies are derived similar to that in in (12.5.1).

Example 12.5.3: L is the loss at issue random variable for a fully discrete whole life insurance of 1 issued to (49). Calculate P and E(L) on the basis of the following information. (i) $A_{49} = 0.23882$, $\ddot{a}_{49} = 13.4475$, ${}^2A_{49} = 0.08873$, (ii) i = 0.06 (iii) Var(L) = 0.10.

Solution: Suppose that the annual premium is P. Since the policy is whole life, the loss at issue random variable is $L = Z – PY = Z - P((1 - Z)/d) = [1+(P/d)]Z - (P/d)$. Then,

$$E(L) = E(Z) - PE(Y) = A_{49} - P\ddot{a}_{49} = 0.23882\text{-}13.4475P.$$

The variance of L is 0.10, hence we have,

$$0.10 = Var(L) = [1+(P/d)]^2 Var(Z)$$

$$=[1+(P/d)]^2\,[{}^2A_{49}-(A_{49})^2] = [1+P/0.0556]^2\,[0.0873 - (0.2382)^2],$$

from which we get P = 0.044. Then E(L) = 0.23882 – 13.4475(0.044) = -0.35.

Example 12.5.5: Express in international actuarial notation the premiums corresponding to the following insurances.

i. If (40) dies within 25 years, a death benefit of Rs.50000 is payable at the end of the year of death, otherwise a sum of Rs.2000 is payable yearly in advance from age 65. Premiums are payable as a 25-year temporary life annuity due.

ii. ii. An insurance issued to (50) provides a guaranteed benefit of 20000, at the end of a term of 10 years. Annual premiums are payable as a 10 year temporary annuity due.

iii. iii. (30) signs an insurance contract, in which the annual payment of a 15 year annuity immediate is 12000 for the first 5 years and 24000 for the remaining term. Annual premiums are payable as a 5-year temporary annuity due.

Solution:

i. $(50000\bar{A}^{\,1}_{40:\overline{25|}} + 2000\ {}_{25|}\ddot{a}_{40})/\ \ddot{a}_{40:\overline{25|}}$

ii. $(20000v^{10})/\ddot{a}_{50:\overline{10|}}$

iii. $(120000a_{30:\overline{5|}} + {}_{5}E_{30}24000a_{35:\overline{10|}})/\ddot{a}_{30:\overline{5|}}$

In practice, sometimes in fully discrete policies and semi-continuous policies, premiums are payable m times in a policy year rather than annually. We discuss such a setup in the following section.

12.6 Gross Premiums

Insurance models discussed so far have not incorporated several aspects of insurance practice. For example, an insurer has cash outflows other than claim payments. These are expenses to run the business. Expenditures of various types include salaries paid to the staff, commissions paid to the agents, taxes and license fees, rent paid for the offices, cost of establishing and maintaining the offices, as well as those for selling and servicing insurance policies. There are settlement expenses, expenses of payment of benefits and the expenses of maintaining records of policies with continuing benefits after premiums have ceased. Some expenses are related to the premium and benefit amount and some are not related to either of them. For example, commissions are related to the premium or the benefit amount while cost of purchasing the office or rent of the office does not depend on the benefit or premium. When expenses are related to benefit, these are specified as a percentage of premium or benefit. Some insurance companies employ a policy fee system, whereby a fixed number is added to the annual premium. For example, the gross annual premium for a policy with benefit of Rs. 20,000/- may be quoted as 'Rs.50 per Rs.1000 plus a policy fee of Rs.50/-'. Thus the premium for a benefit of Rs. 20,000/- will be $20 \times 50 + 50 = 1050$ rupees. This system reflects the fact that certain administrative costs do not depend on the size of the benefit. The expenses of running the business must be met from premiums and investment income. Consequently, the insurance company has to add an amount to the net premium to cover all these operating costs and to provide itself with some profit. The total amount added to the net premium to cover all of the insurer's cost of doing business is called a loading. The net premium with the loading added is called the gross premium, which is the premium amount the insurer charges the policy owner to keep the policy in force. The difference between gross premium and the net premium is known as expense loading. In the following, we incorporate expenses into the equivalence principle for net premiums to find the gross premiums.

The equivalence principle to determine net premiums is

Expected present value of inflow = Expected present value of outflow,

Where, inflow to the company is via net premiums and outflow of the company is via benefit claims. This equivalence principle is extended to take into account the administrative expenses. Thus, the expected present value of the outflow of the insurance company considers output corresponding to death benefits as well as administrative expenses. We assume that administrative expenses are not random variables but are fixed by the company at the time of issue of the policy. The expense augmented equivalence principle is as follows:

Expected present value of gross premiums Expected present value of benefits and expenses.

To elaborate on such expense augmented principle, as an illustration, consider a 15-year endowment insurance issued to (30), with a benefit of 1000, payable at the end of the year of death. Expenses are Rs 50 per renewal premium payment (including the first), plus Rs 100 at the time of signing the contract. Then the present value of benefit and expenses is $1000A_{30:\overline{15|}}+ 50\ddot{a}_{30:\overline{15|}} + 100$. The present value of annual premiums, with gross premium G, is G $\ddot{a}_{30:\overline{15|}}$. Hence the expenses augmented equivalence principle gives the equation

$$1000A_{30:\overline{15|}}+ + 50\ddot{a}_{30:\overline{15|}}+ 100 = \mathrm{G}\ddot{a}_{30:\overline{15|}}$$

Solving the equation, we get the gross premium G. Thus, to determine the gross premium G, along with interest and mortality pattern, it is essential to have information regarding expenses of the insurance company. In the following we take a hypothetical situation describing the expenses, in one of the examples of determining the net premium and discuss the computation of gross premium.

Example 12.6.1: A 10-year endowment insurance is sold to (40). A benefit of Rs. 10000 is payable at the end of the year of death or at the end of the term of 10 years, whichever is earlier. Suppose there is an initial expense of Rs. 200 and renewal expenses of 0.1 % of the benefit insured, incurred at the beginning of the year, including the first year. There are settlement expenses of Rs. 50. Find the purchase price of the insurance, if the force of interest is 5% and the mortality of (40) is modeled by Gompertz's law given by, $\mu_x = BC^x$, with B = 0.0001151 and C = 1.096.

Solution: Purchase price of the 10-year endowment insurance is the actuarial present value of the benefit and the expenses. The initial expense is Rs. 200; the renewal expenses of 0.1% of the benefit insured, incurred at the beginning of the year form a 10-year temporary life annuity due, with actuarial present value $0.001 \times 10000\ddot{a}_{40:\overline{10|}}$ The settlement expense may be at the end of the first year or the second year or at the end of 10 years, depending on when the death occurs. Thus, the actuarial present value of the settlement expense is given by $50A_{40:\overline{10|}}$. For the specified mor- tality law and the force of interest, we have, $A_{40:\overline{10|}}$ = 0.61611 and from example 6.4.5, $\ddot{a}_{40:\overline{10|}}$ =7.8713. From this information we write the actuarial present value of the benefit and expenses as follows. It is the purchase price of the policy and we denote it by G. Thus,

$$G = 10000A_{40:\overline{10|}} + 200 + 0.001 \times 10000\, \ddot{a}_{40:\overline{10|}} + 50A_{40:\overline{10|}}$$

$$= 6161.10 + 200 + 78.71 + 50 \times 0.6161 = 6470.62.$$

Without expenses, the purchase price is 6161.10. Thus, loading for expenses is 309.52.

Example 12.6.2: An insurance company offers a whole life annuity product that pays the annuitant 12,000 at the beginning of each year and a death benefit of 10000 that will be paid at the end of the year of death. Expenses, payable at the beginning of the year are as follows:

a. Taxes are 0.5% of the expense-loaded premium.
b. Commissions are 2% of the expense-loaded premium.
c. Other expenses are 100 in the first year and 50 in following years till the contract is in force.

Suppose such a contract is signed by (25) who pays the premium at the beginning of each year, including the first, for at most 10 years. Assume that the life length of (25) is modeled by Gompertz's law with force of mortality $\mu_x = BC^x$, $x > 0$ with $B = 0.0001151$ and $C = 1.096$. Find the gross premium, if the force of interest is 0.05.

Solution: In this contract, there is a survival benefit of 12,000 per annum and a death benefit of 10,000. Hence for this insurance contract, the actuarial present value of the benefit is given by $12000\, \ddot{a}_{25} + 10000A_{25} = 12000\, \ddot{a}_{25} + 10000(1 - d\ddot{a}_{25})$.

The expenses payable at the beginning of the year form annuity due. Suppose G denotes the gross premium. Thus, the actuarial present value of expenses is given by,

$$0.005G\, \ddot{a}_{25} + 0.02G\, \ddot{a}_{25} + 100+50\, \ddot{a}_{25} - 50.$$

The first term corresponds to expenses due to taxes, second corresponds to expenses due to commission, third term specifies the initial cost and the last two correspond to renewal expenses, 50 is subtracted as the renewal expenses are from second year onwards. For the given mortality and interest pattern we have calculated $\ddot{a}_{25} = 17.4538$. Thus, the actuarial present value of the benefit and the expenses is given by 210933.3 + 0.436345 G + 922.69. Premium G is paid a as 10- year temporary life annuity due. Hence, the actuarial present value of the premium payments is given by, G $\ddot{a}_{25:\overline{10|}}$. For the given mortality and interest pattern, we have calculated $\ddot{a}_{25:\overline{10|}}$ =8.0173. Hence, the equivalence relation to determine G is obtained by equating the actuarial present value of the premium payments to the actuarial present value of the benefit and the expenses. Thus, we get the equation,

$$\ddot{a}_{25:\overline{10|}}G = 8.0173G = 210933.3 + 0.436345\, G + 922.69.$$

Solving it we get, G =27945.82.

By paying this premium for at most 10 years, (25) gets the annual benefit of 12,000 per annum till he dies and a death benefit of 10,000 at the end of the year of death.

12.7 Fully Continuous Reserves

Prospective Reserve: Suppose (x) signs a fully continuous policy. We want to find the future obligations to the insurance company corresponding to this policy after t units of time from the time of the signing the contract. As we have defined the loss at issue random variable in the determination of premiums, we define a random variable specifying the present value at t of the prospective loss to the insurance company corresponding to this policy. It is denoted by ${}_tL$ and defined as,

${}_tL$ = Present value random variable at t of future benefits and of future expenses - Present value random variable at t of future premiums. (12.7.1)

Premiums in this formulation are assumed to be benefit premiums. The benefit reserve, from now on referred to simply as reserve, at time t of the contract, calculated prospectively, that is with reference to future cash flows, is denoted by ${}_tV$ and is defined as follows. The letter V comes from the word policy value.

${}_tV$ = E(${}_tL$) = Actuarial present value at t of future benefits and expenses - Actuarial present value at t of future premiums. (12.7.2)

Here E denotes the conditional expectation of ${}_tL$ given that T(x) > t .

If the expenses are ignored, reserve is the difference between actuarial present values of future benefits and future premiums. This may or may not agree with the premium basis that is assumptions regarding mortality, interest rates and expenses while determining the premium. If these bases agree and if there are no expenses, the reserves obtained are known as net premium reserves. By convention, the reserve ${}_tV$ is calculated just before receipt of any premium then due. The reserve just after payment of this premium is

$${}_tV + P - e,$$

Where P is the premium paid at t and e is the expense incurred for that transaction. When t =0 , ${}_tV$ =0, as the premium is determined using the equivalence principle. For various types of insurance models, appropriate suffixes are attached to the symbol t V . As an illustration, we discuss how to find reserve for a whole life insurance of unit benefit issued to (x) on a fully continuous basis with an annual benefit premium of $\bar{P}\,(\bar{A}_x)$. Using the definition of ${}_tL$ given above, the prospective loss random variable for whole life insurance is defined as follows.

$${}_tL = v^{T(x)-t} - \bar{P}\,(\bar{A}_x)\,\bar{a}_{\overline{T(x)-t|}} \ .$$

Thus, the first term corresponds to the present value at t of the benefit payments payable at the moment of death while the second term corresponds to the present value at t of the future premium payments at the rate of $\bar{P}\,(\bar{A}_x)$ per annum, the period of such payments being the random variable (T(x) – t). The reserve, denoted by ${}_t\bar{V}(\bar{A}_x)$, is obtained as a conditional expectation of ${}_tL$ given T(x) > t . Such a conditional event is in view of the fact that the valuation of the policy will be done only if it is in force. In international actuarial notation,

$${}_t\bar{V}(\bar{A}_x) = E({}_tL \mid T(x) > t)$$

$$= E(V^{T(x)-t} \mid T(x) > t) - \bar{P}(\bar{A}_x) E(\bar{a}_{\overline{T(x)-t|}} \mid T(x) > t)$$

$$= \bar{A}_{x+t} - \bar{P}(\bar{A}_x)\, \bar{a}_{x+t} \qquad (12.7.3)$$

In the last expression, we use the result that in the aggregate mortality setup the conditional distribution of T(x) – t given T(x) > t is the same as the distribution of T(x+t), the future life time of (x+t). For example,

$$P[T(50) > 15 \mid T(50) > 10] = {}_{15}p_{50} / {}_{10}p_{50} = {}_{10}p_{50}\, {}_{5}p_{60} / {}_{10}p_{50} = {}_{5}p_{60} = P[T(60) > 5].$$

In words, the probability that a 50 – year old survives to age 65 given that he has survived to age 60 is simply the probability that a 60 – year old survives to age 65. Similarly,

$$P[K(30) = 12 \mid K(30) \geq 5] = {}_{12|}q_{30} / {}_{5}p_{30} = {}_{12}p_{30}\, q_{42} / {}_{5}p_{30} = {}_{5}p_{30}\, {}_{7}p_{35}\, q_{42} / {}_{5}p_{30}$$

$$= {}_{7}p_{35}\, q_{42} = {}_{7|}q_{35} = P[K(35) = 7].$$

Such results are not true in the select survival setup. In the entire discussion in this chapter, we assume aggregate mortality pattern. By following the steps similar to those adopted to obtain variance of loss random variable, the variance of ${}_tL$ is obtained as follows. The expression for ${}_tL$ can be rewritten as

$${}_tL = v^{T(x)-t}\left[1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right] - \frac{\bar{P}(\bar{A}_x)}{\delta}\ .$$

Hence,

$$\mathrm{Var}(({}_tL\ T(x) > t) = \left(1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2 \mathrm{Var}(V^{T(x)-t} \mid T(x) > t)$$

$$= \left(1 + \frac{\bar{P}(\bar{A}_x)}{\delta}\right)^2 ({}^2\bar{A}_{x+t} - \bar{A}^2_{x+t}). \qquad (12.7.4)$$

The formula, $t\bar{V}(\bar{A}_x) = \bar{A}_{x+t} - \bar{P}(\bar{A}_x) a_{x+t}$, as derived in (12.7.3), can be expressed in different forms as follows.

$$ {}_t\bar{V}(\bar{A}_x) = \left[\frac{\bar{A}_{x+t}}{\bar{a}_{x+t}} - \bar{P}\,(\bar{A}_x)\right]\bar{a}_{x+t} \quad = [\bar{P}\,(\bar{A}_{x+t}) - \bar{P}\,(\bar{A}_x)]\bar{a}_{x+t} \qquad (12.7.5) $$

This formula for reserve is known as premium difference formula. A second formula is obtained by factoring the actuarial present value of future benefits out of the prospective formula. Thus, we have,

$$ {}_t\bar{V}(\bar{A}_x) = \left[1 - \bar{P}\,(\bar{A}_x)\frac{\bar{a}_{x+t}}{\bar{A}_{x+t}}\right]\bar{A}_{x+t} = \left[1 - \frac{\bar{P}\,(\bar{A}_x)}{\bar{P}\,(\bar{A}_{x+t})}\right]\bar{A}_{x+t}. \qquad (12.7.6) $$

This is called a paid -up insurance formula. The formula for reserve can be expressed in terms of a single actuarial function as follows. Expressing $\bar{A}_x$ and $\bar{P}\,(\bar{A}_x)$ in terms of annuity function we obtain,

$$ {}_t\bar{V}(\bar{A}_x) = 1 - \delta\,\bar{a}_{x+t} - \left(\frac{1}{\bar{a}_x} - \delta\right)\bar{a}_{x+t} = 1 - \frac{\bar{a}_{x+t}}{\bar{a}_x}. \qquad (12.7.7) $$

This is known as the annuity function formula. In (12.7.7), using $\bar{a}_{x+t} = (1-\bar{A}_{x+t})/\delta$,
we obtain

$$ {}_t\bar{V}(\bar{A}_x) = (\bar{A}_{x+t} - \bar{A}_x) / (1 - \bar{A}_x) \qquad (12.2.8) $$

This expression of ${}_t\bar{V}(\bar{A}_x)$ is useful to compute reserve when $\bar{A}_{x+t}$is known forvarious values of t. In (12.2.5), if we substitute $\bar{a}_{x+t} = (\bar{P}(\bar{A}_{x+t}) + \delta)^{-1}$, we get,

$$ {}_t\bar{V}(\bar{A}_x) = [\bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]\ \bar{a}_{x+t} = \bar{P}(\bar{A}_{x+t}) - \bar{P}(\bar{A}_x)]/\ \bar{P}(\bar{A}_{x+t}) + \delta\ . \qquad (12.2.9) $$

Example 12.7.1: For a fully continuous whole life insurance of 1000 issued to (25), with premiums payable as whole life annuity, it is given that δ = 0.05 and mortality follows Gompertz's law with B = 0.0001151 and C = 1.096. Calculate prospective reserve for t=5, 10 and 15.

Solution: According to the international actuarial notation, we have to find $1000\,{}_t\bar{V}(\bar{A}_{25})$for t = 5, 10 and 15. We can use any one of the formulae listed above. But it is simple to use (12.7.8), as we need only the values of $\bar{A}_x$ for various x. From (12.7.8), the formula for prospective reserve is

$$ {}_t\bar{V}(\bar{A}_x) = (\bar{A}_{x+t} - \bar{A}_x) / (1 - \bar{A}_x). $$

Here, we have computed $\bar{A}_x$ for x = 25, 30, 35 and 40 for Gompertz'slaw with given values of parameters. These are as shown in the following table. Using it we find ${}_t\bar{V}(\bar{A}_{25})$ for t = 5, 10 and 15. Results are shown in Table 12.7.1.

Table 12.7.1 Gompertz's law: Prospective Reserve

Age	$1000\bar{A}_x$	t	$1000\,{}_t\bar{V}(\bar{A}_{25})$
25	152.54	0	0.00
30	189.12	5	43.16
35	232.70	10	94.59
40	283.72	15	154.79

Values of $1000{}_t\bar{V}(\bar{A}_{25})$ indicate the expected financial liability of the insurance company at different time points in the period when the policy is in force. Thus, at t = 5, the expected fund per surviving policy that the insurer must have is 43.16to pay the claim of 1000 units if it is made at that time. It is to be noted that at t = 0, the reserve is 0, as the premiums are determined using equivalence principle and the prospective loss random variable ${}_tL$ at t = 0 is the loss at issue random variable L. Further, as t increases, the reserve increases, because the chance of claim increases in whole life insurance as t increases.

The prospective loss random variables and the corresponding reserves for an n-year term insurance, n-year pure endowment and n-year endowment insurance are defined on similar lines as those for whole life insurance. To write these expressions, recall the following expressions for the present value random variable Z of the unit benefit. For an n-year term insurance, Z is given by,

$$Z=\begin{cases} v^t, & if\ T \le n \\ 0\ , & if\ T > n\ . \end{cases}$$

For an n-year pure endowment insurance, Z is given by,

$$Z_T = \begin{cases} 0, & if\ T \le n \\ v^n, & if\ T > n\ . \end{cases}$$

For an n-year endowment insurance, Z is given by,

$$Z_T = \begin{cases} v^T, & if\ T \le n \\ v^n, & if\ T > n\,. \end{cases}$$

For all these insurance products, suppose premiums are payable as n-year temporary continuous life annuity. Then, the present value random variable Y corresponding to this annuity is given by,

$$Y = \begin{cases} \bar{a}_{\overline{T}|}, & if\ 0 \le T < n \\ \bar{a}_{\overline{n}|}, & if\ T \ge n\,. \end{cases}$$

As discussed at the beginning of this section, the prospective loss random variable is defined as

$_t L$ = Present value random variable at t of future benefits - Present value random variable at t of future premiums

Using this basic principle, we write expressions for prospective loss random variable $_t L$ for these insurance products.

For an n-year term insurance, $_t L$ for t < n is given by,

$$_t L = \begin{cases} v^{T(x)-t} - \bar{P}\left(\bar{A}^1_{x:\overline{n}|}\right) \bar{a}_{\overline{T(x)-t}|}, & \text{if } t < T(x) < n, \\ 0 - \bar{P}\left(\bar{A}^1_{x:\overline{n}|}\right) \bar{a}_{\overline{n-t}|}, & if\ T(x) > n. \end{cases}$$

At t = n, $_t L$ = 0, as by convention, ā_0 = 0. Thus, the prospective loss random variable for an n-year term insurance is degenerate at 0 if t = n. This is logically appealing as in an n-year term insurance; the insurer has no liability of paying the claims when the term of n years is over. For an n-year pure endowment insurance, $_t L$ for t ≤ n is given by,

$$_t L = \begin{cases} 0 - \bar{P}\left(A_{x:\overline{n}|}^{\ \ 1}\right) \bar{a}_{\overline{T(x)-t}|}, & \text{if } t < T(x) < n, \\ v^{n-t} - \bar{P}\left(\bar{A}_{x:\overline{n}|}^{\ \ 1}\right) \bar{a}_{\overline{n-t}|}, & if\ T(x) > n. \end{cases}$$

Similarly, for an n-year endowment insurance, $_t L$ for t ≤ n is given by,

$$_t L = \begin{cases} v^{T(x)-t} - \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) \bar{a}_{\overline{T(x)-t}|}, & \text{if } t < T(x) < n, \\ v^{n-t} - \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) \bar{a}_{\overline{n-t}|}, & if\ T(x) > n. \end{cases}$$

Note that the prospective loss random variable for an n-year pure endowment insurance and for an n-year endowment insurance is degenerate at 1 if t = n. This is also logically appealing, as in these insurance products, the insurer has a liability of paying 1 unit at the end of the term, corresponding to each surviving policy.

Further, from the expressions of ${}_tL$ for the n-year term insurance, the n-year pure endowment insurance and the n-year endowment insurance, we immediately note that for t = n, ${}_tL$ for an n-year endowment insurance is the addition of ${}_tL$ for an n-year term insurance and ${}_tL$ for an n-year pure endowment insurance. This follows from the fact that the premium for an n-year endowment insurance is the addition of premiums for an n-year term insurance and for an n-year pure endowment insurance. For t < n, we write ${}_tL$ for an n-year endowment insurance as follows.

$$
\begin{aligned}
{}_tL(\text{Endowment}) &= v^{T(x)-t} - \bar{P}(\bar{A}_{x:\overline{n}|})\bar{a}_{\overline{T(x)-t}|} \\
&= v^{T(x)-t} - \left(\bar{P}\left(\bar{A}^{1}_{x:\overline{n}|}\right) + \bar{P}\left(A_{x:\overline{n}|}^{\;\;1}\right)\right)\bar{a}_{\overline{T(x)-t}|} \\
&= v^{T(x)-t} - \bar{P}\left(\bar{A}^{1}_{x:\overline{n}|}\right)\bar{a}_{\overline{T(x)-t}|} - \bar{P}\left(A_{x:\overline{n}|}^{\;\;1}\right)\bar{a}_{\overline{T(x)-t}|} \\
&= {}_tL(\text{Term}) + {}_tL(\text{Pure endowment}). \qquad (12.7.10)
\end{aligned}
$$

Complexity in the definition of ${}_tL$ increases, when the number of premiums paying years is restricted to h years in whole life insurance or in n-year term or endowment insurance, with h less than n. For a whole life insurance, when the premiums are payable as an h-year temporary continuous life annuity, ${}_tL$ is defined as follows. For t < h,

$$
{}_tL = \begin{cases} v^{T(x)-t} - h\bar{P}(\bar{A}_x)\bar{a}_{\overline{T(x)-t}|}, \text{if } t < T(x) < h, \\ v^{T(x)-t} - h\,\bar{P}(\bar{A}_x)\bar{a}_{\overline{h-t}|}, if\ t \le h < T(x). \end{cases}
$$

Suppose t = h. It is implicitly assumed that T(x) > h. In this case, ${}_tL$ is given by,

$$
{}_tL = v^{T(x)-t} - h\,\bar{P}(\bar{A}_x)\bar{a}_{\overline{h-h}|} = v^{T(x)-t}.
$$

Note that it can be combined with the second equation in the previous case of t< h. For t > h, the present value random variable of annuity payments is 0 as the premium paying term is over. Hence, ${}_tL$ is given by,

$$
{}_tL = v^{T(x)-t}
$$

On similar lines, for an n-year endowment insurance, when premiums are payable as an h-year temporary continuous life annuity, ${}_tL$ for the case t ≤ h is given by,

$$
{}_{t}L = \begin{cases} \nu^{T(x)-t} - h\bar{P}(\bar{A}_x)\bar{a}_{\overline{T(x)-t|}}, \text{ if } t < T(x) < h, \\ \nu^{T(x)-t} - h\,\bar{P}(\bar{A}_x)\bar{a}_{\overline{h-t|}}, if\ t \le h < T(x) < n \\ \nu^{n-t} - h\,\bar{P}(\bar{A}_x)\bar{a}_{\overline{h-t|}}, if\ t \le h \le n < T(x). \end{cases}
$$

For, h< t $\le n$, it is given by,

$$
{}_{t}L = \begin{cases} \nu^{T(x)-t}, \text{ if } t < T(x) < n, \\ \nu^{n-t}, \qquad if\ T(x) > n. \end{cases}
$$

For an n-year term insurance, when premiums are payable as an h-year temporary continuous life annuity, ${}_{t}L$ is similar to that for an n-year endowment with the change that υ^(n-t) is replaced by 0, as there is no benefit payment if death occurs after n. For an n-year pure endowment insurance, when premiums are payable as an h-year temporary continuous life annuity, ${}_{t}L$ can be obtained from that of an n-year endowment insurance by replacing $\upsilon^{T(x)-t}$ by 0, as in an n-year pure endowment insurance there is no benefit payment if death occurs before n. In this case also ${}_{t}L$ for endowment is the addition of ${}_{t}L$ for term insurance and pure endowment. This observation helps to write down the expectation of ${}_{t}L$.

The conditional expectation of ${}_{t}L$, conditional on the event that T(x) > t, gives the formula for prospective reserve. We again use the fact that in an aggregate mortality pattern, the conditional distribution of T(x) –t is the same as the distribution of T(x + t). Thus, the expectation of ${}_{t}L$ with respect to the distribution of T(x + t), gives the formulae for prospective reserve.

12.8 Fully Discrete Reserves

Here we will proceed to derive the formulae for reserve for various insurance products. The entire development is similar to that in Section 8.2, with modifications to suit a discrete setup. The main change is that the underlying random variable is K(x) instead of T(x), thus the expectations are with respect to the distribution of K(x). We first study the prospective approach. Suppose (x) signs a fully discrete policy. We want to find the future obligations to the insurance company corresponding to this policy after k units of time from the time of signing the contract. As we have defined ${}_{t}L$, we define a random variable ${}_{k}L$ specifying the present value at k of the prospective loss to the insurance company corresponding to this policy. Prospective loss random variable ${}_{k}L$ for a whole life insurance is defined as,

$$
{}_{k}L = v^{(K(x)-k)+1} - P_x\ddot{a}_{\overline{(K(x)-k)+1|}}.
$$

The benefit reserve, denoted by ${}_kV_x$, is the conditional expectation of the prospective loss, ${}_kL$, at duration k, given that K(x) = k, k + 1, ... More precisely,

$${}_kV_x = E({}_kL|K(x) = k, k + 1, ...). \qquad (12.8.1)$$

As in the previous section, we use the result that in an aggregate mortality setup, the conditional distribution of K(x) – k, given that K(x) > k, is the same as the distribution of K(x+k). Thus, for the whole life insurance policy with unit benefit, payable at the end of the year of death, premiums payable as a whole life annuity due, the prospective formula for the benefit reserve is,

$${}_kV_x = {}_kV_{x+k} - P_x \ddot{a}_{\overline{(x+k)}} \qquad (12.8.2)$$

this formula is the actuarial present value of future benefits less the actuarial present value of future benefit premiums. Here,

$$Var({}_kL\ K(x) = k, k + 1, ...) = Var\left(V^{(K(x)-k)+1}\left(1+\frac{P_x}{d}\right)\middle|\ K(x) = k, k + 1, ..\right)$$

$$= \left(1 + \frac{P_x}{d}\right)^2 Var\left(V^{(K(x)-k)+1}\ |K(x) = k, k + 1, ..\right)$$

$$= \left(1 + \frac{P_x}{d}\right)^2 [{}^2A_{x+k} - (A_{x+k})^2] \qquad (12.8.3)$$

From the formula for reserve given in (12.8.1), we obtain the premium difference formula and paid-up insurance formula. These are as follows.

Premium difference formula: ${}_kV_x = (P_{x+k} - P_x)\ \ddot{a}_{x+k}$. (12.8.4)

Paid up insurance formula: ${}_kV_x = \left(1 + \frac{P_x}{P_{x+k}}\right) A_{x+k}$. (12.8.5)

As in the fully continuous case, formula for reserve given in (12.8.2), can be expressed either in terms of annuity symbol or the premium symbol or the symbol of actuarial present value of benefit. These are given below and are obtained by using the relations $A_x = 1 - d\ddot{a}_x$ and $1/\ddot{a}_x = P_x + d$.

$${}_kV_x = 1 - d\ddot{a}_{x+k} - ((1/\ddot{a}_x)-d)\ \ddot{a}_{x+k} = 1 - \ddot{a}_{x+k}/\ddot{a}_x. \qquad (12.8.6)$$

$$_kV_x = (1 + \frac{P_x + d}{P_{x+k} + d}) = \frac{P_{x+k} - P_x}{P_{x+k} + d} \qquad (12.8.7)$$

and

$$_kV_x = 1 - [(1 - A_{x+k}) / (1 - A_x)] = (A_{x+k} - A_x) / (1 - A_x) \qquad (12.8.8)$$

These expressions are useful in the computation of reserves depending on the available data. For a fully discrete policy, the inflow, and the outflow of money to the insurance company is at discrete time points, hence the policy is also valued at discrete time points, thus we find the expression of reserve at integer values of k.

The conditional expectation of $_kL$, conditional on the event that K(x) +1 > k, gives the formula for prospective reserve. Again, in aggregate mortality pattern, the conditional distribution of K(x) -k is the same as the distribution of K(x+k). Thus, the expectation of $_kL$ with respect to the distribution of K(x+k), gives the formulae for prospective reserve.

As in fully continuous policies, the prospective reserve for an n-year endowment insurance is the addition of reserves for an n-year term insurance and for an n-year pure endowment insurance. The values of the reserves at the end of the term, that is at k = n, are 0, 1 and 1 for an n-year term insurance, an n-year pure endowment insurance and an n-year endowment insurance respectively.

Example 12.8.1: Suppose the force of mortality follows Makeham's law given by $\mu_x = A + BC^x$ with A = 0.0007, B = 0.0001151 and C = 1.096. Suppose i = 0.05 and for the same law and i, A_{40} = 0.27975. Calculate the prospective reserve for a whole life insurance issued to (30), with a benefit of 1000 units, for k = 1 to 10, when premiums are payable as a whole life annuity due.

Solution: Values of net single premium A for x = 30 to 40 are derived in Example5.4.2 for Makeham's law with the given set of parameters and i = 0.05. Hence, we use the formula for prospective reserve as given in (12.8.8). Table 12.8.3 displays values of A_x for x = 30 to 40 and reserve for k = 0 to 10. From the table we note that reserve at k = 0 is 0 as we calculate the premium by equivalence principle. It is clear from (12.8.8) also. Further, the reserve increases as k increases.

Table 12.8.3: Makeham's law: 1000A_x and 1000 $_kV_{30}$

Age x	1000 A_x	k	1000 $_kV_{30}$
30	184.29	0	0.00

31	192.55	1	10.13
32	201.09	2	20.60
33	209.91	3	31.41
34	219.01	4	42.56
35	228.40	5	54.08
36	238.09	6	65.95
37	248.06	7	78.18
38	258.33	8	8
39	268.89	9	103.71
40	279.75	10	117.03

Retrospective Reserve: Computation of the retrospective reserve for a 5-year endowment insurance is nothing but the derivation of the formula for retrospective reserve is illustrated for a 3-year endowment insurance. We follow the same procedure to derive the formulae for a retrospective reserve for various discrete policies. We begin with whole life insurance with a benefit of 1 unit to be paid at the end of the year of death and premiums payable as whole life annuity due. The formula is derived by accumulating the funds of l_x identical policies until time k and sharing the money among the survivors. Interest is assumed to be earned at the rate of i per annum. The accumulated funds at time k are as given below.

$l_xP_x(1+i)^k + l_{x+1}P_x(1+i)^{(k-1)} + ... + l_{x+k-1}P_x(1+i)- d_x(1+i)^{(k-1)} + d_{x+1}(1+i)^{(k-2)} + ... + d_{x+k-1}$

$= \quad l_xP_x(1+i)^k \; [\, 1 + vp_x + v^2\; P_xP_{x+1} + ... + v^{(k-1)}{}_{(k-1)}P_xP_{x+1}\,]\text{- } l_x\;(1+i)^k\;[\; vq_x + v^2p_x\; q_{x+1} + ... + v^k{}_{(k-1)}p_x\; q_{x+1}\;]$

$= \quad l_xP_x(1+i)^k\sum_{j=0}^{k-1} v^j{}_jP_x - l_xp_x(1+i)^k\sum_{j=0}^{k-1} v^{j+1}{}_jP_x\; q_{x+j}$

$= \quad l_x\;(1+i)^k\;[P_x\ddot{a}_{x:\bar{k}|}\text{-}A^1_{x:\bar{k}|}]$

Dividing this expression by l_{x+k}, we get the retrospective reserve as

$${}_kV_x = \frac{l_x(1+i)^k}{l_{x+k}}\;[p_x\ddot{a}_{x:\bar{k}|}\text{-}A^1_{x:\bar{k}|}] = (1/{}_kE_x)\;[P_x\ddot{a}_{x:\bar{k}|}\text{-}A^1_{x:\bar{k}|}]= P_x\ddot{S}_{x:\bar{k}|} - {}_kK_x, \qquad (12.8.9)$$

Note that,

$$\ddot{S}_{x:\bar{k}|} = \frac{1}{A^{1}_{x:\bar{k}|}/\ddot{a}_{x:\bar{k}|}} = \frac{1}{A_{x:\bar{k}|}^{\;\;1}}$$

Further from (12.3.9) we have,

$$ {}_{k}V_{x} = (1/{}_{k}E_{x})\, [P_x \ddot{a}_{x:\bar{k}|} - A^{1}_{x:\bar{k}|}] = \frac{P_x - P^{1}_{x:\bar{n}|}}{P_{x:\bar{n}|}^{\;\;1}} \qquad (12.8.10)$$

The formula in (12.8.10) expresses ${}_{k}V$ in terms of premium symbols.

The retrospective reserves for other insurance products are obtained on similar lines as those for whole life insurance by considering the accumulated value at k of the premiums received and the accumulated value at k of the benefits paid. The expression for premium changes from product to product but the rest of the arguments remain the same. For an h-payment whole life insurance and for h-payment n-year endowment insurance, the derivation is similar to that for fully continuous policies. The accumulation of the fund is for h years only, with the payment of premiums at the beginning of the year.

Example 12.8.2: Suppose the force of mortality follows Makeham's law given by $\mu_x = A + BC^x$ with A = 0.0007, B = 0.0001151 and C = 1.096. Suppose i = 0.05 and for the same law and i, A_{40} = 0.27975. Calculate the retrospective reserve for a whole life insurance issued to (30) with a benefit of 1000, for k = 1 to 10, when premiums are payable as a whole life annuity due.

Solution: To find the reserve using the retrospective approach, we have

$$ {}_{k}V_{x} = (1/{}_{k}E_{x})\, [P_x \ddot{a}_{x:\bar{k}|} - A^{1}_{x:\bar{k}|}]$$

We have the values of A_x for x = 30 to 40. Note that

$$P_{30} = d\, A_{30}/(1 - A_{30}) = 0.01075838$$

with d = 0.04761905 corresponding to i = 0.05. $A^{1}_{30:\bar{k}|}$ and $A_{30:\bar{k}|}$can be expressed as

$$A^{1}_{30:\bar{k}|} = A_{30} - {}_{k}E_{30}\, A_{30+k} \text{ and } A_{30:\bar{k}|} = A_{30} - {}_{k}E_{30}\, A_{x+30} + {}_{k}E_{30}.$$

From $A_{30:\bar{k}|}$, we find $\ddot{a}_{30:\bar{k}|}$ = (1 - $A_{30:\bar{k}|}$)/d. Thus, it is enough to compute ${}_{k}E_{30} = \upsilon^{k}\,{}_{k}P_{30} = e^{-\delta k}\,{}_{k}p_{30}$ for Makeham's law for x = 30 and k = 1 to 10. Using the survival function of Makeham's law, we get ${}_{k}E_{30}$ for these k values. Using these values and values of A_x for a = 30 to 40, we find ${}_{k}V_{30}$. Table 12.8.5 reports these values.

Table 12.8.5: Makeham's law: ${}_kE_{30}$ and 1000 ${}_kV_{30}$

k	${}_kE_{30}$	1000 ${}_kV_{30}$
1	0.9513	10.13
2	0.9058	20.60
3	0.8625	31.41
4	0.8212	42.56
5	0.7819	54.08
6	0.7445	65.95
7	0.7089	78.18
8	0.6749	90.77
9	0.6425	103.7
10	0.6117	117.03

Comparing these reserve values with those in Table 12.8.3, we note that the values of reserve using both the approaches are the same.

Example 12.8.3: A fully discrete whole life insurance of 1 is issued to (25). Premiums are paid annually till age 65. The net premium during the first 10 years is P_{25}, followed by a different level annual premium payable during the next 30 years. It is given that $A_{35} = 0.3$, $P_{25} = 0.01$ and $d = 0.06$. Calculate the reserve at the end of the tenth year.

Solution: Using the prospective approach to compute reserve, we have,

$$ {}_{10}V = A_{35} - P_{25}\ddot{a}_{35} = 0.3 - (0.01) *[(1 - 0.3) / 0.06] = 0.183. $$

In the calculation of reserve for t = 10, the different level premium does not matter. Note that the reserve is simply denoted by ${}_{10}V$ as the underlying insurance product is not among the standard ones. The prefix t = 10 indicates the valuation of the policy at t = 10.

Example 12.8.4: ${}_kL$ is the prospective loss random variable for a fully discrete life insurance of 1000 issued to (x). It is given that $A_x = 0.125$, $A_{x+k} = 0.4$, $2A_{x+k} = 0.2$. $d = 0.05$. Assume that premiums are calculated on the basis of the equivalence principle.

a. Calculate $E({}_kL)$, $Var({}_kL)$ and the aggregate reserve at time k for 100 policies of this type.

b. An insurer has 100 policies of this type at time k. Assume the losses on the100 policies are independent. Use the approximation that the aggregate loss is normally distributed in (i) and (ii) below.

i. Calculate the aggregate amount the insurer must have at time k so that the probability of meeting its obligations for these policies is 0.95.

ii. The premium can be changed at time k. Calculate the level annual premium the insurer must charge beginning at time k on each policy so that together with the reserve calculated in (a), the probability of meeting its obligations for these policies is 0.95.

Solution:

a. From the given information

$$\ddot{a}_x = (1 - A_x) / d = 0.875 / 0.05 = 17.5, \text{ hence } 1000P_x = (1000A_x) / \ddot{a}_x = 7.142860.$$

Further,

$$\ddot{a}_{x+k} = (1 - A_{x+k}) / d = 0.6/0.05 = 12.$$

By definition,

$$E({}_kL) = 1000A_{x+k} - 1000P\ \ddot{a}_{x+k} = 314.2857,$$

$$Var({}_kL) = 1000^2\ (1+(P_x/d))^2\ [\ {}^2A_{x+k} - A_{x+k}^2] = 52,\ 244.898$$

and reserve for 100 policies is$100E({}_kL) = 31428.57$.

b. i. Suppose, $S = \sum_{j=1}^{100} 1{}_kL^{(j)}$. Then,

$$\mu = E(S) = 100E({}_kL) = 31428.57,\ \sigma^2 = Var(S) = 100Var({}_kL) = 5224,\ 489.8.$$

By normal approximation, $P[S > c] = 0.05$ gives, $(c-\mu)/\sigma = 1.645$ and hence,

$c = 35,\ 188.57$ is the amount that the insurer must have at time k so that the probability of meeting its obligations for these policies is 0.95.

ii. Suppose P is the premium to be determined for the insurance of 1000units. Then,

$$E({}_k\hat{L}) = 1000A_{x+k} - \hat{P}\ \ \ddot{a}_{x+k} = 400 - 12\hat{P}.$$

$$\text{Var}({}_k\hat{L})) = (100+(\hat{P}/\text{d}))^2\ ({}^2\text{A}_{x+k}–A_{x+k}^2) = (200 + 4\hat{P})^2.$$

Suppose $\hat{S} = \sum_{j=1}^{100} 1({}_k\hat{L}^{(j)})$, then

$$\mu = \text{E}(\hat{\text{S}}) = 100\text{E}(\ {}_k\hat{L}) = 40{,}000 – 1200\hat{P} \text{ and } \hat{\sigma}^2 = \text{Var(s)} = 100(200 + 4\hat{P})^2.$$

Again, by normal approximation,

$$\hat{c}=\hat{\mu}+1.645\hat{\sigma} \quad \text{but by (a) } \hat{\text{S}} = 100\text{E}({}_k\text{L}) = 31428.57.$$

Solving $\hat{c}=\hat{\mu}+1.645\hat{\sigma}$, we get $\hat{P}$ = 10.46.

12.5 Summary

In actuarial statistics, the Loss at Issue random variable represents the insurer's profit or loss at policy inception, defined as the present value of future benefits minus premiums. Fully continuous premiums assume that both premiums and benefits are paid continuously over time, providing a smooth theoretical model. In contrast, fully discrete premiums involve premiums and benefits being paid at discrete intervals (e.g., annually), aligning more closely with real-world insurance practices. Gross premiums are the total premiums charged to the policyholder, including loadings for expenses and profit, unlike net premiums which only cover expected benefits. Fully continuous reserves calculate the insurer's liability assuming continuous payment of benefits and premiums, while fully discrete reserves determine this liability based on discrete cash flows. These reserves represent the expected value of future liabilities minus future premiums and are essential for ensuring financial stability of insurance contracts over time.

12.7 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

12.8 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.

UNIT:13 SOME PRATICAL CONSIDERATIONS

Structure

13.1 Introduction

In the field of actuarial science and risk management, practical problems often arise that involve future lifetime random variables, life tables, and financial instruments such as life insurance and life annuities. These tools are essential in modeling and managing the uncertainties associated with human life spans. Future lifetime random variables provide the probabilistic foundation for predicting the duration of an individual's remaining life, while life tables offer empirical data to support these predictions across populations. These concepts are crucial in pricing life insurance policies and annuities, where accurate estimation of premiums and reserves ensures financial sustainability and fairness for both insurers and policyholders. Premiums must reflect the risk of the insured event, while reserves are maintained to meet future obligations. Together, these elements form the backbone of actuarial calculations, guiding decisions in product design, pricing, and risk assessment in life-contingent financial contracts.

13.2 Objectives

After reading this unit the learner should able to understand:

1. Interest rate assumptions and inflation effects.
2. Modeling payments for non-integer ages or multiple lives.
3. Regulatory constraints on premium structures.
4 Using appropriate reserve methods (e.g., gross premium, net premium).
5. Handling reserves for dynamic products like unit-linked or with-profits.

13.3 Practical based on Future Lifetime Random Variables, Life Tables, Life Insurance, Life Annuity, Premiums and Reserves

Practical 13.3.1: Suppose life length random variable X has a distribution with survival function S(x) = 1 - $x^2/100$, $0 \le x \le 10$. Find (i) $F_x(x)$, (ii) ${}_5p_4$, (iii) probability density function of T(4) and (iv) median future life time of (5).

Solution:

i. $$F(x) = 1 - S(x) = \begin{cases} 1, & if\ x < 0 \\ \frac{x^2}{100}, & if\ 0 \le x < 10 \\ 0, & if\ x \ge 10 \end{cases}$$

ii. ${}_tp_4 = S(4+t)/S(4) = 1-(4+t)^2/100 / 0.84 = 1- 8t+t^2/84.$

Hence, ${}_5p_4 = 1 - 65/84 = 0.2262$

iii. $f_{T(4)}(t) = \mu_{4+t t}p_4 = -$ d/dt {log S(4+t)} S(4+t)/S(4) = - d/dt {-log S(4+t)} S(4+t)/S(4)

= 2(4+t)/100 * 1/0.84 = 4+t/42, 0 < t < 6.

Note that $\int_0^6 4 + t/42$ dt = 1. Upper limit is 6 in view of the fact that upper limit of X is 10 and $f_{T(4)}(t)$ is the probability density function of T(4), future life time of (4). The survival distribution in this example is a suitable model for the life span of some birds and animals.

iv. The median future life time of (x) is the solution of the equation

$${}_tq_x = {}_tp_x = 0.50 \Leftrightarrow S(x + t) = (0.5)S(x).$$

With given S(x) and x = 5, we solve the equation,

$$1 - (5+t)^2/100 = 1/2(1-5^2/100).$$

The solution is t = 2.9056. Thus, the median future life time of (5) is 2.9056 years.

Practical 13.3.2: Suppose the life length random variable X is modeled by an exponential distribution with parameter μ. Find the probability density function g(t) of T(x).

Solution: If X is modeled by an exponential distribution with parameter μ, then its probability density function is $f(x) = \mu e^{-\mu x}$ for $x > 0$, its survival function is $S(x) = e^{-\mu x}$ for $x > 0$ and its force of mortality is $\mu_x = \mu$ for $x > 0$. Consequently, the probability density function g(t) of T(x) is given by,

$$g(t) = {}_tP_x \, \mu_{x+t} = \mu e^{-\mu t} \text{ for } t > 0.$$

Thus, T(x) again has exponential distribution with parameter μ.

Practical 13.3.3: Suppose the life length random variable is modeled by a distribution with force of mortality as specified below.

$$\mu_s = \begin{cases} 0.04, & if 0 \leq s < 15 \\ 0.08, & if\ 15 \leq s < 25 \\ 0.12, & if\ 25 \leq s > 35 \\ 0.18 & if\ \ s \geq 35\ . \end{cases}$$

Find the probability density function g(t) of T(30). Check whether it is a density function.

Solution: Proceeding exactly on similar lines as in the previous example, we get the survival function as,

$$S(x) = \begin{cases} e^{-0.04x}, & if\ \ 0 \leq x < 15 \\ e^{-0.08x+0.6}, & if\ 15 \leq x < 25 \\ e^{-0.12x+1.6}, & if\ 25\ \leq x < 35 \\ e^{-0.18x+3.7} & if\ \ \ x \geq 35\ \ . \end{cases}$$

To find the probability density function g(t) of T(30), we use the formula,

$$g(t) = \mu_{30+t}\,{}_tp_{30} = \mu_{30+t}\, S(30 + t)/S(30).$$

The form of S(30 + t) changes for $0 < t < 5$ as (30 + t) varies between 30 to 35 and for $t \geq 5$, $(30+t) > 35$. Thus, with appropriate choice of S(30 + t), we get

$$g(t) = \begin{cases} 0.12e^{-0.12t}, & if\ 0 < t < 5 \\ 0.18e^{-0.18t+0.3}, & if\ t \geq 5. \end{cases}$$

It is easy to check that g(t) is always non-negative and $\int_0^\infty g(t)\,dt = 1$.

Further, $P[T(30) \leq 55]$ is 0.9999189 and $P[T(30) \leq 71]$ is 0.9999954. Thus, the future life span of (30) is at most 71. So, it seems to be a feasible model for human life length.

Practical 13.3.4: Suppose life length random variable X has a distribution with survival function $S(x) = 1 - x^2/100$, $0 \leq x \leq 10$. Find ${}_{2|2}q_4$.

Solution:

2|2q4 = 2P4 296

= (S(6)/S(4)) (1 − S(8)/S(6))

= (S(6) − S(8))/S(4)

= (0.64 − 0.36}/0.84

= 1/3.

Practical 13.3.5: Find the distribution of K(30) when the life length random variable is modelled by a force of mortality as specified below.

$$\mu_s = \begin{cases} 0.04, & if 0 \leq s < 15 \\ 0.08, & if\ 15 \leq s < 25 \\ 0.12, & if\ 25 \leq s > 35 \\ 0.18 & if\ \ s \geq 35\ . \end{cases}$$

Solution: We have obtained the survival function corresponding to this mortality law. It is given by,

$$S(x) = \begin{cases} e^{-0.04x}, & if\ 0 \leq x < 15 \\ e^{-0.08x+0.6}, & if\ 15 \leq x < 25 \\ e^{-0.12x+1.6}, & if\ 25\ \leq x < 35 \\ e^{-0.18x+3.7} & if\ \ \ x \geq 35\ . \end{cases}$$

The probability mass function of K(30) involves S(30 + k) and S(30 + k + 1). Its form changes depending on the value of k. Thus, for k ≤ 4 we have,

$P[K(30) = k] = {}_kP_{30}\ q_{30+k} = S(30 + k) - S(30 + k + 1)/S(30)$

$= \exp[-0.12(30 + k) + 1.6] - \exp[-0.12(30 + k + 1) + 1.6]/\exp[-0.12(30) + 1.6]$

$= \exp[-0.12k] - \exp[-0.12(k + 1)].$

For k ≥ 5 we have,

P[K(30) = k] = kP30 q30+k = S(30 + k) − S(30 + k + 1)/S(30)

$= \exp[-0.18(30 + k) + 3.7] - \exp[-0.18(30 + k + 1) + 3.7]/\exp[-0.12(30) + 1.6]$

$= \exp[-0.18k + 0.3] - \exp[-0.18k + 0.12]$.

Table 13.3.1 displays the probability distribution of K(30). Probabilities beyond 60 are 0 up to five decimal places and hence are taken as 0. The sum of the probabilities from 0 to 59 is 0.999977 which is almost 1. Thus, for the specified mortality pattern, the curtate future life of (30) has support 0 to 60 (approximately).

Practical 13.3.6: Find the distribution of K(x) when the life length random variable is modelled by Gompertz's law and Makeham's law.

Solution: The probability mass function of K(x) for Gompertz's law is given by,

$$P[K(x) = k] = kPx\ qx{+}k = S(x + k) - S(x + k + 1)/S(x)$$

$$= \exp[-m(C^{x+k} - 1)] - \exp[-m(C^{x+k+1} - 1)]/\exp[-m(C^{x} - 1)]$$

$$= \exp[-mC^{x}(C^{k} - 1)] - \exp[-mC^{x}(C^{k+1} - 1)].$$

Table 13.3.1 displays the probability distribution of K(30) under Gompertz's law, with B = 0.0001151 and C = 1.096.

Table 13.2.1 *Distribution of K(30)*

k	*P[K(30)=kJ*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*
0	0.11308	15	0.01494	30	0.00100	45	0.00007
1	0.10029	16	0.01248	31	0.00084	46	0.00006
2	0.08895	17	0.01043	32	0.00070	47	0.00005
3	0.07889	18	0.00871	33	0.00059	48	0.00004
4	0.06997	19	0.00727	34	0.00049	49	0.00003
5	0.09041	20	0.00608	35	0.00041	50	0.00003
6	0.07551	21	0.00507	36	0.00034	51	0.00002
7	0.06307	22	0.00424	37	0.00028	52	0.00002
8	0.05268	23	0.00354	38	0.00024	53	

9	0.04401	24	0.00296	39	0.00020	54	0.00002
10	0.03676	25	0.00247	40	0.00017	55	0.00001
11	0.03070	26	0.00206	41	0.00014	56	0.00001
12	0.02564	27	0.00172	42	0.00012	57	0.00001
13	0.02142	28	0.00144	43	0.00010	58	0.00001
14	0.01789	29	0.00120	44	0.00008	59	0.00001
							0.00001

Table 13.3.1 *Gompertz's law: Distribution of K (30)*

k	*P[K(30)=k]*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*
0	0.00188	18	0.00900	36	0.02984	54	0.01489
1	0.00206	19	0.00976	37	0.03100	55	0.01235
2	0.00225	20	0.01057	38	0.03204	56	0.00998
3	0.00246	21	0.01145	39	0.03293	57	0.00783
4	0.00269	22	0.01238	40	0.03363	58	0.00595
5	0.00294	23	0.01337	41	0.03412	59	0.00437
6	0.00322	24	0.01441	42	0.03436	60	0.00308
7	0.00351	25	0.01552	43	0.03432	61	0.00209
8	0.00384	26	0.01668	44	0.03398	62	0.00135
9	0.00419	27	0.01789	45	0.03331	63	0.00083
10	0.00457	**28**	0.01915	46	0.03231	64	0.00048
11	0.00498	**29**	0.02046	47	0.03098	65	0.00026
12	0.00543	30	0.02180	48	0.02932	66	0.00014
13	0.00592	31	0.02317	49	0.02736	67	0.00006
14	0.00644	32	0.02454	50	0.02514	68	0.00003
15	0.00701	33	0.02592	51	0.02272	69	0.00001
16	0.00762	34	0.02728	52	0.02015		

17	0.00828	·35	0.02859	53	0.01752		

Probabilities beyond 69 are 0 up to five decimal places and hence are taken as 0. Sum of the probabilities from 0 to 69 is 0.9999939 which is almost 1. Thus, for the Gompertz's mortality pattern the curtate future life of (30) has support 0 to 69 (approximately).

Using the same approach as for Gompertz's law, the probability mass function of K(x) for Makeham's law is given by,

$$P[K(x) = k] = kPx\ qx{+}k = S(x + k) - S(x + k + 1)/S(x)$$

$$= \exp[-A(x + k) - m(C^{x+k} - 1)] - \exp[-A(x + k + 1) - m(C^{x+k+1} - 1)]/\exp[-Ax - m(C^{x} - 1)]$$

$$= \exp[-Ak - mC^{x}(C^{k} - 1)] - \exp[-A(k + 1) - mC^{x}(C^{k+1} - 1)].$$

Table 13.3.1 displays the probability distribution of K(30) under Makeham's law when A = 0.0007, B = 0.0001151 and C = 1.096. Probabilities beyond 69 are 0 up to five decimal places and hence are taken as 0. The sum of the probabilities from 0 to 69 is 0.9999936 which is almost 1.

Table 13.3.1 *Makeham's law,d iistribution.of K(30)*

k	*P[K(30)=k]*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*	*k*	*P[K(30)=k]*
0	0.00118	18	0.00846	36	0.03019	54	0.01543
1	0.00136	19	0.02887	37	0.03143	55	0.01281
2	0.00156	20	0.00925	38	0.03254	56	0.01036
3	0.00177	21	0.01009	39	0.03350	57	0.00814
4	0.00201	22	0.01099	40	0.03427	58	0.00619
5	0.00.226	23	0.01195	41	0.03482	59	0.00454
6	0.00254	24	0.01298	42	0.03512	60	0.00321
7	0.00284	25	0.01406	43	0.03513	61	0.00218
8	0.00317	26	0.01520				

9	0.00353	27	0.01641	44	0.03482	62	0.00141
10	0.00392	28	0.01767	45	0.03418	63	0.00087
11	0.00434	29	0.01898	46	0.03320	64	0.00050
12	0.00480	30	0.02034	47	0.03186	65	0.00028
13	0.00530	31	0.02174	48	0,03019	66	0.00014
14	0.00584	32	0.02317	49	0.02821	67	0.00007
15	0.00642	33	0.02461	50	0.02595	68	0.00003
16	0.00705	34	0.02606	51	0.02347	69	0.00001•
17	0.00773	35	0.02748	52	0.02084		
				53	0.01813		

Practical 13.3.7: Suppose the life length random variable is modeled by a distribution with force of mortality as specified below.

$$\mu_s = \begin{cases} 0.04 \; if \; 0 \leq s < 15 \\ 0.08 \; if \; 15 \leq s < 25 \\ 0.12 \; if \; 25 \; \leq s < 35 \\ 0.18 \;\; if \;\;\; s \geq 35. \end{cases}$$

i. Find $A_{30:\frac{1}{5}|}$ .Take δ = 0.06.

ii. Find the net single premium in an n year pure endowment for a benefit of 1000, for n = 1,2,..., 10 and for δ = 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12. Comment on the changes in values as n changes and as d changes.

Solution: The probability density function g(t) of T(30) is given by,

$$g(t) = \begin{cases} e^{-0.12t}, \; if \;\; 0 \leq t < 5 \\ e^{-0.18t+0.3}, \;\; if \;\; t \geq 15 \end{cases}$$

i. By definition $A_{30:\frac{1}{5}|} = v^5 \; {}_5p_{30} = e^{-5\delta} {}_5p_{30} = 0.4066$. Thus, the actuarial present value of benefit of 1 unit in a 5-year pure endowment is 0.4066 when the force of interest is 0.06. It is interpreted as, if (30) signs a 5 year pure endowment contract for a benefit of 1 unit and if he is willing to pay a onetime insurance premium at the time of policy issue then the net single premium or the purchase price is 0.4066.

ii. To find the net single premium for an n year pure endowment for a benefit of 1000, for various values of n and δ, we first find the formula for $A_{30:\frac{1}{n}|} = \nu^{n}$ ${}_{n}p_{30}$ as a function on n and δ. Note that the form of g(t) changes depending on whether t < 5 or t > 5. So the formula for ${}_{n}p_{30}$ and $\bar{A}_{30:\frac{1}{n}|}$ also changes accordingly.

$$ {}_{n}p_{30} = 1 - {}_{n}q_{30} = 1 - \int_0^n g(t)dt = e^{-0.12n}, \quad \text{for } n < 5, $$

$$ = \int_n^\infty g(t)dt = e^{-0.18n+0.3}, \quad \text{for } n \geq 5 $$

Using these formulae for ${}_{n}p_{30}$, we get,

$$ A_{30:\frac{1}{n}|} = e^{-\delta n-0.12n} \quad \text{,for } n < 5, $$

$$ = e^{-\delta n-0.18n+0.3} \quad \text{, for } n \geq 5 $$

Note that $A_{30:\frac{1}{n}|}$ being an integral is a continuous function of n. Thus for n = 5, the value of

$A_{30:\frac{1}{5}|}$ using any one of the above two expressions comes out to be the same. Values of 1000 $A_{30:\frac{1}{5}|}$ for various values of n and δ are reported in Table 13.2.2

Table 13.3.2 $1000A_{30:\frac{1}{5}|}$

n	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
1	843.66	835.27	826.96	818.73	810.58	802.52	794.53	786.63
2	711.77	697.68	683.86	670.32	657.05	644.04	631.28	618.78
3	600.50	582.75	565.53	548.81	532.59	516.85	501.58	486.75
4	506.62	486.75	467.67	449.33	431.71	414.78	398.52	382.89
5	427.41	406.57	386.74	367.88	349.94	332.87	316.64	301.19
6	339.60	319.82	301.19	283.65	267.14	251.58	236.93	223.13

7	269.82	251.58	234.57	218.71	203.93	190.14	177.28	165.30
8	214.38	197.90	182.68	168.64	155.67	143.70	132.66	122.46
9	170.33	155.67	142.27	130.03	118.84	108.61	99.26	90.72
10	135.34	122.46	110.80	100.26	90.72	82.08	74.27	67.21

It is to be noted that as the force of interest increases, values of net single premium decrease, as expected. However, the decrease in net single premium with increase in force of interest is not significant. For example, if the force of interest is doubled, the values of net single premium are not halved. Further as the term of insurance contract increases, the net single premium decreases in contrast to that for an n year term insurance. This is in view of the fact that as the term increases, the chance of survival within the term decreases and the liability of the company also decreases as the benefit is not to be paid before the term ends.

Practical 13.3.8: Suppose the life length random variable is modeled by a distribution with force of mortality as specified below.

$$\mu_s = \begin{cases} 0.04, & if 0 \leq s < 15 \\ 0.08, & if\ 15 \leq s < 25 \\ 0.12, & if\ 25 \leq s > 35 \\ 0.18 & if\ \ s \geq 35\ . \end{cases}$$

Find the net single premium for an n year endowment insurance for a benefit of 1000 to be paid at the moment of death, for n = 1,2,..., 10 and for 8 = 0.05, 0.06, 0.07, 0.08, 0.09, 0.10, 0.11, 0.12. Comment on the changes in values as n changes and as δ changes.

Solution: As stated above the net single premium $\bar{A}_{30:\overline{n|}}$ is given by the formula

$$\bar{A}_{30:\overline{n|}} = \bar{A}^{1}_{30:\,\overline{n|}} + \bar{A}_{30:\overset{\,1}{\overline{n|}}}\ .$$

We thus combine these results to find $\bar{A}_{30:\overline{n|}}$ for various values of n and δ. These are reported in Table 13.3.3

Table 13.3.3 *1000*$\bar{A}_{30:\overline{n|}}$

n	δ=0.05	δ=0.06	δ=0.07	δ=0.08	δ=0.09	δ=0.10	δ=0.11	δ=0.12

1	954.02	945.09	936.25	927.49	918.82	910.24	901.73	893.31
2	915.23	899.23	883.53	868.13	853.02	838.20	823.66	809.39
3	882.50	860.92	839.93	819.52	799.68	780.39	761.62	743.38
4	854.89	828.92	803.88	779.73	756.45	733.99	712.34	691.45
5	831.59	802.19	774.06	747.15	721.40	696.76	673.17	650.60
6	812.50	780.50	750.11	721.24	693.80	667.73	642.94	619.37
7	797.33	763.44	731.45	701.25	672.73	645.78	620.32	596.24
8	785.28	750.02	716.93	685.85	656.65	629.20	603.39	579.10
9	775.71	739.47	705.61	673.97	644.37	616.67	590.72	566.41
10	768.10	731.16	696.80	664.81	635.00	607.19	581.24	557.00

Net single premiums for an n year endowment insurance are high as compared to those for an n year term insurance and for an n year pure endowment because, in an n year endowment insurance, the insurer has to pay the benefit amount, may be before the term ends or if not, definitely at the end of the term. Net single premiums are high for low force of interest and for small terms, as expected.

Year(n)	Term $\bar{A}^1_{30:\,\overline{n}|}$	Pure Endowment $\bar{A}_{30:\frac{1}{\overline{n}|}}$	Endowment $\bar{A}_{30:\overline{n}|}$
1	109.82	835.27	945.09
2	201.55	697.68	899.23
3	278.17	582.75	860.92
4	342.17	486.75	828.92
5	395.62	406.57	802.19
6	460.68	319.82	780.50
7	511.86	251.58	763.44
8	552.12	197.90	750.02
9	583.79	155.67	739.47
10	608.71	122.46	731.16

It is to be noted that for any year n, actuarial present value of benefit 1000 is small for term insurance as compared to that for endowment insurance. Such a difference is reasonable in the light of the nature of assurance in these two products. For term insurance, the insurance company has no liability after the term of the insurance, while in endowment insurance; the insurance company has to pay the benefit amount of 1000 at some time during the term or definitely at the end of the term. Thus for an insurance company, it is essential to have larger funds for a portfolio of endowment insurance products than that for term insurance products.

Practical 13.3.9: Suppose the force of mortality follows Makeham's law given by $\mu_x = A + BC^x$.

i. Find the values of the net single premium for a benefit of 1000 to be paid at the end of the year of death, in whole life insurance for ages 30 to 50 using the backward recurrence relation.

ii. Find the value of the net single premium for a benefit of 1000 to be paid at the end of the year of death, in a 5-year deferred whole life insurance issued to (40). Take A50 = 0.40356, δ = 0.05, A = 0.0007, B = 0.0001151 and C = 1.096.

Solution:

i. Survival function corresponding to Makeham's law is derived in Example 4.2.11. For the specified values of the parameters, we find the survival function S(x) for values of x ranging from 30 to 50 and then the values of p_x for x = 30 to 49. These values are reported in Table 5.4.2. It is given that A_{50} = 0.40356. We then use the backward recurrence relation $A_x = vq_x + vp_xA_{x+1}$ to find net single premium for benefit of 1000 to be paid at the end of the year of death, in whole life insurance for ages 30 to 49. These values are reported in Table 13.3.4

Table 13.3.4 : p_x for Makeham's law

Age x	p_x	Age x	p_x
30	0.9988	40	0.9960
31	0.9986	41	0.9955
32	0.9984	42	0.9950
33	0.9982	43	0.9945
34	0.9980.	44	0.9939
35	0.9977	45	0.9933
36	0.9974	46	0.9926

37	0.9971	47	0.9918
38	0.9968	48	0.9909
39	0.9964	49	0.9900

Table 13.3.5 1000 A_x for Makeham's law

Age x	1000Ax	Age x	1000Ax
30	180.30	41	287.95
31	188.58	42	299.59
32	197.15	43	311.54
33	206.02	44	323.80
34	215.19	45	336.36
35	224.65	46	349.23
36	234.43	47	362.38
37	244.51	48	375.83
38	254.90	49	389.56
39	265.61	50	403.56
40	276.62		

ii. We have to find 1000 $_{5|}A_{40}$. From Example 5.4.1, we have $_{5|}A_{40}= {}_5E_{40}\ A_{45}$. From (i) we have A_{45} = 0.33636. Further, $_5E_{40}= v^5\ {}_5p_{40}$ = 0.759525. Hence 1000 $_{5|}A_{40}$= 255.48.

Practical 13.3.10: Find the present value and the accumulated value of a 10-year annuity immediate of 1000 per annum if the effective rate of interest is 5% for the first 6 years and 6% for the next four years.

Solution: It is to be noted that the rate of interest changes after 6 years, hence the present value of 1 paid at the end of the 7th year is $v_1\ v^6$, where v corresponds to i = 0.05 and v_1 corresponds to i = 0.06. We have to make similar changes for the rest of the years. Thus, the present value is given by

$$10000a_{\bar{n}|} = 1000\left\{\frac{1-v^6}{i} + v^6\{v_1 + v_1^2 + v_1^3 + v_1^4\}\right\} = 7661.41.$$

Accumulated value for the first 6 years will be $S_{\overline{6|}}$ with rate 5% and this will get accumulated for the next four years with rate 6%. Accumulated value for the next 4 years will be $S_{\overline{4|}}$ with rate 6%. Thus, the required answer is

$$1000\{S_{\overline{6|}}\ (1+0.06)^4 + S_{\overline{4|}}\}$$

$$= 1000\ \{[\frac{(1+0.05)^6 - 1}{0.05}]\ [1 + 0.06]^4 + \frac{(1+0.06)^4 - 1}{0.06}]\ \} = 12961.82.$$

Practical 13.3.11: Rs. 3000/- is deposited in a bank on January 1 of each year from 1991 to 1999. What is the accumulated value of this fund on December 31, 1999 at 3% annual rate of interest?

Solution: The term of this annuity due is 9 years. Hence, the accumulated value on December 31,1999 is 3000 $\ddot{S}_{\overline{9|}}$ = 3000 (1.03) ((1.03)^9 – 1)/0.03 = 31391.64.

Practical 13.3.12: Mr. Bapat deposited Rs. 5000 in a savings bank account on July 21 of each year from 1983 to 1998, both years inclusive. On July 21, 2002, he withdrew his savings. Assuming that the rate of interest for all these years is 6%, find the amount withdrawn by Mr. Bapat.

Solution: Mr. Bapat deposited Rs. 5000 16 times starting from 1983 to 1998. Thus, the value of the series of 16 payments on July 21, 1999 is 5000$\ddot{S}_{\overline{16|}}$) = 136064.40. From 1999 to 2002 this amount earned interest at 6% rate, hence the accumulated amount is 5000$\ddot{S}_{\overline{16|}}$) (1.06)3 = 162054.90. Thus, the amount withdrawn by Mr. Bapat is 162054.90.

Practical 13.3.13: A loan of Rs.10 lakhs is taken on January 1, 2000. It has to be repaid in equal monthly installments payable at the beginning of the month for 10 years. Based on a 6% annual rate of interest, determine the amount of the installment.

Solution: Denoting the monthly installment by I, 12I is the annual payment made in 12 parts. Its present value for 10 years is 12I $\ddot{a}_{\overline{10|}}^{(12)}$. Hence, we get the equation as,

$$10{,}00000 = 12I\ \ddot{a}_{\overline{10|}}^{(12)} = 12I\ (7.5972).$$

Solving we have, I = 10969.01 is the monthly installment.

Practical 13.3.14: An annuity of Rs 25000 per annum payable monthly in advance is purchased for 5 years. Find its price if the annual rate of interest is 4%. Find its accumulated value at the end of 5 years.

Solution: Purchase price of the annuity contract is the present value of mthly, with m = 12, annuity due. It is given by

$$25000\ddot{a}_{5}^{(12)} = 25000\,\frac{d}{(d)^{12}}\,\ddot{a}_{\overline{5}|} = 25000\,(0.03846/0.03956)(4.63008) = 112533.40.$$

Further, the corresponding accumulated value is given by,

$$25000\ddot{S}_{5}^{(12)} = 25000\,(1+i)^{5}\ddot{a}_{5}^{(12)} = (1.04)^{5}\ 112533.40 = 136914.10.$$

Practical 13.3.15: An amount of Rs. 10000 is payable on December 31, for 10 years. The first payment is due in 2010. Find the purchase price of this annuity on January1, 2000 with 6 % annual effective rate of interest.

Solution: The first payment is deferred by 10 years. Thus, it is a 10-year deferred,10-year annuity certain immediate. Hence the purchase price is given by,

$$10000\ 10|a_{\overline{10}|} = 10000v^{10}\,a_{\overline{10}|} = 10000\,(1.06)^{-10}\left\{\frac{1-(1.06)^{-10}}{0.06}\right\} = 41098.34.$$

Thus, 41098.34 invested on January 1, 2000 will accumulate to a fund from which payment of 10000 can be made for 10 years from 2010. In other words Rs 41098.34is the purchase price of a 10-year deferred, 10-year annuity certain immediate.

Practical 13.3.16: A general insurance company sells a 1-year insurance of benefit 1000, to be paid at the end of the year. The insurance is to cover damage to the car due to accidents. Suppose the chance of accident in a year is 0.01. Determine the premium P to be paid at the issue of the policy, so that the insurer is indifferent to two options-accepting or not accepting the risk under a utility of wealth function, $u(w) = -e^{-0.007w}$.

Solution: With the expected utility principle to find P, the utility of wealth of the insurer assuming that the insurance is not issued and the expected utility of wealth of the insurer assuming that the insurance is issued, are equal. The premium is found from such an indifference equation. The insurer's utility of wealth assuming that the insurance is not issued is $u(w) = -e^{-0.007w}$. If the insurance is issued and if the accident does not occur, the insurer's wealth will be w + P, with utility $-e^{-0.007\,(w+P)}$. If the accident occurs, the insurer's wealth will be w + P – 1000, with utility $-e^{-0.007(w+P-1000)}$. We are given that the probability of an accident is 0.01, so if insurance is issued, the insurer's expected utility of wealth is $0.99[-e^{-0.007(w+P)}]$ - $0.01/e^{-0.007(w+P-1000)}]$. The indifference equation is therefore given by,

$$-e^{-0.007w} = 0.99[-e^{-0.007(w+P)}] - 0.01[e^{-0.007(w+P-1000)}].$$

This equation simplifies to $P = (1000/7)\log_e(0.99+0.01e^7) = 354.47$. Thus, 354.47 is the 7 annual premium for a benefit of Rs.1000/-.

Practical 13.3.17: An insurance company offers a whole life insurance product with a specified benefit amount to be paid at the end of the year of death. If death occurs during a term of n-years then along with the death benefit, the bonus is also paid.

If death occurs after the end of the term, no bonus is paid. The amount of bonus is not fixed a priori. At the end of each year, depending on the reserve calculations and after accounting for the company's expenses and profits, the bonus is determined. A loading on premium is charged to pay for the bonus. The premium loading is 2.25 per thousand benefit if the term is 5 to 9 years, Rs. 1.5 per thousand benefit if the term is 10 to 14 years, Rs. 1.25 per thousand if the term is 15 to 19 years, Rs.1.15 per thousand benefit if the term is 20 to 24 years and Rs. 1 per thousand benefit if the term is 25 years or more. There is a rebate on premium, with the rate of rebate depending on the sum assured. The rebate is Rs.1 per thousand, if the benefit amount is more than or equal to 1.5 lakhs but less than 5 lakhs, Rs. 1.5 per thousand, if the benefit amount is more than or equal to 5 lakhs but less than 10 lakhs, Rs. 1.75 per thousand, if the benefit amount is more than or equal to 10 lakhs. Suppose (40) signs this contract with premiums payable as an n-year temporary annuity due. Determine the annual premium in the following two setups. i. benefit amount Rs.2 lakhs and premium paying period of 5 years and ii. benefit amount Rs.5 lakhs and premium paying period of 10 years. Assume that life length of (40) is modeled by Gompertz's law with force of mortality $\mu_x = BC^x$, x > 0 with B = 0.0001151 and C = 1.096 and the force of interest is 0.05.

Solution: In this contract the actuarial present value of the premiums is given by G 240:元), where G denotes the annual premium. This premium has to take into account the bonus payments and the rebate given on the premiums. The bonus amount is available only through premium loading. Thus, the actuarial present value of the premiums paid should equal the actuarial present value of the benefit, allowance for the bonus and rebate. The actuarial present value of the benefit of b amount is b A_{40}. If c denotes the loading for bonus per 1000, then loading for bonus for benefit b will be c (b/1000). It is payable at the beginning of each year for at most n years. Hence, its actuarial present value is c (b/1000) $\ddot{a}_{40:\overline{n|}}$. Suppose r denotes the rebate on the premium, when benefit is of 1000. For b benefit rebate will be 1000, which again r (b/1000) forms an n-year temporary life annuity due. Hence, its actuarial present value is r (b/1000) $\ddot{a}_{40:\overline{n|}}$. Hence, the equivalence relation to determine G is given by,

$$G\,\ddot{a}_{40:\overline{n|}} = bA_{40} + c\,(b/1000)\ddot{a}_{40:\overline{n|}} + r\,(b/1000)\ddot{a}_{40:\overline{n|}}$$

Hence, G is given by,

$$G = bA_{40}/\ddot{a}_{40:\overline{n|}} + c\,(b/1000) + r\,(b/1000)$$

For Gompertz's law with force of mortality $\mu_x = BC^x$, with B = 0.0001151 and C = 1.096 and the force of interest 0.05, we have computed in Example 13.4.4 and

$$A_{40} = 0.27673,\ \ddot{a}_{40:\overline{5|}} = 4.4912 \text{ and } \ddot{a}_{40:\overline{10|}} = 7.8713$$

Using these values, we find premium in (i) and (ii). Thus,

(i) With b = 2,00,000 and n = 5, G = 12973.21.

(ii) With b = 5,00,000 and n = 10, G = 19078.42.

Practical 13.3.18: $_kL$ is the prospective loss random variable for a fully discrete life insurance of 1000 issued to (x). It is given that $A_x = 0.125$, $A_{x+k} = 0.4$, $2A_{x+k} = 0.2$. d = 0.05. Assume that premiums are calculated on the basis of the equivalence principle.

a. Calculate $E(_kL)$, $Var(_kL)$ and the aggregate reserve at time k for 100 policies of this type.

b. An insurer has 100 policies of this type at time k. Assume the losses on the100 policies are independent. Use the approximation that the aggregate loss is normally distributed in (i) and (ii) below.

   i. Calculate the aggregate amount the insurer must have at time k so that the probability of meeting its obligations for these policies is 0.95.

   ii. The premium can be changed at time k. Calculate the level annual premium the insurer must charge beginning at time k on each policy so that together with the reserve calculated in (a), the probability of meeting its obligations for these policies is 0.95.

Solution:

a. From the given information

$$\ddot{a}_x = (1 - A_x) / d = 0.875 / 0.05 = 17.5, \text{ hence } 1000P_x = (1000A_x) / \ddot{a}_x = 7.142860.$$

Further,

$$\ddot{a}_{x+k} = (1 - A_{x+k}) / d = 0.6/0.05 = 12.$$

By definition,

$$E({}_kL) = 1000A_{x+k} - 1000P\ \ddot{a}_{x+k} = 314.2857,$$

$$Var({}_kL) = 1000^2\ (1+(P_x/d))^2\ [\ {}^2A_{x+k} - A_{x+k}^2] = 52,\ 244.898$$

and reserve for 100 policies is100E(${}_kL$) = 31428.57.

b. i. Suppose, $S = \sum_{j=1}^{100} 1_kL^{(j)}$. Then,

$$\mu = E(S) = 100E({}_kL) = 31428.57,\ \sigma^2 = Var(S) = 100Var({}_kL) = 5224,\ 489.8.$$

By normal approximation, P[S > c] = 0.05 gives, (c-μ)/σ = 1.645 and hence, c = 35, 188.57 is the amount that the insurer must have at time k so that the probability of meeting its obligations for these policies is 0.95.

ii. Suppose P is the premium to be determined for the insurance of 1000units. Then,

$$E({}_k\hat{L}) = 1000A_{x+k} - \hat{P}\ \ \ddot{a}_{x+k} = 400 - 12\hat{P}.$$

$$Var({}_k\hat{L})) = (100+(\hat{P}/d))^2\ ({}^2A_{x+k} - A_{x+k}^2) = (200 + 4\hat{P})^2.$$

Suppose $\hat{S} = \sum_{j=1}^{100} 1({}_k\hat{L}^{(j)})$, then

$$\mu = E(\hat{S}) = 100E(\ {}_k\hat{L}) = 40{,}000 - 1200\hat{P} \text{ and } \hat{\sigma}^2 = Var(s) = 100(200 + 4\hat{P})^2.$$

Again, by normal approximation,

$$\hat{c}=\hat{\mu}+1.645\hat{\sigma} \quad \text{but by (a) } \hat{S} = 100\ E({}_kL) = 31428.57.$$

Solving $\hat{c}=\hat{\mu}+1.645\hat{\sigma}$, we get $\hat{P}$ = 10.46.

13.4 Summary

Practical problems in actuarial science often revolve around future lifetime random variables, which model the uncertainty of an individual's remaining lifetime. These variables are fundamental in constructing life tables, which provide statistical data on mortality and survival rates for various ages. Life tables are crucial in pricing and evaluating life insurance and life annuity products. Life insurance involves payments upon death, while life annuities provide periodic income while the individual is alive—both requiring careful calculation based on survival probabilities and interest rates. Determining premiums—the payments policyholders make to insurers—requires balancing expected benefits and company profitability, often using the equivalence principle. Meanwhile, reserves represent the funds insurers must hold to ensure they can meet future obligations, accounting for the timing and amount of future claims and benefits. Together, these components form the backbone of actuarial models used in real-world financial decision-making for risk management and product pricing.

13.5 References

- Sinha, S. K. and Kale, B. K. (1983): Life Testing and Reliability Estimation, Wiley Eastern Limited.
- Zacks, S. (1992). Introduction to Reliability Analysis: Probability Models and Statistics Methods, Springer-Verlag, New York, 1992.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.

13.6 Further Readings

- Nelson, W. (1982): Applied Life Data Analysis, John Willey.
- Deshmukh, S. R. (2009). Actuarial statistics: An introduction using R. Universities Press (India). Hyderabad.
- Mann, N.R., Schafer, R.E. and Singpurwala, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data, John Wiley, New York.